Sato-Tate in dimension 3

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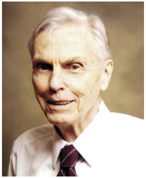
Mikio Sato



John Tate

Joint work with F. Fité, K.S. Kedlaya, and V. Rotger, and with D. Harvey.

Sato-Tate in dimension 1

Let $E/\mathbb{Q}$ be an elliptic curve, say,

$$y^2 = x^3 + Ax + B,$$

and let p be a prime of good reduction (so $p \nmid \Delta(E)$).

The number of $\mathbb{F}_p$-points on the reduction E_p of E modulo p is

$$\#E_p(\mathbb{F}_p) = p + 1 - t_p,$$

where the trace of Frobenius t_p is an integer in $[-2\sqrt{p}, 2\sqrt{p}]$.

We are interested in the limiting distribution of $x_p = -t_p/\sqrt{p} \in [-2, 2]$, as p varies over primes of good reduction up to $N \to \infty$.

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Sato-Tate distributions in dimension 1

1. Typical case (no CM)

Elliptic curves $E/\mathbb{Q}$ w/o CM have the semi-circular trace distribution. (This is also known for E/k, where k is a totally real number field).

[Barnet-Lamb, Clozel, Geraghty, Harris, Shepherd-Barron, Taylor]

2. Exceptional cases (CM)

Elliptic curves E/k with CM have one of two distinct trace distributions, depending on whether k contains the CM field or not.

[classical (Hecke, Deuring)]

Sato-Tate groups in dimension 1

The *Sato-Tate group* of E is a closed subgroup G of $\mathrm{SU}(2) = \mathrm{USp}(2)$ derived from the ℓ-adic Galois representation attached to E.

A refinement of the Sato-Tate conjecture implies that the distribution of normalized Frobenius traces of E converges to the distribution of traces in its Sato-Tate group G (under its Haar measure).

G	G/G^0	E	k	$\mathrm{E}[a_1^0], \mathrm{E}[a_1^2], \mathrm{E}[a_1^4] \ldots$
$\mathrm{SU}(2)$	C_1	$y^2 = x^3 + x + 1$	$\mathbb{Q}$	$1, 1, 2, 5, 14, 42, \ldots$
$N(\mathrm{U}(1))$	C_2	$y^2 = x^3 + 1$	$\mathbb{Q}$	$1, 1, 3, 10, 35, 126, \ldots$
$\mathrm{U}(1)$	C_1	$y^2 = x^3 + 1$	$\mathbb{Q}(\sqrt{-3})$	$1, 2, 6, 20, 70, 252, \ldots$

In dimension 1 there are three possible Sato-Tate groups, two of which arise for elliptic curves defined over $\mathbb{Q}$.

Zeta functions and L-polynomials

For a smooth projective curve $C/\mathbb{Q}$ of genus g and each prime p of good reduction for C we have the *zeta function*

$$Z(C_p/\mathbb{F}_p;T) := \exp\left(\sum_{k=1}^{\infty} \#C_p(\mathbb{F}_{p^k})T^k/k\right) = \frac{L_p(T)}{(1-T)(1-pT)},$$

where $L_p \in \mathbb{Z}[T]$ has degree $2g$. The normalized L-polynomial

$$\bar{L}_p(T) := L_p(T/\sqrt{p}) = \sum_{i=0}^{2g} a_i T^i \in \mathbb{R}[T]$$

is monic, reciprocal, and unitary, with $|a_i| \le \binom{2g}{i}$.

We now consider the limiting distribution of $a_1, a_2, \dots, a_g$ over all primes $p \le N$ of good reduction, as $N \to \infty$.

