Sato-Tate distributions

Andrew V. Sutherland

Massachusetts Institute of Technology

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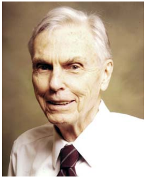
Mikio Sato



John Tate

Joint work with F. Fité, K.S. Kedlaya, and V. Rotger (part 1), and D. Harvey (part 2).

Sato-Tate in dimension 1

Let $E/\mathbb{Q}$ be an elliptic curve, which we can write in the form

$$y^2 = x^3 + ax + b,$$

and let p be a prime of good reduction ($4a^3 + 27b^2 \not\equiv 0 \bmod p$).

The number of $\mathbb{F}_p$-points on the reduction E_p of E modulo p is

$$\#E_p(\mathbb{F}_p) = p + 1 - t_p,$$

where the trace of Frobenius $t_p \in \mathbb{Z}$ lies in the interval $[-2\sqrt{p}, 2\sqrt{p}]$.

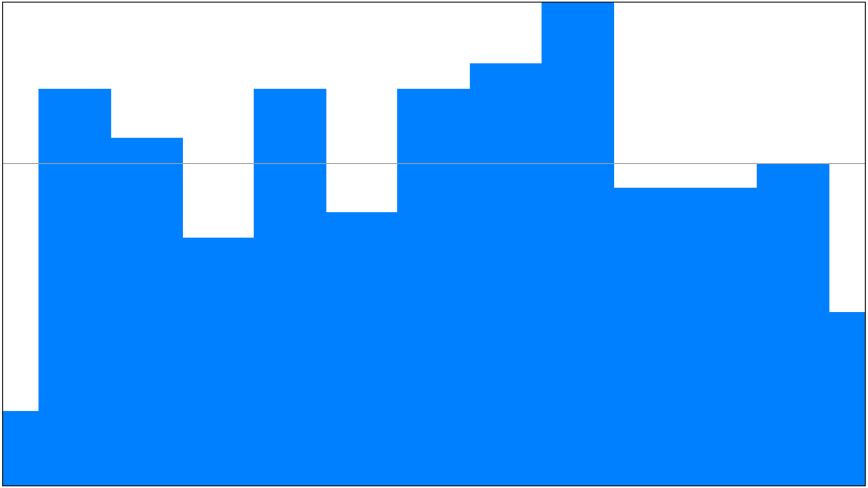
We are interested in the limiting distribution of $x_p = -t_p/\sqrt{p} \in [-2, 2]$, as p varies over primes of good reduction up to N, as $N \to \infty$.

Example: $y^2 = x^3 + x + 1$

p	t_p	x_p	p	t_p	x_p	p	t_p	x_p
3	0	**0.000000**	71	13	**−1.542816**	157	−13	**1.037513**
5	−3	**1.341641**	73	2	**−0.234082**	163	−25	**1.958151**
7	3	**−1.133893**	79	−6	**0.675053**	167	24	**−1.857176**
11	−2	**0.603023**	83	−6	**0.658586**	173	2	**−0.152057**
13	−4	**1.109400**	89	−10	**1.059998**	179	0	**0.000000**
17	0	**0.000000**	97	1	**−0.101535**	181	−8	**0.594635**
19	−1	**0.229416**	101	−3	**0.298511**	191	−25	**1.808937**
23	−4	**0.834058**	103	17	**−1.675060**	193	−7	**0.503871**
29	−6	**1.114172**	107	3	**−0.290021**	197	−24	**1.709929**
37	−10	**1.643990**	109	−13	**1.245174**	199	−18	**1.275986**
41	7	**−1.093216**	113	−11	**1.034793**	211	−11	**0.757271**
43	10	**−1.524986**	127	2	**−0.177471**	223	−20	**1.339299**
47	−12	**1.750380**	131	4	**−0.349482**	227	0	**0.000000**
53	−4	**0.549442**	137	12	**−1.025229**	229	−2	**0.132164**
59	−3	**0.390567**	139	14	**−1.187465**	233	−3	**0.196537**
61	12	**−1.536443**	149	14	**−1.146925**	239	−22	**1.423062**
67	12	**−1.466033**	151	−2	**0.162758**	241	22	**−1.417145**

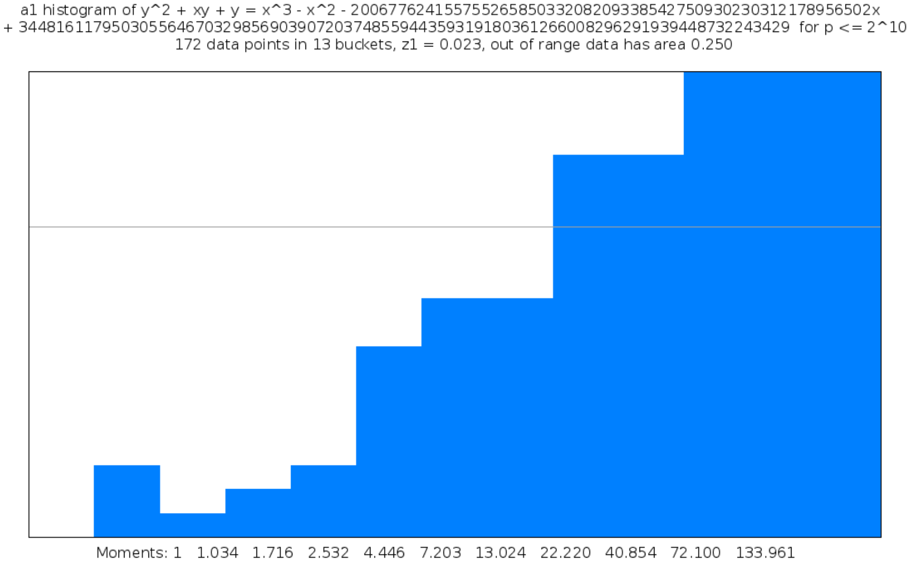
`http://math.mit.edu/~drew/g1SatoTateDistributions.html`

a1 histogram of y^2 = x^3 + x + 1 for p <= 2^10
170 data points in 13 buckets, z1 = 0.029, out of range data has area 0.018



Moments: 1   0.051   1.039   0.081   2.060   0.294   4.971   1.134   13.278   4.308   37.954

click histogram to animate (requires adobe reader)



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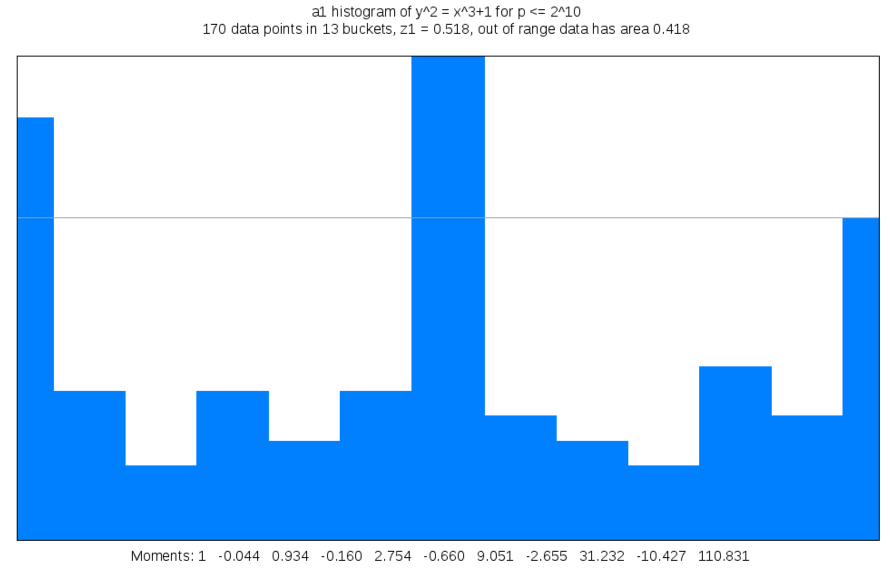
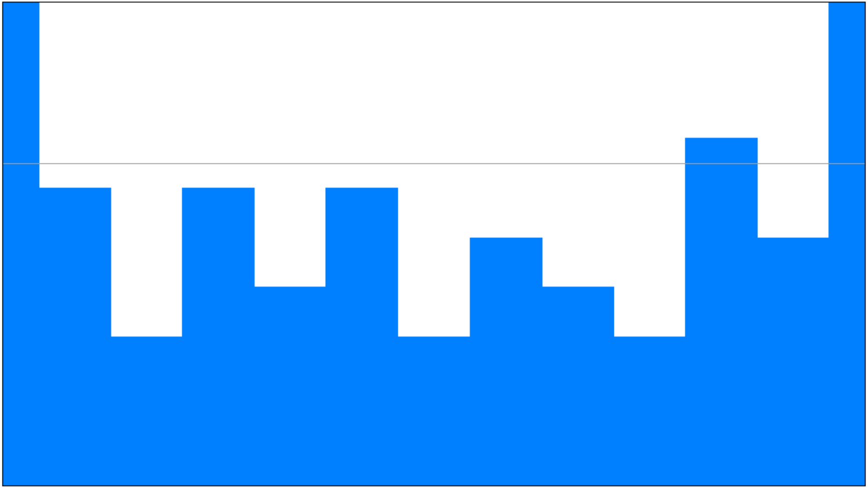


click histogram to animate (requires adobe reader)

a1 histogram of y^2 = x^3+1 over Q(sqrt(-3)) for split p <= 2^10
164 data points in 13 buckets, out of range data has area 0.122



