

Sato-Tate groups of abelian threefolds

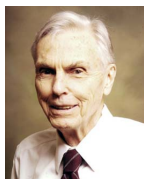
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Joint work with Francesc Fité and Kiran Kedlaya

Sato-Tate in dimension 1

Let E/\mathbb{Q} be an elliptic curve, say,

$$y^2 = x^3 + Ax + B,$$

and let p be a prime of good reduction (so $p \nmid \Delta(E)$).

The number of \mathbb{F}_p -points on the reduction E_p of E modulo p is

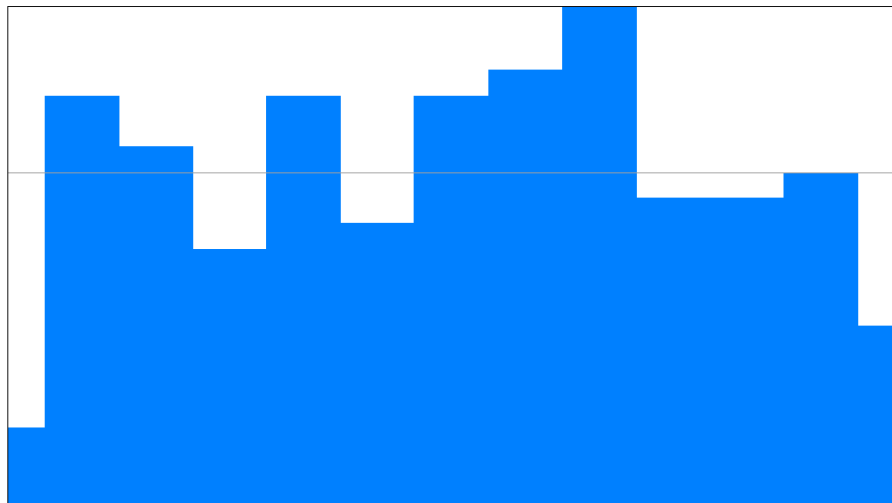
$$\#E_p(\mathbb{F}_p) = p + 1 - t_p,$$

where the trace of Frobenius t_p is an integer in $[-2\sqrt{p}, 2\sqrt{p}]$.

We are interested in the limiting distribution of $x_p = -t_p/\sqrt{p} \in [-2, 2]$, as p varies over primes of good reduction up to $N \rightarrow \infty$.

Sato-Tate distribution of a typical elliptic curve

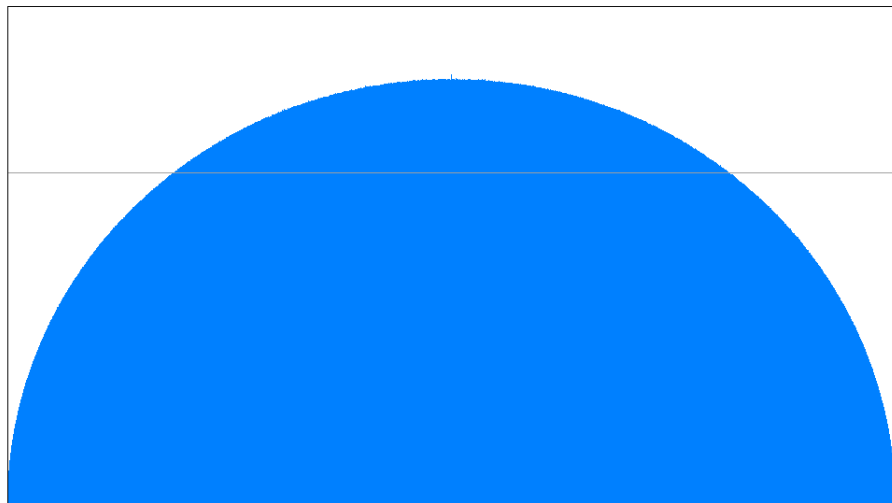
a1 histogram of $y^2 = x^3 + x + 1$ for $p \leq 2^{10}$
170 data points in 13 buckets, $z_1 = 0.029$, out of range data has area 0.018



Moments: 1 0.051 1.039 0.081 2.060 0.294 4.971 1.134 13.278 4.308 37.954

Sato-Tate distribution of a typical elliptic curve

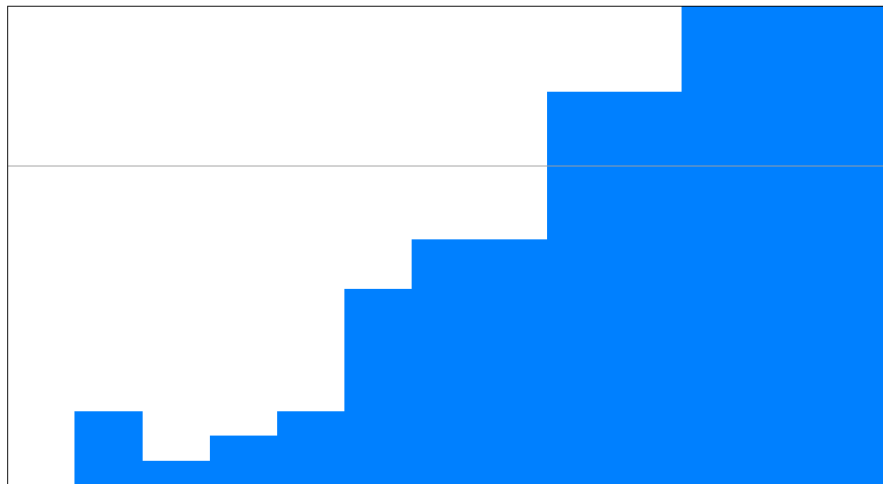
a1 histogram of $y^2 = x^3 + x + 1$ for $p \leq 2^{40}$
41203088794 data points in 202985 buckets



Moments: 1 0.000 1.000 0.000 2.000 0.000 5.000 0.000 14.000 0.000 41.999

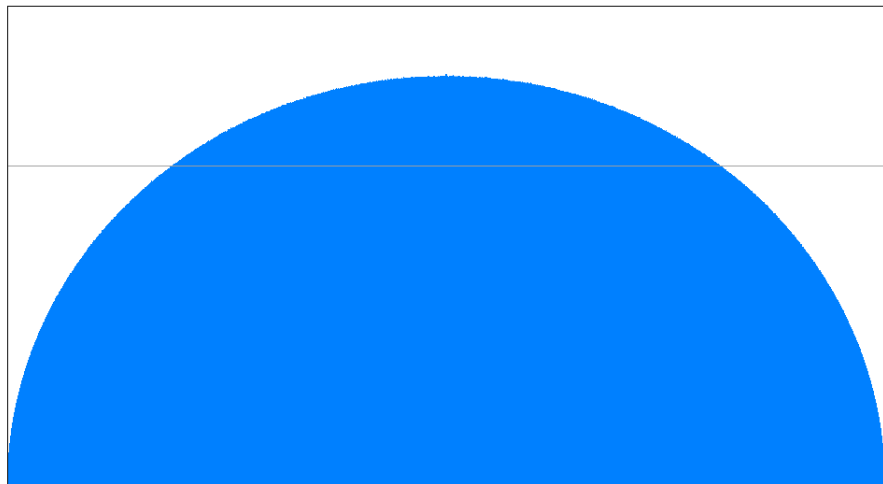
Sato-Tate distribution of another typical elliptic curve

a1 histogram of $y^2 + xy + y = x^3 - x^2 - 20067762415575526585033208209338542750930230312178956502x + 344816111795030556467032985690390720374855944359319180361266008296291939448732243429$ for $p \leq 2^{10}$
172 data points in 13 buckets, $z_1 = 0.023$, out of range data has area 0.250



Sato-Tate distribution of another typical elliptic curve

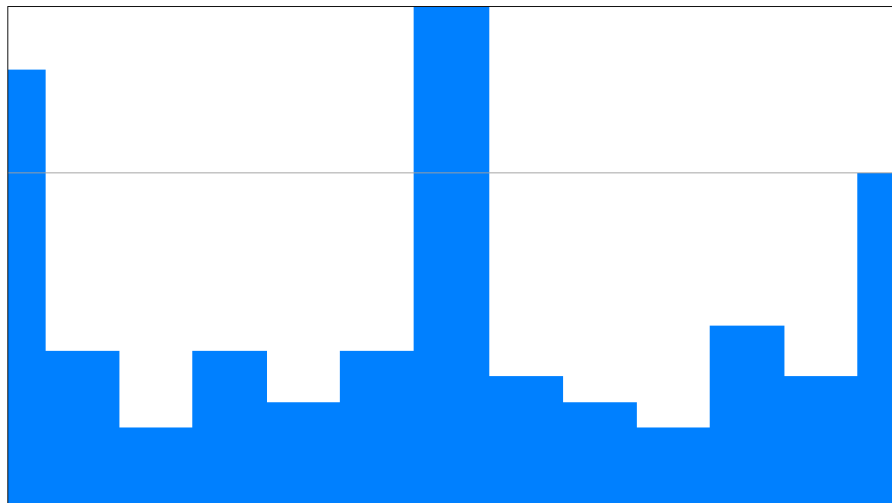
a1 histogram of $y^2 + xy + y = x^3 - x^2 - 20067762415575526585033208209338542750930230312178956502x + 34481611795030556467032985690390720374855944359319180361266008296291939448732243429$ for $p \leq 2^{40}$
41203088796 data points in 202985 buckets



