Telescopes for mathematicians

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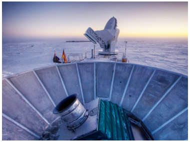
  

Sato-Tate in dimension 1

Let $E/\mathbb{Q}$ be an elliptic curve, which we can write in the form

$$y^2 = x^3 + ax + b,$$

and let p be a prime of good reduction ($4a^3 + 27b^2 \not\equiv 0 \bmod p$).

The number of $\mathbb{F}_p$-points on the reduction E_p of E modulo p is

$$\#E_p(\mathbb{F}_p) = p + 1 - t_p,$$

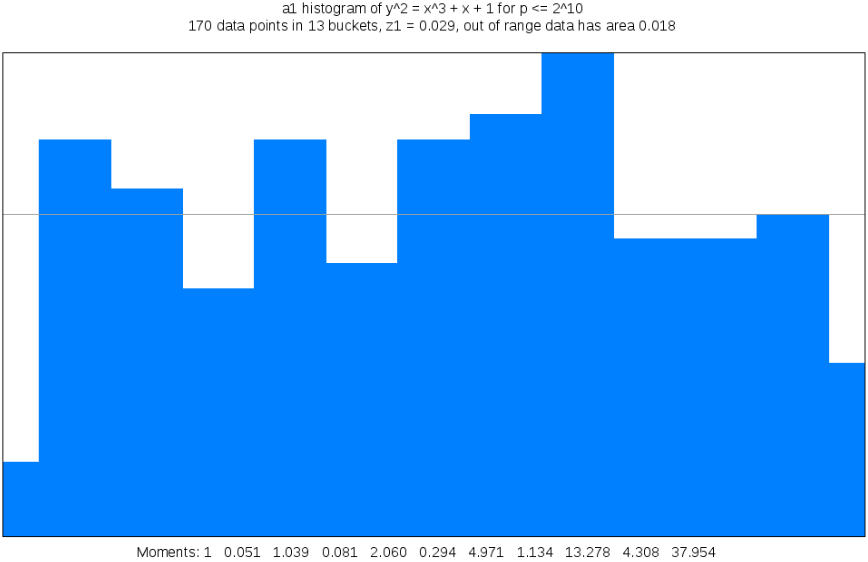
where the trace of Frobenius t_p is an integer in $[-2\sqrt{p}, 2\sqrt{p}]$.

We are interested in the limiting distribution of $x_p = -t_p/\sqrt{p} \in [-2, 2]$, as p varies over primes of good reduction.

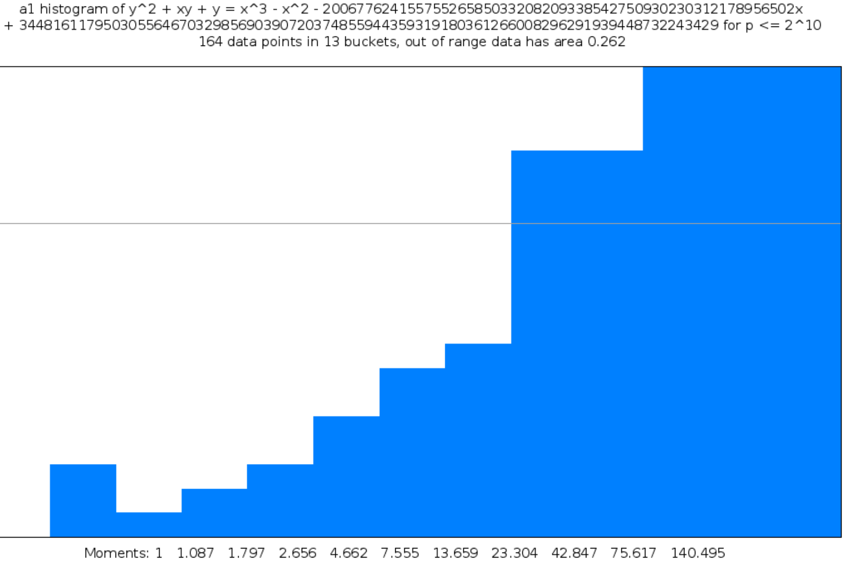
Example: $y^2 = x^3 + x + 1$

p	t_p	x_p	p	t_p	x_p	p	t_p	x_p
3	0	**0.000000**	71	13	**−1.542816**	157	−13	**1.037513**
5	−3	**1.341641**	73	2	**−0.234082**	163	−25	**1.958151**
7	3	**−1.133893**	79	−6	**0.675053**	167	24	**−1.857176**
11	−2	**0.603023**	83	−6	**0.658586**	173	2	**−0.152057**
13	−4	**1.109400**	89	−10	**1.059998**	179	0	**0.000000**
17	0	**0.000000**	97	1	**−0.101535**	181	−8	**0.594635**
19	−1	**0.229416**	101	−3	**0.298511**	191	−25	**1.808937**
23	−4	**0.834058**	103	17	**−1.675060**	193	−7	**0.503871**
29	−6	**1.114172**	107	3	**−0.290021**	197	−24	**1.709929**
37	−10	**1.643990**	109	−13	**1.245174**	199	−18	**1.275986**
41	7	**−1.093216**	113	−11	**1.034793**	211	−11	**0.757271**
43	10	**−1.524986**	127	2	**−0.177471**	223	−20	**1.339299**
47	−12	**1.750380**	131	4	**−0.349482**	227	0	**0.000000**
53	−4	**0.549442**	137	12	**−1.025229**	229	−2	**0.132164**
59	−3	**0.390567**	139	14	**−1.187465**	233	−3	**0.196537**
61	12	**−1.536443**	149	14	**−1.146925**	239	−22	**1.423062**
67	12	**−1.466033**	151	−2	**0.162758**	241	22	**−1.417145**