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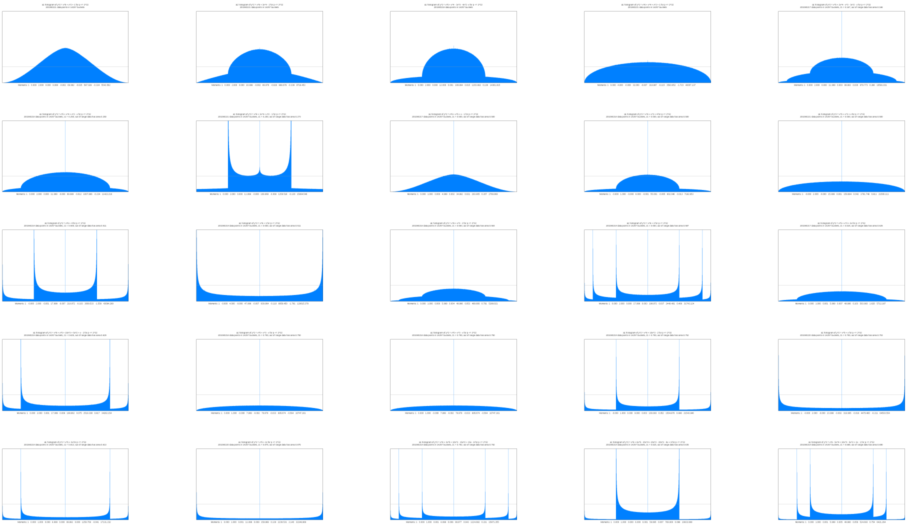
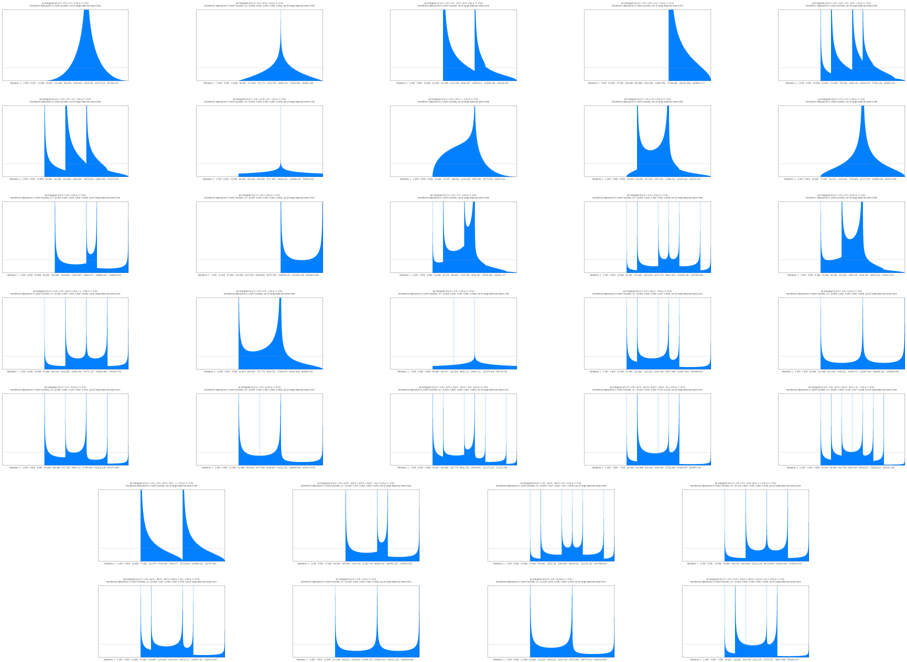
Exceptional distributions for abelian surfaces over $\mathbb{Q}$:





L-polynomials of Abelian varieties

Let A be an abelian variety over a number field k. Fix a prime ℓ.
The action of $\mathrm{Gal}(\bar{k}/k)$ on the ℓ-adic Tate module

$$V_\ell(A) := \varprojlim A[\ell^n] \otimes_{\mathbb{Z}} \mathbb{Q}$$

gives rise to a Galois representation

$$\rho_\ell \colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$$

For each prime $\mathfrak{p}$ of good reduction for A we have the *L-polynomial*

$$L_\mathfrak{p}(T) := \det(1 - \rho_\ell(\mathrm{Frob}_\mathfrak{p})T), \qquad \bar{L}_\mathfrak{p}(T) := L_\mathfrak{p}(T/\sqrt{\|\mathfrak{p}\|}),$$

which appears as an Euler factor in the L-series

$$L(A,s) := \prod_\mathfrak{p} L_\mathfrak{p}(\|\mathfrak{p}\|^{-s})^{-1}.$$

The Sato-Tate group of an abelian variety

The Zariski closure of the image of

$$\rho_\ell\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$$

is a $\mathbb{Q}_\ell$-algebraic group $G_\ell^{\mathrm{zar}} \subseteq \mathrm{GSp}_{2g}$ that determines a $\mathbb{C}$-algebraic group $G_{\ell,\iota}^{1,\mathrm{zar}} \subseteq \mathrm{Sp}_{2g}$ after fixing $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ and intersecting with Sp_{2g}.