Moments: 1 0.000 1.000 0.000 2.000 0.000 5.000 0.001 14.000 0.003 42.000

Sato-Tate distribution of an atypical elliptic curve

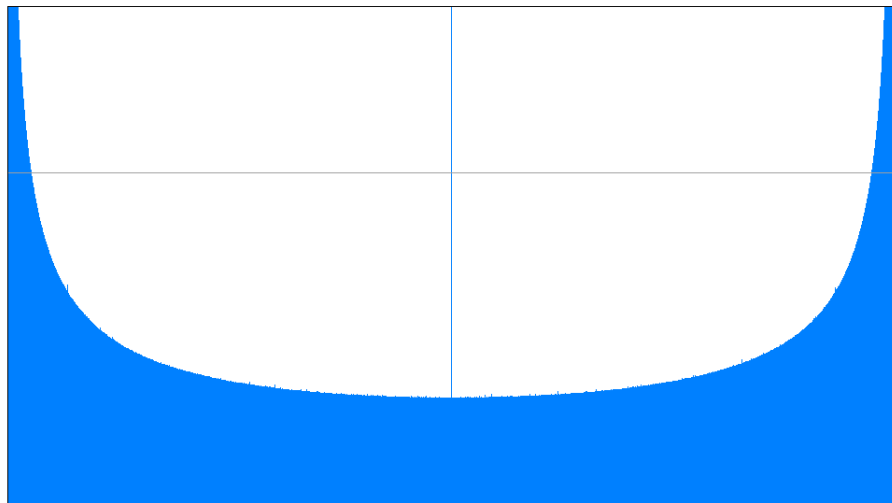
a1 histogram of $y^2 = x^3 + 1$ for $p \leq 2^{10}$
170 data points in 13 buckets, $z_1 = 0.518$, out of range data has area 0.418



Moments: 1 -0.044 0.934 -0.160 2.754 -0.660 9.051 -2.655 31.232 -10.427 110.831

Sato-Tate distribution of an atypical elliptic curve

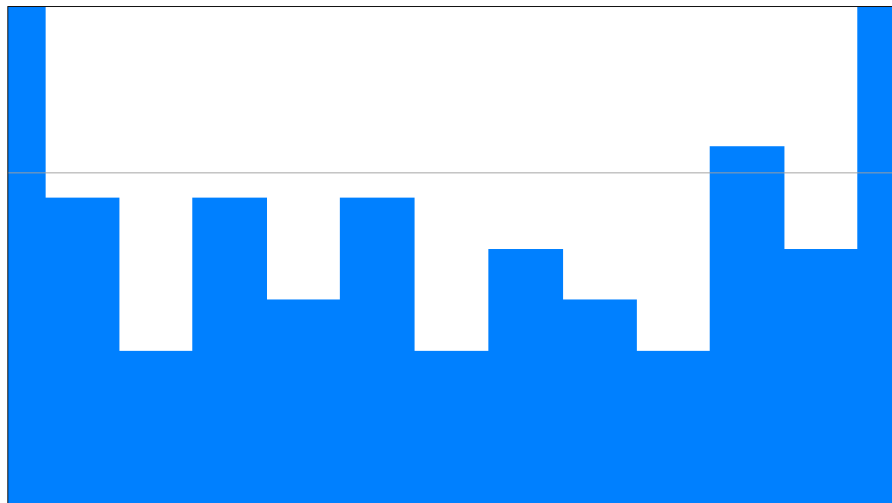
a1 histogram of $y^2 = x^3 + 1$ for $p \leq 2^{40}$
41203088794 data points in 202985 buckets, $z_1 = 0.500$, out of range data has area 0.534



Moments: 1 -0.000 1.000 -0.000 3.000 -0.000 10.000 -0.000 35.000 -0.000 126.000

Sato-Tate distribution of another atypical elliptic curve

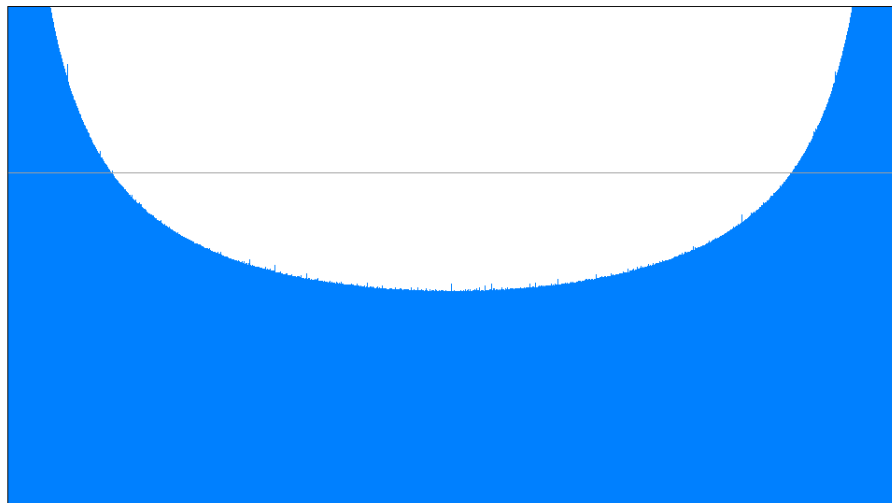
a1 histogram of $y^2 = x^3 + 1$ over $\mathbb{Q}(\sqrt{-3})$ for split $p \leq 2^{10}$
164 data points in 13 buckets, out of range data has area 0.122



Moments: 1 -0.092 1.935 -0.331 5.710 -1.368 18.765 -5.504 64.750 -21.616 229.771

Sato-Tate distribution of another atypical elliptic curve

a_1 histogram of $y^2 = x^3 + 1$ over $\mathbb{Q}(\sqrt{-3})$ for split $p \leq 2^{40}$
41203047020 data points in 202985 buckets, out of range data has area 0.137



Moments: 1 -0.000 2.000 -0.000 6.000 -0.000 20.000 -0.000 70.000 -0.000 252.000

Sato-Tate distributions in dimension 1

1. Typical case (no CM)

Elliptic curves E/\mathbb{Q} w/o CM have the semi-circular trace distribution. (Also known for E/k , where k is a totally real or CM number field).

[CHT08, Taylor08, HST10, BGG11, BGHT11, ACCGHHNSTT18]

2. Exceptional cases (CM)

Elliptic curves E/k with CM have one of two distinct trace distributions, depending on whether k contains the CM field or not.

[Hecke, Deuring, early 20th century]

Sato-Tate groups in dimension 1

The **Sato-Tate group** of E is a closed subgroup G of $SU(2) = USp(2)$ that is determined by the ℓ -adic Galois representation attached to E .

A refinement/generalization of the Sato-Tate conjecture states that the distribution of normalized Frobenius traces of E converges to the distribution of traces in its Sato-Tate group G (under its Haar measure).

| G | G/G^0 | E | k | $E[x_p^0], E[x_p^2], E[x_p^4] \dots$ |
|-----------|---------|---------------------|-------------------------|--------------------------------------|
| $SU(2)$ | C_1 | $y^2 = x^3 + x + 1$ | \mathbb{Q} | $1, 1, 2, 5, 14, 42, \dots$ |
| $N(U(1))$ | C_2 | $y^2 = x^3 + 1$ | \mathbb{Q} | $1, 1, 3, 10, 35, 126, \dots$ |
| $U(1)$ | C_1 | $y^2 = x^3 + 1$ | $\mathbb{Q}(\sqrt{-3})$ | $1, 2, 6, 20, 70, 252, \dots$ |

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Fun fact: in the non-CM case the Sato-Tate conjecture implies that $E[x_p^n] = \frac{1}{2\pi} \int_0^\pi (2 \cos \theta)^n \sin^2 \theta d\theta$ is the $\frac{n}{2}$ th Catalan number.

Zeta functions and L -polynomials

For a smooth projective curve X/\mathbb{Q} of genus g and each prime p of good reduction for X we have the **zeta function**

$$Z(X_p/\mathbb{F}_p; T) := \exp \left(\sum_{k=1}^{\infty} \#X_p(\mathbb{F}_{p^k}) T^k / k \right) = \frac{L_p(T)}{(1-T)(1-pT)},$$

where $L_p \in \mathbb{Z}[T]$ has degree $2g$. The **normalized L -polynomial**

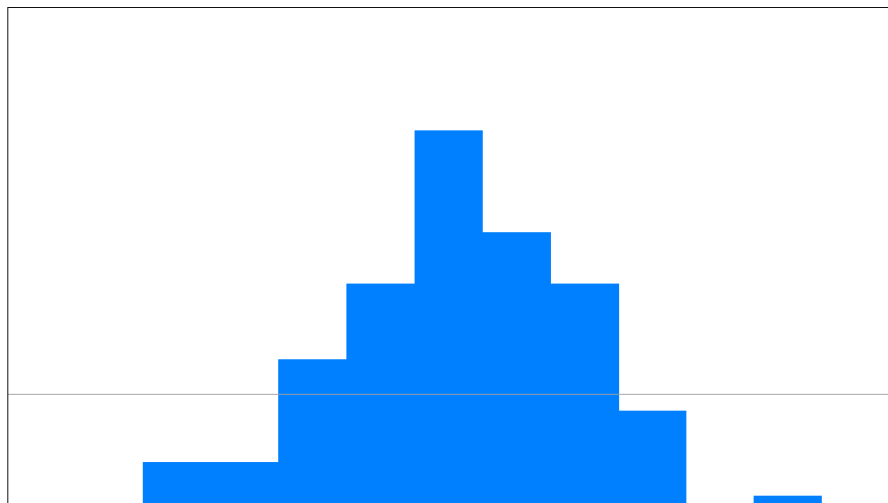
$$\bar{L}_p(T) := L_p(T/\sqrt{p}) = \sum_{i=0}^{2g} a_i T^i \in \mathbb{R}[T]$$

is monic, reciprocal, and unitary, with $|a_i| \leq \binom{2g}{i}$.