`http://math.mit.edu/~drew/g1SatoTateDistributions.html`



click histogram to animate (requires adobe reader)



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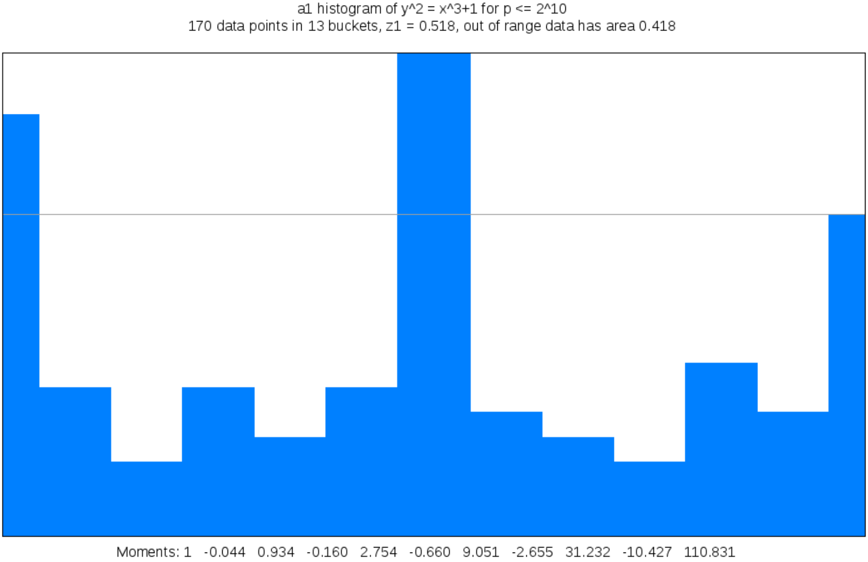

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Sato-Tate distributions in dimension 1

1. Typical case (no CM)

Elliptic curves $E/\mathbb{Q}$ w/o CM have the semi-circular trace distribution. (This is also known for E/k, where k is a totally real number field).

[Taylor et al.]

2. Exceptional cases (CM)

Elliptic curves E/k with CM have one of two distinct trace distributions, depending on whether k contains the CM field or not.

[classical]

Sato-Tate groups in dimension 1

The *Sato-Tate group* of E is a closed subgroup G of $\mathrm{SU}(2) = \mathrm{USp}(2)$ derived from the ℓ-adic Galois representation attached to E.

The refined Sato-Tate conjecture implies that the normalized trace distribution of E converges to the distribution of traces in G given by the Haar measure.

G	G/G^0	E	k	$\mathrm{E}[a_1^0], \mathrm{E}[a_1^2], \mathrm{E}[a_1^4] \dots$
$\mathrm{U}(1)$	C_1	$y^2 = x^3 + 1$	$\mathbb{Q}(\sqrt{-3})$	$1, 2, 6, 20, 70, 252, \dots$
$N(\mathrm{U}(1))$	C_2	$y^2 = x^3 + 1$	$\mathbb{Q}$	$1, 1, 3, 10, 35, 126, \dots$
$\mathrm{SU}(2)$	C_1	$y^2 = x^3 + x + 1$	$\mathbb{Q}$	$1, 1, 2, 5, 14, 42, \dots$

In dimension 1 there are three possible Sato-Tate groups, two of which arise for elliptic curves defined over $\mathbb{Q}$.

Zeta functions and L-polynomials

For a smooth projective curve $C/\mathbb{Q}$ of genus g and each prime p of good redution for C we have the *zeta function*

$$Z(C_p/\mathbb{F}_p;T) := \exp\left(\sum_{k=1}^{\infty} N_k T^k/k\right),$$

where $N_k = \#C_p(\mathbb{F}_{p^k})$. This is a rational function of the form

$$Z(C_p/\mathbb{F}_p;T) = \frac{L_p(T)}{(1-T)(1-pT)},$$

where $L_p(T)$ is an integer polynomial of degree $2g$.

For $g = 1$ we have $L_p(t) = pT^2 + c_1T + 1$, and for $g = 2$,

$$L_p(T) = p^2T^4 + c_1pT^3 + c_2T^2 + c_1T + 1.$$

Normalized L-polynomials

The normalized polynomial

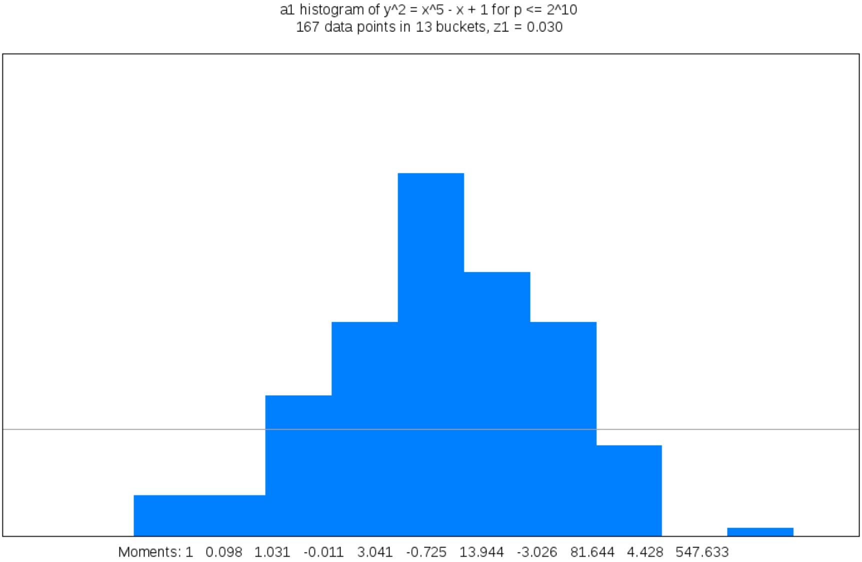
$$\bar{L}_p(T) := L_p(T/\sqrt{p}) = \sum_{i=0}^{2g} a_i T^i \in \mathbb{R}[T]$$

is monic, reciprocal ($a_i = a_{2g-i}$), and unitary (roots on the unit circle). The coefficients a_i necessarily satisfy $|a_i| \le \binom{2g}{i}$.

