

Stronger arithmetic equivalence

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Dedekind zeta functions

Let K be a number field. The **Dedekind zeta** function

$$\zeta_K(s) := \sum_{n \geq 1} a_n n^{-s} := \sum_I N(I)^{-s} = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}$$

is holomorphic and nonvanishing on $\operatorname{Re}(s) > 1$ and extends to a meromorphic function on \mathbb{C} with a simple pole at $s = 1$.

Each $a_n = \#\{I : N(I) = n\}$ is determined by $\zeta_K(s)$ (independent of K).

If we define $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s} \Gamma(s)$, let r_1 and r_2 count the number of real and complex places of K , then

$$Z_K(s) := |D_K|^{s/2} \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_K(s),$$

satisfies the functional equation

$$Z_K(s) = Z_K(1 - s).$$

Decomposition types

Let $K = \mathbb{Q}(\alpha)$ be a number field with $G := \text{Gal}(L/\mathbb{Q})$ and $H := \text{Gal}(L/K)$, and let S the set of $\bar{\mathbb{Q}}$ -conjugates of α .

The **decomposition type** of a rational prime p in K is the multiset $\{f_{\mathfrak{p}} : \mathfrak{p}|p\}$, where $f_{\mathfrak{p}} := [\mathbb{F}_{\mathfrak{p}} : \mathbb{F}_p] = \log_p N(\mathfrak{p})$ is the residue field degree.

Each of the following is uniquely determined by the others

- The Dedekind zeta function $\zeta_K(s)$;
- The integers $a_p, a_{p^2}, \dots, a_{p^d}$ for all primes p , where $d := [K : \mathbb{Q}]$.
- The decomposition types of all rational primes p ;
- The cycle types of the permutations of S given by the action of absolute Frobenius elements σ_p .

One can replace “all” with “all but finitely many”, a_p, \dots, a_{p^d} with a_p , and absolute Frobenii with their projections to G .

Arithmetic equivalence

Definition

Number fields K_1 and K_2 are **arithmetically equivalent** if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$. The fields $K_1 \sim K_2$ must have the same degree and Galois closure L .

Let $G := \text{Gal}(L/\mathbb{Q})$, $H_1 := \text{Gal}(L/K_1)$, and $H_2 := \text{Gal}(L/K_2)$.

Definition

A **Gassmann triple** (G, H_1, H_2) consists of finite groups $H_1, H_2 \leq G$ that satisfy $\#(H_1 \cap C) = \#(H_2 \cap C)$ for every G -conjugacy class C . We then say that $H_1 \sim H_2$ are **Gassmann equivalent** (as subgroups of G).

Theorem (Gassmann 1926)

$K_1 \sim K_2$ if and only if $H_1 \sim H_2$.

Note that K_1, K_2 are conjugate if and only if H_1, H_2 are conjugate.

Examples

Let $G = \mathrm{GL}_2(\mathbb{F}_3)$, let $H_1 = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \in G \right\}$, and let $H_2 = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in G \right\}$. Then (G, H_1, H_2) is a non-trivial Gassmann triple (de Smit, ANTS III).

Let E/\mathbb{Q} be an elliptic curve with surjective mod-3 Galois image and let $L = \mathbb{Q}(E[3])$. then $\mathrm{Gal}(L/\mathbb{Q}) \simeq G$ and the fields $K_1 := L^{H_1}$ and $K_2 = L^{H_2}$ are non-conjugate arithmetically equivalent number fields of degree 8.

One can achieve degree 7 using similar subgroups of $\mathrm{SL}_3(\mathbb{F}_2)$, which is best possible (Bosma–de Smit, ANTS V, de Smit–Lenstra 2000).

Lemma

Finite groups H_1 and H_2 occur as elements of a Gassmann triple (G, H_1, H_2) if and only if they have the same order statistics.

It follows that Gassmann equivalence does not imply isomorphism: take $(\mathbb{Z}/p\mathbb{Z})^3$ and $H_3(\mathbb{F}_p)$ ($p \geq 3$), or $\langle 16, 3 \rangle$ and $\langle 16, 10 \rangle$, for example.