We can now consider the limiting distribution of a_1, a_2, \dots, a_g over all primes $p \leq N$ of good reduction, as $N \rightarrow \infty$.

Sato-Tate a_1 -distribution of a typical genus 2 curve

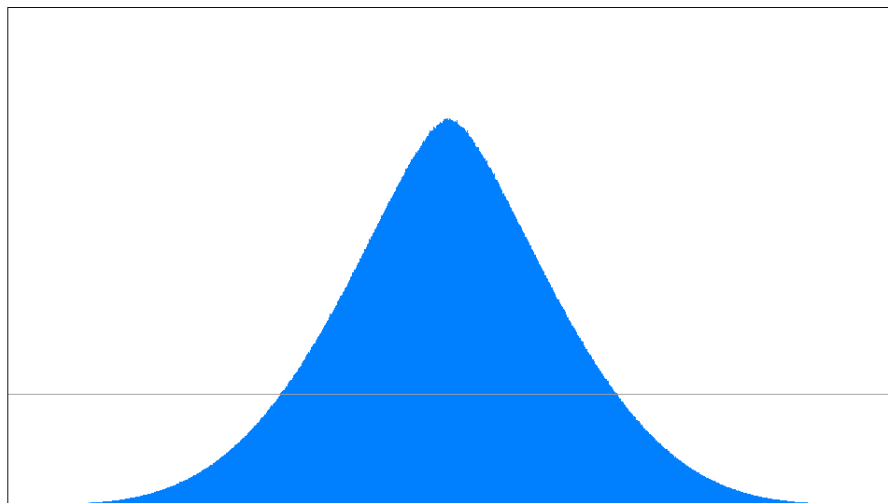
a_1 histogram of $y^2 = x^5 - x + 1$ for $p \leq 2^{10}$
167 data points in 13 buckets, $z_1 = 0.030$



Moments: 1 0.098 1.031 -0.011 3.041 -0.725 13.944 -3.026 81.644 4.428 547.633

Sato-Tate a_1 -distribution of a typical genus 2 curve

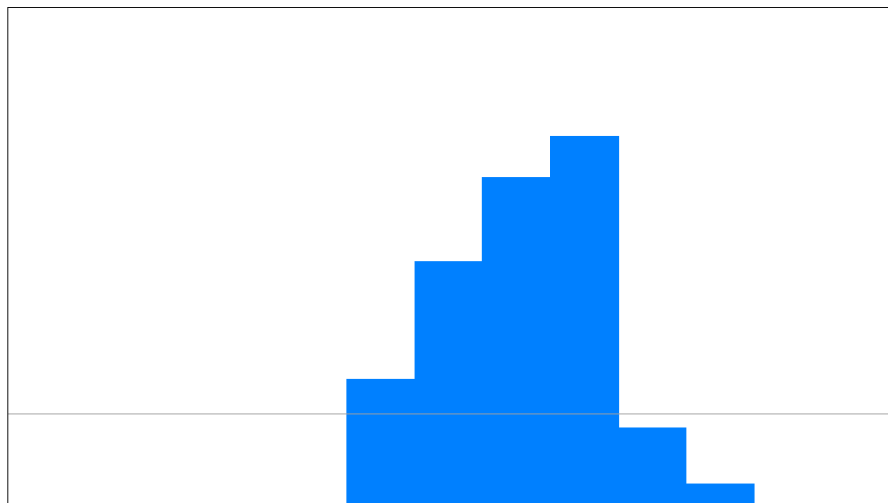
a_1 histogram of $y^2 = x^5 - x + 1$ for $p \leq 2^{32}$
203280216 data points in 14257 buckets



Moments: 1 0.000 1.000 0.000 2.999 0.002 13.999 0.009 84.014 0.054 594.283

Sato-Tate a_2 -distribution of a typical genus 2 curve

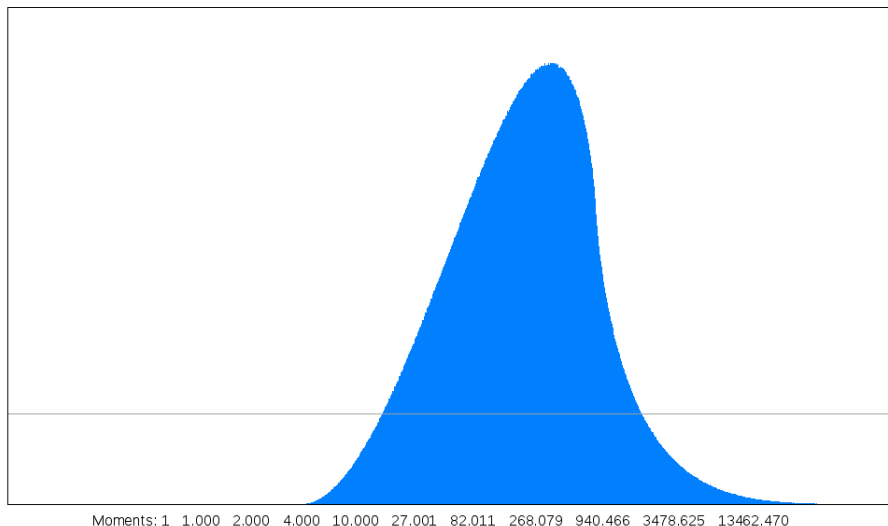
a_2 histogram of $y^2 = x^5 - x + 1$ for $p \leq 2^{10}$
167 data points in 13 buckets



Moments: 1 0.996 2.058 4.129 10.085 26.401 75.879 231.863 746.430 2496.195 8595.192

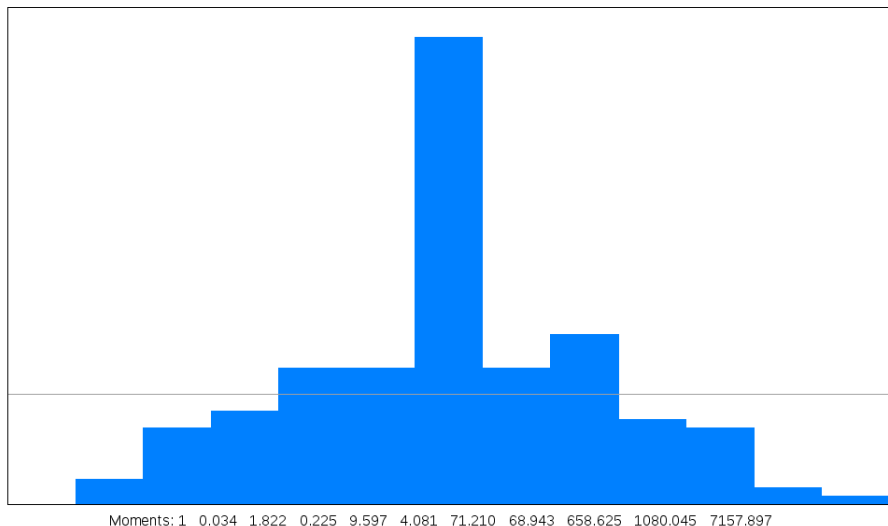
Sato-Tate a_2 -distribution of a typical genus 2 curve

a_2 histogram of $y^2 = x^5 - x + 1$ for $p \leq 2^{32}$
203280216 data points in 14257 buckets



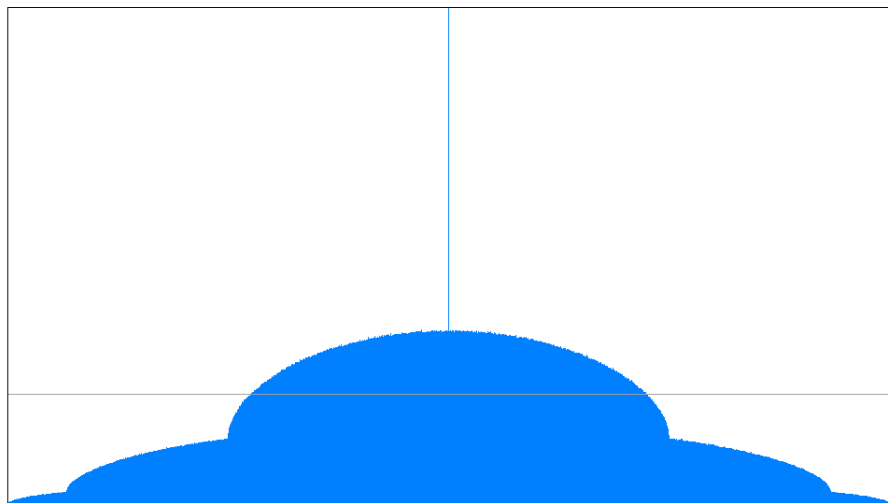
Sato-Tate a_1 -distribution of an atypical genus 2 curve

a_1 histogram of $y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x$ for $p \leq 2^{10}$
168 data points in 13 buckets, $z_1 = 0.196$