We now consider the limiting distribution of $a_1, a_2, \ldots, a_g$ over all primes $p \le N$ of good reduction, as $N \to \infty$.

In this talk we will focus primarily on the case $g = 2$.

`http://math.mit.edu/~drew/g2SatoTateDistributions.html`



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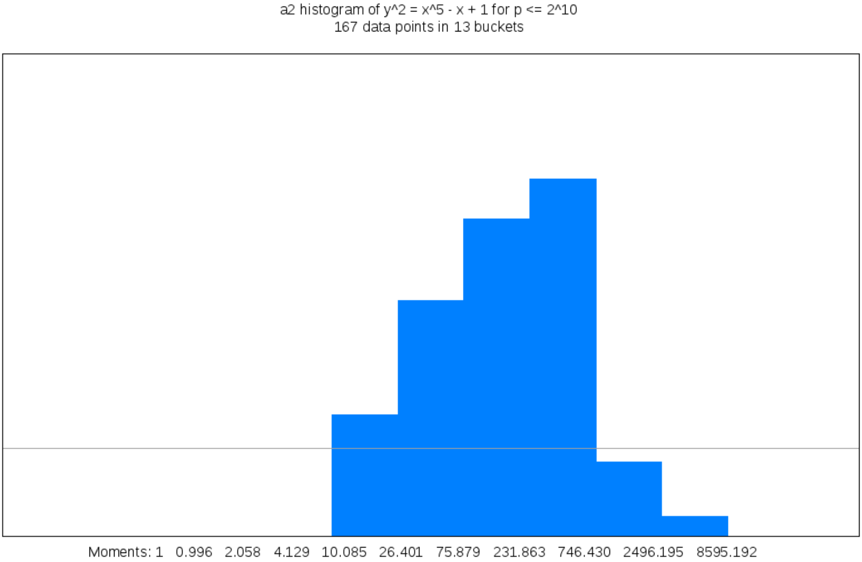

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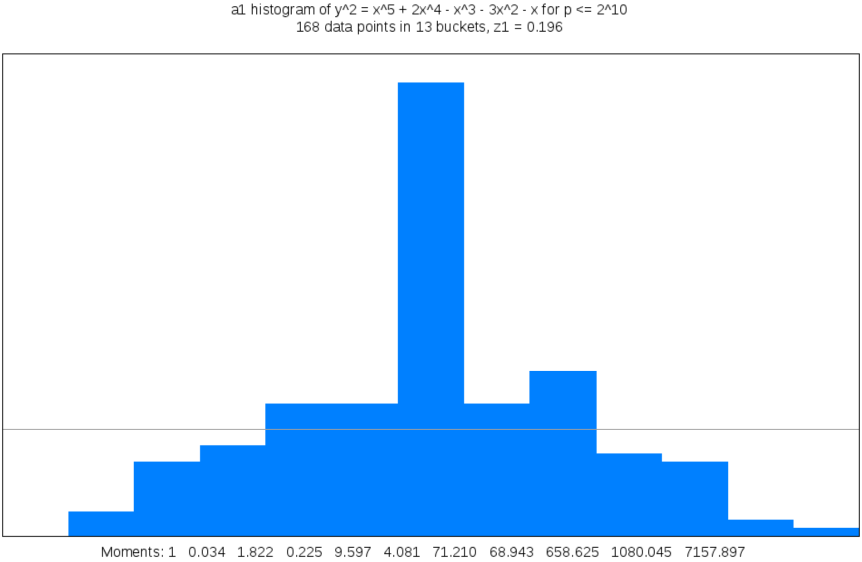

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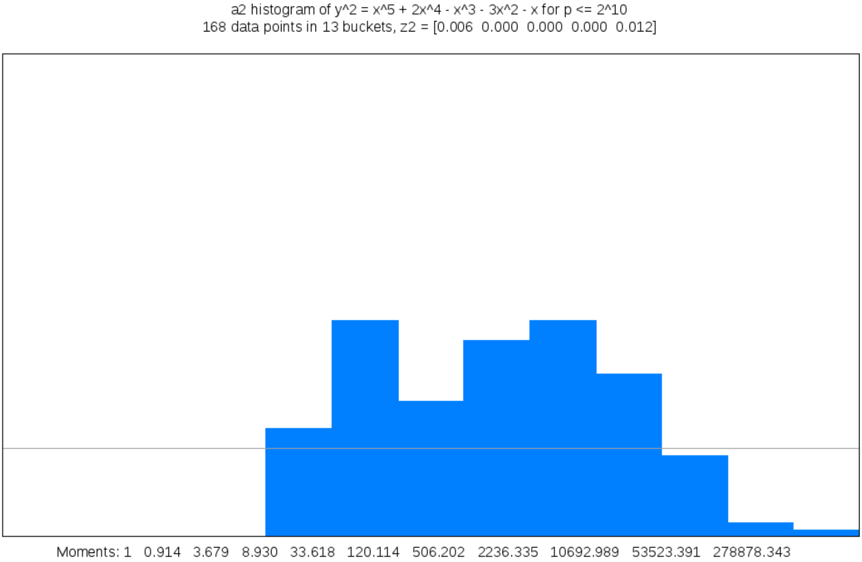