Definition [Serre]

$\mathrm{ST}(A) \subseteq \mathrm{USp}(2g)$ is a maximal compact subgroup of $G_{\ell,\iota}^{1,\mathrm{zar}}(\mathbb{C})$.

Conjecture [Mumford-Tate, Algebraic Sato-Tate]

$(G_\ell^{\mathrm{zar}})^0 = \mathrm{MT}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, equivalently, $(G_\ell^{1,\mathrm{zar}})^0 = \mathrm{Hg}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.
More generally, $G_\ell^{1,\mathrm{zar}} = \mathrm{AST}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

This conjecture is known for $g \leq 3$ (see Banaszak-Kedlaya 2015).

A refined Sato-Tate conjecture

Let $s(\mathfrak{p})$ denote the conjugacy class of $\|\mathfrak{p}\|^{-1/2}M_{\mathfrak{p}}$ in $\mathrm{ST}(A)$, where $M_{\mathfrak{p}}$ is the image of $\mathrm{Frob}_{\mathfrak{p}}$ in $G_{\ell,\iota}^{\mathrm{zar}}(\mathbb{C})$ (semisimple, by a theorem of Tate), and let $\mu_{\mathrm{ST}(A)}$ denote the pushforward of the Haar measure to $\mathrm{Conj}(\mathrm{ST}(A))$.

Conjecture

The conjugacy classes $s(\mathfrak{p})$ are equidistributed with respect to $\mu_{\mathrm{ST}(A)}$.

In particular, the distribution of normalized Euler factors $\bar{L}_{\mathfrak{p}}(T)$ matches the distribution of characteristic polynomials in $\mathrm{ST}(A)$.

We can test this numerically by comparing statistics of the coefficients $a_1, \ldots, a_g$ of $\bar{L}_{\mathfrak{p}}(T)$ over $\|\mathfrak{p}\| \le N$ to the predictions given by $\mu_{\mathrm{ST}(A)}$.

Galois endomorphism modules

Let A be an abelian variety defined over a number field k.
Let K be the minimal extension of k for which $\mathrm{End}(A_K) = \mathrm{End}(A_{\bar{k}})$.
$\mathrm{Gal}(K/k)$ acts on the $\mathbb{R}$-algebra $\mathrm{End}(A_K)_{\mathbb{R}} = \mathrm{End}(A_K) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition

The *Galois endomorphism type* of A is the isomorphism class of $[\mathrm{Gal}(K/k), \mathrm{End}(A_K)_{\mathbb{R}}]$, where $[G, E] \simeq [G', E']$ iff there are isomorphisms $G \simeq G'$ and $E \simeq E'$ that are compatible with the Galois action.

Theorem [Fité, Kedlaya, Rotger, S 2012]

For abelian varieties A/k of dimension $g \leq 3$ there is a one-to-one correspondence between Sato-Tate groups and Galois types.

More precisely, the identity component G^0 is uniquely determined by $\mathrm{End}(A_K)_{\mathbb{R}}$ and $G/G^0 \simeq \mathrm{Gal}(K/k)$ (with corresponding actions).

Real endomorphism algebras of abelian surfaces

abelian surface	$\mathbf{End}(A_K)_{\mathbb{R}}$	$\mathbf{ST}(A)^{\mathbf{0}}$
square of CM elliptic curve	$\mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1)_2$
• QM abelian surface • square of non-CM elliptic curve	$\mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2)_2$
• CM abelian surface • product of CM elliptic curves	$\mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1)$
product of CM and non-CM elliptic curves	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2)$
• RM abelian surface • product of non-CM elliptic curves	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
generic abelian surface	$\mathbb{R}$	$\mathrm{USp}(4)$

(factors in products are assumed to be non-isogenous)

Sato-Tate groups in dimension 2

Theorem [Fité-Kedlaya-Rotger-S 2012]

Up to conjugacy in $\mathrm{USp}(4)$, there are 52 Sato-Tate groups $\mathrm{ST}(A)$ that arise for abelian surfaces A/k over number fields; 34 occur for $k = \mathbb{Q}$.