Sato-Tate a_1 -distribution of an atypical genus 2 curve

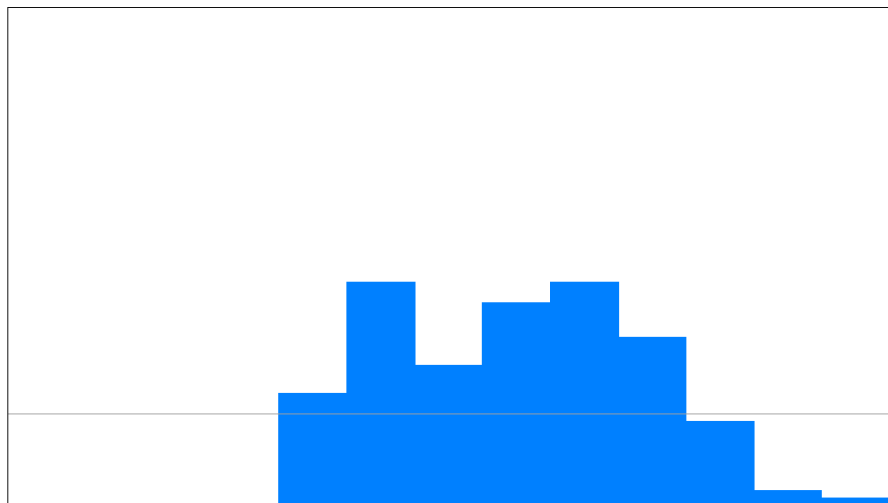
a_1 histogram of $y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x$ for $p \leq 2^{32}$
203280217 data points in 14257 buckets, $z_1 = 0.167$, out of range data has area 0.166



Moments: 1 0.000 2.000 0.000 11.999 0.003 99.983 0.030 979.773 0.286 10581.031

Sato-Tate a_2 -distribution of an atypical genus 2 curve

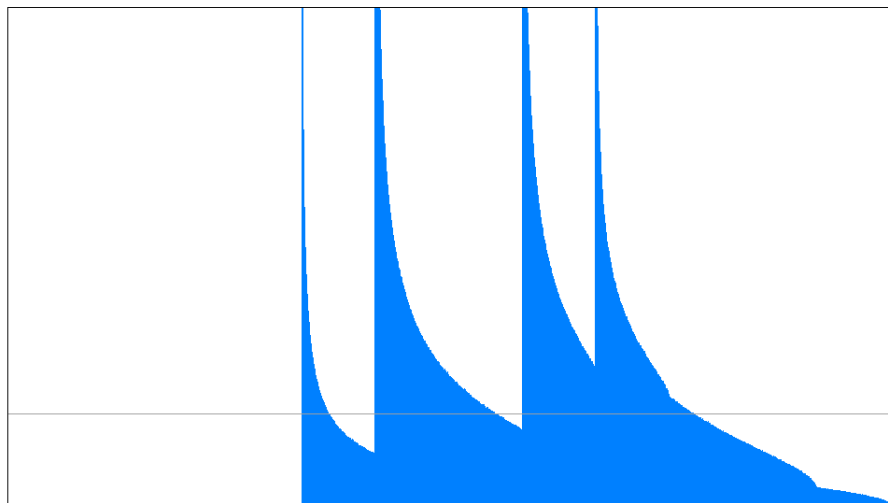
a_2 histogram of $y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x$ for $p \leq 2^{10}$
168 data points in 13 buckets, $z_2 = [0.006 \ 0.000 \ 0.000 \ 0.000 \ 0.012]$



Moments: 1 0.914 3.679 8.930 33.618 120.114 506.202 2236.335 10692.989 53523.391 278878.343

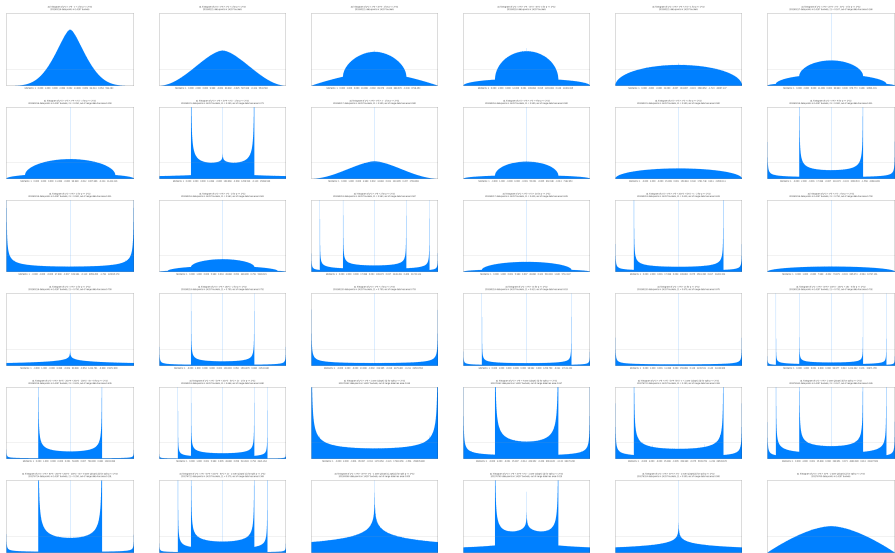
Sato-Tate a_2 -distribution of an atypical genus 2 curve

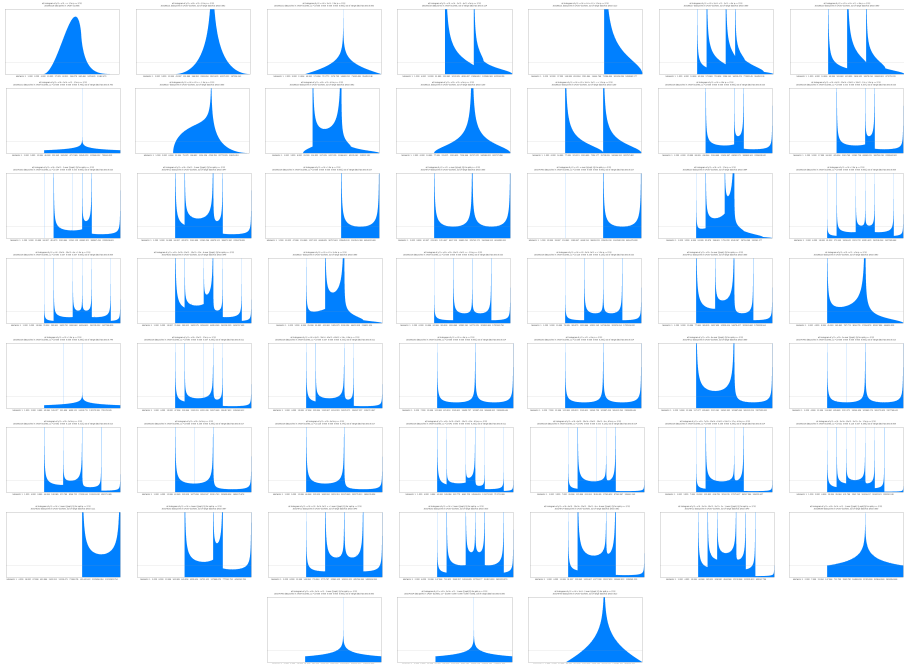
a_2 histogram of $y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x$ for $p \leq 2^{32}$
203280217 data points in 14257 buckets, out of range data has area 0.069



Moments: 1 1.000 4.000 10.999 43.994 171.969 753.838 3396.141 16015.474 77492.145 384452.151

Sato-Tate trace distributions of genus 2 curves:





L -polynomials of Abelian varieties

Let A be an abelian variety over a number field k and fix a prime ℓ . The action of $\text{Gal}(\bar{k}/k)$ on the ℓ -adic Tate module

$$V_\ell(A) := \varprojlim A[\ell^n] \otimes_{\mathbb{Z}} \mathbb{Q}$$

gives rise to a Galois representation

$$\rho_\ell: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \text{GSp}_{2g}(\mathbb{Q}_\ell).$$

For each prime \mathfrak{p} of good reduction for A we have the L -polynomial

$$L_{\mathfrak{p}}(T) := \det(1 - \rho_\ell(\text{Frob}_{\mathfrak{p}})T), \quad \bar{L}_{\mathfrak{p}}(T) := L_{\mathfrak{p}}(T/\sqrt{\|\mathfrak{p}\|}),$$

which appears as an Euler factor in the L -series

$$L(A, s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\|\mathfrak{p}\|^{-s})^{-1}.$$

The Sato-Tate group of an abelian variety

The Zariski closure of the image of

$$\rho_\ell: G_k \rightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$$

is a \mathbb{Q}_ℓ -algebraic group $G_\ell^{\mathrm{zar}} \subseteq \mathrm{GSp}_{2g}$, and we let $G_\ell^{1,\mathrm{zar}} := G_\ell^{\mathrm{zar}} \cap \mathrm{Sp}_{2g}$.
Now fix $\iota: \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, and let $G_{\ell,\iota}^{\mathrm{zar}}$ and $G_{\ell,\iota}^{1,\mathrm{zar}}$ denote base changes to \mathbb{C} .