Gassmann triples in other contexts

Gassmann triples (G, H_1, H_2) arise in many other contexts involving potentially non-isomorphic objects with the same “zeta function”:

- If $\pi: M \rightarrow M_0$ is a normal finite Riemannian covering with deck group G then M/H_1 , and M/H_2 are isospectral (Sunada 1985).
- If Γ is a finite graph with $G = \text{Aut}(\Gamma)$ then Γ/H_1 and Γ/H_2 are isospectral (Halbeisen–Hungerbühler 1995).
- If X/k is a projective curve with $G = \text{Aut}(X)$, then X/H_1 and X/H_2 have isogenous Jacobians (Prasad–Rajan 2003).
- If $\pi: X \rightarrow Y$ is a Galois étale cover of k -varieties then X/H_1 and X/H_2 have isomorphic Chow motives (Arapura et al. 2017).

Unlike the number field case, non-trivial Gassmann triples may yield isomorphic objects (imposing further conditions prevents this), and zeta function equality does not always force Gassmann equivalence.

Characterizations of Gassmann triples

Let $[G/H]$ be the transitive G -set consisting of cosets of H .

Let $\chi_H: G \rightarrow \mathbb{Z}$ be the permutation character $g \mapsto \#[G/H]^g$ (equivalently, the character of 1_H^G), and define $\chi_H(K) := \#[G/H]^K$ for all subgroups K of G . Then

$$\chi_H(K) \neq 0 \iff K \leq_G H$$

(indeed, $HgK = Hg \iff gKg^{-1} \subseteq H$).

Proposition

For $H_1, H_2 \leq G$ the following are equivalent:

- 1 $\#(H_1 \cap C) = \#(H_2 \cap C)$ for all $C \in \text{conj}(G)$.
- 2 There is a G -conjugacy preserving bijection $H_1 \leftrightarrow H_2$.
- 3 $\chi_{H_1}(K) = \chi_{H_2}(K)$ for all cyclic subgroups $K \leq G$ (or all $K \leq H_1, H_2$).
- 4 $\mathbb{Q}[G/H_1] \simeq \mathbb{Q}[G/H_2]$ (as $\mathbb{Q}[G]$ modules).

How strong is arithmetic equivalence?

Let K_1 and K_2 be arithmetically equivalent number fields.

Theorem (Perlis 1977)

The number fields K_1 and K_2 have the same degree, discriminant, signature, roots of unity, normal closure, and normal core.

The analytic class number formula

$$\lim_{s \rightarrow 1^+} (s-1)\zeta_K(s) = \frac{2^{r_1} (2\pi)^{r_2} h_K R_K}{\#\mu_K |D_K|^{1/2}}$$

implies $R_{K_1} h_{K_1} = R_{K_2} h_{K_2}$, but the class numbers and regulators may differ.

There is a bijection of the places of K_1 and K_2 that preserves residue field degrees, but not necessarily ramification indices.

The adèle rings of K_1 and K_2 need not be isomorphic.

Stronger notions of arithmetic equivalence

Definition

Two number fields are **locally isomorphic** if there is a bijection of their places such that corresponding completions are isomorphic.

Locally isomorphic fields are arithmetically equivalent (Klingen 1998).

Proposition (Iwasawa 1953)

Two number fields K_1, K_2 are locally isomorphic if and only if they have isomorphic rings of adèles $\mathbb{A}_{K_1} \simeq \mathbb{A}_{K_2}$.

Proposition (Linowitz–McReynolds–Miller 2017)

Locally isomorphic number fields have isomorphic Brauer groups.

But locally isomorphic fields may have distinct class numbers, as happens with $\mathbb{Q}(\sqrt[8]{-33})$ and $\mathbb{Q}(\sqrt[8]{-33 \cdot 16})$ (de Smit–Perlis, 1994).

Local integral equivalence

A finite group K is *p -cyclic* (or *p -hypoelementary*) if the quotient of K by the intersection of its p -Sylow subgroups (its p -core) is cyclic.

