Sato-Tate groups of abelian threefolds

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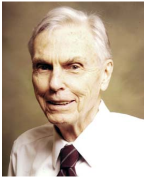
Mikio Sato



John Tate

Joint work with Francesc Fité and Kiran Kedlaya

Sato-Tate in dimension 1

Let $E/\mathbb{Q}$ be an elliptic curve, say,

$$y^2 = x^3 + Ax + B,$$

and let p be a prime of good reduction (so $p \nmid \Delta(E)$).

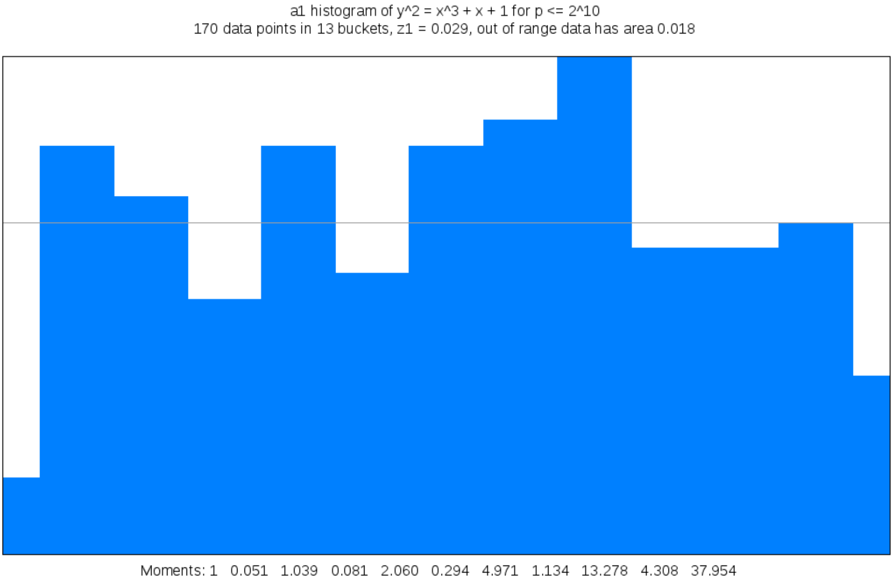
The number of $\mathbb{F}_p$-points on the reduction E_p of E modulo p is

$$\#E_p(\mathbb{F}_p) = p + 1 - t_p,$$

where the trace of Frobenius t_p is an integer in $[-2\sqrt{p}, 2\sqrt{p}]$.

We are interested in the limiting distribution of $x_p = -t_p/\sqrt{p} \in [-2, 2]$, as p varies over primes of good reduction up to $N \to \infty$.

Sato-Tate distribution of a typical elliptic curve



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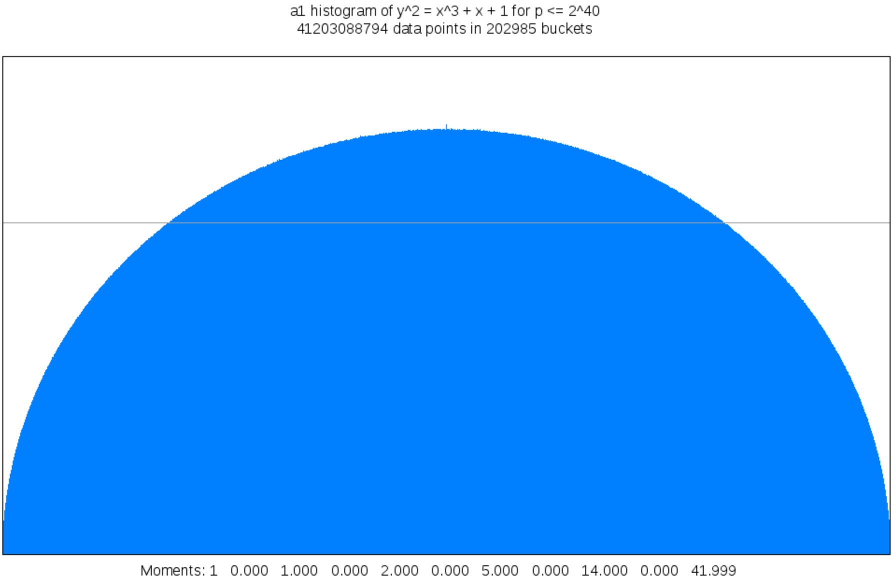
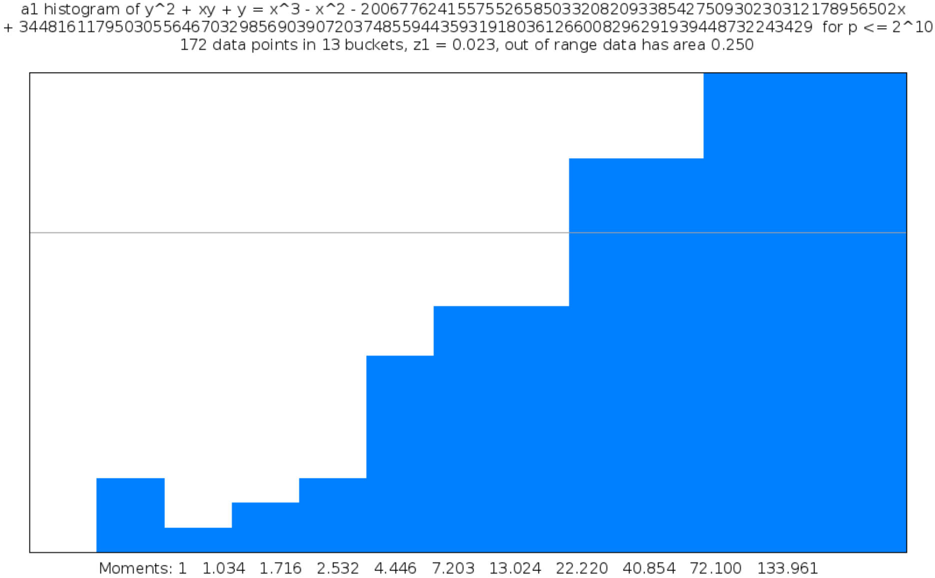


Sato-Tate distribution of another typical elliptic curve



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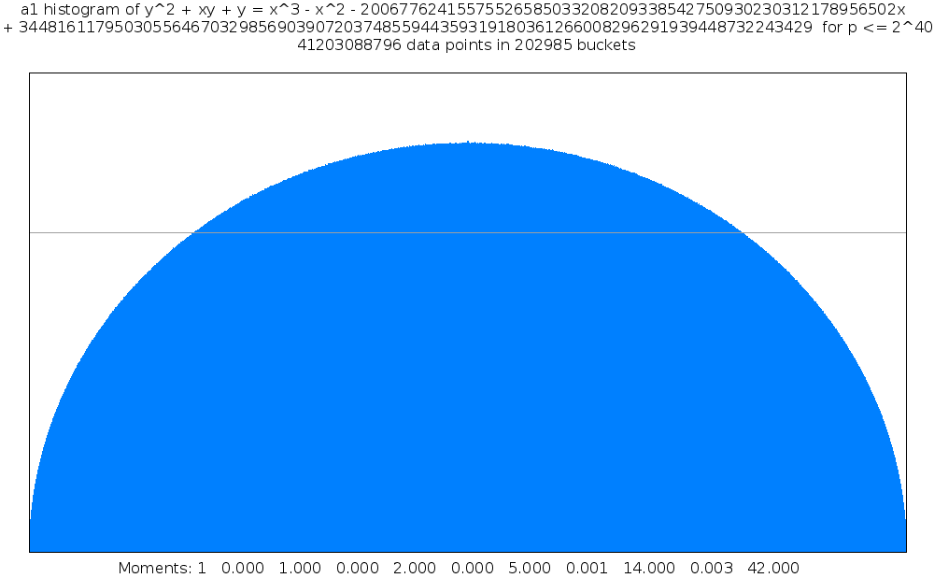

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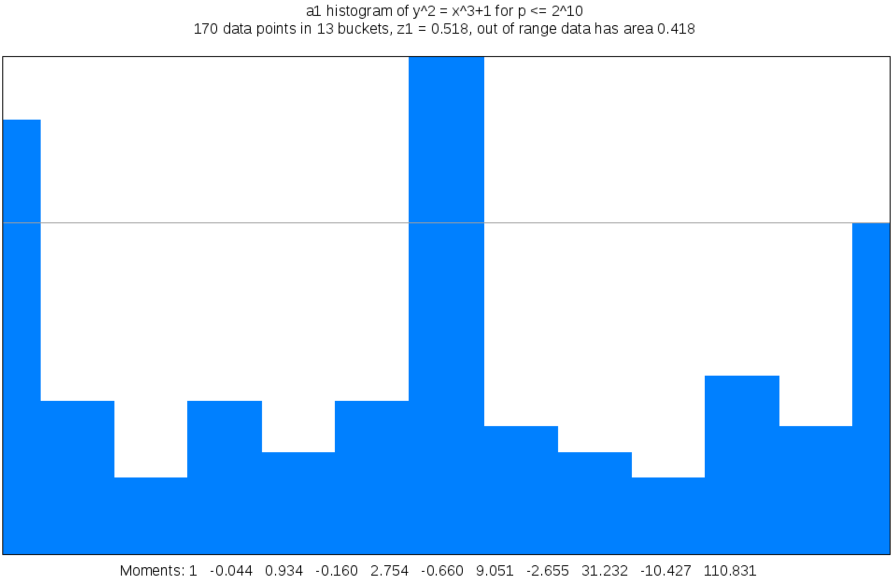