Moments: 1 -0.092 1.935 -0.331 5.710 -1.368 18.765 -5.504 64.750 -21.616 229.771

click histogram to animate (requires adobe reader)

Sato-Tate distributions in dimension 1

1. Typical case (no CM)

Elliptic curves $E/\mathbb{Q}$ without CM have the semicircular trace distribution. (This is also known for E/k, where k is a totally real number field).

[Barnet-Lamb, Clozel, Geraghty, Harris, Shepherd-Barron, Taylor]

2. Exceptional cases (CM)

Elliptic curves E/k with CM have one of two distinct trace distributions, depending on whether k contains the CM field or not.

[classical (Hecke, Deuring)]

Sato-Tate groups in dimension 1

The *Sato-Tate group* of E is a closed subgroup G of $\mathrm{SU}(2) = \mathrm{USp}(2)$ derived from the ℓ-adic Galois representation attached to E.

The refined Sato-Tate conjecture implies that the distribution of normalized traces of E_p converges to the distribution of traces in the Sato-Tate group of G, under the Haar measure.

G	G/G^0	E	k	$\mathrm{E}[a_1^0], \mathrm{E}[a_1^2], \mathrm{E}[a_1^4] \ldots$
$\mathrm{U}(1)$	C_1	$y^2 = x^3 + 1$	$\mathbb{Q}(\sqrt{-3})$	$1, 2, 6, 20, 70, 252, \ldots$
$N(\mathrm{U}(1))$	C_2	$y^2 = x^3 + 1$	$\mathbb{Q}$	$1, 1, 3, 10, 35, 126, \ldots$
$\mathrm{SU}(2)$	C_1	$y^2 = x^3 + x + 1$	$\mathbb{Q}$	$1, 1, 2, 5, 14, 42, \ldots$

In dimension 1 there are three possible Sato-Tate groups, two of which arise for elliptic curves defined over $\mathbb{Q}$.

Zeta functions and L-polynomials

Let $C/\mathbb{Q}$ be a nice curve of genus g and p a prime of good reduction. Define the *zeta function*

$$Z_p(T) := \exp\left(\sum_{r=1}^{\infty} N_r T^r / r\right),$$

where $N_r = \#C_p(\mathbb{F}_{p^r})$. This is a rational function of the form

$$Z_p(T) = \frac{L_p(T)}{(1-T)(1-pT)},$$

where $L_p(T)$ is an integer polynomial of degree $2g$.

For $g = 1$ we have $L_p(t) = pT^2 + c_1 T + 1$, and for $g = 2$,

$$L_p(T) = p^2T^4 + c_1 pT^3 + c_2 T^2 + c_1 T + 1.$$

Normalized L-polynomials

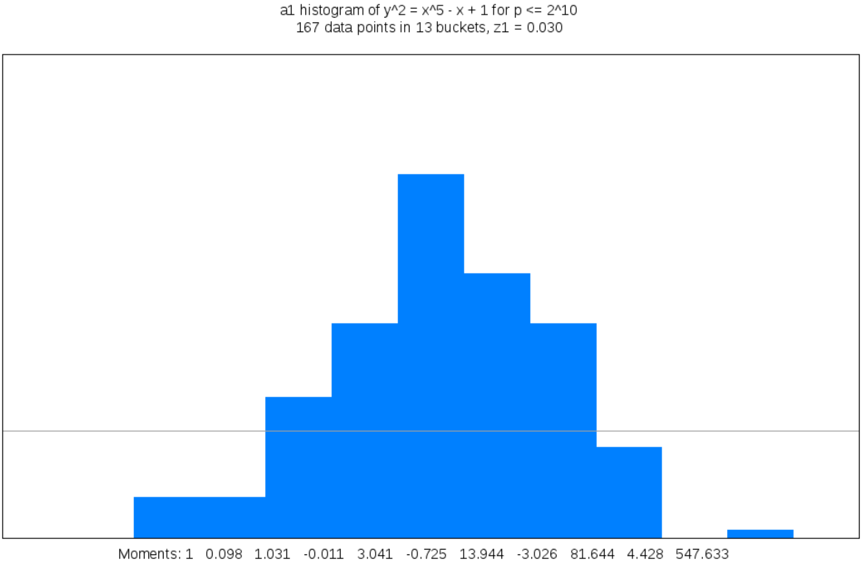
The normalized L-polynomial

$$\bar{L}_p(T) := L_p(T/\sqrt{p}) = \sum_{i=0}^{2g} a_i T^i \in \mathbb{R}[T]$$

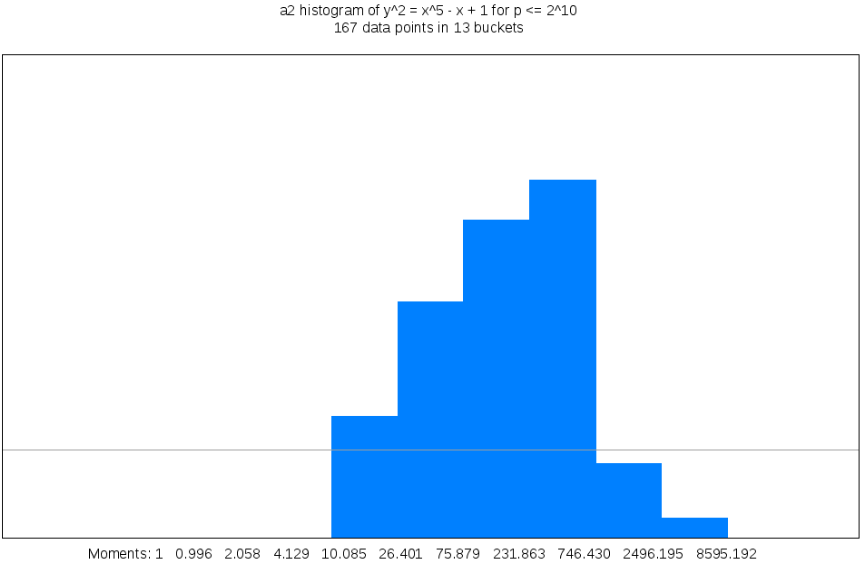
is monic, reciprocal ($a_i = a_{2g-i}$), and unitary (roots on the unit circle). The coefficients a_i satisfy the Weil bounds $|a_i| \le \binom{2g}{i}$.

We now consider the limiting distribution of $a_1, a_2, \ldots, a_g$ over all primes $p \le N$ of good reduction, as $N \to \infty$.