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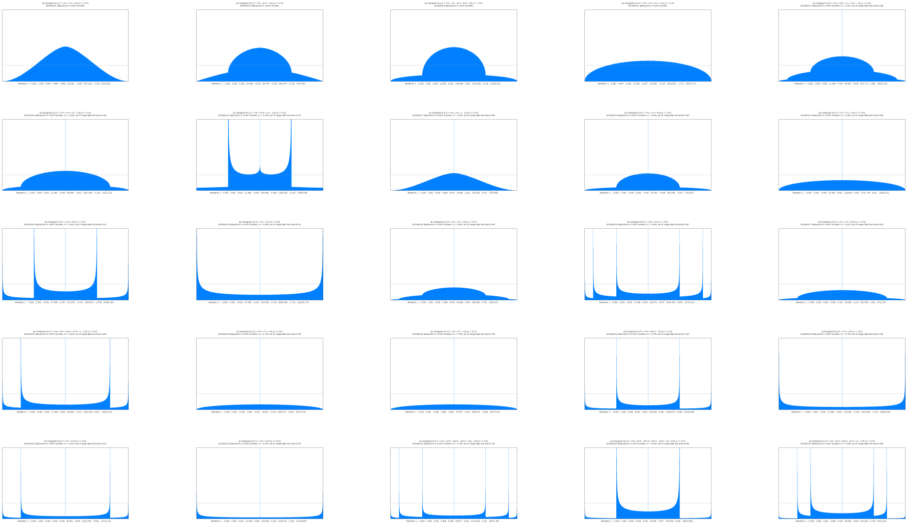
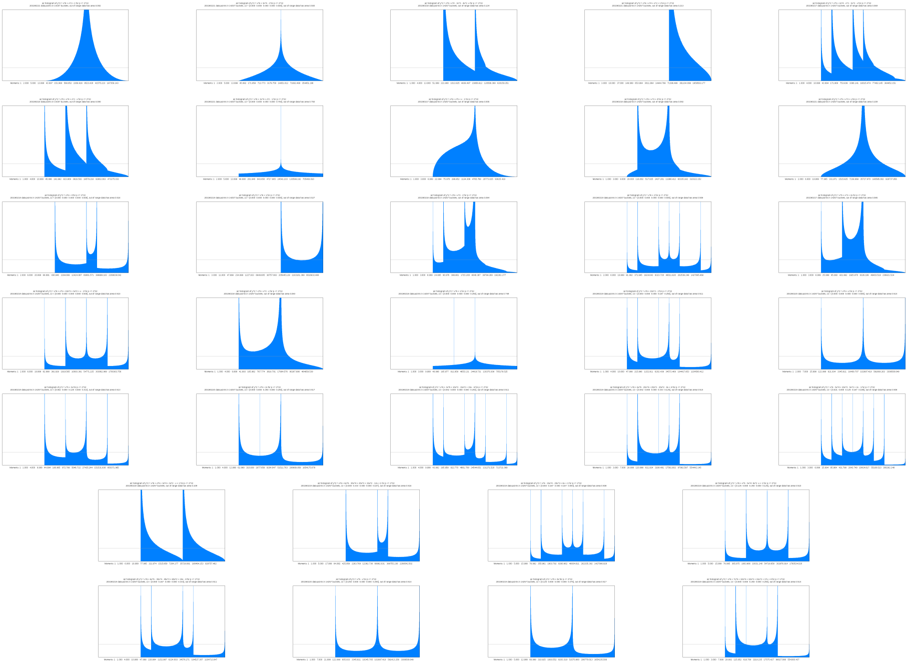
Exceptional trace distributions of genus 2 curves $C/\mathbb{Q}$





The Sato-Tate group

For each prime ℓ and abelian variety A/k, the action of $G_k := \mathrm{Gal}(\bar{k}/k)$ on $V_\ell(A) := (\varprojlim A[\ell^n]) \otimes \mathbb{Q}_\ell$ induces a Galois representation

$$\rho_\ell \colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell).$$

Fixing an embedding $\iota \colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we now apply

$$\ker(G_k \to \mathbb{Q}_\ell^\times) \xrightarrow{\rho_\ell} \mathrm{Sp}_{2g}(\mathbb{Q}_\ell) \xrightarrow{\iota} \mathrm{Sp}_{2g}(\mathbb{C}),$$

and define ST_A to be a maximal compact subgroup of the image.

Conjecturally, ST_A does not depend on ℓ or ι; this is known for $g \leq 3$.

Definition [Serre]

$\mathrm{ST}_A \subseteq \mathrm{USp}(2g)$ is the *Sato-Tate group* of A.

The Sato-Tate conjecture

Let $s(\mathfrak{p})$ denote the conjugacy class of $\rho_\ell(\mathrm{Frob}_\mathfrak{p})/\sqrt{\|\mathfrak{p}\|} \in \mathrm{ST}_A$.

Let μ_{ST_A} denote the image of the Haar measure on $\mathrm{Conj}(\mathrm{ST}_A)$, which does not depend on the choice of ℓ or ι.

Conjecture

The conjugacy classes $s(\mathfrak{p})$ are equidistributed with respect to μ_{ST_A}.

In particular, the distribution of $\bar{L}_\mathfrak{p}(T)$ matches the distribution of characteristic polynomials of random matrices in ST_A.

We can test this numerically by comparing statistics of the coefficients $a_1, \ldots, a_g$ of $\bar{L}_\mathfrak{p}(T)$ over $\|\mathfrak{p}\| \le N$ to the predictions given by μ_{ST_A}.

`https://hensel.mit.edu:8000/home/pub/6`

The Sato-Tate axioms

The Sato-Tate axioms for abelian varieties (weight-1 motives):

1. G is closed subgroup of $\mathrm{USp}(2g)$.
2. **Hodge condition**: G contains a Hodge circle[1] whose conjugates generate a dense subset of G.
3. **Rationality condition**: for each component H of G and each irreducible character χ of $\mathrm{GL}_{2g}(\mathbb{C})$ we have $\mathrm{E}[\chi(\gamma) : \gamma \in H] \in \mathbb{Z}$.