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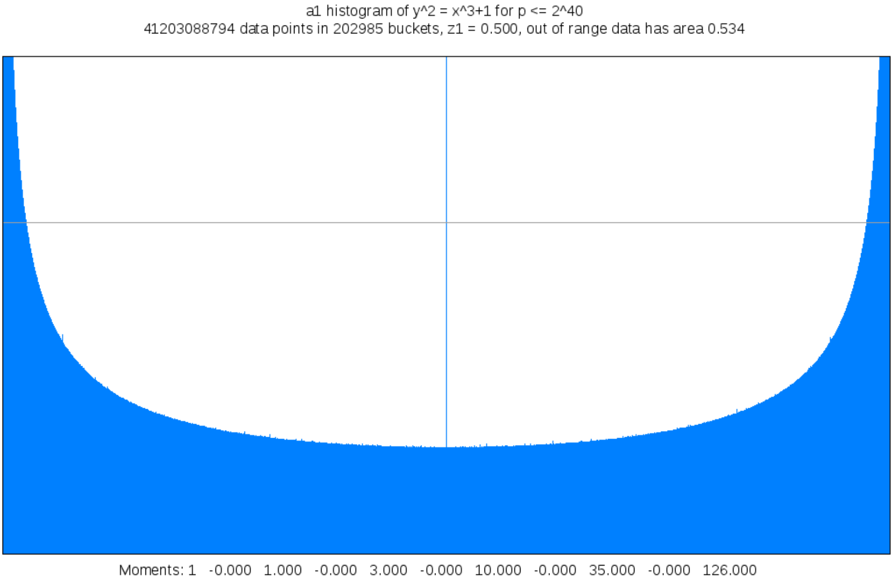

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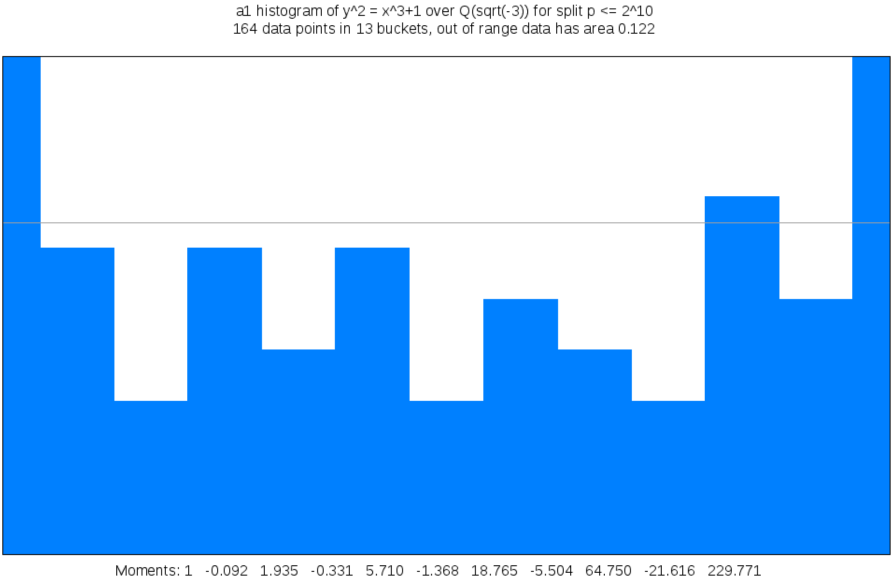

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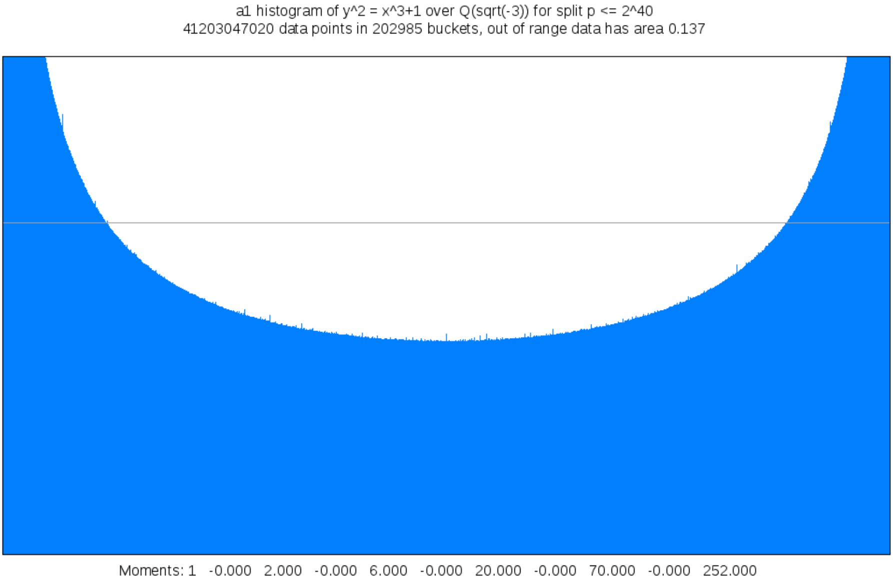

Sato-Tate distributions in dimension 1

1. Typical case (no CM)

Elliptic curves $E/\mathbb{Q}$ w/o CM have the semi-circular trace distribution. (Also known for E/k, where k is a totally real or CM number field).

[CHT08, Taylor08, HST10, BGG11, BGHT11, ACCGHHNSTT18]

2. Exceptional cases (CM)

Elliptic curves E/k with CM have one of two distinct trace distributions, depending on whether k contains the CM field or not.

[Hecke, Deuring, early 20th century]

Sato-Tate groups in dimension 1

The Sato-Tate group of E is a closed subgroup G of $\mathrm{SU}(2) = \mathrm{USp}(2)$ that is determined by the ℓ-adic Galois representation attached to E.

A refinement/generalization of the Sato-Tate conjecture states that the distribution of normalized Frobenius traces of E converges to the distribution of traces in its Sato-Tate group G (under its Haar measure).

G	G/G^0	E	k	$\mathrm{E}[x_p^0], \mathrm{E}[x_p^2], \mathrm{E}[x_p^4] \dots$
$\mathrm{SU}(2)$	C_1	$y^2 = x^3 + x + 1$	$\mathbb{Q}$	$1, 1, 2, 5, 14, 42, \dots$
$N(\mathrm{U}(1))$	C_2	$y^2 = x^3 + 1$	$\mathbb{Q}$	$1, 1, 3, 10, 35, 126, \dots$
$\mathrm{U}(1)$	C_1	$y^2 = x^3 + 1$	$\mathbb{Q}(\sqrt{-3})$	$1, 2, 6, 20, 70, 252, \dots$

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Fun fact: in the non-CM case the Sato-Tate conjecture implies that $E[x_p^n] = \frac{1}{2\pi}\int_0^\pi (2\cos\theta)^n \sin^2\theta\, d\theta$ is the $\frac{n}{2}$th Catalan number.

Zeta functions and L-polynomials

For a smooth projective curve $X/\mathbb{Q}$ of genus g and each prime p of good reduction for X we have the zeta function

$$Z(X_p/\mathbb{F}_p;T) := \exp\left(\sum_{k=1}^{\infty} \#X_p(\mathbb{F}_{p^k})T^k/k\right) = \frac{L_p(T)}{(1-T)(1-pT)},$$

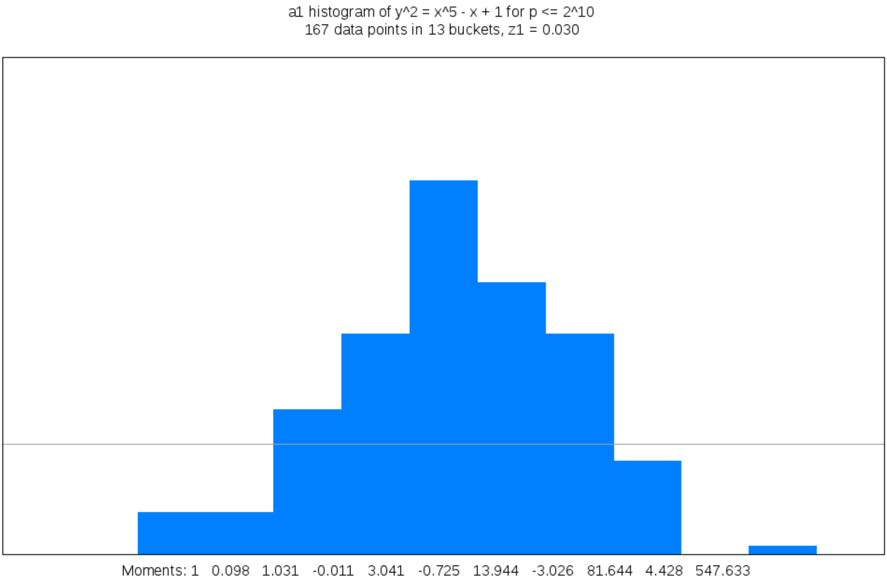
where $L_p \in \mathbb{Z}[T]$ has degree $2g$. The normalized L-polynomial

$$\bar{L}_p(T) := L_p(T/\sqrt{p}) = \sum_{i=0}^{2g} a_i T^i \in \mathbb{R}[T]$$

is monic, reciprocal, and unitary, with $|a_i| \leq \binom{2g}{i}$.