Proposition

Let p be a prime. For $H_1, H_2 \leq G$ the following are equivalent:

- $\chi_{H_1}(K) = \chi_{H_2}(K)$ for all p -cyclic $K \leq G$ (or all $K \leq H_1, H_2$);
- $\mathbb{Z}_p[G/H_1] \simeq \mathbb{Z}_p[G/H_2]$;
- $\mathbb{F}_p[G/H_1] \simeq \mathbb{F}_p[G/H_2]$;
- $\det(\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H_1], \mathbb{Z}[G/H_2])) \notin p\mathbb{Z}$.

If $\mathbb{Z}_p[G/H_1] \simeq \mathbb{Z}_p[G/H_2]$ for all p we have *local integral equivalence*.

Theorem (Perlis 1978)

Number fields K_1, K_2 corresponding to a locally integrally equivalent $H_1, H_2 \leq G$ have isomorphic class groups.

Integral equivalence

Definition

Subgroups $H_1, H_2 \leq G$ are **integrally equivalent** if $\mathbb{Z}[G/H_1] \simeq \mathbb{Z}[G/H_2]$.

Let $H_1, H_2 \leq G$ have index n , let $\rho_1, \rho_2: G \rightarrow S_n$ be the representations corresponding to the permutation modules $\mathbb{Z}[G/H_1], \mathbb{Z}[G/H_2]$.

Fix an ordering of $[G/H_1]$ and $[G/H_2]$. We may represent elements of $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H_1], \mathbb{Z}[G/H_2])$ by matrices $M \in \mathbb{Z}^{n \times n}$ that satisfy

$$M_{ij} = M_{\rho_1(g)(i), \rho_2(g)(j)} \quad \text{for all } g \in G.$$

- **rational equivalence**: $\exists M \det(M) \neq 0$
- **local integral equivalence**: $\exists M_i \gcd(\det(M_1), \dots, \det(M_r)) = 1$
- **integral equivalence**: $\exists M \det(M) = \pm 1$

What we know about integral equivalence

Theorem (Prasad 2017)

Let $\pi: X \rightarrow Y$ be a Galois cover of nice curves over k with Galois group G . If $H_1, H_2 \leq G$ are integrally equivalent then $\text{Jac}(X/H_1) \simeq \text{Jac}(X/H_2)$.

Remark: Infinite families of non-isomorphic curves of low genus with isomorphic Jacobians were previously known (Howe 2005).

Essentially only one non-trivial example of integral equivalence is known: $G = \text{PSL}_2(\mathbb{F}_{29})$ with $H_1, H_2 \simeq A_5$ non-conjugate of index 203.

This example is due to Leonard Scott, who proved it by explicitly exhibiting $M \in \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H_1], \mathbb{Z}[G/H_2]) \subseteq \mathbb{Z}^{203 \times 203}$ with $\det M = 1$.

Similar triples exist for $p \equiv \pm 29 \pmod{120} \dots$

\dots but for $p = 149$ we already have to work with $M \in \mathbb{Z}^{27565 \times 27565}$.

What we don't know about integral equivalence

Question 1: Must integrally equivalent $H_1, H_2 \leq G$ be isomorphic?
How about locally integrally equivalent $H_1, H_2 \leq G$?

Both necessarily hold if $G = \mathrm{PSL}_2(\mathbb{F}_p)$. In fact, Gassmann equivalent subgroups of $\mathrm{PSL}_2(\mathbb{F}_p)$, $\mathrm{SL}_2(\mathbb{F}_p)$, $\mathrm{GL}_2(\mathbb{F}_p)$ are isomorphic (S 2016).

Scott's triple gives rise to infinitely many arithmetically equivalent non-isomorphic number fields of degree 203 that are also locally isomorphic, hence have isomorphic adèle rings.

But (as noted by Prasad), it is not clear that integral equivalence alone guarantees that we will get locally isomorphic number fields.

Question 2: Do locally integrally equivalent $H_1, H_2 \leq G$ give rise to locally isomorphic number fields? If not, does integral equivalence?