For any fixed g, the set of subgroups $G \subseteq \mathrm{USp}(2g)$ that satisfy the *Sato-Tate axioms* is **finite** up to conjugacy.

[1] An embedding $\theta\colon \mathrm{U}(1) \to G^0$ where $\theta(u)$ has eigenvalue u with multiplicity g.

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For any fixed g, the set of subgroups $G \subseteq \mathrm{USp}(2g)$ that satisfy the *Sato-Tate axioms* is **finite** up to conjugacy.

Theorem

For $g \leq 3$, the group ST_A satisfies the Sato-Tate axioms.

This is expected to hold for all g.

[1] An embedding $\theta\colon \mathrm{U}(1) \to G^0$ where $\theta(u)$ has eigenvalue u with multiplicity g.

Sato-Tate groups in dimension 2

Theorem 1 [FKRS 2012]

Up to conjugacy, 55 subgroups of $\mathrm{USp}(4)$ satisfy the Sato-Tate axioms:

$$
\begin{aligned}
\mathrm{U}(1):\quad & C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O,\\
& J(C_1), J(C_2), J(C_3), J(C_4), J(C_6),\\
& J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O),\\
& C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1\\
\mathrm{SU}(2):\quad & E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6)\\
\mathrm{U}(1)\times\mathrm{U}(1):\quad & F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c}\\
\mathrm{U}(1)\times\mathrm{SU}(2):\quad & \mathrm{U}(1)\times\mathrm{SU}(2),\ N(\mathrm{U}(1)\times\mathrm{SU}(2))\\
\mathrm{SU}(2)\times\mathrm{SU}(2):\quad & \mathrm{SU}(2)\times\mathrm{SU}(2),\ N(\mathrm{SU}(2)\times\mathrm{SU}(2))\\
\mathrm{USp}(4):\quad & \mathrm{USp}(4)
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\mathrm{U}(1)\times\mathrm{U}(1): & F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c} \\
\mathrm{U}(1)\times\mathrm{SU}(2): & \mathrm{U}(1)\times\mathrm{SU}(2),\ N(\mathrm{U}(1)\times\mathrm{SU}(2)) \\
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\end{array}$$

Of these, exactly 52 arise as ST_A for an abelian surface A (34 over $\mathbb{Q}$).

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& J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O), \\
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\mathrm{SU}(2)\times\mathrm{SU}(2): & \mathrm{SU}(2)\times\mathrm{SU}(2),\ N(\mathrm{SU}(2)\times\mathrm{SU}(2)) \\
\mathrm{USp}(4): & \mathrm{USp}(4)
\end{array}$$

Of these, exactly 52 arise as ST_A for an abelian surface A (34 over $\mathbb{Q}$).

This theorem says nothing about equidistribution, however this is now known in many special cases [FS 2012, Johansson 2013].

Sato-Tate groups in dimension 2 with $G^0 = \mathrm{U}(1)$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
1	1	C_1	C_1	0	0, 0, 0, 0, 0	8, 96, 1280, 17920	4, 18, 88, 454
1	2	C_2	C_2	1	0, 0, 0, 0, 0	4, 48, 640, 8960	2, 10, 44, 230
1	3	C_3	C_3	0	0, 0, 0, 0, 0	4, 36, 440, 6020	2, 8, 34, 164
1	4	C_4	C_4	1	0, 0, 0, 0, 0	4, 36, 400, 5040	2, 8, 32, 150
1	6	C_6	C_6	1	0, 0, 0, 0, 0	4, 36, 400, 4900	2, 8, 32, 148
1	4	D_2	D_2	3	0, 0, 0, 0, 0	2, 24, 320, 4480	1, 6, 22, 118
1	6	D_3	D_3	3	0, 0, 0, 0, 0	2, 18, 220, 3010	1, 5, 17, 85
1	8	D_4	D_4	5	0, 0, 0, 0, 0	2, 18, 200, 2520	1, 5, 16, 78
1	12	D_6	D_6	7	0, 0, 0, 0, 0	2, 18, 200, 2450	1, 5, 16, 77
1	2	$J(C_1)$	C_2	1	1, 0, 0, 0, 0	4, 48, 640, 8960	1, 11, 40, 235
1	4	$J(C_2)$	D_2	3	1, 0, 0, 0, 1	2, 24, 320, 4480	1, 7, 22, 123
1	6	$J(C_3)$	C_6	3	1, 0, 0, 2, 0	2, 18, 220, 3010	1, 5, 16, 85
1	8	$J(C_4)$	$\mathrm{C}_4 \times \mathrm{C}_2$	5	1, 0, 2, 0, 1	2, 18, 200, 2520	1, 5, 16, 79
1	12	$J(C_6)$	$\mathrm{C}_6 \times \mathrm{C}_2$	7	1, 2, 0, 2, 1	2, 18, 200, 2450	1, 5, 16, 77
1	8	$J(D_2)$	$\mathrm{D}_2 \times \mathrm{C}_2$	7	1, 0, 0, 0, 3	1, 12, 160, 2240	1, 5, 13, 67
1	12	$J(D_3)$	D_6	9	1, 0, 0, 2, 3	1, 9, 110, 1505	1, 4, 10, 48
1	16	$J(D_4)$	$\mathrm{D}_4 \times \mathrm{C}_2$	13	1, 0, 2, 0, 5	1, 9, 100, 1260	1, 4, 10, 45
1	24	$J(D_6)$	$\mathrm{D}_6 \times \mathrm{C}_2$	19	1, 2, 0, 2, 7	1, 9, 100, 1225	1, 4, 10, 44
1	2	$C_{2,1}$	C_2	1	0, 0, 0, 0, 1	4, 48, 640, 8960	3, 11, 48, 235
1	4	$C_{4,1}$	C_4	3	0, 0, 2, 0, 0	2, 24, 320, 4480	1, 5, 22, 115
1	6	$C_{6,1}$	C_6	3	0, 2, 0, 0, 1	2, 18, 220, 3010	1, 5, 18, 85
1	4	$D_{2,1}$	D_2	3	0, 0, 0, 0, 2	2, 24, 320, 4480	2, 7, 26, 123
1	8	$D_{4,1}$	D_4	7	0, 0, 2, 0, 2	1, 12, 160, 2240	1, 4, 13, 63
1	12	$D_{6,1}$	D_6	9	0, 2, 0, 0, 4	1, 9, 110, 1505	1, 4, 11, 48
1	6	$D_{3,2}$	D_3	3	0, 0, 0, 0, 3	2, 18, 220, 3010	2, 6, 21, 90
1	8	$D_{4,2}$	D_4	5	0, 0, 0, 0, 4	2, 18, 200, 2520	2, 6, 20, 83
1	12	$D_{6,2}$	D_6	7	0, 0, 0, 0, 6	2, 18, 200, 2450	2, 6, 20, 82
1	12	T	A_4	3	0, 0, 0, 0, 0	2, 12, 120, 1540	1, 4, 12, 52
1	24	O	S_4	9	0, 0, 0, 0, 0	2, 12, 100, 1050	1, 4, 11, 45
1	24	O_1	S_4	15	0, 0, 6, 0, 6	1, 6, 60, 770	1, 3, 8, 30
1	24	$J(T)$	$\mathrm{A}_4 \times \mathrm{C}_2$	15	1, 0, 0, 8, 3	1, 6, 60, 770	1, 3, 7, 29
1	48	$J(O)$	$\mathrm{S}_4 \times \mathrm{C}_2$	33	1, 0, 6, 8, 9	1, 6, 50, 525	1, 3, 7, 26