We can now consider the limiting distribution of $a_1, a_2, \ldots, a_g$ over all primes $p \leq N$ of good reduction, as $N \to \infty$.

Sato-Tate a_1-distribution of a typical genus 2 curve



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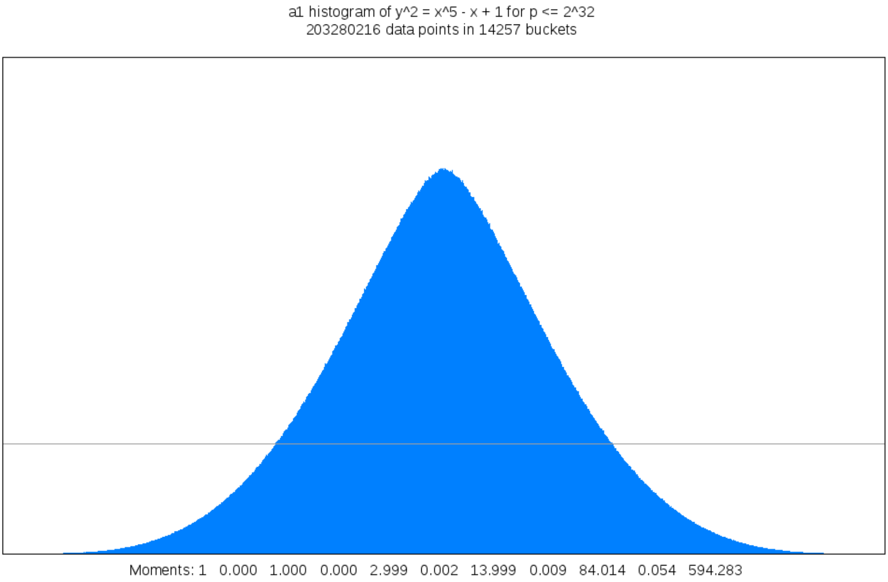

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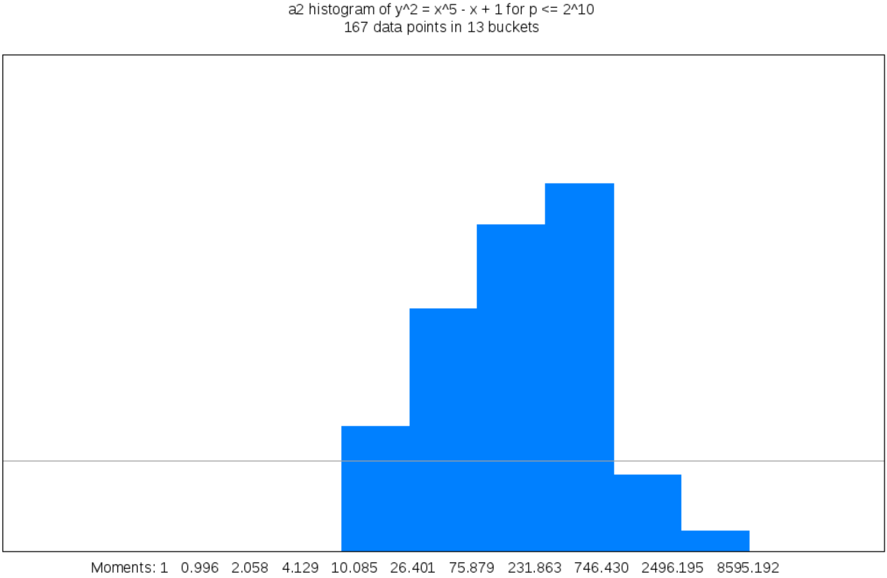

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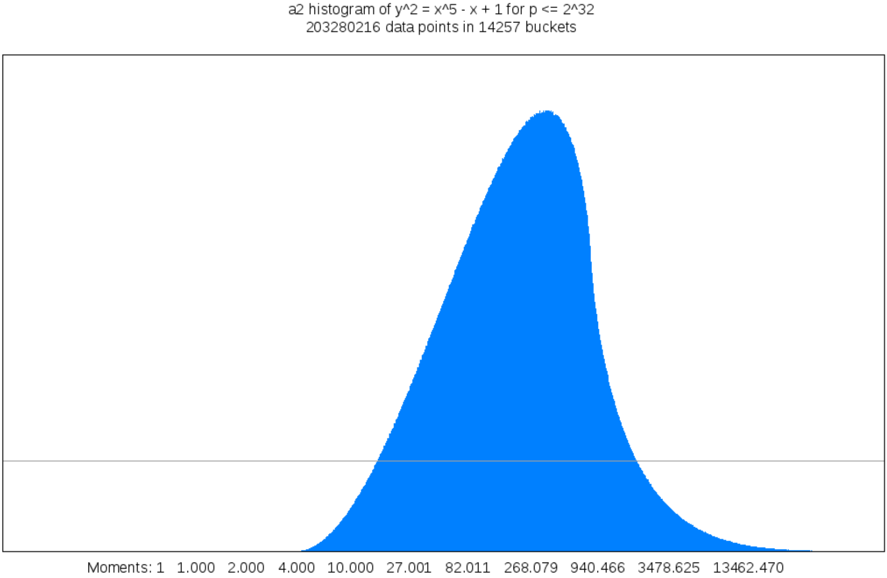

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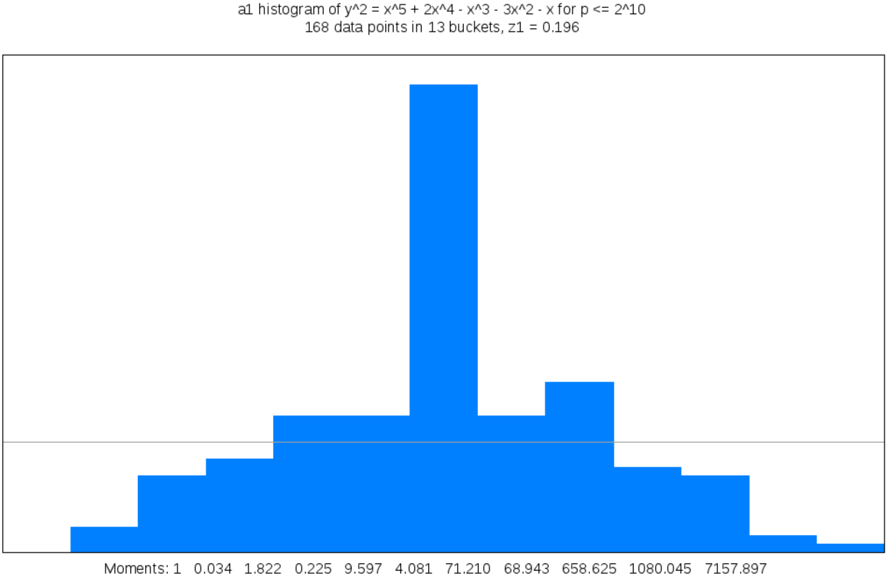