Solvable integral equivalence

Definition

Subgroups $H_1, H_2 \leq G$ are **solvably equivalent** if $\chi_{H_1}(K) = \chi_{H_2}(K)$ for all solvable subgroups $K \leq G$.

Solvable equivalence implies local integral equivalence and that corresponding number fields are locally isomorphic.

Proposition

Number fields K_1, K_2 corresponding to solvably equivalent $H_1, H_2 \leq G$ are arithmetically equivalent, locally isomorphic, and have the same class number. Moreover, there is a bijection of the places of K_1 and K_2 that preserves residue degrees and ramification indices.

Remark: Solvable equivalence is stronger than necessary

Results

Proposition

There are infinitely many non-isomorphic pairs of degree 32 number fields arising from locally (not globally) integrally equivalent $H_1, H_2 \leq G$.

Proposition

There are infinitely many non-isomorphic pairs of degree 96 number fields arising from solvably (not integrally) equivalent $H_1, H_2 \leq G$.

These results are effective; we can construct explicit examples.

The fact that $\mathrm{PSL}_2(\mathbb{F}_p)$ contains non-conjugate solvably equivalent subgroups H_1, H_2 for all primes $p \equiv \pm 29 \pmod{120}$ implies that there are infinitely many non-isomorphic pairs of number fields arising from infinitely many solvably equivalent $H_1, H_2 \leq G$, but the degrees of these fields is at least 203 and they hard to construct explicitly.

First example

An exhaustive search of the 11,759,892 groups of order less than 1024 finds exactly 74 groups G that contain non-conjugate locally integrally equivalent subgroups H_1, H_2 .

The smallest two have GAP ids [384, 18050] and [384, 18046], isomorphic to transitive permutation groups $32T9403$ and $32T9408$. Both are 2-extensions of $D_4 \times S_4$, which makes it easy to construct explicit examples.

For instance, the polynomials

$$\begin{aligned}x^{32} + 12x^{28} + 72x^{24} + 120x^{20} - 234x^{16} + 108x^{12} + 396x^8 - 432x^4 + 81, \\x^{32} - 12x^{28} + 72x^{24} - 120x^{20} - 234x^{16} - 108x^{12} + 396x^8 + 432x^4 + 81\end{aligned}$$

both have Galois group $G = 32T9403$. They define non-isomorphic fields K_1, K_2 corresponding to locally integrally equivalent $H_1, H_2 \leq G$.

First example (continued)

We can view each $M \in \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H_1], \mathbb{Z}[G/H_2])$ as 32×32 matrix with entries $x_1, \dots, x_8 \in \mathbb{Z}$. corresponding to the decomposition of G into eight double cosets $H_1 g H_2$. A (non-trivial) calculation finds that

$$\begin{aligned} \det M = & - (2(x_2 - x_3)^2 + 3(x_5 - x_6)^2)^8 \\ & \cdot (2(x_1 - x_4) + (x_5 + x_6 - 2x_7))^6 \\ & \cdot (2(x_1 + x_2 + x_3 + x_4) - (x_5 + x_6 + 2x_7 + 4x_8))^3 \\ & \cdot (2(x_1 - x_2 - x_3 + x_4) - (x_5 + x_6 + 2x_7 - 4x_8))^3 \\ & \cdot (2(x_1 - x_4) - 3(x_5 + x_6 - 2x_7))^2 \\ & \cdot (2(x_1 + x_2 + x_3 + x_4) + 3(x_5 + x_6 + 2x_7 + 4x_8)) \\ & \cdot (2(x_1 - x_2 - x_3 + x_4) + 3(x_5 + x_6 + 2x_7 - 4x_8)). \end{aligned}$$

One can choose the x_i so that $\det M = 2^{32}$, and so that $\det M = 3^{12}$. Thus H_1 and H_2 are locally integrally equivalent, but they not integrally equivalent because there is no choice of the x_i for which $\det M = \pm 1$.

This negatively answers a question of Guralnick–Weiss from 1993.