Sato-Tate groups in dimension 2 with $G^0 \neq \mathrm{U}(1)$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
3	1	E_1	C_1	0	0, 0, 0, 0, 0	4, 32, 320, 3584	3, 10, 37, 150
3	2	E_2	C_2	1	0, 0, 0, 0, 0	2, 16, 160, 1792	1, 6, 17, 78
3	3	E_3	C_3	0	0, 0, 0, 0, 0	2, 12, 110, 1204	1, 4, 13, 52
3	4	E_4	C_4	1	0, 0, 0, 0, 0	2, 12, 100, 1008	1, 4, 11, 46
3	6	E_6	C_6	1	0, 0, 0, 0, 0	2, 12, 100, 980	1, 4, 11, 44
3	2	$J(E_1)$	C_2	1	0, 0, 0, 0, 0	2, 16, 160, 1792	2, 6, 20, 78
3	4	$J(E_2)$	D_2	3	0, 0, 0, 0, 0	1, 8, 80, 896	1, 4, 10, 42
3	6	$J(E_3)$	D_3	3	0, 0, 0, 0, 0	1, 6, 55, 602	1, 3, 8, 29
3	8	$J(E_4)$	D_4	5	0, 0, 0, 0, 0	1, 6, 50, 504	1, 3, 7, 26
3	12	$J(E_6)$	D_6	7	0, 0, 0, 0, 0	1, 6, 50, 490	1, 3, 7, 25
2	1	F	C_1	0	0, 0, 0, 0, 0	4, 36, 400, 4900	2, 8, 32, 148
2	2	F_a	C_2	0	0, 0, 0, 0, 1	3, 21, 210, 2485	2, 6, 20, 82
2	2	F_c	C_2	1	0, 0, 0, 0, 0	2, 18, 200, 2450	1, 5, 16, 77
2	2	F_{ab}	C_2	1	0, 0, 0, 0, 1	2, 18, 200, 2450	2, 6, 20, 82
2	4	F_{ac}	C_4	3	0, 0, 2, 0, 1	1, 9, 100, 1225	1, 3, 10, 41
2	4	$F_{a,b}$	D_2	1	0, 0, 0, 0, 3	2, 12, 110, 1260	2, 5, 14, 49
2	4	$F_{ab,c}$	D_2	3	0, 0, 0, 0, 1	1, 9, 100, 1225	1, 4, 10, 44
2	8	$F_{a,b,c}$	D_4	5	0, 0, 2, 0, 3	1, 6, 55, 630	1, 3, 7, 26
4	1	G_4	C_1	0	0, 0, 0, 0, 0	3, 20, 175, 1764	2, 6, 20, 76
4	2	$N(G_4)$	C_2	0	0, 0, 0, 0, 1	2, 11, 90, 889	2, 5, 14, 46
6	1	G_6	C_1	0	0, 0, 0, 0, 0	2, 10, 70, 588	2, 5, 14, 44
6	2	$N(G_6)$	C_2	1	0, 0, 0, 0, 0	1, 5, 35, 294	1, 3, 7, 23
10	1	$\mathrm{USp}(4)$	C_1	0	0, 0, 0, 0, 0	1, 3, 14, 84	1, 2, 4, 10

Galois types

Let A be an abelian variety defined over a number field k.

Let K be the minimal extension of k for which $\mathrm{End}(A_K) = \mathrm{End}(A_{\bar{\mathbb{Q}}})$.

$\mathrm{Gal}(K/k)$ acts on the $\mathbb{R}$-algebra $\mathrm{End}(A_K)_{\mathbb{R}} = \mathrm{End}(A_K) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition

The *Galois type* of A is the isomorphism class of $[\mathrm{Gal}(K/k), \mathrm{End}(A_K)_{\mathbb{R}}]$, where $[G, E] \simeq [G', E']$ iff there are isomorphisms $G \simeq G'$ and $E \simeq E'$ that are compatible with the the Galois action.