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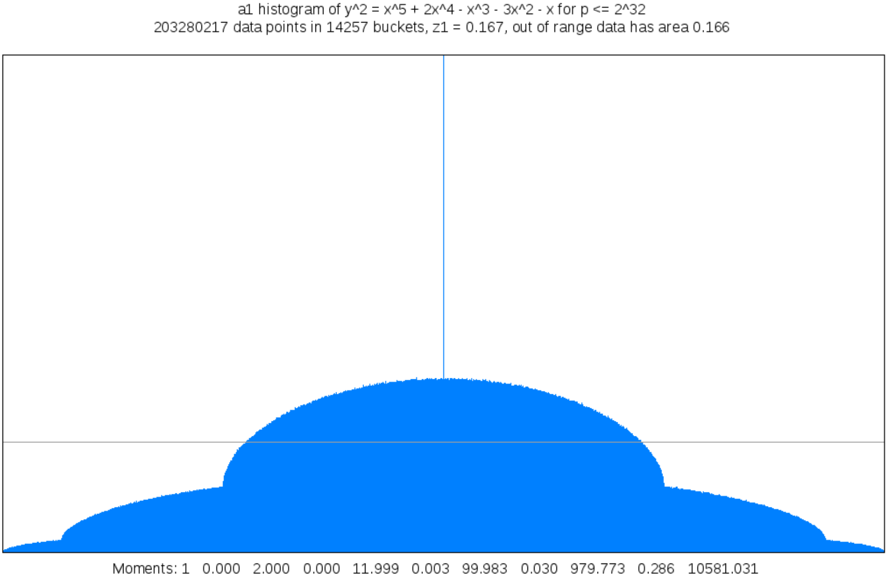
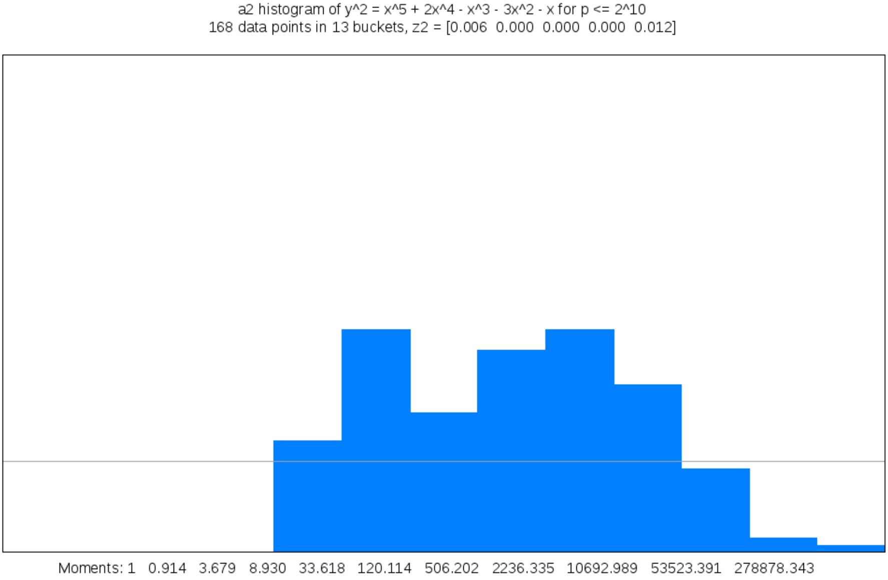


Sato-Tate a_2-distribution of an atypical genus 2 curve



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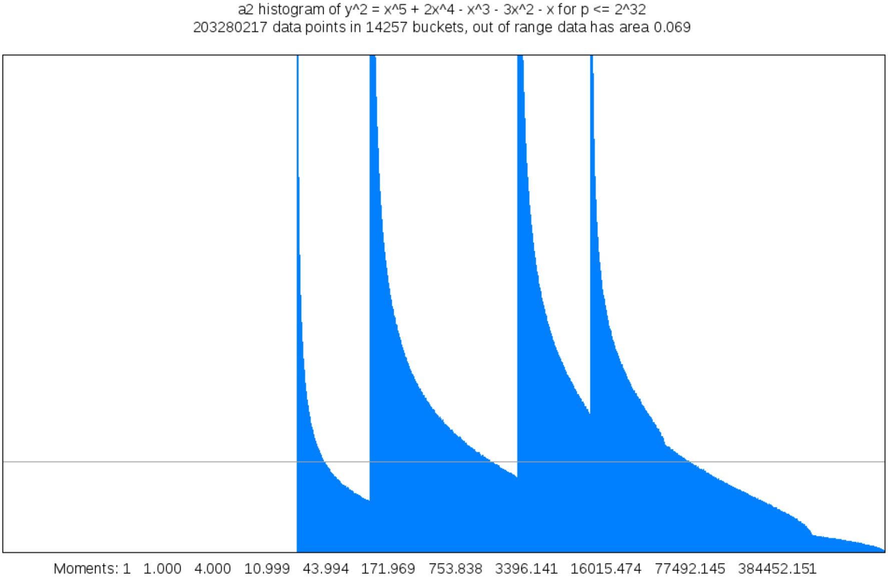
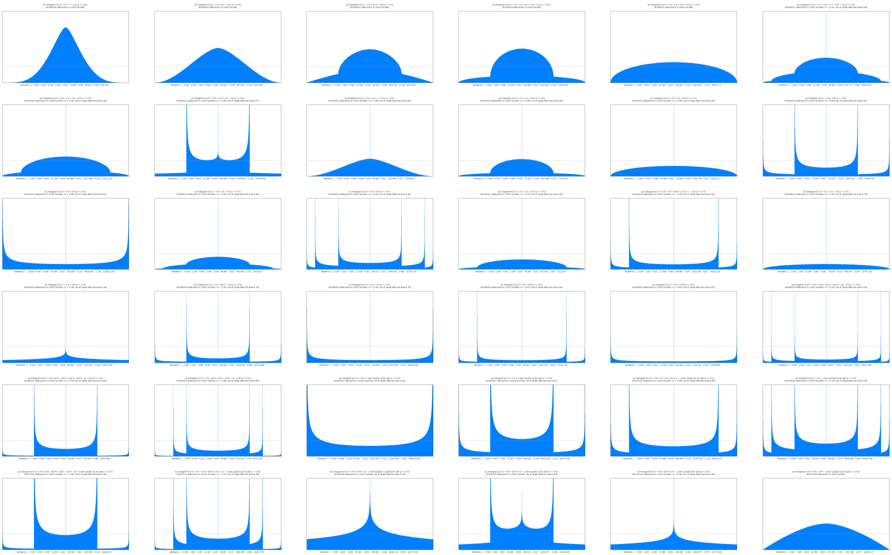
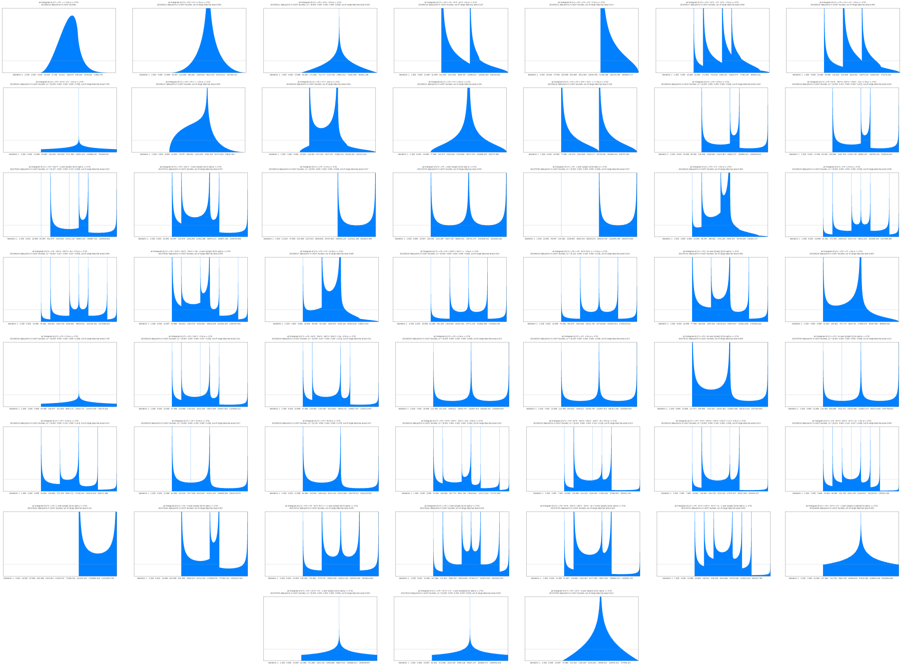


Sato-Tate trace distributions of genus 2 curves:





L-polynomials of Abelian varieties

Let A be an abelian variety over a number field k and fix a prime ℓ.
The action of $\mathrm{Gal}(\bar{k}/k)$ on the ℓ-adic Tate module

$$V_\ell(A) := \varprojlim A[\ell^n] \otimes_{\mathbb{Z}} \mathbb{Q}$$

gives rise to a Galois representation

$$\rho_\ell \colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell).$$

For each prime $\mathfrak{p}$ of good reduction for A we have the L-polynomial

$$L_\mathfrak{p}(T) := \det(1 - \rho_\ell(\mathrm{Frob}_\mathfrak{p})T), \qquad \bar{L}_\mathfrak{p}(T) := L_\mathfrak{p}(T/\sqrt{\|\mathfrak{p}\|}),$$

which appears as an Euler factor in the L-series

$$L(A,s) := \prod_\mathfrak{p} L_\mathfrak{p}(\|\mathfrak{p}\|^{-s})^{-1}.$$

The Sato-Tate group of an abelian variety

The Zariski closure of the image of

$$\rho_\ell\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$$

is a $\mathbb{Q}_\ell$-algebraic group $G_\ell^{\mathrm{zar}} \subseteq \mathrm{GSp}_{2g}$, and we let $G_\ell^{1,\mathrm{zar}} := G_\ell^{\mathrm{zar}} \cap \mathrm{Sp}_{2g}$. Now fix $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, and let $G_{\ell,\iota}^{\mathrm{zar}}$ and $G_{\ell,\iota}^{1,\mathrm{zar}}$ denote base changes to $\mathbb{C}$.