`http://math.mit.edu/~drew/g2SatoTateDistributions.html`



click histogram to animate (requires adobe reader)



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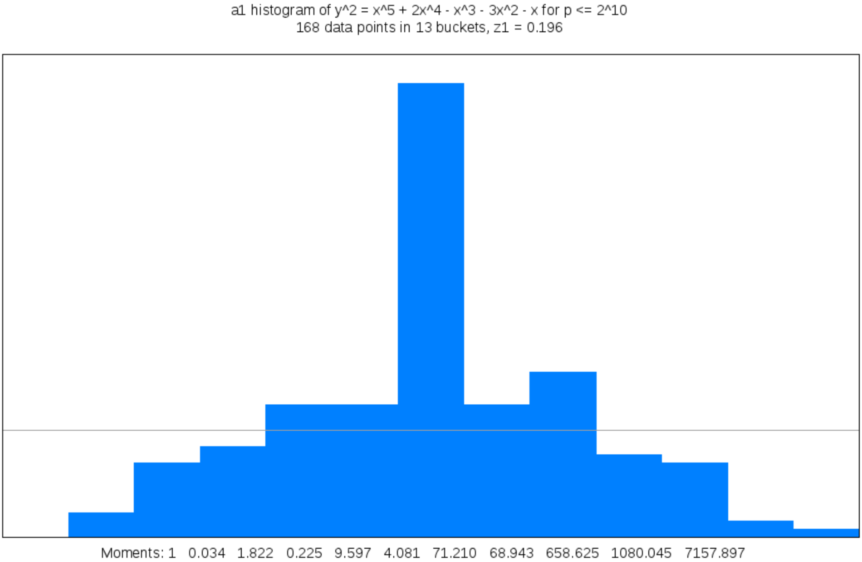

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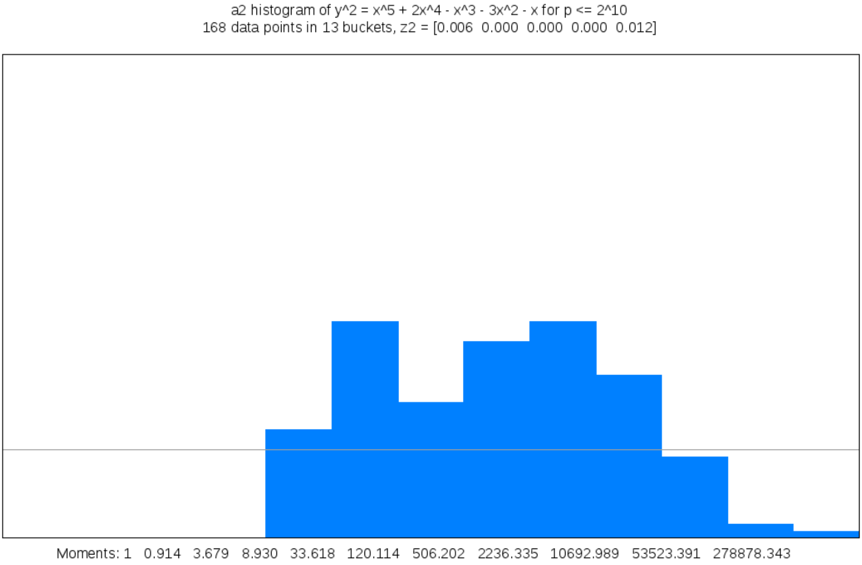

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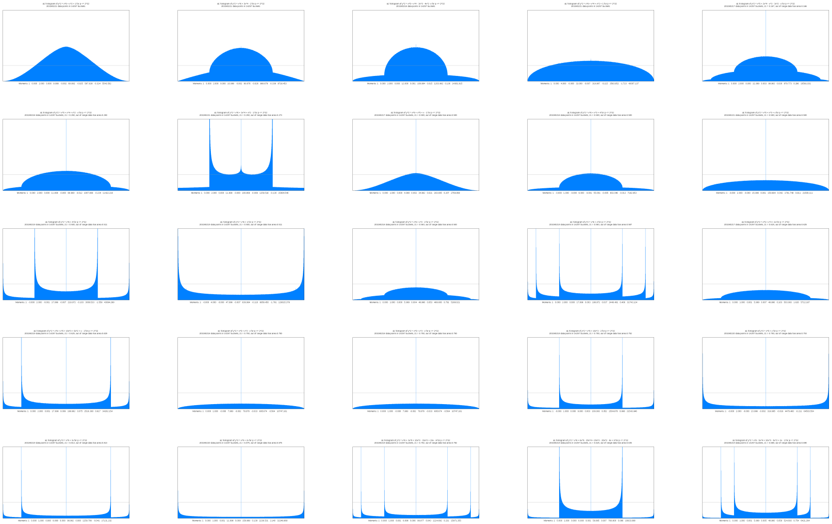
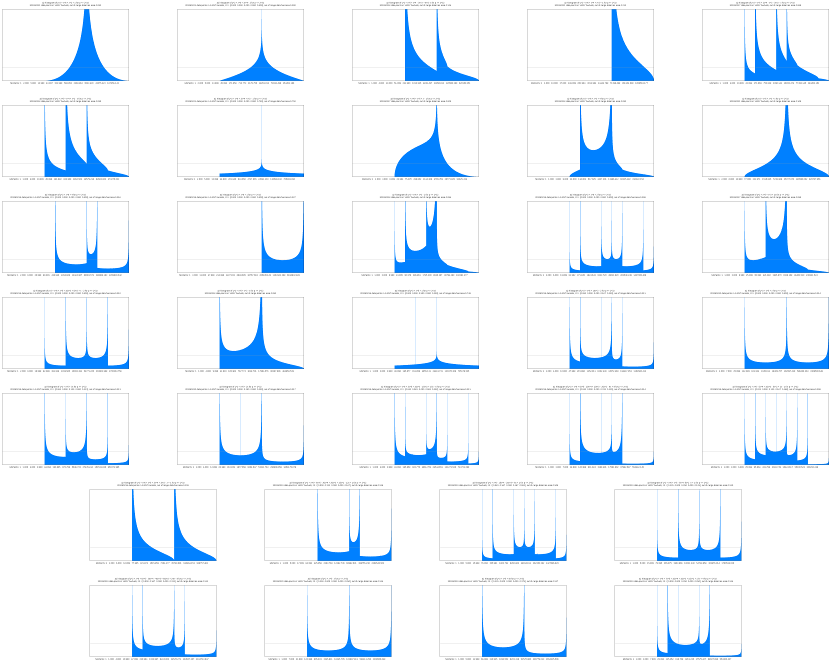
Exceptional distributions for abelian surfaces over $\mathbb{Q}$:





L-polynomials of Abelian varieties

Let A be an abelian variety of dimension $g \geq 1$ over a number field k, and let us fix a prime ℓ.

Let $\rho_\ell \colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ be the Galois representation arising from the action of $G_k := \mathrm{Gal}(\bar{k}/k)$ on the ℓ-adic Tate module

$$V_\ell(A) := \varprojlim A[\ell^n] \otimes \mathbb{Q}.$$

For each prime $\mathfrak{p}$ of good reduction for A we have the *L-polynomial*

$$L_\mathfrak{p}(T) := \det(1 - \rho_\ell(\mathrm{Frob}_\mathfrak{p})T),$$
$$\bar{L}_\mathfrak{p}(T) := L_\mathfrak{p}(T/\sqrt{\|\mathfrak{p}\|}) = \sum a_i T^i.$$

When A is the Jacobian of a genus g curve C, this agrees with our earlier definition of $L_\mathfrak{p}(T)$ as the numerator of the zeta function $Z_\mathfrak{p}(T)$.

The Sato-Tate problem for an abelian variety

The $\bar{L}_{\mathfrak{p}} \in \mathbb{R}[T]$ are monic, symmetric, unitary polynomials of degree $2g$.

Every such polynomial arises as the characteristic polynomial of a conjugacy class in the unitary symplectic group $\mathrm{USp}(2g)$.

Each probability measure on $\mathrm{USp}(2g)$ determines a distribution of conjugacy classes (hence a distribution of characteristic polynomials).

The *Sato-Tate problem*, in its simplest form, is to find a measure for which these classes are equidistributed.

Conjecturally, such a measure arises as the Haar measure of a compact subgroup ST_A of $\mathrm{USp}(2g)$.

The Sato-Tate group

Recall that the action of G_k on $V_\ell(A)$ induces the representation

$$\rho_\ell \colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell).$$

Let $G_\ell^{1,\mathsf{zar}}$ denote the kernel of the similitude character of $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ on the Zariski closure of $\rho_\ell(G_k)$. Now fix $\iota \colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, and define ST_A to be a maximal compact subgroup of the image $G_\ell^{1,\mathsf{zar}}$ under

$$\mathrm{Sp}_{2g}(\mathbb{Q}_\ell) \xrightarrow{\otimes_\iota \mathbb{C}} \mathrm{Sp}_{2g}(\mathbb{C}).$$

Conjecturally, ST_A does not depend on ℓ or ι; this is known for $g \leq 3$.

Definition [Serre]

$\mathrm{ST}_A \subseteq \mathrm{USp}(2g)$ is the *Sato-Tate group* of A.

The refined Sato-Tate conjecture

Let $s(\mathfrak{p})$ denote the conjugacy class of the image of $\mathrm{Frob}_{\mathfrak{p}}$ in ST_A.

Let μ_{ST_A} denote the image of the Haar measure on $\mathrm{Conj}(\mathrm{ST}_A)$, which does not depend on the choice of ℓ or ι.

Conjecture

The conjugacy classes $s(\mathfrak{p})$ are equidistributed with respect to μ_{ST_A}.