Note: in several cases $G \simeq G'$ and $E \simeq E'$, but $[G, E] \not\simeq [G', E']$.

Galois types and Sato-Tate groups in dimension 2

Theorem 2 [FKRS 2012]

Up to conjugacy, the Sato-Tate group G of an abelian surface A is uniquely determined by its Galois type, and vice versa.

We also have $G/G^0 \simeq \mathrm{Gal}(K/k)$, and G^0 is uniquely determined by the isomorphism class of $\mathrm{End}(A_K)_{\mathbb{R}}$, and vice versa:

$$\begin{array}{rlrl} \mathrm{U}(1) & \mathrm{M}_2(\mathbb{C}) & \mathrm{U}(1)\times\mathrm{SU}(2) & \mathbb{C}\times\mathbb{R} \\ \mathrm{SU}(2) & \mathrm{M}_2(\mathbb{R}) & \mathrm{SU}(2)\times\mathrm{SU}(2) & \mathbb{R}\times\mathbb{R} \\ \mathrm{U}(1)\times\mathrm{U}(1) & \mathbb{C}\times\mathbb{C} & \mathrm{USp}(4) & \mathbb{R} \end{array}$$

There are 52 distinct Galois types of abelian surfaces.

The proof uses the *algebraic Sato-Tate group* of Banaszak and Kedlaya, which, for $g \leq 3$, uniquely determines ST_A.

Searching for curves

We surveyed the $\bar{L}$-polynomial distributions of genus 2 curves

$$y^2 = x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

$$y^2 = x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

with integer coefficients $|c_i| \leq 128$, over 2^{48} curves.

We found over 10 million non-isogenous curves with exceptional distributions, including at least 3 apparent matches for all of our target Sato-Tate groups.

Representative examples were computed to high precision $N = 2^{30}$.

For each example, the field K was then determined, allowing the Galois type, and hence the Sato-Tate group, to be **provably** identified.

Exhibiting Sato-Tate groups of abelian surfaces

Remarkably, the 34 Sato-Tate groups that can arise for an abelian surface over $\mathbb{Q}$ can all be realized via Jacobians of genus 2 curves.

By extending the base field from $\mathbb{Q}$ to a suitable subfield k of K, we can restrict $G/G^0 \simeq \mathrm{Gal}(K/k)$ to any normal subgroup of $\mathrm{Gal}(K/k)$ (this does not change the identity component G^0).

This allows us to realize all 52 Sato-Tate groups using 34 curves.

In fact, these 52 Sato-Tate groups can be realized using just 9 hyperelliptic curves over varying base fields.

Genus 2 curves realizing Sato-Tate groups with $G^0 = \mathrm{U}(1)$

Group	Curve $y^2 = f(x)$	k	K
C_1	$x^6 + 1$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3})$
C_2	$x^5 - x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2})$
C_3	$x^6 + 4$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
C_4	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a);\, a^4 + 17a^2 + 68 = 0$
C_6	$x^6 + 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
D_2	$x^5 + 9x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
D_3	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
D_4	$x^5 + 3x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
D_6	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3}, a);\, a^3 + 3a - 2 = 0$
T	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a, b);$
			$a^3 - 7a + 7 = b^4 + 4b^2 + 8b + 8 = 0$
O	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-11}, a, b);$
			$a^3 - 4a + 4 = b^4 + 22b + 22 = 0$
$J(C_1)$	$x^5 - x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_2)$	$x^5 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(C_4)$	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}$	see entry for C_4
$J(C_6)$	$x^6 - 15x^4 - 20x^3 + 6x + 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{3}, a);\, a^3 + 3a^2 - 1 = 0$
$J(D_2)$	$x^5 + 9x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
$J(D_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(D_4)$	$x^5 + 3x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
$J(D_6)$	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}$	see entry for D_6
$J(T)$	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}$	see entry for T
$J(O)$	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}$	see entry for O
$C_{2,1}$	$x^6 + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3})$
$C_{4.1}$	$x^5 + 2x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt[4]{2})$
$C_{6,1}$	$x^6 + 6x^5 - 30x^4 + 20x^3 + 15x^2 - 12x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, a);\, a^3 - 3a + 1 = 0$
$D_{2,1}$	$x^5 + x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$D_{4,1}$	$x^5 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$D_{6,1}$	$x^6 + 6x^5 - 30x^4 - 40x^3 + 60x^2 + 24x - 8$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-3}, a);\, a^3 - 9a + 6 = 0$
$D_{3,2}$	$x^6 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$D_{4,2}$	$x^6 + x^5 + 10x^3 + 5x^2 + x - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a);\, a^4 - 14a^2 + 28a - 14 = 0$
$D_{6,2}$	$x^6 + 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
O_1	$x^6 + 7x^5 + 10x^4 + 10x^3 + 15x^2 + 17x + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a, b);$
			$a^3 + 5a + 10 = b^4 + 4b^2 + 8b + 2 = 0$