Definition [Serre]

$\mathrm{ST}(A) \subseteq \mathrm{USp}(2g)$ is a maximal compact subgroup of $G_{\ell,\iota}^{1,\mathrm{zar}}(\mathbb{C})$ equipped with the map $s\colon \mathfrak{p} \mapsto \mathrm{conj}(\|\mathfrak{p}\|^{-1/2}\rho_{\ell,\iota}(\mathrm{Frob}_\mathfrak{p})) \in \mathrm{Conj}(\mathrm{ST}(A))$.

Note that the characteristic polynomial of $s(\mathfrak{p})$ is $\bar{L}_\mathfrak{p}(T)$.

The Sato-Tate conjecture for abelian varieties

Conjecture [Mumford-Tate, Algebraic Sato-Tate]

$(G_\ell^{\mathrm{zar}})^0 = \mathrm{MT}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, equivalently, $(G_\ell^{1,\mathrm{zar}})^0 = \mathrm{Hg}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.
More generally, $(G_\ell^{\mathrm{zar}}) = \mathrm{AST}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

The algebraic Sato-Tate conjecture is known for $g \leq 3$ [BK15].

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Sato-Tate conjecture for abelian varieties.

The conjugacy classes $s(\mathfrak{p})$ are equidistributed with respect to $\mu_{\mathrm{ST}(A)}$, the pushforward of the Haar measure to $\mathrm{Conj}(\mathrm{ST}(A))$.

The Sato-Tate conjecture implies that the distribution $\bar{L}_{\mathfrak{p}}(T)$ is given by the distribution of characteristic polynomials in $\mathrm{ST}(A)$.

Sato-Tate axioms for abelian varieties

$G \subseteq \mathrm{USp}(2g)$ satisfies the Sato-Tate axioms (for abelian varieties) if:

1. **Compact**: G is closed;
2. **Hodge**: G contains a Hodge circle $\theta\colon \mathrm{U}(1) \to G^0$ whose elements $\theta(u)$ have eigenvalues u, $1/u$ with multiplicity g, such that the conjugates of θ conjugates generate a dense subset of G;
3. **Rationality**: for each component H of G and each irreducible character χ of $\mathrm{GL}_{2g}(\mathbb{C})$ we have $\mathrm{E}[\chi(\gamma) : \gamma \in H] \in \mathbb{Z}$;
4. **Lefschetz**: The subgroup of $\mathrm{USp}(2g)$ fixing $\mathrm{End}(\mathbb{C}^{2g})^{G_0}$ is G^0.

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Theorem [FKRS12, FKS19]

Let A/k be an abelian variety of dimension $g \leq 3$.
Then $\mathrm{ST}(A)$ satisfies the Sato-Tate axioms.

Axioms 1-3 are expected to hold in general, but Axiom 4 fails for $g = 4$.
For any g, the set of G satisfying axioms 1-3 is **finite**.

Galois endomorphism types

Let A be an abelian variety defined over a number field k.
Let K be the minimal extension of k for which $\mathrm{End}(A_K) = \mathrm{End}(A_{\bar{k}})$.
$\mathrm{Gal}(K/k)$ acts on the $\mathbb{R}$-algebra $\mathrm{End}(A_K)_{\mathbb{R}} = \mathrm{End}(A_K) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition

The *Galois endomorphism type* of A is the isomorphism class of $[\mathrm{Gal}(K/k), \mathrm{End}(A_K)_{\mathbb{R}}]$, where $[G, E] \simeq [G', E']$ iff there are isomorphisms $G \simeq G'$ and $E \simeq E'$ compatible with the group actions.

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Theorem [FKRS12]

For abelian varieties A/k of dimension $g \leq 3$ there is a one-to-one correspondence between Sato-Tate groups and Galois types.

More precisely, the identity component G^0 is uniquely determined by $\mathrm{End}(A_K)_{\mathbb{R}}$ and $G/G^0 \simeq \mathrm{Gal}(K/k)$ (with corresponding actions).

Real endomorphism algebras of abelian surfaces

abelian surface	$\mathbf{End}(A_K)_{\mathbb{R}}$	$\mathbf{ST}(A)^0$
square of CM elliptic curve	$\mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1)_2$
• QM abelian surface • square of non-CM elliptic curve	$\mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2)_2$
• CM abelian surface • product of CM elliptic curves	$\mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1)$
product of CM and non-CM elliptic curves	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2)$
• RM abelian surface • product of non-CM elliptic curves	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
generic abelian surface	$\mathbb{R}$	$\mathrm{USp}(4)$

(factors in products are assumed to be non-isogenous)

Sato-Tate groups of abelian surfaces

Theorem [FKRS12]

Up to conjugacy in $\mathrm{USp}(4)$, there are 52 Sato-Tate groups $\mathrm{ST}(A)$ that arise for abelian surfaces A/k over number fields; 34 occur for $k = \mathbb{Q}$.

$$
\begin{aligned}
\mathrm{U}(1)_2: &\quad C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O,\\
&\quad J(C_1), J(C_2), J(C_3), J(C_4), J(C_6),\\
&\quad J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O),\\
&\quad C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1\\
\mathrm{SU}(2)_2: &\quad E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6)\\
\mathrm{U}(1)\times\mathrm{U}(1): &\quad F, F_a, F_{a,b}, F_{ab}, F_{ac}\\
\mathrm{U}(1)\times\mathrm{SU}(2): &\quad \mathrm{U}(1)\times\mathrm{SU}(2),\ N(\mathrm{U}(1)\times\mathrm{SU}(2))\\
\mathrm{SU}(2)\times\mathrm{SU}(2): &\quad \mathrm{SU}(2)\times\mathrm{SU}(2),\ N(\mathrm{SU}(2)\times\mathrm{SU}(2))\\
\mathrm{USp}(4): &\quad \mathrm{USp}(4)
\end{aligned}
$$

This theorem says nothing about equidistribution, however this is now known in many special cases [FS12, Johansson13, Taylor18].

Maximal Sato-Tate groups of abelian surfaces

G_0	G/G_0	X
$\mathrm{USp}(4)$	C_1	$y^2 = x^5 - x + 1$
$\mathrm{SU}(2) \times \mathrm{SU}(2)$	C_2	$y^2 = x^6 + x^5 + x - 1$
$\mathrm{U}(1) \times \mathrm{SU}(2)$	C_2	$y^2 = x^6 + 3x^4 - 2$
$\mathrm{U}(1) \times \mathrm{U}(1)$	D_2	$y^2 = x^6 + 3x^4 + x^2 - 1$
	C_4	$y^2 = x^5 + 1$
$\mathrm{SU}(2)_2$	D_4	$y^2 = x^5 + x^3 + 2x$
	D_6	$y^2 = x^6 + x^3 - 2$
$\mathrm{U}(1)_2$	$\mathrm{D}_6 \times \mathrm{C}_2$	$y^2 = x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$
	$\mathrm{S}_4 \times \mathrm{C}_2$	$y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$

Each of the 9 maximal Sato-Tate groups in dimension 2 can be realized by the Jacobian of a genus 2 curve $X/\mathbb{Q}$.
One can now verify this using the algorithm of [CMSV19].

Maximal Sato-Tate groups of abelian surfaces

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$\mathrm{U}(1) \times \mathrm{U}(1)$	D_2	$y^2 = x^6 + 3x^4 + x^2 - 1$
	C_4	$y^2 = x^5 + 1$
$\mathrm{SU}(2)_2$	D_4	$y^2 = x^5 + x^3 + 2x$
	D_6	$y^2 = x^6 + x^3 - 2$
$\mathrm{U}(1)_2$	$\mathrm{D}_6 \times \mathrm{C}_2$	$y^2 = x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$
	$\mathrm{S}_4 \times \mathrm{C}_2$	$y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$

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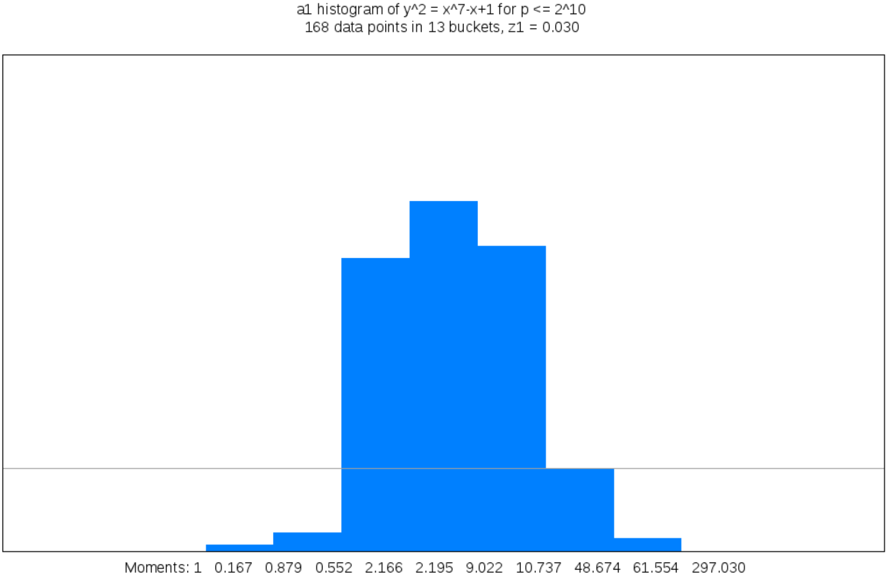
There are 3 subgroups of $N(\mathrm{U}(1) \times \mathrm{U}(1))$ that satisfy the Sato-Tate axioms but do not occur as Sato-Tate groups of abelian surfaces.