In particular, the distribution of $\bar{L}_{\mathfrak{p}}(T)$ matches the distribution of characteristic polynomials of random matrices in ST_A.

We can test this numerically by comparing statistics of the coefficients $a_1, \ldots, a_g$ of $\bar{L}_{\mathfrak{p}}(T)$ over $\|\mathfrak{p}\| \leq N$ to the predictions given by μ_{ST_A}.

`https://hensel.mit.edu:8000/home/pub/6`

The Sato-Tate axioms

The Sato-Tate axioms for abelian varieties (weight-1 motives):

1. G is closed subgroup of $\mathrm{USp}(2g)$.
2. **Hodge condition**: G contains a Hodge circle[1] whose conjugates generate a dense subset of G.
3. **Rationality condition**: for each component H of G and each irreducible character χ of $\mathrm{GL}_{2g}(\mathbb{C})$ we have $\mathrm{E}[\chi(\gamma) : \gamma \in H] \in \mathbb{Z}$.

For any fixed g, the set of subgroups $G \subseteq \mathrm{USp}(2g)$ that satisfy the *Sato-Tate axioms* is **finite** up to conjugacy (3 for $g = 1$, 55 for $g = 2$).

[1] An embedding $\theta\colon \mathrm{U}(1) \to G^0$ where $\theta(u)$ has eigenvalues u, u^{-1} with multiplicity g.

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Theorem

For $g \leq 3$, the group ST_A satisfies the Sato-Tate axioms.

This is expected to hold for all g.

[1] An embedding $\theta\colon \mathrm{U}(1) \to G^0$ where $\theta(u)$ has eigenvalues u, u^{-1} with multiplicity g.

Galois endomorphism modules

Let A be an abelian variety defined over a number field k.
Let K be the minimal extension of k in $\bar{k}$ for which $\mathrm{End}(A_K) = \mathrm{End}(A_{\bar{k}})$.
$\mathrm{Gal}(K/k)$ acts on the $\mathbb{R}$-algebra $\mathrm{End}(A_K)_{\mathbb{R}} := \mathrm{End}(A_K) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition

The *Galois (endomorphism module) type* of A is the isomorphism class of $[\mathrm{Gal}(K/k), \mathrm{End}(A_K)_{\mathbb{R}}]$, where $[G, E] \simeq [G', E']$ iff there are isomorphisms $G \simeq G'$ and $E \simeq E'$ that are compatible with the Galois action.

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Theorem [FKRS 2012]

For abelian varieties A/k of dimension $g \leq 3$ there is a one-to-one correspondence between Sato-Tate groups and Galois types.

More precisely, the identity component ST_A^0 is determined by $\mathrm{End}(A_K)_{\mathbb{R}}$, and there is a natural isomorphism $\mathrm{ST}_A/\mathrm{ST}_A^0 \simeq \mathrm{Gal}(K/k)$.

Real endomorphism algebras of abelian surfaces

abelian surface	$\mathbf{End}(A_K)_\mathbb{R}$	$\mathbf{ST}_A^0$
square of CM elliptic curve	$\mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1)_2$
• QM abelian surface • square of non-CM elliptic curve	$\mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2)_2$
• CM abelian surface • product of CM elliptic curves	$\mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1)$
product of CM and non-CM elliptic curves	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2)$
• RM abelian surface • product of non-CM elliptic curves	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
generic abelian surface	$\mathbb{R}$	$\mathrm{USp}(4)$

(factors in products are assumed to be non-isogenous)

Sato-Tate groups in dimension 2

Theorem [Fité-Kedlaya-Rotger-S 2012]

Up to conjugacy, 55 subgroups of $\mathrm{USp}(4)$ satisfy the Sato-Tate axioms:

$$\begin{array}{rl}
\mathrm{U}(1)_2\colon & C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O, \\
& J(C_1), J(C_2), J(C_3), J(C_4), J(C_6), \\
& J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O), \\
& C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1 \\
\mathrm{SU}(2)_2\colon & E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6) \\
\mathrm{U}(1)\times\mathrm{U}(1)\colon & F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c} \\
\mathrm{U}(1)\times\mathrm{SU}(2)\colon & \mathrm{U}(1)\times\mathrm{SU}(2),\ N(\mathrm{U}(1)\times\mathrm{SU}(2)) \\
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Of these, exactly 52 arise as ST_A for an abelian surface A (34 over $\mathbb{Q}$).

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& C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1 \\
\mathrm{SU}(2)_2: & E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6) \\
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\end{array}$$

Of these, exactly 52 arise as ST_A for an abelian surface A (34 over $\mathbb{Q}$).

This theorem says nothing about equidistribution, however this is now known in many special cases [FS 2012, Johansson 2013].

Sato-Tate groups in dimension 2 with $G^0 = \mathrm{U}(1)_2$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
1	1	C_1	C_1	0	$0, 0, 0, 0, 0$	$8, 96, 1280, 17920$	$4, 18, 88, 454$
1	2	C_2	C_2	1	$0, 0, 0, 0, 0$	$4, 48, 640, 8960$	$2, 10, 44, 230$
1	3	C_3	C_3	0	$0, 0, 0, 0, 0$	$4, 36, 440, 6020$	$2, 8, 34, 164$
1	4	C_4	C_4	1	$0, 0, 0, 0, 0$	$4, 36, 400, 5040$	$2, 8, 32, 150$
1	6	C_6	C_6	1	$0, 0, 0, 0, 0$	$4, 36, 400, 4900$	$2, 8, 32, 148$
1	4	D_2	D_2	3	$0, 0, 0, 0, 0$	$2, 24, 320, 4480$	$1, 6, 22, 118$
1	6	D_3	D_3	3	$0, 0, 0, 0, 0$	$2, 18, 220, 3010$	$1, 5, 17, 85$
1	8	D_4	D_4	5	$0, 0, 0, 0, 0$	$2, 18, 200, 2520$	$1, 5, 16, 78$
1	12	D_6	D_6	7	$0, 0, 0, 0, 0$	$2, 18, 200, 2450$	$1, 5, 16, 77$
1	2	$J(C_1)$	C_2	1	$1, 0, 0, 0, 0$	$4, 48, 640, 8960$	$1, 11, 40, 235$
1	4	$J(C_2)$	D_2	3	$1, 0, 0, 0, 1$	$2, 24, 320, 4480$	$1, 7, 22, 123$
1	6	$J(C_3)$	C_6	3	$1, 0, 0, 2, 0$	$2, 18, 220, 3010$	$1, 5, 16, 85$
1	8	$J(C_4)$	$\mathrm{C}_4 \times \mathrm{C}_2$	5	$1, 0, 2, 0, 1$	$2, 18, 200, 2520$	$1, 5, 16, 79$
1	12	$J(C_6)$	$\mathrm{C}_6 \times \mathrm{C}_2$	7	$1, 2, 0, 2, 1$	$2, 18, 200, 2450$	$1, 5, 16, 77$
1	8	$J(D_2)$	$\mathrm{D}_2 \times \mathrm{C}_2$	7	$1, 0, 0, 0, 3$	$1, 12, 160, 2240$	$1, 5, 13, 67$
1	12	$J(D_3)$	D_6	9	$1, 0, 0, 2, 3$	$1, 9, 110, 1505$	$1, 4, 10, 48$
1	16	$J(D_4)$	$\mathrm{D}_4 \times \mathrm{C}_2$	13	$1, 0, 2, 0, 5$	$1, 9, 100, 1260$	$1, 4, 10, 45$
1	24	$J(D_6)$	$\mathrm{D}_6 \times \mathrm{C}_2$	19	$1, 2, 0, 2, 7$	$1, 9, 100, 1225$	$1, 4, 10, 44$
1	2	$C_{2,1}$	C_2	1	$0, 0, 0, 0, 1$	$4, 48, 640, 8960$	$3, 11, 48, 235$
1	4	$C_{4,1}$	C_4	3	$0, 0, 2, 0, 0$	$2, 24, 320, 4480$	$1, 5, 22, 115$
1	6	$C_{6,1}$	C_6	3	$0, 2, 0, 0, 1$	$2, 18, 220, 3010$	$1, 5, 18, 85$
1	4	$D_{2,1}$	D_2	3	$0, 0, 0, 0, 2$	$2, 24, 320, 4480$	$2, 7, 26, 123$
1	8	$D_{4,1}$	D_4	7	$0, 0, 2, 0, 2$	$1, 12, 160, 2240$	$1, 4, 13, 63$
1	12	$D_{6,1}$	D_6	9	$0, 2, 0, 0, 4$	$1, 9, 110, 1505$	$1, 4, 11, 48$
1	6	$D_{3,2}$	D_3	3	$0, 0, 0, 0, 3$	$2, 18, 220, 3010$	$2, 6, 21, 90$
1	8	$D_{4,2}$	D_4	5	$0, 0, 0, 0, 4$	$2, 18, 200, 2520$	$2, 6, 20, 83$
1	12	$D_{6,2}$	D_6	7	$0, 0, 0, 0, 6$	$2, 18, 200, 2450$	$2, 6, 20, 82$
1	12	T	A_4	3	$0, 0, 0, 0, 0$	$2, 12, 120, 1540$	$1, 4, 12, 52$
1	24	O	S_4	9	$0, 0, 0, 0, 0$	$2, 12, 100, 1050$	$1, 4, 11, 45$
1	24	O_1	S_4	15	$0, 0, 6, 0, 6$	$1, 6, 60, 770$	$1, 3, 8, 30$
1	24	$J(T)$	$\mathrm{A}_4 \times \mathrm{C}_2$	15	$1, 0, 0, 8, 3$	$1, 6, 60, 770$	$1, 3, 7, 29$
1	48	$J(O)$	$\mathrm{S}_4 \times \mathrm{C}_2$	33	$1, 0, 6, 8, 9$	$1, 6, 50, 525$	$1, 3, 7, 26$