$M :=$

x_8	x_8	x_5	x_8	x_8	x_7	x_8	x_5	x_8	x_2	x_1	x_8	x_6	x_8	x_8	x_7	x_7	x_8	x_7	x_4	x_3	x_3	x_1	x_7	x_8	x_6	x_2	x_4	x_5	x_6	x_7	x_8
x_8	x_8	x_7	x_8	x_8	x_5	x_8	x_7	x_8	x_3	x_4	x_8	x_7	x_8	x_8	x_5	x_6	x_8	x_5	x_4	x_2	x_2	x_1	x_6	x_8	x_7	x_3	x_1	x_7	x_7	x_6	x_8
x_7	x_6	x_8	x_3	x_7	x_8	x_5	x_1	x_2	x_8	x_7	x_5	x_8	x_3	x_7	x_8	x_8	x_2	x_1	x_5	x_8	x_8	x_6	x_8	x_6	x_4	x_8	x_7	x_8	x_8	x_4	x_7
x_8	x_1	x_6	x_8	x_8	x_2	x_8	x_5	x_8	x_7	x_8	x_4	x_5	x_8	x_1	x_7	x_7	x_8	x_7	x_8	x_5	x_6	x_8	x_2	x_8	x_6	x_7	x_8	x_3	x_3	x_7	x_4
x_8	x_8	x_6	x_8	x_8	x_7	x_8	x_6	x_8	x_2	x_4	x_8	x_5	x_8	x_8	x_7	x_7	x_8	x_7	x_1	x_3	x_3	x_4	x_7	x_8	x_5	x_2	x_1	x_6	x_5	x_7	x_8
x_5	x_7	x_8	x_2	x_6	x_8	x_7	x_4	x_3	x_8	x_5	x_7	x_8	x_2	x_5	x_8	x_8	x_3	x_1	x_7	x_8	x_8	x_7	x_8	x_7	x_1	x_8	x_6	x_8	x_8	x_4	x_6
x_8	x_4	x_7	x_8	x_8	x_3	x_8	x_7	x_8	x_5	x_8	x_1	x_7	x_8	x_1	x_5	x_6	x_8	x_6	x_8	x_7	x_7	x_8	x_3	x_8	x_7	x_6	x_8	x_2	x_2	x_5	x_4
x_8	x_8	x_7	x_8	x_8	x_6	x_8	x_7	x_8	x_3	x_1	x_8	x_7	x_8	x_8	x_6	x_5	x_8	x_6	x_1	x_2	x_2	x_4	x_5	x_8	x_7	x_3	x_4	x_7	x_7	x_5	x_8
x_5	x_7	x_8	x_7	x_6	x_8	x_7	x_8	x_6	x_1	x_3	x_7	x_8	x_7	x_6	x_8	x_8	x_5	x_8	x_2	x_1	x_4	x_2	x_8	x_7	x_8	x_4	x_3	x_8	x_8	x_8	x_5
x_4	x_8	x_2	x_8	x_1	x_6	x_4	x_7	x_8	x_5	x_8	x_8	x_2	x_8	x_8	x_3	x_3	x_8	x_5	x_8	x_7	x_7	x_8	x_5	x_1	x_7	x_6	x_8	x_7	x_6	x_8	
x_7	x_3	x_8	x_5	x_7	x_1	x_6	x_8	x_7	x_8	x_7	x_3	x_8	x_6	x_2	x_8	x_8	x_7	x_8	x_5	x_8	x_8	x_6	x_4	x_5	x_8	x_6	x_8	x_7	x_1	x_4	x_8
x_4	x_8	x_3	x_8	x_1	x_7	x_1	x_6	x_8	x_7	x_8	x_3	x_8	x_8	x_8	x_2	x_2	x_8	x_7	x_8	x_5	x_6	x_8	x_7	x_4	x_5	x_7	x_8	x_5	x_6	x_7	x_8
x_7	x_5	x_8	x_3	x_7	x_8	x_6	x_4	x_2	x_8	x_7	x_6	x_8	x_3	x_7	x_8	x_8	x_2	x_4	x_6	x_8	x_8	x_5	x_8	x_5	x_1	x_8	x_7	x_8	x_8	x_1	x_7
x_8	x_4	x_5	x_8	x_8	x_2	x_8	x_6	x_8	x_7	x_8	x_1	x_6	x_8	x_4	x_7	x_7	x_8	x_7	x_8	x_6	x_5	x_8	x_2	x_8	x_5	x_7	x_8	x_3	x_3	x_7	x_1
x_7	x_5	x_8	x_5	x_7	x_8	x_5	x_8	x_7	x_1	x_2	x_6	x_8	x_6	x_7	x_8	x_7	x_8	x_3	x_4	x_1	x_3	x_8	x_6	x_8	x_4	x_2	x_8	x_8	x_8	x_7	x_8
x_1	x_8	x_3	x_8	x_4	x_7	x_4	x_5	x_8	x_7	x_8	x_3	x_8	x_8	x_8	x_2	x_2	x_8	x_7	x_8	x_6	x_5	x_8	x_7	x_1	x_6	x_7	x_8	x_6	x_5	x_7	x_8