Sato-Tate a_1-distribution of a typical genus 3 curve



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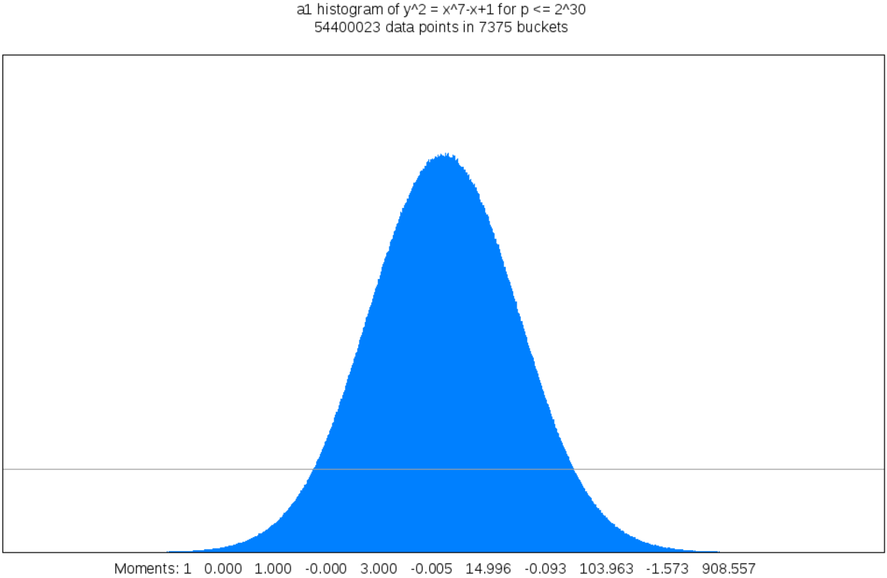
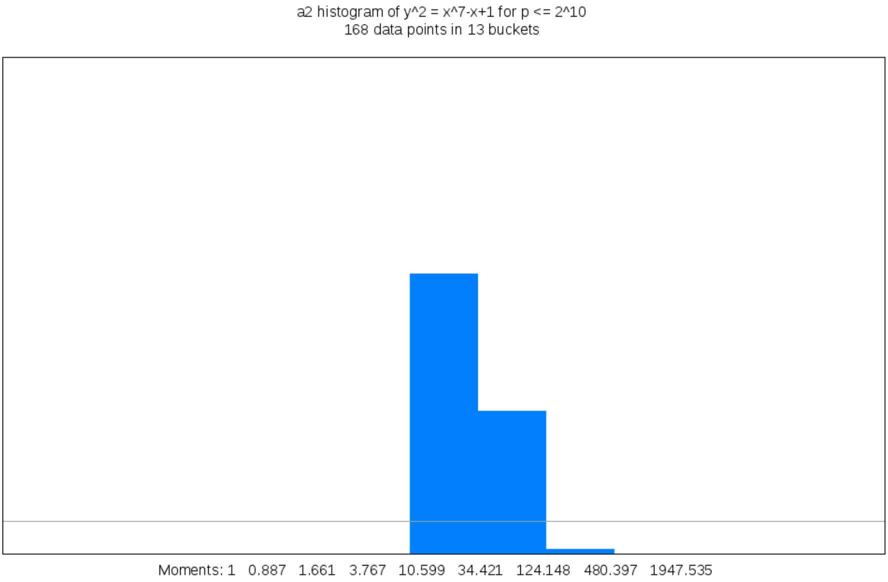


Sato-Tate a_2-distribution of a typical genus 3 curve



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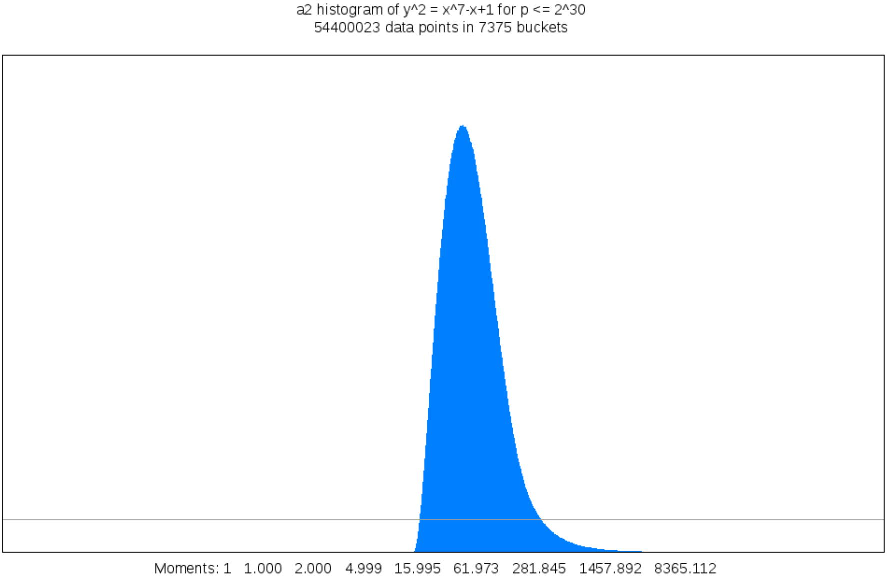

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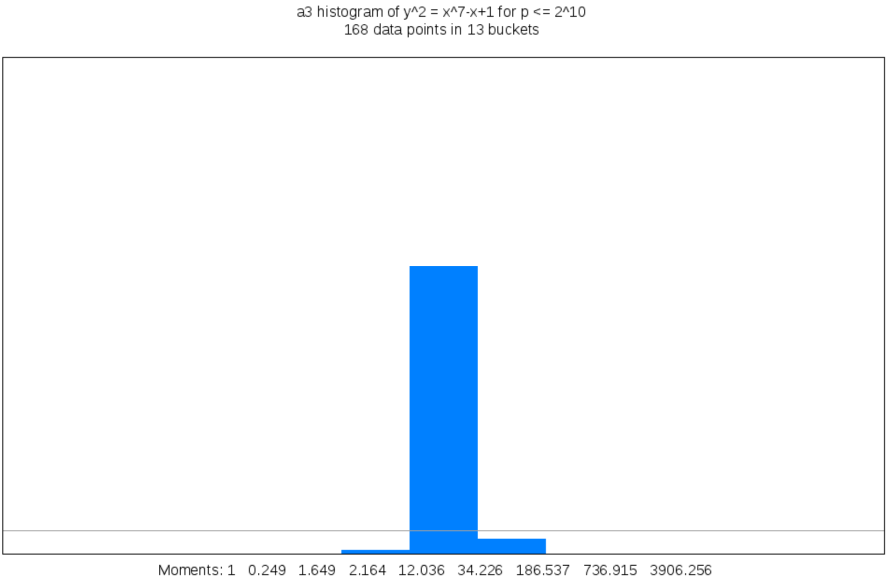