Sato-Tate groups in dimension 2 with $G^0 \neq \mathrm{U}(1)_2$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
3	1	E_1	C_1	0	$0, 0, 0, 0, 0$	$4, 32, 320, 3584$	$3, 10, 37, 150$
3	2	E_2	C_2	1	$0, 0, 0, 0, 0$	$2, 16, 160, 1792$	$1, 6, 17, 78$
3	3	E_3	C_3	0	$0, 0, 0, 0, 0$	$2, 12, 110, 1204$	$1, 4, 13, 52$
3	4	E_4	C_4	1	$0, 0, 0, 0, 0$	$2, 12, 100, 1008$	$1, 4, 11, 46$
3	6	E_6	C_6	1	$0, 0, 0, 0, 0$	$2, 12, 100, 980$	$1, 4, 11, 44$
3	2	$J(E_1)$	C_2	1	$0, 0, 0, 0, 0$	$2, 16, 160, 1792$	$2, 6, 20, 78$
3	4	$J(E_2)$	D_2	3	$0, 0, 0, 0, 0$	$1, 8, 80, 896$	$1, 4, 10, 42$
3	6	$J(E_3)$	D_3	3	$0, 0, 0, 0, 0$	$1, 6, 55, 602$	$1, 3, 8, 29$
3	8	$J(E_4)$	D_4	5	$0, 0, 0, 0, 0$	$1, 6, 50, 504$	$1, 3, 7, 26$
3	12	$J(E_6)$	D_6	7	$0, 0, 0, 0, 0$	$1, 6, 50, 490$	$1, 3, 7, 25$
2	1	F	C_1	0	$0, 0, 0, 0, 0$	$4, 36, 400, 4900$	$2, 8, 32, 148$
2	2	F_a	C_2	0	$0, 0, 0, 0, 1$	$3, 21, 210, 2485$	$2, 6, 20, 82$
2	2	F_c	C_2	1	$0, 0, 0, 0, 0$	$2, 18, 200, 2450$	$1, 5, 16, 77$
2	2	F_{ab}	C_2	1	$0, 0, 0, 0, 1$	$2, 18, 200, 2450$	$2, 6, 20, 82$
2	4	F_{ac}	C_4	3	$0, 0, 2, 0, 1$	$1, 9, 100, 1225$	$1, 3, 10, 41$
2	4	$F_{a,b}$	D_2	1	$0, 0, 0, 0, 3$	$2, 12, 110, 1260$	$2, 5, 14, 49$
2	4	$F_{ab,c}$	D_2	3	$0, 0, 0, 0, 1$	$1, 9, 100, 1225$	$1, 4, 10, 44$
2	8	$F_{a,b,c}$	D_4	5	$0, 0, 2, 0, 3$	$1, 6, 55, 630$	$1, 3, 7, 26$
4	1	G_4	C_1	0	$0, 0, 0, 0, 0$	$3, 20, 175, 1764$	$2, 6, 20, 76$
4	2	$N(G_4)$	C_2	0	$0, 0, 0, 0, 1$	$2, 11, 90, 889$	$2, 5, 14, 46$
6	1	G_6	C_1	0	$0, 0, 0, 0, 0$	$2, 10, 70, 588$	$2, 5, 14, 44$
6	2	$N(G_6)$	C_2	1	$0, 0, 0, 0, 0$	$1, 5, 35, 294$	$1, 3, 7, 23$
10	1	$\mathrm{USp}(4)$	C_1	0	$0, 0, 0, 0, 0$	$1, 3, 14, 84$	$1, 2, 4, 10$

Genus 2 curves realizing Sato-Tate groups with $G^0 = \mathrm{U}(1)_2$

Group	Curve $y^2 = f(x)$	k	K
C_1	$x^6 + 1$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3})$
C_2	$x^5 - x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2})$
C_3	$x^6 + 4$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
C_4	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a)$; $a^4 + 17a^2 + 68 = 0$
C_6	$x^6 + 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
D_2	$x^5 + 9x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
D_3	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
D_4	$x^5 + 3x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
D_6	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3}, a)$; $a^3 + 3a - 2 = 0$
T	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a, b)$;
			$a^3 - 7a + 7 = b^4 + 4b^2 + 8b + 8 = 0$
O	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-11}, a, b)$;
			$a^3 - 4a + 4 = b^4 + 22b + 22 = 0$
$J(C_1)$	$x^5 - x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_2)$	$x^5 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(C_4)$	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}$	see entry for C_4
$J(C_6)$	$x^6 - 15x^4 - 20x^3 + 6x + 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{3}, a)$; $a^3 + 3a^2 - 1 = 0$
$J(D_2)$	$x^5 + 9x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
$J(D_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(D_4)$	$x^5 + 3x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
$J(D_6)$	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}$	see entry for D_6
$J(T)$	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}$	see entry for T
$J(O)$	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}$	see entry for O
$C_{2,1}$	$x^6 + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3})$
$C_{4,1}$	$x^5 + 2x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt[4]{2})$
$C_{6,1}$	$x^6 + 6x^5 - 30x^4 + 20x^3 + 15x^2 - 12x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, a)$; $a^3 - 3a + 1 = 0$
$D_{2,1}$	$x^5 + x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$D_{4,1}$	$x^5 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$D_{6,1}$	$x^6 + 6x^5 - 30x^4 - 40x^3 + 60x^2 + 24x - 8$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-3}, a)$; $a^3 - 9a + 6 = 0$
$D_{3,2}$	$x^6 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$D_{4,2}$	$x^6 + x^5 + 10x^3 + 5x^2 + x - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a)$; $a^4 - 14a^2 + 28a - 14 = 0$
$D_{6,2}$	$x^6 + 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
O_1	$x^6 + 7x^5 + 10x^4 + 10x^3 + 15x^2 + 17x + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a, b)$;
			$a^3 + 5a + 10 = b^4 + 4b^2 + 8b + 2 = 0$

Genus 2 curves realizing Sato-Tate groups with $G^0 \neq \mathrm{U}(1)_2$

Group	Curve $y^2 = f(x)$	k	K
F	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i, \sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_a	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
F_{ab}	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_{ac}	$x^5 + 1$	$\mathbb{Q}$	$\mathbb{Q}(a)$; $a^4 + 5a^2 + 5 = 0$
$F_{a,b}$	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
E_1	$x^6 + x^4 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
E_2	$x^6 + x^5 + 3x^4 + 3x^2 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$
E_3	$x^5 + x^4 - 3x^3 - 4x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a)$; $a^3 - 3a + 1 = 0$
E_4	$x^5 + x^4 + x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a)$; $a^4 - 5a^2 + 5 = 0$
E_6	$x^5 + 2x^4 - x^3 - 3x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{7}, a)$; $a^3 - 7a - 7 = 0$
$J(E_1)$	$x^5 + x^3 + x$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$J(E_2)$	$x^5 + x^3 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(E_3)$	$x^6 + x^3 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$J(E_4)$	$x^5 + x^3 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$J(E_6)$	$x^6 + x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$G_{1,3}$	$x^6 + 3x^4 - 2$	$\mathbb{Q}(i)$	$\mathbb{Q}(i)$
$N(G_{1,3})$	$x^6 + 3x^4 - 2$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$G_{3,3}$	$x^6 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
$N(G_{3,3})$	$x^6 + x^5 + x - 1$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$\mathrm{USp}(4)$	$x^5 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}$

Part Two

Searching for curves

We surveyed the $\bar{L}$-polynomial distributions of genus 2 curves

$$y^2 = x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

$$y^2 = x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

with integer coefficients $|c_i| \leq 128$. More than 2^{48} curves.