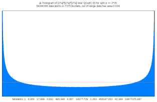
$\mathrm{U}(1)_2$:	$C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O,$ $J(C_1), J(C_2), J(C_3), J(C_4), J(C_6),$ $J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O),$ $C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1$
$\mathrm{SU}(2)_2$:	$E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6)$
$\mathrm{U}(1) \times \mathrm{U}(1)$:	$F, F_a, F_{a,b}, F_{ab}, F_{ac}$
$\mathrm{U}(1) \times \mathrm{SU}(2)$:	$\mathrm{U}(1) \times \mathrm{SU}(2),\ N(\mathrm{U}(1) \times \mathrm{SU}(2))$
$\mathrm{SU}(2) \times \mathrm{SU}(2)$:	$\mathrm{SU}(2) \times \mathrm{SU}(2),\ N(\mathrm{SU}(2) \times \mathrm{SU}(2))$
$\mathrm{USp}(4)$:	$\mathrm{USp}(4)$

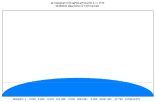
This theorem says nothing about equidistribution, however this is now known in many special cases [Fité-S 2012, Johansson 2013].

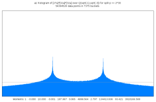
Real endomorphism algebras of abelian threefolds

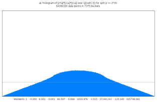
abelian threefold	$\mathbf{End}(A_K)_{\mathbb{R}}$	$\mathbf{ST}(A)^{\mathbf{0}}$
cube of a CM elliptic curve	$\mathrm{M}_3(\mathbb{C})$	$\mathrm{U}(1)_3$
cube of a non-CM elliptic curve	$\mathrm{M}_3(\mathbb{R})$	$\mathrm{SU}(2)_3$
product of CM elliptic curve and square of CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1) \times \mathrm{U}(1)_2$
• product of CM elliptic curve and QM abelian surface • product of CM elliptic curve and square of non-CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{U}(1) \times \mathrm{SU}(2)_2$
product of non-CM elliptic curve and square of CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{SU}(2) \times \mathrm{U}(1)_2$
• product of non-CM elliptic curve and QM abelian surface • product of non-CM elliptic curve and square of non-CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$
• CM abelian threefold • product of CM elliptic curve and CM abelian surface • product of three CM elliptic curves	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$
• product of non-CM elliptic curve and CM abelian surface • product of non-CM elliptic curve and two CM elliptic curves	$\mathbb{C} \times \mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$
• product of CM elliptic curve and RM abelian surface • product of CM elliptic curve and two non-CM elliptic curves	$\mathbb{C} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$
• RM abelian threefold • product of non-CM elliptic curve and RM abelian surface • product of 3 non-CM elliptic curves	$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$
product of CM elliptic curve and abelian surface	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{USp}(4)$
product of non-CM elliptic curve and abelian surface	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{USp}(4)$
quadratic CM abelian threefold	$\mathbb{C}$	$\mathrm{U}(3)$
generic abelian threefold	$\mathbb{R}$	$\mathrm{USp}(6)$

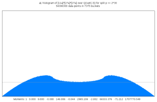
Connected Sato-Tate groups of abelian threefolds:

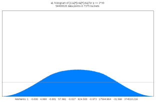

$\mathrm{U}(1)_3$

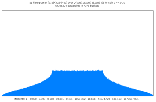

$\mathrm{SU}(2)_3$

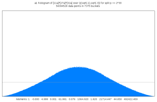

$\mathrm{U}(1)\times\mathrm{U}(1)_2$

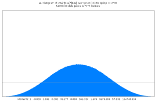

$\mathrm{U}(1)\times\mathrm{SU}(2)_2$

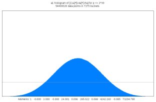

$\mathrm{SU}(2)\times\mathrm{U}(1)_2$

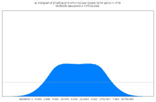

$\mathrm{SU}(2)\times\mathrm{SU}(2)_2$

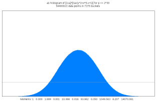

$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$

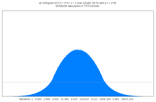

$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)$

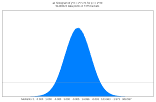

$\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{U}(1)$


$\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$


$\mathrm{U}(1)\times\mathrm{USp}(4)$


$\mathrm{SU}(2)\times\mathrm{USp}(4)$


$\mathrm{U}(3)$


$\mathrm{USp}(6)$

Partial classification of component groups

G_0	$G/G_0 \hookrightarrow$	$\lvert G/G_0\rvert$ divides
$\mathrm{USp}(6)$	C_1	1
$\mathrm{U}(3)$	C_2	2
$\mathrm{SU}(2)\times\mathrm{USp}(4)$	C_1	1
$\mathrm{U}(1)\times\mathrm{USp}(4)$	C_2	2
$\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$	S_3	6
$\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$	D_2	4
$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)$	D_4	8
$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$	$\mathrm{C}_2\wr\mathrm{S}_3$	48
$\mathrm{SU}(2)\times\mathrm{SU}(2)_2$	$\mathrm{D}_4,\ \ \mathrm{D}_6$	$8,\ 12$
$\mathrm{SU}(2)\times\mathrm{U}(1)_2$	$\mathrm{D}_6\times\mathrm{C}_2,\ \ \mathrm{S}_4\times\mathrm{C}_2$	48
$\mathrm{U}(1)\times\mathrm{SU}(2)_2$	$\mathrm{D}_4\times\mathrm{C}_2,\ \ \mathrm{D}_6\times\mathrm{C}_2$	$16,\ 24$
$\mathrm{U}(1)\times\mathrm{U}(1)_2$	$\mathrm{D}_6\times\mathrm{C}_2\times\mathrm{C}_2,\ \ \mathrm{S}_4\times\mathrm{C}_2\times\mathrm{C}_2$	96
$\mathrm{SU}(2)_3$	$\mathrm{D}_6,\ \ \mathrm{S}_4$	24
$\mathrm{U}(1)_3$	(to be determined)	$336,\ 1728$

(disclaimer: work in progress, subject to verification)

Algorithms to compute zeta functions

Given a curve $C/\mathbb{Q}$ of genus g, we want to compute the normalized L-polynomials $\overline{L}_p(T)$ at all good primes $p \le N$.

algorithm	complexity per prime (ignoring factors of $O(\log\log p)$) $g = 1$	 $g = 2$	 $g = 3$
point enumeration	$p\log p$	$p^2\log p$	$p^3(\log p)^2$
group computation	$p^{1/4}\log p$	$p^{3/4}\log p$	$p\log p$
p-adic cohomology	$p^{1/2}(\log p)^2$	$p^{1/2}(\log p)^2$	$p^{1/2}(\log p)^2$
CRT (Schoof-Pila)	$(\log p)^5$	$(\log p)^8$	$(\log p)^{12?}$
average poly-time	$(\log p)^4$	$(\log p)^4$	$(\log p)^4$

Genus 3 curves

The canonical embedding of a genus 3 curve into $\mathbb{P}^2$ is either

1. a degree-2 cover of a smooth conic (hyperelliptic case);
2. a smooth plane quartic (generic case).

Average polynomial-time implementations available for the first case:

- rational hyperelliptic model [Harvey-S 2014];
- no rational hyperelliptic model [Harvey-Massierer-S 2016].

Here we address the second case.

Prior work has all been based on p-adic cohomology:

[Lauder 2004], [Castryck-Denef-Vercauteren 2006], [Abott-Kedlaya-Roe 2006], [Harvey 2010], [Tuitman-Pancrantz 2013], [Tuitman 2015], [Costa 2015], [Tuitman-Castryck 2016], [Shieh 2016]

New algorithm

Let $C_p/\mathbb{F}_p$ be a smooth plane quartic defined by $f(x,y,z)=0$.
For $n\geq 0$ let $f^n_{i,j,k}$ denote the coefficient of $x^iy^jz^k$ in f^n.