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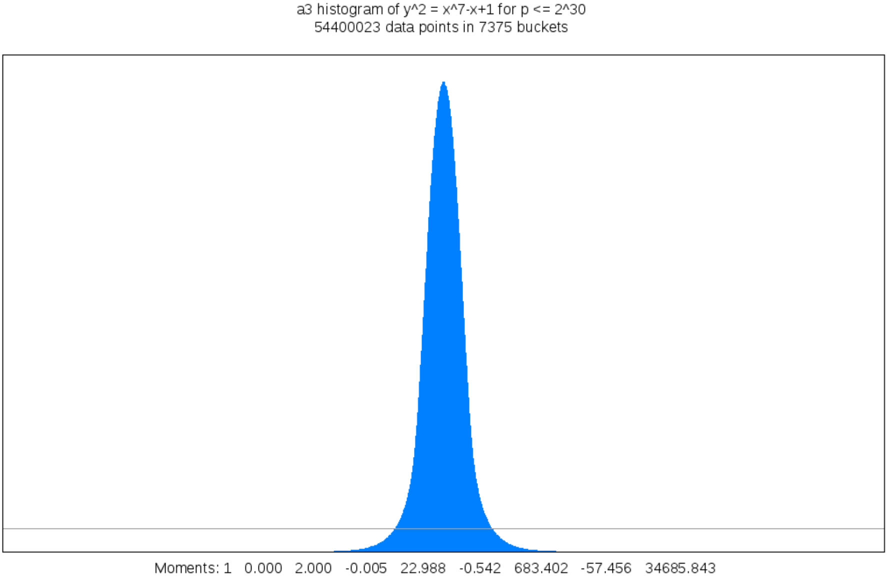
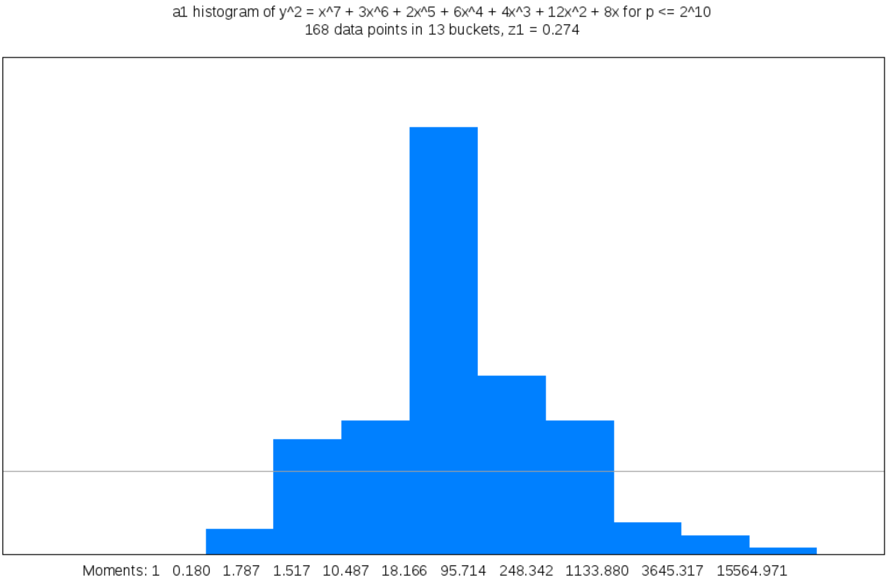
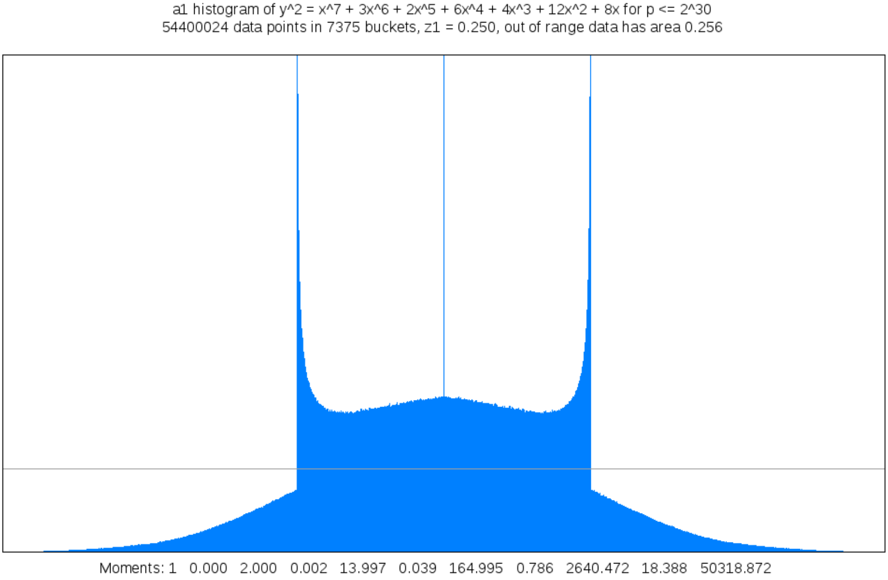


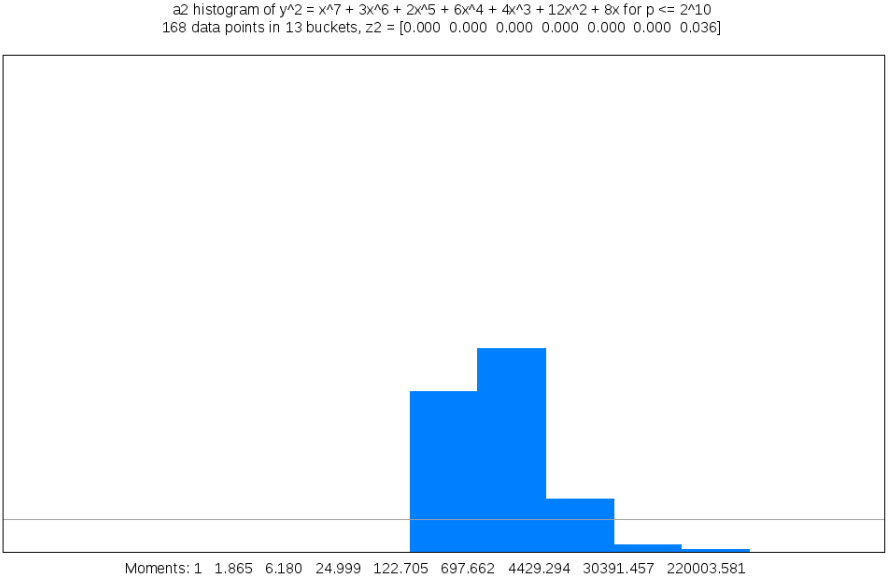
Sato-Tate a_1-distribution of an atypical genus 3 curve



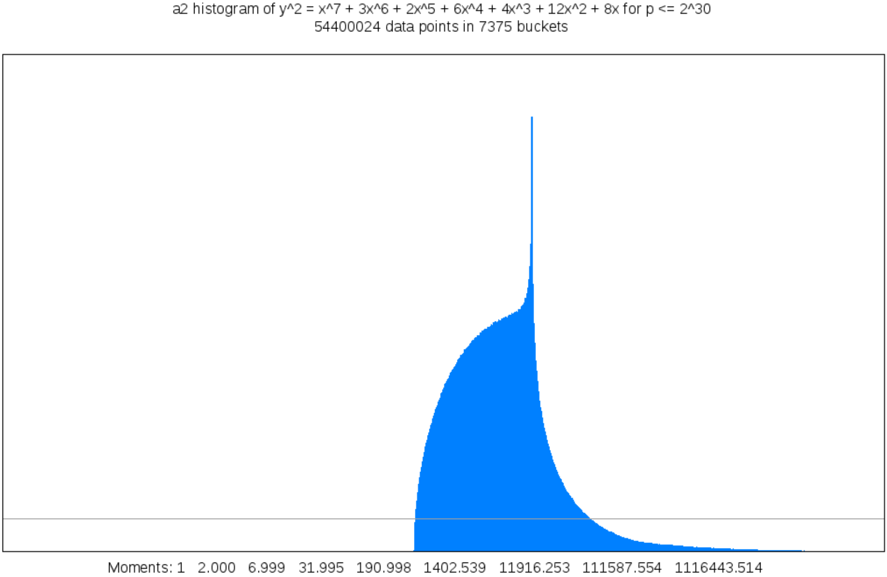
Sato-Tate a_1-distribution of an atypical genus 3 curve



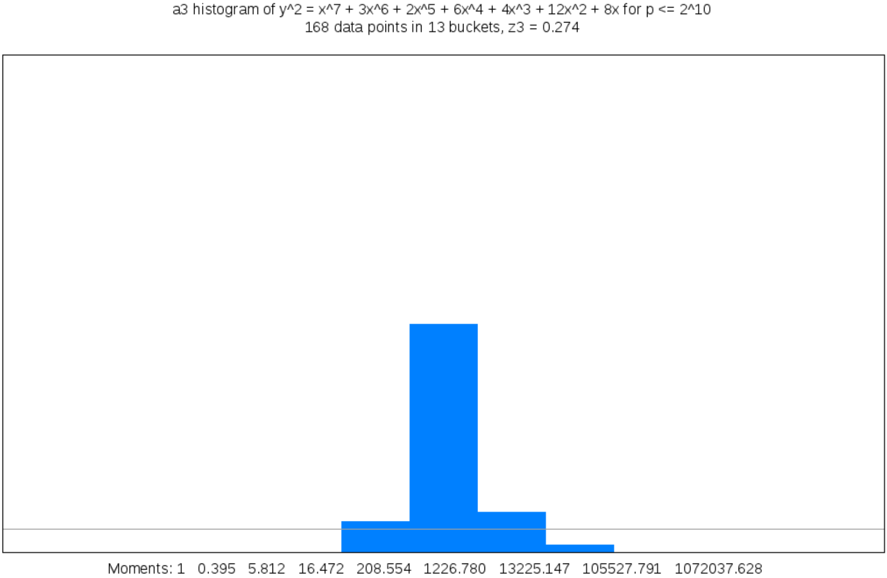
Sato-Tate a_2-distribution of an atypical genus 3 curve