Definition [Serre]

$\mathrm{ST}(A) \subseteq \mathrm{USp}(2g)$ is a maximal compact subgroup of $G_{\ell,\iota}^{1,\mathrm{zar}}(\mathbb{C})$ equipped with the map $s: \mathfrak{p} \mapsto \mathrm{conj}(\|\mathfrak{p}\|^{-1/2} \rho_{\ell,\iota}(\mathrm{Frob}_\mathfrak{p})) \in \mathrm{Conj}(\mathrm{ST}(A))$.

Note that the characteristic polynomial of $s(\mathfrak{p})$ is $\bar{L}_\mathfrak{p}(T)$.

The Sato-Tate conjecture for abelian varieties

Conjecture [Mumford-Tate, Algebraic Sato-Tate]

$(G_\ell^{\text{zar}})^0 = \text{MT}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, equivalently, $(G_\ell^{1,\text{zar}})^0 = \text{Hg}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

More generally, $(G_\ell^{\text{zar}}) = \text{AST}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

The algebraic Sato-Tate conjecture is known for $g \leq 3$ [BK15].

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Sato-Tate conjecture for abelian varieties.

The conjugacy classes $s(\mathfrak{p})$ are equidistributed with respect to $\mu_{\text{ST}(A)}$, the pushforward of the Haar measure to $\text{Conj}(\text{ST}(A))$.

The Sato-Tate conjecture implies that the distribution $\bar{L}_p(T)$ is given by the distribution of characteristic polynomials in $\text{ST}(A)$.

Sato-Tate axioms for abelian varieties

$G \subseteq \mathrm{USp}(2g)$ satisfies the Sato-Tate axioms (for abelian varieties) if:

- 1 **Compact:** G is closed;
- 2 **Hodge:** G contains a **Hodge circle** $\theta: \mathrm{U}(1) \rightarrow G^0$ whose elements $\theta(u)$ have eigenvalues $u, 1/u$ with multiplicity g , such that the conjugates of θ conjugates generate a dense subset of G ;
- 3 **Rationality:** for each component H of G and each irreducible character χ of $\mathrm{GL}_{2g}(\mathbb{C})$ we have $\mathbb{E}[\chi(\gamma) : \gamma \in H] \in \mathbb{Z}$;
- 4 **Lefschetz:** The subgroup of $\mathrm{USp}(2g)$ fixing $\mathrm{End}(\mathbb{C}^{2g})^{G^0}$ is G^0 .

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Theorem [FKRS12, FKS19]

Let A/k be an abelian variety of dimension $g \leq 3$.

Then $\mathrm{ST}(A)$ satisfies the Sato-Tate axioms.

Axioms 1-3 are expected to hold in general, but Axiom 4 fails for $g = 4$.
For any g , the set of G satisfying axioms 1-3 is **finite**.

Galois endomorphism types

Let A be an abelian variety defined over a number field k .

Let K be the minimal extension of k for which $\text{End}(A_K) = \text{End}(A_{\bar{k}})$.

$\text{Gal}(K/k)$ acts on the \mathbb{R} -algebra $\text{End}(A_K)_{\mathbb{R}} = \text{End}(A_K) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition

The *Galois endomorphism type* of A is the isomorphism class of $[\text{Gal}(K/k), \text{End}(A_K)_{\mathbb{R}}]$, where $[G, E] \simeq [G', E']$ iff there are isomorphisms $G \simeq G'$ and $E \simeq E'$ compatible with the group actions.

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Theorem [FKRS12]

For abelian varieties A/k of dimension $g \leq 3$ there is a one-to-one correspondence between Sato-Tate groups and Galois types.

More precisely, the identity component G^0 is uniquely determined by $\text{End}(A_K)_{\mathbb{R}}$ and $G/G^0 \simeq \text{Gal}(K/k)$ (with corresponding actions).

Real endomorphism algebras of abelian surfaces

| abelian surface | $\text{End}(A_K)_{\mathbb{R}}$ | $\text{ST}(A)^0$ |
|---|--------------------------------|----------------------|
| square of CM elliptic curve | $M_2(\mathbb{C})$ | $U(1)_2$ |
| <ul style="list-style-type: none">• QM abelian surface• square of non-CM elliptic curve | $M_2(\mathbb{R})$ | $SU(2)_2$ |
| <ul style="list-style-type: none">• CM abelian surface• product of CM elliptic curves | $\mathbb{C} \times \mathbb{C}$ | $U(1) \times U(1)$ |
| product of CM and non-CM elliptic curves | $\mathbb{C} \times \mathbb{R}$ | $U(1) \times SU(2)$ |
| <ul style="list-style-type: none">• RM abelian surface• product of non-CM elliptic curves | $\mathbb{R} \times \mathbb{R}$ | $SU(2) \times SU(2)$ |
| generic abelian surface | \mathbb{R} | $USp(4)$ |

(factors in products are assumed to be non-isogenous)

Sato-Tate groups of abelian surfaces

Theorem [FKRS12]

Up to conjugacy in $\mathrm{USp}(4)$, there are 52 Sato-Tate groups $\mathrm{ST}(A)$ that arise for abelian surfaces A/k over number fields; 34 occur for $k = \mathbb{Q}$.

$$\begin{aligned} \mathrm{U}(1)_2: & \quad C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O, \\ & \quad J(C_1), J(C_2), J(C_3), J(C_4), J(C_6), \\ & \quad J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O), \end{aligned}$$

$$\begin{aligned} & \quad C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1 \\ \mathrm{SU}(2)_2: & \quad E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6) \end{aligned}$$

$$\mathrm{U}(1) \times \mathrm{U}(1): \quad F, F_a, F_{a,b}, F_{ab}, F_{ac}$$

$$\mathrm{U}(1) \times \mathrm{SU}(2): \quad \mathrm{U}(1) \times \mathrm{SU}(2), N(\mathrm{U}(1) \times \mathrm{SU}(2))$$

$$\mathrm{SU}(2) \times \mathrm{SU}(2): \quad \mathrm{SU}(2) \times \mathrm{SU}(2), N(\mathrm{SU}(2) \times \mathrm{SU}(2))$$

$$\mathrm{USp}(4): \quad \mathrm{USp}(4)$$

This theorem says nothing about equidistribution, however this is now known in many special cases [FS12, Johansson13, Taylor18].

Maximal Sato-Tate groups of abelian surfaces

| G_0 | G/G_0 | X |
|--|------------------|--|
| $\mathrm{USp}(4)$ | C_1 | $y^2 = x^5 - x + 1$ |
| $\mathrm{SU}(2) \times \mathrm{SU}(2)$ | C_2 | $y^2 = x^6 + x^5 + x - 1$ |
| $\mathrm{U}(1) \times \mathrm{SU}(2)$ | C_2 | $y^2 = x^6 + 3x^4 - 2$ |
| $\mathrm{U}(1) \times \mathrm{U}(1)$ | D_2 | $y^2 = x^6 + 3x^4 + x^2 - 1$ |
| | C_4 | $y^2 = x^5 + 1$ |
| $\mathrm{SU}(2)_2$ | D_4 | $y^2 = x^5 + x^3 + 2x$ |
| | D_6 | $y^2 = x^6 + x^3 - 2$ |
| $\mathrm{U}(1)_2$ | $D_6 \times C_2$ | $y^2 = x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$ |
| | $S_4 \times C_2$ | $y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$ |

Each of the 9 maximal Sato-Tate groups in dimension 2 can be realized by the Jacobian of a genus 2 curve X/\mathbb{Q} .

One can now verify this using the algorithm of [CMSV19].

Maximal Sato-Tate groups of abelian surfaces

| G_0 | G/G_0 | X |
|--|------------------|--|
| $\mathrm{USp}(4)$ | C_1 | $y^2 = x^5 - x + 1$ |
| $\mathrm{SU}(2) \times \mathrm{SU}(2)$ | C_2 | $y^2 = x^6 + x^5 + x - 1$ |
| $\mathrm{U}(1) \times \mathrm{SU}(2)$ | C_2 | $y^2 = x^6 + 3x^4 - 2$ |
| $\mathrm{U}(1) \times \mathrm{U}(1)$ | D_2 | $y^2 = x^6 + 3x^4 + x^2 - 1$ |
| | C_4 | $y^2 = x^5 + 1$ |
| $\mathrm{SU}(2)_2$ | D_4 | $y^2 = x^5 + x^3 + 2x$ |
| | D_6 | $y^2 = x^6 + x^3 - 2$ |
| $\mathrm{U}(1)_2$ | $D_6 \times C_2$ | $y^2 = x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$ |
| | $S_4 \times C_2$ | $y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$ |

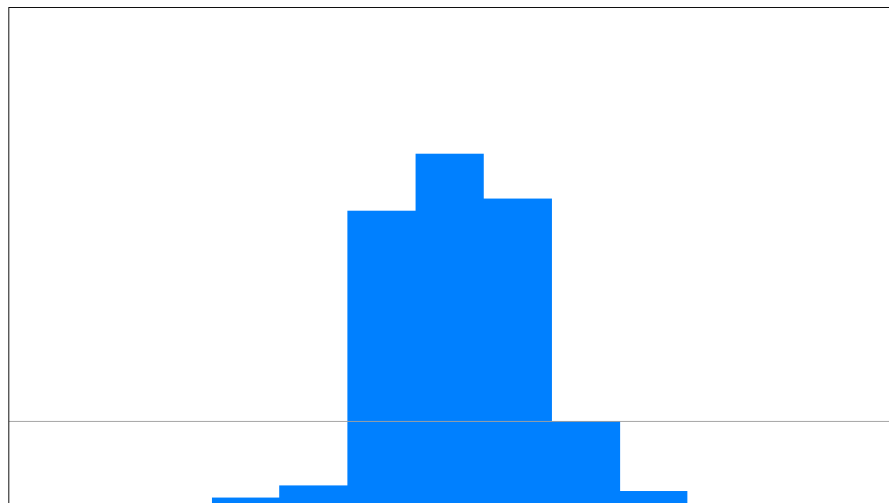
Each of the 9 maximal Sato-Tate groups in dimension 2 can be realized by the Jacobian of a genus 2 curve X/\mathbb{Q} .