The *Hasse–Witt matrix* of C_p is the 3×3 matrix

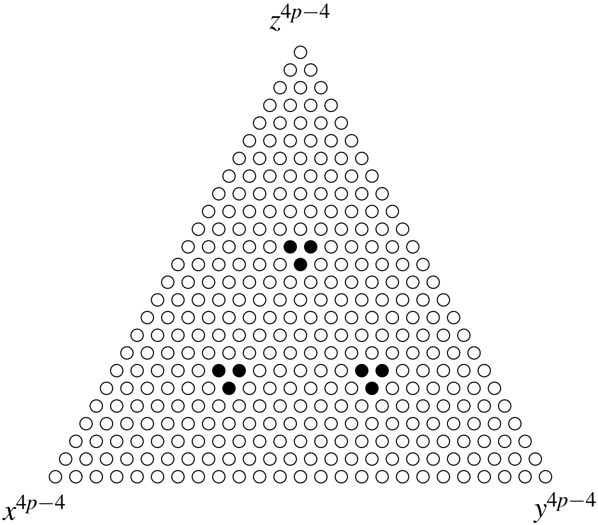
$$W_p := \begin{bmatrix} f^{p-1}_{p-1,p-1,2p-2} & f^{p-1}_{2p-1,p-1,p-2} & f^{p-1}_{p-1,2p-1,p-2} \\ f^{p-1}_{p-2,p-1,2p-1} & f^{p-1}_{2p-2,p-1,p-1} & f^{p-1}_{p-2,2p-1,p-1} \\ f^{p-1}_{p-1,p-2,2p-1} & f^{p-1}_{2p-1,p-2,p-1} & f^{p-1}_{p-1,2p-2,p-1} \end{bmatrix}.$$

This is the matrix of the p-power Frobenius acting on $H^1(C_p,\mathcal{O}_{C_p})$ (and the Cartier-Manin operator acting on the space of regular differentials). As proved by Manin, we have

$$L_p(T)\equiv\det(I-TW_p)\bmod p,$$

Our strategy is to compute W_p then lift $L_p(T)$ from $(\mathbb{Z}/p\mathbb{Z})[T]$ to $\mathbb{Z}[T]$.

Target coefficients of f^{p-1} for $p = 7$:



Coefficient relations

Let $\partial_x = x\frac{\partial}{\partial x}$ (degree-preserving). The relations

$$f^{p-1} = f \cdot f^{p-2} \qquad \text{and} \qquad \partial_x f^{p-1} = -(\partial_x f) f^{p-2}$$

yield the relation

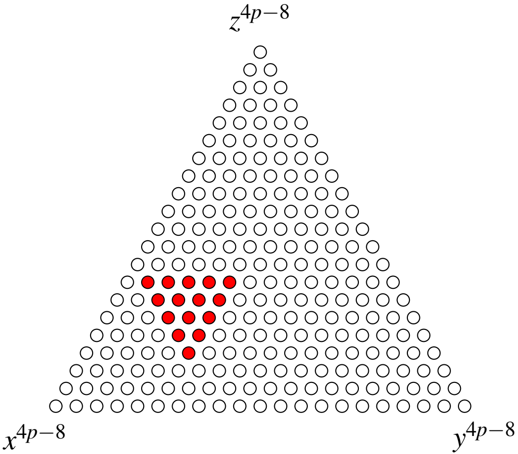
$$\sum_{i'+j'+k'=4} (i+i') f_{i',j',k'} f^{p-2}_{i-i',j-j',k-k'} = 0.$$

among nearby coefficients of f^{p-2} (a triangle of side length 5).

Replacing ∂_x by ∂_y yields a similar relation (replace $i+i'$ with $j+j'$).