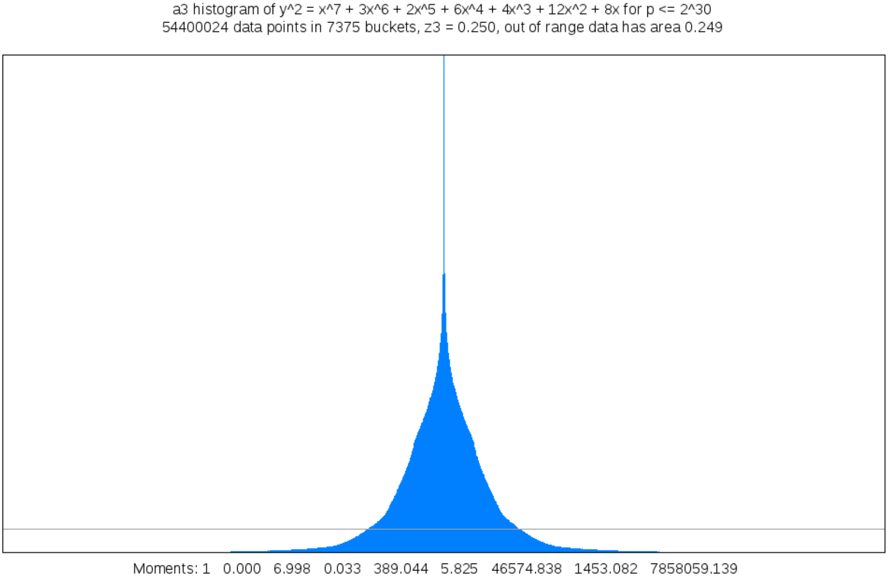
Sato-Tate a_2-distribution of an atypical genus 3 curve



Sato-Tate a_3-distribution of a typical genus 3 curve



Sato-Tate a_3-distribution of a typical genus 3 curve



Sato-Tate groups of abelian threefolds

Theorem [FKS19]

Up to conjugacy in $\mathrm{USp}(6)$, 433 groups satisfy the Sato-Tate axioms for $g = 3$, but 23 cannot arise as Sato-Tate groups of abelian threefolds.

Sato-Tate groups of abelian threefolds

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Theorem [FKS19]

Up to conjugacy in $\mathrm{USp}(6)$ there are 410 Sato-Tate groups of abelian threefolds over number fields, of which 33 are maximal.

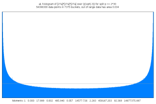
The 33 maximal groups all arise as the Sato-Tate group of an abelian threefold defined over $\mathbb{Q}$; the rest can be realized via base change.

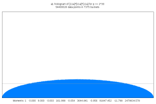
There are 14 distinct identity components that arise, and the order of every component group always divides one of the following integers: $192 = 2^6 \cdot 3, \;\; 336 = 2^4 \cdot 3 \cdot 7, \;\; 432 = 2^4 \cdot 3^3$.

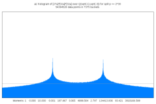
Real endomorphism algebras of abelian threefolds

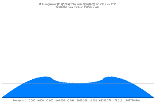
abelian threefold	$\mathbf{End}(A_K)_{\mathbb{R}}$	$\mathbf{ST}(A)^0$
cube of a CM elliptic curve	$\mathrm{M}_3(\mathbb{C})$	$\mathrm{U}(1)_3$
cube of a non-CM elliptic curve	$\mathrm{M}_3(\mathbb{R})$	$\mathrm{SU}(2)_3$
product of CM elliptic curve and square of CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1) \times \mathrm{U}(1)_2$
product of non-CM elliptic curve and square of CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{SU}(2) \times \mathrm{U}(1)_2$
• product of CM elliptic curve and QM abelian surface • product of CM elliptic curve and square of non-CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{U}(1) \times \mathrm{SU}(2)_2$
• product of non-CM elliptic curve and QM abelian surface • product of non-CM elliptic curve and square of non-CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$
• CM abelian threefold • product of CM elliptic curve and CM abelian surface • product of three CM elliptic curves	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$
• product of non-CM elliptic curve and CM abelian surface • product of non-CM elliptic curve and two CM elliptic curves	$\mathbb{C} \times \mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$
• product of CM elliptic curve and RM abelian surface • product of CM elliptic curve and two non-CM elliptic curves	$\mathbb{C} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$
• RM abelian threefold • product of non-CM elliptic curve and RM abelian surface • product of 3 non-CM elliptic curves	$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$
product of CM elliptic curve and abelian surface	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{USp}(4)$
product of non-CM elliptic curve and abelian surface	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{USp}(4)$
quadratic CM abelian threefold	$\mathbb{C}$	$\mathrm{U}(3)$
generic abelian threefold	$\mathbb{R}$	$\mathrm{USp}(6)$

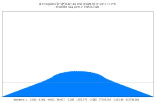
Connected Sato-Tate groups of abelian threefolds:

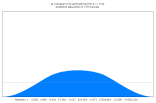

$\mathrm{U}(1)_3$

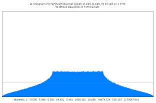

$\mathrm{SU}(2)_3$

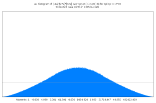

$\mathrm{U}(1) \times \mathrm{U}(1)_2$

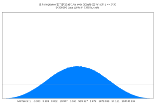

$\mathrm{SU}(2) \times \mathrm{U}(1)_2$

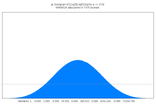

$\mathrm{U}(1) \times \mathrm{SU}(2)_2$

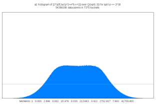

$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$

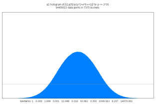

$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$

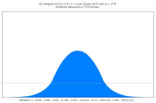

$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$

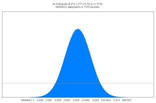

$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$


$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$


$\mathrm{U}(1) \times \mathrm{USp}(4)$


$\mathrm{SU}(2) \times \mathrm{USp}(4)$


$\mathrm{U}(3)$


$\mathrm{USp}(6)$

Maximal Sato-Tate groups of abelian threefolds

G_0	G/G_0	$\lvert G/G_0\rvert$
$\mathrm{USp}(6)$	C_1	1
$\mathrm{U}(3)$	C_2	2
$\mathrm{SU}(2)\times\mathrm{USp}(4)$	C_1	1
$\mathrm{U}(1)\times\mathrm{USp}(4)$	C_2	2
$\mathrm{SU}(2)^3$	S_3	6
$\mathrm{U}(1)\times\mathrm{SU}(2)^2$	D_2	4
$\mathrm{U}(1)^2\times\mathrm{SU}(2)$	$\mathrm{C}_2,\ \mathrm{D}_2$	4
$\mathrm{U}(1)^3$	$\mathrm{S}_3,\ {\mathrm{C}_2}^3,\ \mathrm{C}_2\times\mathrm{C}_4$	6, 8
$\mathrm{SU}(2)\times\mathrm{SU}(2)_2$	$\mathrm{D}_4,\ \ \mathrm{D}_6$	8, 12
$\mathrm{U}(1)\times\mathrm{SU}(2)_2$	$\mathrm{D}_4\times\mathrm{C}_2,\ \ \mathrm{D}_6\times\mathrm{C}_2$	16, 24
$\mathrm{SU}(2)\times\mathrm{U}(1)_2$	$\mathrm{D}_6\times\mathrm{C}_2,\ \ \mathrm{S}_4\times\mathrm{C}_2$	48
$\mathrm{U}(1)\times\mathrm{U}(1)_2$	$\mathrm{D}_6\times{\mathrm{C}_2}^2,\ \ \mathrm{S}_4\times{\mathrm{C}_2}^2$	48, 96
$\mathrm{SU}(2)_3$	$\mathrm{D}_6,\ \ \mathrm{S}_4$	12, 24
$\mathrm{U}(1)_3$	see below	$48^{\times 4}$, 96, $144^{\times 2}$, $192^{\times 2}$, 336, $432^{\times 2}$

$$\langle 48,15\rangle, \langle 48,15\rangle, \langle 48,38\rangle, \langle 48,41\rangle, \langle 96,193\rangle, \langle 144,125\rangle,$$
$$\langle 144,127\rangle, \langle 192,988\rangle, \langle 192,956\rangle, \langle 336,208\rangle, \langle 432,523\rangle, \langle 432,734\rangle.$$

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