One can now verify this using the algorithm of [CMSV19].

There are 3 subgroups of $N(\mathrm{U}(1) \times \mathrm{U}(1))$ that satisfy the Sato-Tate axioms but do not occur as Sato-Tate groups of abelian surfaces.

Sato-Tate a_1 -distribution of a typical genus 3 curve

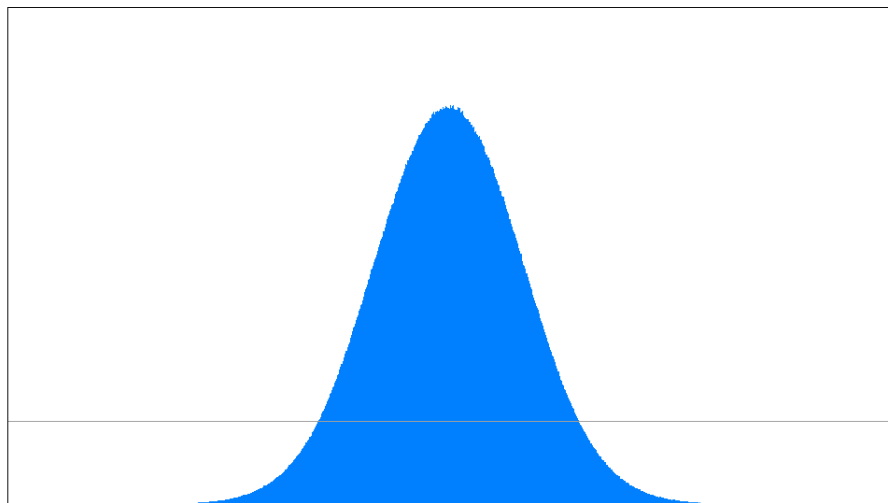
a_1 histogram of $y^2 = x^7 - x + 1$ for $p \leq 2^{10}$
168 data points in 13 buckets, $z_1 = 0.030$



Moments: 1 0.167 0.879 0.552 2.166 2.195 9.022 10.737 48.674 61.554 297.030

Sato-Tate a_1 -distribution of a typical genus 3 curve

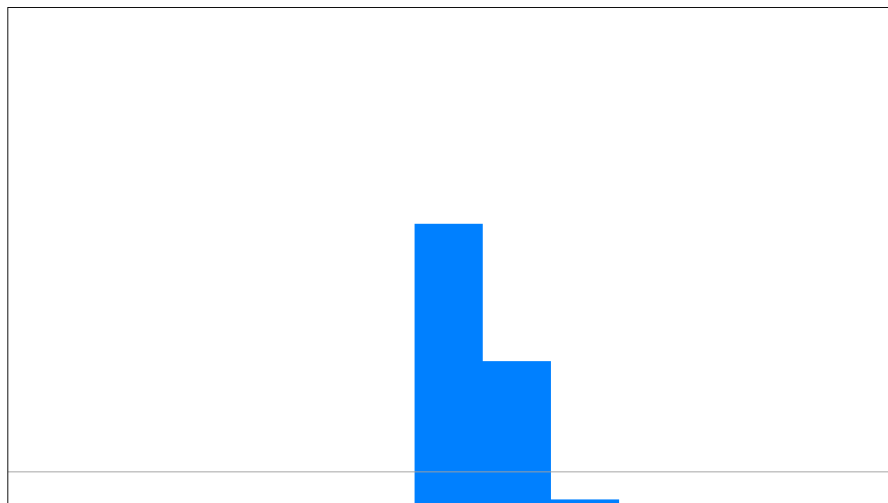
a1 histogram of $y^2 = x^7 - x + 1$ for $p \leq 2^{30}$
54400023 data points in 7375 buckets



Moments: 1 0.000 1.000 -0.000 3.000 -0.005 14.996 -0.093 103.963 -1.573 908.557

Sato-Tate a_2 -distribution of a typical genus 3 curve

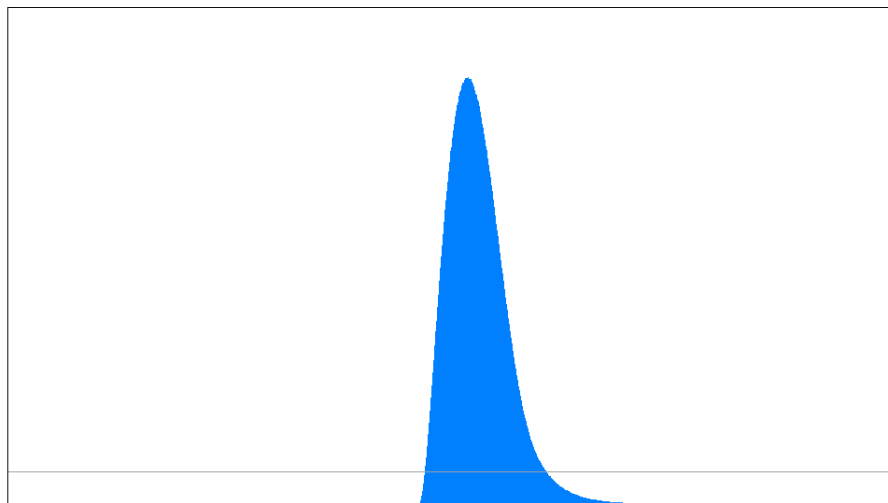
a_2 histogram of $y^2 = x^7 - x + 1$ for $p \leq 2^{10}$
168 data points in 13 buckets



Moments: 1 0.887 1.661 3.767 10.599 34.421 124.148 480.397 1947.535

Sato-Tate a_2 -distribution of a typical genus 3 curve

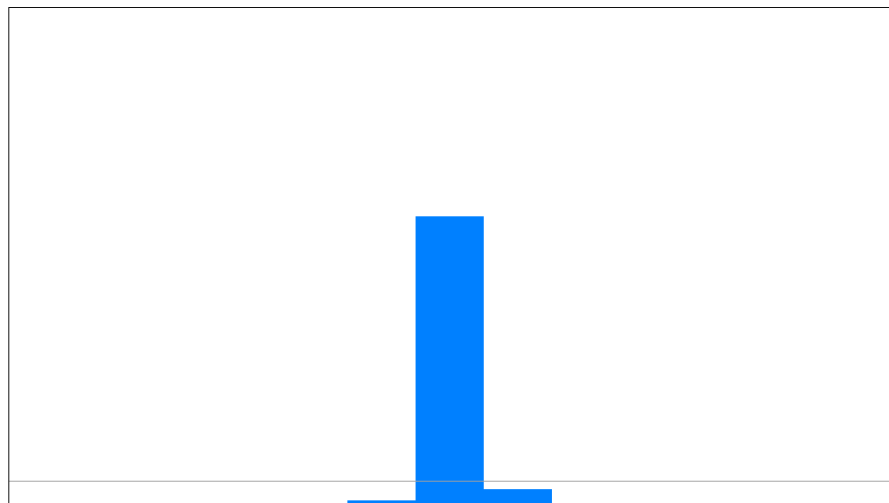
a_2 histogram of $y^2 = x^7 - x + 1$ for $p \leq 2^{30}$
54400023 data points in 7375 buckets



Moments: 1 1.000 2.000 4.999 15.995 61.973 281.845 1457.892 8365.112

Sato-Tate a_3 -distribution of a typical genus 3 curve

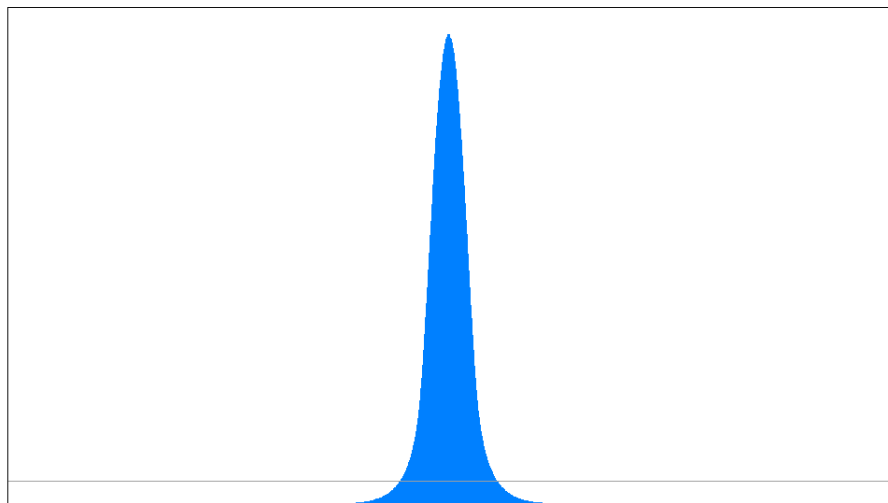
a_3 histogram of $y^2 = x^7 - x + 1$ for $p \leq 2^{10}$
168 data points in 13 buckets