We found over 10 million non-isomorphic curves with exceptional distributions, including at least 3 apparent matches for each of the 34 Sato-Tate groups that can occur over $\mathbb{Q}$.

Representative examples were computed to high precision $N = 2^{30}$.

For each example, the field K was then determined, allowing the Galois type, and hence the Sato-Tate group, to be **provably** identified.

Exhibiting Sato-Tate groups of abelian surfaces

The 34 Sato-Tate groups that can arise for an abelian surface over $\mathbb{Q}$ are all realized by Jacobians of genus 2 curves.

By extending the base field from $\mathbb{Q}$ to a suitable subfield k of K, we can restrict $G/G^0 \simeq \mathrm{Gal}(K/k)$ to any normal subgroup of $\mathrm{Gal}(K/k)$ (base extension does not change the identity component G^0).

This allows us to realize all 52 Sato-Tate groups using base extensions of 34 curves defined over $\mathbb{Q}$ (in fact, 9 suffice).

Serre asks: can all 52 can be realized over a single base field k?

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Serre asks: can all 52 can be realized over a single base field k?

Theorem (Fité-Guitart 2015)

All 52 possible Sato-Tate groups arise for abelian surfaces defined over

$$k := \mathbb{Q}(\sqrt{-10}, \sqrt{-51}, \sqrt{-163}, \sqrt{-67}, \sqrt{817}, \sqrt{-57}).$$

Computing zeta functions

Algorithms to compute $L_p(T)$ for low genus hyperelliptic curves

	complexity (ignoring factors of $O(\log\log p)$)		
algorithm	$g=1$	$g=2$	$g=3$
point enumeration	$p\log p$	$p^2\log p$	$p^3\log p$
group computation	$p^{1/4}\log p$	$p^{3/4}\log p$	$p^{5/4}\log p$
p-adic cohomology	$p^{1/2}\log^2 p$	$p^{1/2}\log^2 p$	$p^{1/2}\log^2 p$
CRT (Schoof-Pila)	$\log^5 p$	$\log^8 p$	$\log^{12} p$

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(see [Kedlaya-S 2008]).

An average polynomial-time algorithm

All of these methods perform separate computations for each p.
But we want to compute $L_p(T)$ for all good $p \leq N$ using reductions of *the same curve* in each case. Can we take advantage of this?

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But we want to compute $L_p(T)$ for all good $p \le N$ using reductions of *the same curve* in each case. Can we take advantage of this?

Theorem (Harvey 2012)

There exists a deterministic algorithm that, given a hyperelliptic curve $y^2 = f(x)$ of genus g with a rational Weierstrass point and an integer N, computes $L_p(T)$ for all good primes $p \le N$ in time

$$O\big(g^{8+\epsilon} N \log^{3+\epsilon} N\big),$$

assuming the coefficients of $f \in \mathbb{Z}[x]$ have size bounded by $O(\log N)$.

Average time is $O\big(g^{8+\epsilon} \log^{4+\epsilon} N\big)$ per prime, polynomial in g and $\log p$.
Recently generalized to arithmetic schemes.

An average polynomial-time algorithm

	complexity (ignoring factors of $O(\log\log p)$)		
algorithm	$g=1$	$g=2$	$g=3$
point enumeration	$p\log p$	$p^2\log p$	$p^3\log p$
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CRT (Schoof-Pila)	$\log^5 p$	$\log^8 p$	$\log^{12} p$
Average polytime	$\log^4 p$	$\log^4 p$	$\log^4 p$

But is it practical?

The Hasse-Witt matrix of a hyperelliptic curve

The *Hasse-Witt* matrix of a hyperelliptic curve $y^2 = f(x)$ over $\mathbb{F}_p$ of genus g is the $g \times g$ matrix $W_p = [w_{ij}]$ with entries

$$w_{ij} = f_{pi-j}^{(p-1)/2} \bmod p \qquad (1 \le i, j \le g).$$

The w_{ij} can each be computed using recurrence relations between the coefficients of f^n and those of f^{n-1}.

The congruence

$$L_P(T) \equiv \det(I - TW_p) \bmod p$$

allows us to determine the coefficients $a_1, \ldots, a_g$ of $L_p(T)$ modulo p. This is enough to compute $\#C_p(\mathbb{F}_p)$ for $p > 16g^2$.

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The algorithm can be extended to compute $L_p(T)$ modulo higher powers of p (and thereby obtain $L_p \in \mathbb{Z}[T]$), but for $g \le 3$ it's easier to derive $L_p(T)$ from $L_p(T) \bmod p$ using computations in $\mathrm{Jac}(C)$.

Complexity

Theorem (Harvey-S 2014)

Given a hyperelliptic curve $y^2 = f(x)$ of genus g, and an integer N, one can compute the Hasse-Witt matrices W_p for all good primes $p \le N$ in

$$O\big(g^3 N \log^3 N \log\log N\big) \text{ time} \qquad \text{and} \qquad O(g^2 N) \text{ space},$$

assuming g and the bit-size of each coefficient of f are $O(\log N)$.

The complexity is close to optimal (nearly quasi-linear in output size).

Extends to computing $L_p \in \mathbb{Z}[T]$ in $O(g^{4+\epsilon} N \log^{3+\epsilon} N)$ time.

In progress: smooth plane quartics.

	genus 2		genus 3	
N	`smalljac`	`hwlpoly`	`hypellfrob`	`hwlpoly`
2^{14}	0.2	0.1	7.2	0.4
2^{15}	0.6	0.3	16.3	1.0
2^{16}	1.7	0.9	39.1	2.9
2^{17}	5.5	2.2	98.3	7.8
2^{18}	19.2	5.3	255	18.3
2^{19}	78.4	12.5	695	43.2
2^{20}	271	27.8	1950	98.8
2^{21}	1120	64.5	5600	229
2^{22}	2820	155	16700	537
2^{23}	9840	357	51200	1240
2^{24}	31900	823	158000	2800
2^{25}	105000	1890	501000	6280
2^{26}	349000	4250	1480000	13900
2^{27}	1210000	9590	4360000	31100
2^{28}	4010000	21200	12500000	69700
2^{29}	13200000	48300	39500000	155000
2^{30}	45500000	108000	120000000	344000

(Intel Xeon E5-2697v2 2.7 GHz CPU seconds).

Naïve approach

For each good prime $p < N$ we want to compute the entries

$$w_{ij} = f_{pi-j}^{(p-1)/2} \bmod p \qquad (1 \leq i,\ j \leq g).$$

of the Hasse-Witt matrix $W_p = [w_{ij}]$.

So we could iteratively compute $f, f^2, f^3, \ldots, f^{(N-1)/2}$ in $\mathbb{Z}[x]$ and just reduce the x^{pi-j} coefficients of $f(x)^{(p-1)/2}$ mod p for each prime $p \leq N$.

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But the polynomials f^n are huge, each has $\Omega(n^2)$ bits.
It would take $\Omega(N^3)$ time to compute $f, \ldots, f^{(N-1)/2}$ in $\mathbb{Z}[x]$.

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So this is a terrible idea...

But we don't need all the coefficients of f^n, we only need one,
and we only need to know its value modulo $p = 2n + 1$.

A better approach

For any integer $n \geq 0$ the equations

$$f^{n+1} = f \cdot f^n \qquad \text{and} \qquad (f^{n+1})' = (n+1) f' f^n$$

yield the relations

$$f_k^{n+1} = \sum_{j=0}^{d} f_j f_{k-j}^n \qquad \text{and} \qquad k f_k^{n+1} = (n+1) \sum_{j=0}^{d} j f_j f_{k-j}^n,$$

where f_k^n denotes the coefficient of x^k in f^n. Subtracting k times the first from the second and solving for f_k^n yields the identity

$$f_k^n = \frac{1}{k f_0} \sum_{j=1}^{d} (nj + j - k) f_j f_{k-j}^n, \tag{1}$$

which is valid for all positive integers k and n (assuming $f_0 \neq 0$).