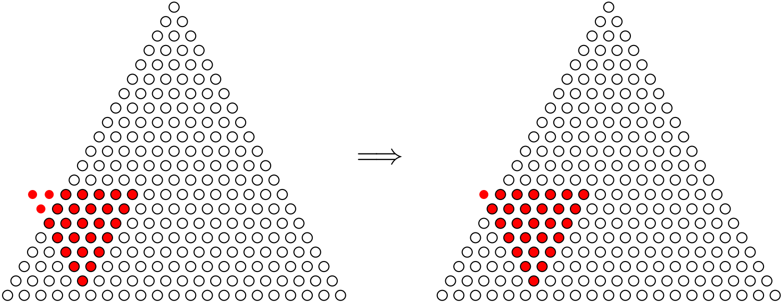
Coefficient triangle

For $p = 7$ with $i = 12, j = 5, k = 7$ the related coefficients of f^{p-2} are:



Moving the triangle

Now consider a bigger triangle with side length 7.
Our relations allow us to move the triangle around:



An initial "triangle" at the edge can be efficiently computed using coefficients of $f(x,0,z)^{p-2}$.

Computing one Hasse-Witt matrix

Nondegeneracy: we need $f(1,0,0), f(0,1,0), f(0,0,1)$ nonzero and $f(0,y,z), f(x,0,z), f(x,y,0)$ squarefree (easily achieved for large p).

The basic strategy to compute W_p is as follows:

- There is a 28×28 matrix M_j that shifts our 7-triangle from y-coordinate j to $j+1$; its coefficients depend on j and f. In fact a 16×16 matrix M_i suffices (use smoothness of C).
- Applying the product $M_0 \cdots M_{p-2}$ to an initial triangle on the edge and applying a final adjustment to shift from f^{p-2} to f^{p-1} gets us one column of the Hasse-Witt matrix W_p.
- By applying the same product (or its inverse) to different initial triangles we can compute all three columns of W_p.

We have thus reduced the problem to computing $M_1 \cdots M_{p-2} \bmod p$.

An average polynomial-time algorithm

Now let $C/\mathbb{Q}$ be smooth plane quartic $f(x,y,z)=0$ with $f\in\mathbb{Z}[x,y,z]$. We want to compute W_p for all good $p\le N$.

Key idea

The matrices M_j do not depend on p; view them as integer matrices. It suffices to compute $M_0\cdots M_{p-2} \mod p$ for all good $p\le N$.

Using an *accumulating remainder tree* we can compute all of these partial products in time $O(N(\log N)^{3+o(1)})$.

This yields an average time of $O((\log p)^{4+o(1)})$ per prime to compute the W_p for all good $p\le N$.*

*We may need to skip $O(1)$ primes p where C_p is degenerate; these can be handled separately using an $\tilde{O}(p^{1/2})$ algorithm based on the same ideas.

Accumulating remainder tree

Given matrices $M_0, \ldots, M_{n-1}$ and moduli $m_1, \ldots, m_n$, to compute

$$\begin{aligned} M_0 &\bmod m_1 \\ M_0 M_1 &\bmod m_2 \\ M_0 M_1 M_2 &\bmod m_3 \\ M_0 M_1 M_2 M_3 &\bmod m_4 \\ &\cdots \\ M_0 M_1 \cdots M_{n-2} M_{n-1} &\bmod m_n \end{aligned}$$

multiply adjacent pairs and recursively compute

$$\begin{aligned} (M_0 M_1) &\bmod m_2 m_3 \\ (M_0 M_1)(M_2 M_3) &\bmod m_4 m_5 \\ &\cdots \\ (M_0 M_1) \cdots (M_{n-2} M_{n-1}) &\bmod m_{n-1} m_n \end{aligned}$$

and adjust the results as required.

Timings for genus 3 curves

N	`costa-AKR`	`non-hyp-avgpoly`	`hyp-avgpoly`
2^{12}	18.2	1.1	0.1
2^{13}	49.1	2.6	0.2
2^{14}	142	5.8	0.5
2^{15}	475	13.6	1.5
2^{16}	1,670	30.6	4.6
2^{17}	5,880	70.9	12.6
2^{18}	22,300	158	25.9
2^{19}	78,100	344	62.1
2^{20}	297,000	760	147
2^{21}	1,130,000	1,710	347
2^{22}	4,280,000	3,980	878
2^{23}	16,800,000	8,580	1,950
2^{24}	66,800,000	18,600	4,500
2^{25}	244,000,000	40,800	10,700
2^{26}	972,000,000	91,000	24,300

(Intel Xeon E7-8867v3 3.3 GHz CPU seconds).