Moments: 1 0.249 1.649 2.164 12.036 34.226 186.537 736.915 3906.256

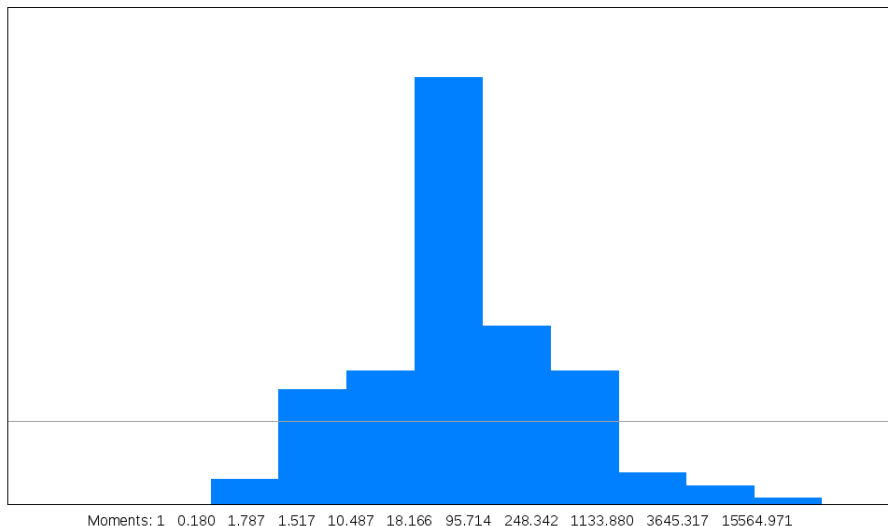
Sato-Tate a_3 -distribution of a typical genus 3 curve

a_3 histogram of $y^2 = x^7 - x + 1$ for $p \leq 2^{30}$
54400023 data points in 7375 buckets



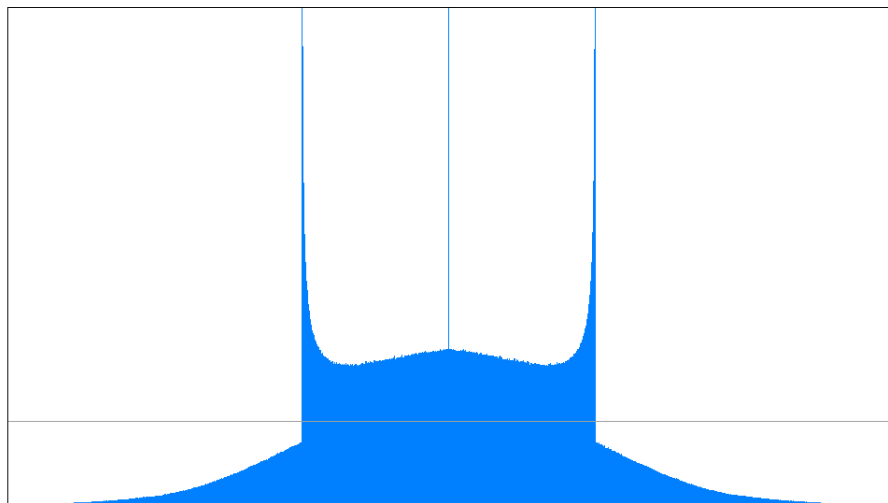
Sato-Tate a_1 -distribution of an atypical genus 3 curve

a_1 histogram of $y^2 = x^7 + 3x^6 + 2x^5 + 6x^4 + 4x^3 + 12x^2 + 8x$ for $p \leq 2^{10}$
168 data points in 13 buckets, $z_1 = 0.274$



Sato-Tate a_1 -distribution of an atypical genus 3 curve

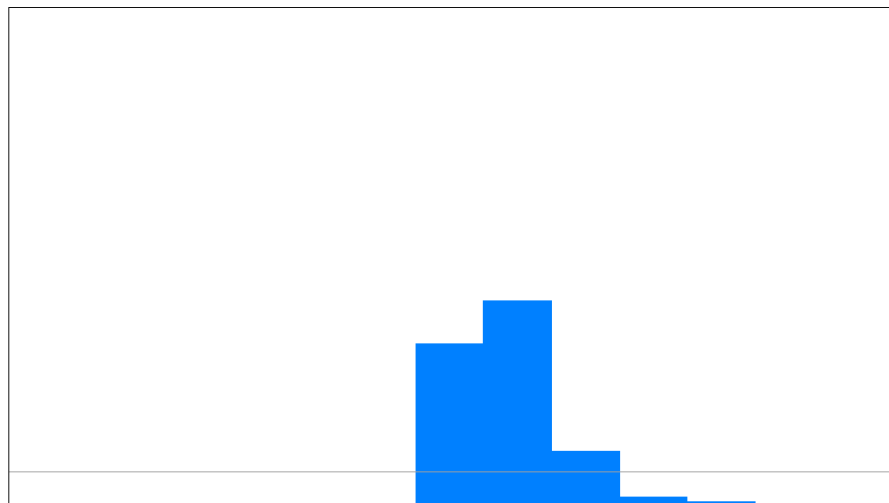
a_1 histogram of $y^2 = x^7 + 3x^6 + 2x^5 + 6x^4 + 4x^3 + 12x^2 + 8x$ for $p \leq 2^{30}$
54400024 data points in 7375 buckets, $z_1 = 0.250$, out of range data has area 0.256



Moments: 1 0.000 2.000 0.002 13.997 0.039 164.995 0.786 2640.472 18.388 50318.872

Sato-Tate a_2 -distribution of an atypical genus 3 curve

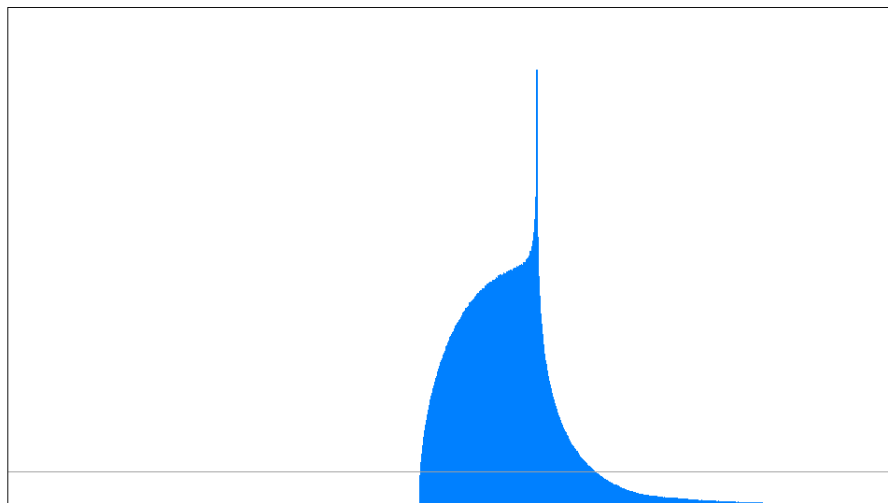
a_2 histogram of $y^2 = x^7 + 3x^6 + 2x^5 + 6x^4 + 4x^3 + 12x^2 + 8x$ for $p \leq 2^{10}$
168 data points in 13 buckets, $z = [0.000 \ 0.000 \ 0.000 \ 0.000 \ 0.000 \ 0.000 \ 0.036]$



Moments: 1 1.865 6.180 24.999 122.705 697.662 4429.294 30391.457 220003.581

Sato-Tate a_2 -distribution of an atypical genus 3 curve

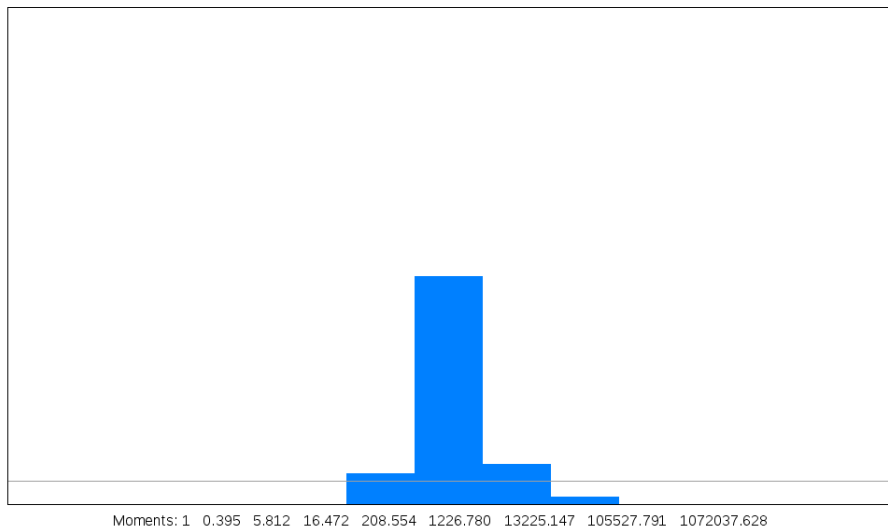
a_2 histogram of $y^2 = x^7 + 3x^6 + 2x^5 + 6x^4 + 4x^3 + 12x^2 + 8x$ for $p \leq 2^{30}$
54400024 data points in 7375 buckets