If we now define

$$v_k^n := [f_{k-d+1}^n, \ldots, f_k^n] \in \mathbb{Z}^d,$$

then the last g entries of $v_{p-1}^{(p-1)/2} \bmod p$ form the first row of W_p, and

$$f_k^n \equiv \frac{1}{2kf_0} \sum_{j=1}^{d} (j-2k) f_j f_{k-j}^n \bmod p,$$

holds for $k \leq p-1 = 2n$. Starting from $v_0^n = [0, \ldots, 0, f_0^n]$, we compute

$$v_{p-1}^n \equiv \frac{v_0^n}{2^{p-1}(p-1)! f_0^{p-1}} \prod_{k=1}^{p-1} M_k \equiv -v_0^n \prod_{i=1}^{p-1} M_k \bmod p,$$

where the $d \times d$ matrices

$$M_k := \begin{bmatrix} 0 & \cdots & 0 & (d-2k)f_d \\ 2kf_0 & \cdots & 0 & (d-1-2k)f_{d-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 2kf_0 & (1-2k)f_1 \end{bmatrix}$$

do not depend on p!

Computing a sequence of reduced partial products

Computing the first row of W_p for all $p < N$ reduces to compute the sequence of reduced partial products

$$\begin{aligned} M_1 M_2 \bmod 3 \\ M_1 M_2 M_3 M_4 \bmod 5 \\ M_1 M_2 M_3 M_4 M_5 M_6 \bmod 7 \\ \vdots \\ M_1 M_2 M_3 M_4 M_5 M_6 \cdots M_{N-2} \bmod N-1 \end{aligned}$$

Doing this naïvely would take time quasi-quadratic in N.

But quasi-linear time is achieved with an *accumulating remainder tree*.

Accumulating remainder trees

Input: integer matrices $M_0, \ldots, M_{N-1}$ and moduli $m_0, \ldots, m_{N-1}$.

Output: $A_0, A_1, \ldots, A_{N-1}$, where $A_i := \prod_{j<i} M_j \bmod m_i$.

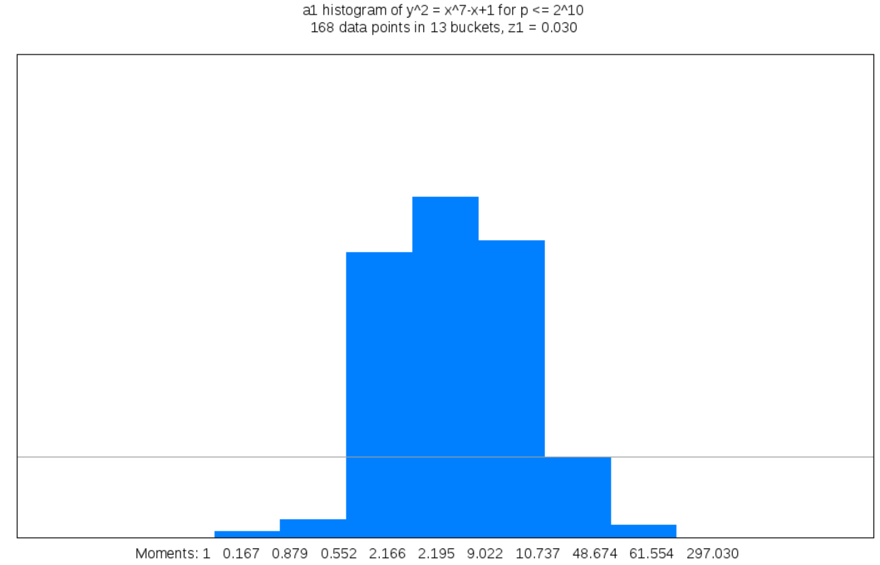
Algorithm:

1. If $N = 1$ then output $A_0 := 1$ and terminate (base case).
2. Use $M'_i := M_{2i} M_{2i+1}$ and $m'_i := m_{2i} m_{2i+1}$ to recursively compute $A'_1, \ldots, A'_{N/2}$.
3. Output

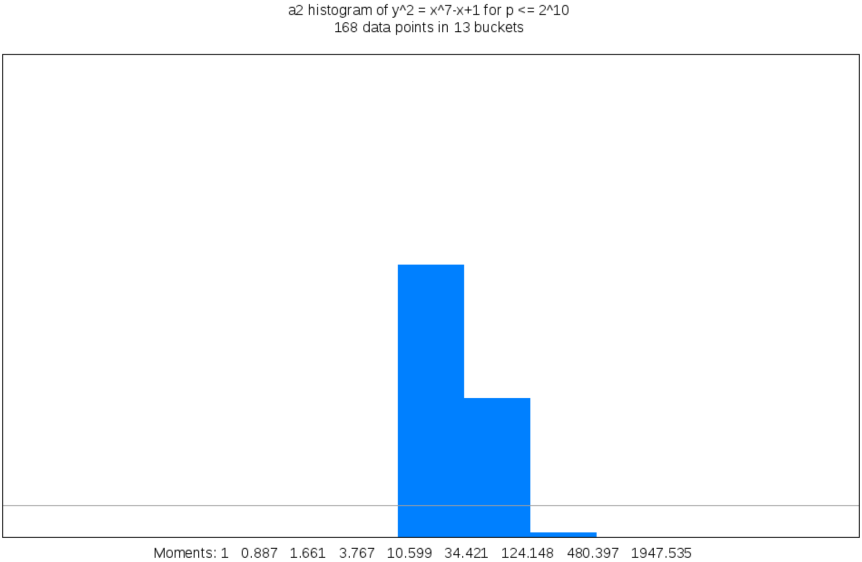
$$A_i := \begin{cases} A'_{i/2} \bmod m_i & i \text{ even;} \\ A'_{(i-1)/2} M_{i-1} \bmod m_i & i \text{ odd.} \end{cases}$$

Using FFT-multiplication, this runs in quasi-linear time.

The space complexity can be improved using a *remainder forest*.



click histogram to animate (requires adobe reader)



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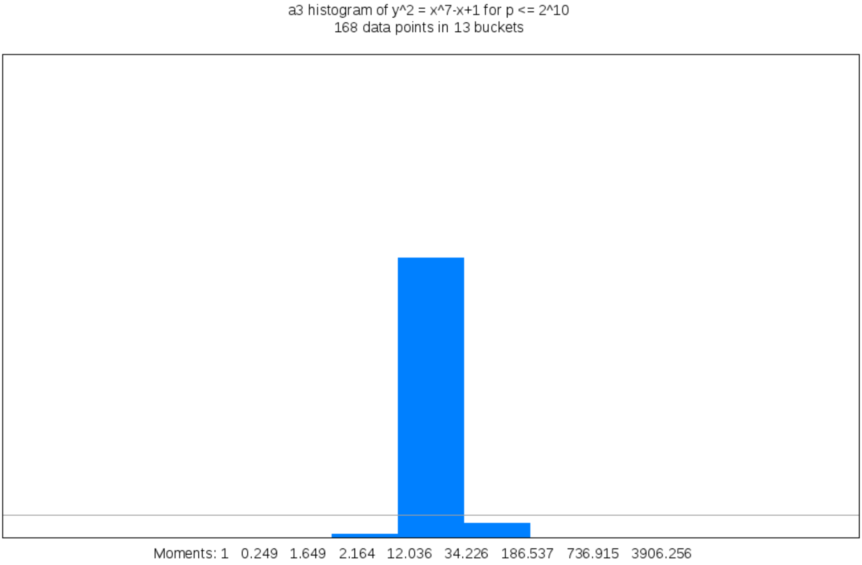