x_5	x_2	x_8	x_7	x_6	x_4	x_7	x_8	x_5	x_8	x_6	x_2	x_8	x_7	x_3	x_8	x_8	x_6	x_8	x_7	x_8	x_8	x_7	x_1	x_7	x_8	x_7	x_8	x_5	x_1	x_4	x_8
x_1	x_8	x_2	x_8	x_4	x_5	x_1	x_7	x_8	x_6	x_8	x_8	x_2	x_8	x_8	x_3	x_3	x_8	x_6	x_8	x_7	x_7	x_8	x_6	x_4	x_7	x_5	x_8	x_7	x_7	x_5	x_8
x_6	x_7	x_8	x_2	x_5	x_8	x_7	x_1	x_3	x_8	x_6	x_7	x_8	x_2	x_6	x_8	x_8	x_3	x_4	x_7	x_8	x_8	x_7	x_8	x_7	x_4	x_8	x_5	x_8	x_8	x_1	x_5
x_8	x_1	x_7	x_8	x_8	x_3	x_8	x_7	x_8	x_6	x_8	x_4	x_7	x_8	x_4	x_6	x_5	x_8	x_5	x_8	x_7	x_7	x_8	x_3	x_8	x_7	x_5	x_8	x_2	x_2	x_6	x_1
x_8	x_8	x_5	x_1	x_8	x_7	x_8	x_3	x_1	x_7	x_8	x_8	x_6	x_4	x_8	x_7	x_7	x_4	x_2	x_8	x_5	x_6	x_8	x_7	x_8	x_3	x_7	x_8	x_6	x_5	x_2	x_8
x_3	x_7	x_4	x_7	x_3	x_8	x_2	x_8	x_5	x_8	x_5	x_7	x_1	x_7	x_6	x_1	x_4	x_6	x_8	x_7	x_8	x_7	x_8	x_7	x_8	x_2	x_8	x_6	x_8	x_8	x_8	x_5
x_8	x_8	x_7	x_4	x_8	x_5	x_8	x_2	x_1	x_5	x_8	x_8	x_7	x_1	x_8	x_6	x_5	x_4	x_3	x_8	x_7	x_7	x_8	x_6	x_8	x_6	x_8	x_2	x_6	x_8	x_7	x_3
x_2	x_6	x_4	x_5	x_2	x_8	x_3	x_8	x_7	x_8	x_7	x_5	x_1	x_6	x_7	x_4	x_1	x_7	x_8	x_6	x_8	x_8	x_5	x_8	x_3	x_8	x_8	x_7	x_8	x_8	x_8	x_7
x_8	x_8	x_6	x_4	x_8	x_7	x_8	x_3	x_4	x_7	x_8	x_8	x_5	x_1	x_8	x_7	x_7	x_1	x_2	x_8	x_6	x_5	x_8	x_7	x_8	x_3	x_7	x_8	x_5	x_6	x_2	x_8
x_6	x_7	x_8	x_7	x_5	x_8	x_7	x_8	x_5	x_4	x_3	x_7	x_8	x_7	x_5	x_8	x_8	x_6	x_8	x_2	x_4	x_1	x_2	x_8	x_7	x_8	x_1	x_3	x_8	x_8	x_8	x_6
x_7	x_3	x_8	x_6	x_7	x_4	x_5	x_8	x_7	x_8	x_7	x_3	x_8	x_5	x_2	x_8	x_7	x_8	x_7	x_8	x_6	x_8	x_5	x_1	x_6	x_8	x_8	x_7	x_4	x_1	x_8	x_2
x_6	x_2	x_8	x_7	x_5	x_1	x_7	x_8	x_6	x_8	x_5	x_2	x_8	x_7	x_3	x_8	x_8	x_5	x_8	x_7	x_8	x_8	x_7	x_4	x_7	x_8	x_6	x_4	x_1	x_8	x_3	
x_2	x_5	x_1	x_6	x_2	x_8	x_3	x_8	x_7	x_8	x_7	x_6	x_4	x_5	x_7	x_1	x_4	x_7	x_8	x_5	x_8	x_8	x_6	x_8	x_3	x_8	x_8	x_7	x_8	x_8	x_8	x_7
x_3	x_7	x_1	x_7	x_3	x_8	x_2	x_8	x_6	x_8	x_6	x_7	x_4	x_7	x_5	x_4	x_1	x_5	x_8	x_7	x_8	x_8	x_7	x_8	x_2	x_8	x_8	x_5	x_8	x_8	x_8	x_6
x_8	x_8	x_7	x_1	x_8	x_6	x_8	x_2	x_4	x_6	x_8	x_8	x_7	x_4	x_8	x_5	x_6	x_1	x_3	x_8	x_7	x_7	x_8	x_5	x_8	x_2	x_5	x_8	x_7	x_7	x_3	x_8
x_7	x_6	x_8	x_6	x_7	x_8	x_6	x_8	x_7	x_4	x_2	x_5	x_8	x_5	x_7	x_8	x_8	x_7	x_8	x_3	x_1	x_4	x_3	x_8	x_5	x_8	x_1	x_2	x_8	x_8	x_8	x_7