Moments: 1 2.000 6.999 31.995 190.998 1402.539 11916.253 111587.554 1116443.514

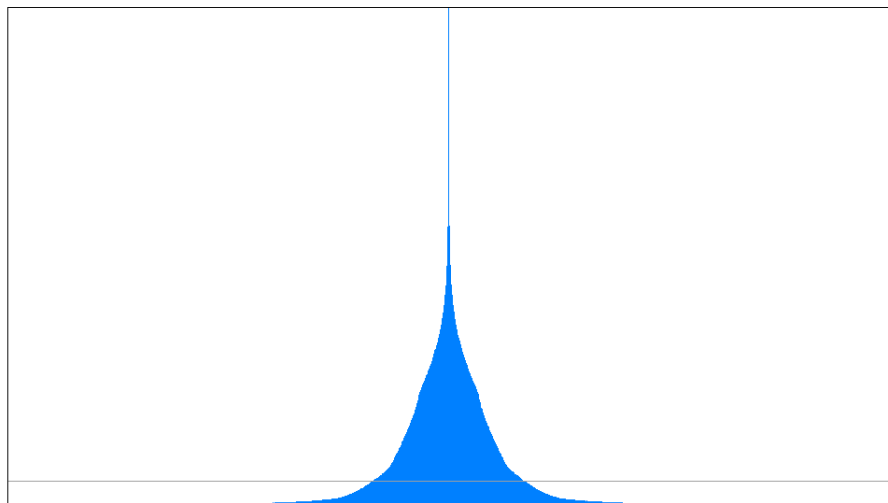
Sato-Tate a_3 -distribution of a typical genus 3 curve

a_3 histogram of $y^2 = x^7 + 3x^6 + 2x^5 + 6x^4 + 4x^3 + 12x^2 + 8x$ for $p \leq 2^{10}$
168 data points in 13 buckets, $z_3 = 0.274$



Sato-Tate a_3 -distribution of a typical genus 3 curve

a_3 histogram of $y^2 = x^7 + 3x^6 + 2x^5 + 6x^4 + 4x^3 + 12x^2 + 8x$ for $p \leq 2^{30}$
54400024 data points in 7375 buckets, $z_3 = 0.250$, out of range data has area 0.249



Moments: 1 0.000 6.998 0.033 389.044 5.825 46574.838 1453.082 7858059.139

Sato-Tate groups of abelian threefolds

Theorem [FKS19]

Up to conjugacy in $\mathrm{USp}(6)$, 433 groups satisfy the Sato-Tate axioms for $g = 3$, but 23 cannot arise as Sato-Tate groups of abelian threefolds.

Sato-Tate groups of abelian threefolds

Theorem [FKS19]

Up to conjugacy in $USp(6)$, 433 groups satisfy the Sato-Tate axioms for $g = 3$, but 23 cannot arise as Sato-Tate groups of abelian threefolds.

Theorem [FKS19]

Up to conjugacy in $USp(6)$ there are 410 Sato-Tate groups of abelian threefolds over number fields, of which 33 are maximal.

The 33 maximal groups all arise as the Sato-Tate group of an abelian threefold defined over \mathbb{Q} ; the rest can be realized via base change.

There are 14 distinct identity components that arise, and the order of every component group always divides one of the following integers:
 $192 = 2^6 \cdot 3$, $336 = 2^4 \cdot 3 \cdot 7$, $432 = 2^4 \cdot 3^3$.

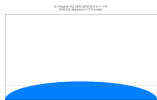
Real endomorphism algebras of abelian threefolds

| abelian threefold | $\text{End}(A_K)_{\mathbb{R}}$ | $\text{ST}(A)^0$ |
|---|--|-----------------------------------|
| cube of a CM elliptic curve | $M_3(\mathbb{C})$ | $U(1)_3$ |
| cube of a non-CM elliptic curve | $M_3(\mathbb{R})$ | $SU(2)_3$ |
| product of CM elliptic curve and square of CM elliptic curve | $\mathbb{C} \times M_2(\mathbb{C})$ | $U(1) \times U(1)_2$ |
| product of non-CM elliptic curve and square of CM elliptic curve | $\mathbb{R} \times M_2(\mathbb{C})$ | $SU(2) \times U(1)_2$ |
| <ul style="list-style-type: none"> product of CM elliptic curve and QM abelian surface product of CM elliptic curve and square of non-CM elliptic curve | $\mathbb{C} \times M_2(\mathbb{R})$ | $U(1) \times SU(2)_2$ |
| <ul style="list-style-type: none"> product of non-CM elliptic curve and QM abelian surface product of non-CM elliptic curve and square of non-CM elliptic curve | $\mathbb{R} \times M_2(\mathbb{R})$ | $SU(2) \times SU(2)_2$ |
| <ul style="list-style-type: none"> CM abelian threefold product of CM elliptic curve and CM abelian surface product of three CM elliptic curves | $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ | $U(1) \times U(1) \times U(1)$ |
| <ul style="list-style-type: none"> product of non-CM elliptic curve and CM abelian surface product of non-CM elliptic curve and two CM elliptic curves | $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ | $U(1) \times U(1) \times SU(2)$ |
| <ul style="list-style-type: none"> product of CM elliptic curve and RM abelian surface product of CM elliptic curve and two non-CM elliptic curves | $\mathbb{C} \times \mathbb{R} \times \mathbb{R}$ | $U(1) \times SU(2) \times SU(2)$ |
| <ul style="list-style-type: none"> RM abelian threefold product of non-CM elliptic curve and RM abelian surface product of 3 non-CM elliptic curves | $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ | $SU(2) \times SU(3) \times SU(3)$ |
| product of CM elliptic curve and abelian surface | $\mathbb{C} \times \mathbb{R}$ | $U(1) \times USp(4)$ |
| product of non-CM elliptic curve and abelian surface | $\mathbb{R} \times \mathbb{R}$ | $SU(2) \times USp(4)$ |
| quadratic CM abelian threefold | \mathbb{C} | $U(3)$ |
| generic abelian threefold | \mathbb{R} | $USp(6)$ |

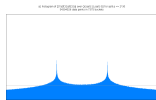
Connected Sato-Tate groups of abelian threefolds:



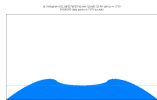
$U(1)_3$



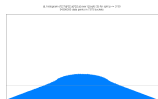
$SU(2)_3$



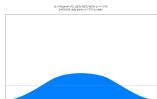
$U(1) \times U(1)_2$



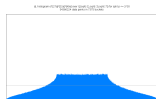
$SU(2) \times U(1)_2$



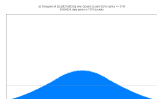
$U(1) \times SU(2)_2$



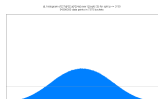
$SU(2) \times SU(2)_2$



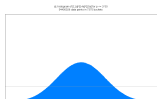
$U(1) \times U(1) \times U(1)$



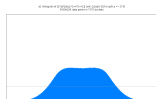
$U(1) \times U(1) \times SU(2)$



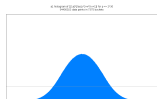
$U(1) \times SU(2) \times U(1)$



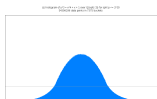
$SU(2) \times SU(2) \times SU(2)$



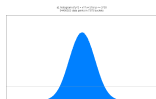
$U(1) \times USp(4)$



$SU(2) \times USp(4)$



$U(3)$



$USp(6)$

Maximal Sato-Tate groups of abelian threefolds

| G_0 | G/G_0 | $ G/G_0 $ |
|--|--------------------------------------|---|
| $\mathrm{USp}(6)$ | C_1 | 1 |
| $\mathrm{U}(3)$ | C_2 | 2 |
| $\mathrm{SU}(2) \times \mathrm{USp}(4)$ | C_1 | 1 |
| $\mathrm{U}(1) \times \mathrm{USp}(4)$ | C_2 | 2 |
| $\mathrm{SU}(2)^3$ | S_3 | 6 |
| $\mathrm{U}(1) \times \mathrm{SU}(2)^2$ | D_2 | 4 |
| $\mathrm{U}(1)^2 \times \mathrm{SU}(2)$ | C_2, D_2 | 4 |
| $\mathrm{U}(1)^3$ | $S_3, C_2^3, C_2 \times C_4$ | 6, 8 |
| $\mathrm{SU}(2) \times \mathrm{SU}(2)_2$ | D_4, D_6 | 8, 12 |
| $\mathrm{U}(1) \times \mathrm{SU}(2)_2$ | $D_4 \times C_2, D_6 \times C_2$ | 16, 24 |
| $\mathrm{SU}(2) \times \mathrm{U}(1)_2$ | $D_6 \times C_2, S_4 \times C_2$ | 48 |
| $\mathrm{U}(1) \times \mathrm{U}(1)_2$ | $D_6 \times C_2^2, S_4 \times C_2^2$ | 48, 96 |
| $\mathrm{SU}(2)_3$ | D_6, S_4 | 12, 24 |
| $\mathrm{U}(1)_3$ | see below | $48^{\times 4}, 96, 144^{\times 2},$ $192^{\times 2}, 336, 432^{\times 2}$ |

$\langle 48, 15 \rangle, \langle 48, 15 \rangle, \langle 48, 38 \rangle, \langle 48, 41 \rangle, \langle 96, 193 \rangle, \langle 144, 125 \rangle,$
 $\langle 144, 127 \rangle, \langle 192, 988 \rangle, \langle 192, 956 \rangle, \langle 336, 208 \rangle, \langle 432, 523 \rangle, \langle 432, 734 \rangle.$

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