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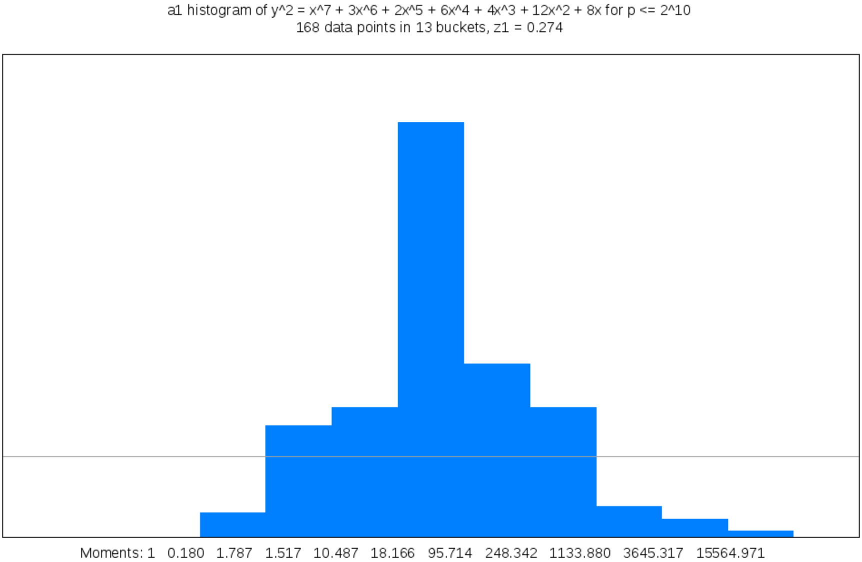
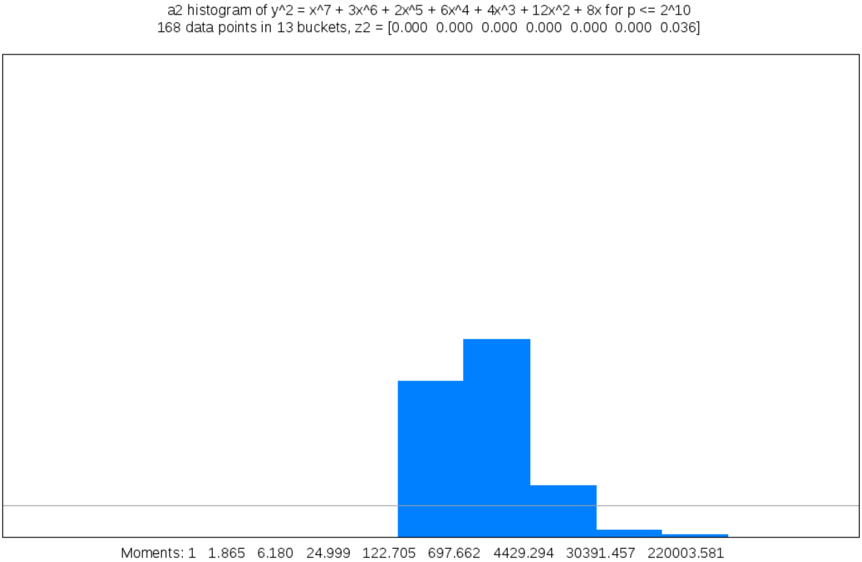
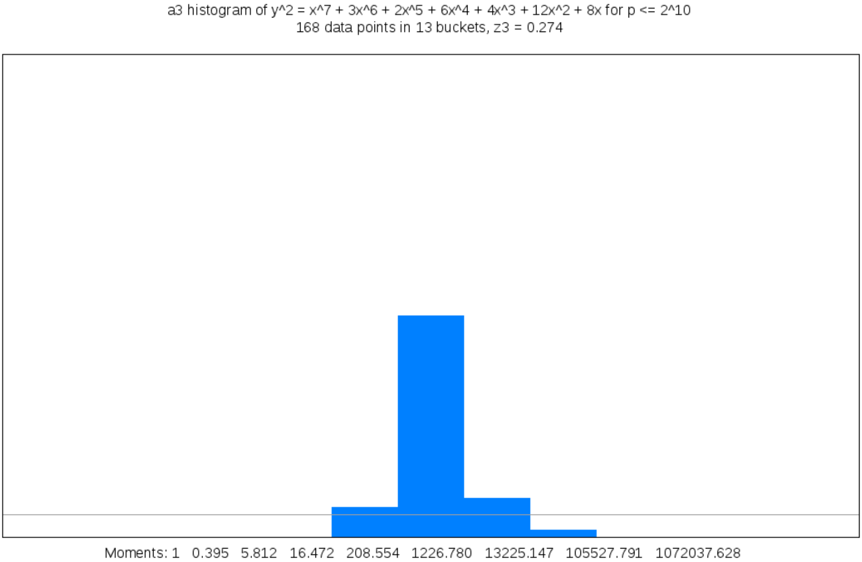


click histogram to animate (requires adobe reader)



click histogram to animate (requires adobe reader)

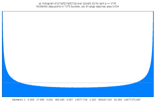


click histogram to animate (requires adobe reader)

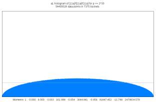
Real endomorphism algebras of abelian threefolds

abelian threefold	$\mathbf{End}(A_K)_{\mathbb{R}}$	$\mathbf{ST}_A^0$
cube of a CM elliptic curve	$\mathrm{M}_3(\mathbb{C})$	$\mathrm{U}(1)_3$
cube of a non-CM elliptic curve	$\mathrm{M}_3(\mathbb{R})$	$\mathrm{SU}(2)_3$
product of CM elliptic curve and square of CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1) \times \mathrm{U}(1)_2$
• product of CM elliptic curve and QM abelian surface • product of CM elliptic curve and square of non-CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{U}(1) \times \mathrm{SU}(2)_2$
product of non-CM elliptic curve and square of CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{SU}(2) \times \mathrm{U}(1)_2$
• product of non-CM elliptic curve and QM abelian surface • product of non-CM elliptic curve and square of non-CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$
• CM abelian threefold • product of CM elliptic curve and CM abelian surface • product of three CM elliptic curves	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$
• product of non-CM elliptic curve and CM abelian surface • product of non-CM elliptic curve and two CM elliptic curves	$\mathbb{C} \times \mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$
• product of CM elliptic curve and RM abelian surface • product of CM elliptic curve and two non-CM elliptic curves	$\mathbb{C} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$
• RM abelian threefold • product of non-CM elliptic curve and RM abelian surface • product of 3 non-CM elliptic curves	$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$
product of CM elliptic curve and abelian surface	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{USp}(4)$
product of non-CM elliptic curve and abelian surface	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{USp}(4)$
quadratic CM abelian threefold	$\mathbb{C}$	$\mathrm{U}(3)$
generic abelian threefold	$\mathbb{R}$	$\mathrm{USp}(6)$

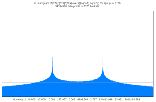
Connected Sato-Tate groups of abelian threefolds:



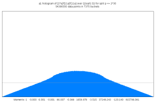
$\mathrm{U}(1)_3$



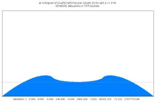
$\mathrm{SU}(2)_3$



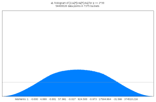
$\mathrm{U}(1) \times \mathrm{U}(1)_2$



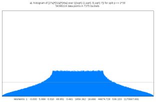
$\mathrm{U}(1) \times \mathrm{SU}(2)_2$



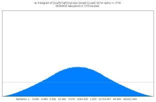
$\mathrm{SU}(2) \times \mathrm{U}(1)_2$



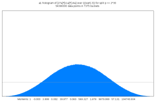
$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$



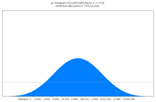
$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$



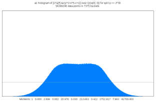
$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$



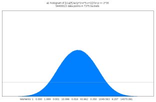
$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$



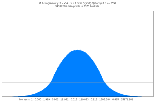
$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$



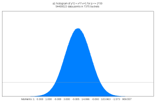
$\mathrm{U}(1) \times \mathrm{USp}(4)$



$\mathrm{SU}(2) \times \mathrm{USp}(4)$



$\mathrm{U}(3)$



$\mathrm{USp}(6)$

Partial classification of component groups

G_0	$G/G_0 \hookrightarrow$	$\lvert G/G_0 \rvert$ divides
$\mathrm{USp}(6)$	C_1	1
$\mathrm{U}(3)$	C_2	2
$\mathrm{SU}(2) \times \mathrm{USp}(4)$	C_1	1
$\mathrm{U}(1) \times \mathrm{USp}(4)$	C_2	2
$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	S_3	6
$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	D_2	4
$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$	D_4	8
$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$	$\mathrm{C}_2 \wr \mathrm{S}_3$	48
$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$	$\mathrm{D}_4, \quad \mathrm{D}_6$	$8, \; 12$
$\mathrm{SU}(2) \times \mathrm{U}(1)_2$	$\mathrm{D}_6 \times \mathrm{C}_2, \quad \mathrm{S}_4 \times \mathrm{C}_2$	48
$\mathrm{U}(1) \times \mathrm{SU}(2)_2$	$\mathrm{D}_4 \times \mathrm{C}_2, \quad \mathrm{D}_6 \times \mathrm{C}_2$	$16, \; 24$
$\mathrm{U}(1) \times \mathrm{U}(1)_2$	$\mathrm{D}_6 \times \mathrm{C}_2 \times \mathrm{C}_2, \quad \mathrm{S}_4 \times \mathrm{C}_2 \times \mathrm{C}_2$	96
$\mathrm{SU}(2)_3$	$\mathrm{D}_6, \quad \mathrm{S}_4$	24
$\mathrm{U}(1)_3$	$\cdots$	$336, \; 1728$

(disclaimer: this is work in progress subject to verification)