Here $x_1, \dots, x_8 \in \mathbb{Z}$ correspond to the decomposition of G into eight double cosets $H_1 g H_2$, consisting of 2, 2, 2, 2, 3, 3, 6, 12 right cosets of H_1 , respectively.

Second example

Let $G = 16T1654$ of order 5760. It contains non-conjugate $H_1, H_2 \simeq A_5$ of index 96 such that every proper subgroup of H_1 is a proper subgroup of H_2 .

The group G is the Galois group of an extension of $\mathbb{Q}[T]$; Hilbert irreducibility gives infinitely many examples of corresponding number fields, including:

$$x^{16} - 2x^{15} + 3x^{14} - 16x^{13} + 18x^{12} - 10x^{10} + 40x^9 - 39x^8 + 54x^7 + 23x^6 + 16x^5 - 140x^4 - 188x^3 - 28x^2 + 104x - 4,$$

Each $M \in \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H_1], \mathbb{Z}[G/H_2])$ has entries $x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}$, and

$$\begin{aligned} \det M = & - (5x_1 + 6x_2 + 10x_3 + 15x_4 + 60x_5) \\ & \cdot (x_1 - 6x_2 - 10x_3 + 3x_4 + 12x_5)^5 \\ & \cdot (3x_1 + 2x_2 - 2x_3 - 7x_4 + 4x_5)^{15} \\ & \cdot (3x_1 - 2x_2 + 2x_3 + x_4 - 4x_5)^{30} \\ & \cdot (x_1 + 2x_2 - 2x_3 + 3x_4 - 4x_5)^{45} \end{aligned}$$

No assignment of $x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}$ makes every factor in $\det M$ equal to ± 1 , so H_1 and H_2 are not integrally equivalent. One also has solvably integrally equivalent subgroups of $SL(2, p)$ for all $p \equiv \pm 29 \pmod{120}$.