Genus 2 curves realizing Sato-Tate groups with $G^0 \neq \mathrm{U}(1)$

Group	Curve $y^2 = f(x)$	k	K
F	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i, \sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_a	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
F_{ab}	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_{ac}	$x^5 + 1$	$\mathbb{Q}$	$\mathbb{Q}(a); a^4 + 5a^2 + 5 = 0$
$F_{a,b}$	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
E_1	$x^6 + x^4 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
E_2	$x^6 + x^5 + 3x^4 + 3x^2 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$
E_3	$x^5 + x^4 - 3x^3 - 4x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a); a^3 - 3a + 1 = 0$
E_4	$x^5 + x^4 + x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a); a^4 - 5a^2 + 5 = 0$
E_6	$x^5 + 2x^4 - x^3 - 3x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{7}, a); a^3 - 7a - 7 = 0$
$J(E_1)$	$x^5 + x^3 + x$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$J(E_2)$	$x^5 + x^3 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(E_3)$	$x^6 + x^3 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$J(E_4)$	$x^5 + x^3 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$J(E_6)$	$x^6 + x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$G_{1,3}$	$x^6 + 3x^4 - 2$	$\mathbb{Q}(i)$	$\mathbb{Q}(i)$
$N(G_{1,3})$	$x^6 + 3x^4 - 2$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$G_{3,3}$	$x^6 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
$N(G_{3,3})$	$x^6 + x^5 + x - 1$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$\mathrm{USp}(4)$	$x^5 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}$

Computing zeta functions

Algorithms to compute $L_p(T)$ for low genus hyperelliptic curves:

	complexity (ignoring factors of $O(\log\log p)$)		
algorithm	$g = 1$	$g = 2$	$g = 3$
point enumeration	$p \log p$	$p^2 \log p$	$p^3 \log p$
group computation	$p^{1/4} \log p$	$p^{3/4} \log p$	$p^{5/4} \log p$
p-adic cohomology	$p^{1/2} \log^2 p$	$p^{1/2} \log^2 p$	$p^{1/2} \log^2 p$
CRT (Schoof-Pila)	$\log^5 p$	$\log^8 p$	$\log^{12} p$

An average polynomial-time algorithm

All of the methods above perform separate computations for each p. But we want to compute $L_p(T)$ for all good $p \leq N$ using reductions of *the same curve* in each case.

An average polynomial-time algorithm

All of the methods above perform separate computations for each p. But we want to compute $L_p(T)$ for all good $p \leq N$ using reductions of *the same curve* in each case.

Theorem (Harvey 2012)

There exists a deterministic algorithm that, given a hyperelliptic curve $y^2 = f(x)$ of genus g with a rational Weierstrass point and an integer N, computes $L_p(T)$ for all good primes $p \leq N$ in time

$$O\big(g^{8+\epsilon} N \log^{3+\epsilon} N\big),$$

assuming the coefficients of $f \in \mathbb{Z}[x]$ have size bounded by $O(\log N)$.

Average time is $O\big(g^{8+\epsilon}\log^{4+\epsilon} N\big)$ per prime, polynomial in g and $\log p$. Recently generalized to arbitrary arithmetic schemes (in principle).

An average polynomial-time algorithm

algorithm	complexity (ignoring factors of $O(\log\log p)$)		
	$g=1$	$g=2$	$g=3$
point enumeration	$p\log p$	$p^2\log p$	$p^3\log p$
group computation	$p^{1/4}\log p$	$p^{3/4}\log p$	$p^{5/4}\log p$
p-adic cohomology	$p^{1/2}\log^2 p$	$p^{1/2}\log^2 p$	$p^{1/2}\log^2 p$
CRT (Schoof-Pila)	$\log^5 p$	$\log^8 p$	$\log^{12} p$
Average polytime	$\log^4 p$	$\log^4 p$	$\log^4 p$

But is it practical?

	$d = 5$			$d = 6$	
N	**new** (2+)	**new** (1)	**smalljac**	**new** (0)	**smalljac**
2^{14}	0.1	0.1	0.2	0.2	0.3
2^{15}	0.2	0.3	0.4	0.5	0.5
2^{16}	0.3	0.8	1.2	1.4	1.4
2^{17}	0.8	1.8	3.7	3.4	4.4
2^{18}	2.1	4.7	13.3	8.6	15.2
2^{19}	5.0	11.1	56.2	17.5	66.4
2^{20}	12.6	26.0	257	46.3	284
2^{21}	30.5	61.4	828	108	914
2^{22}	72.4	142	2630	254	2900
2^{23}	170	321	8570	583	9520
2^{24}	400	729	28000	1340	31100
2^{25}	923	1660	92300	3030	102000
2^{26}	2140	3800	316000	6827	349000

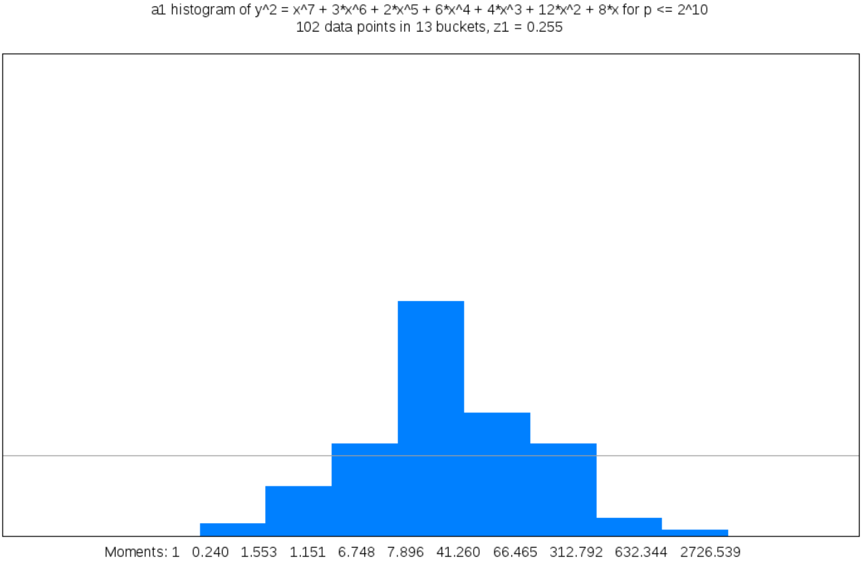
Genus 2 comparison of **new** algorithm (# rat Weierstrass pts) with **smalljac**.

(Times in CPU seconds).

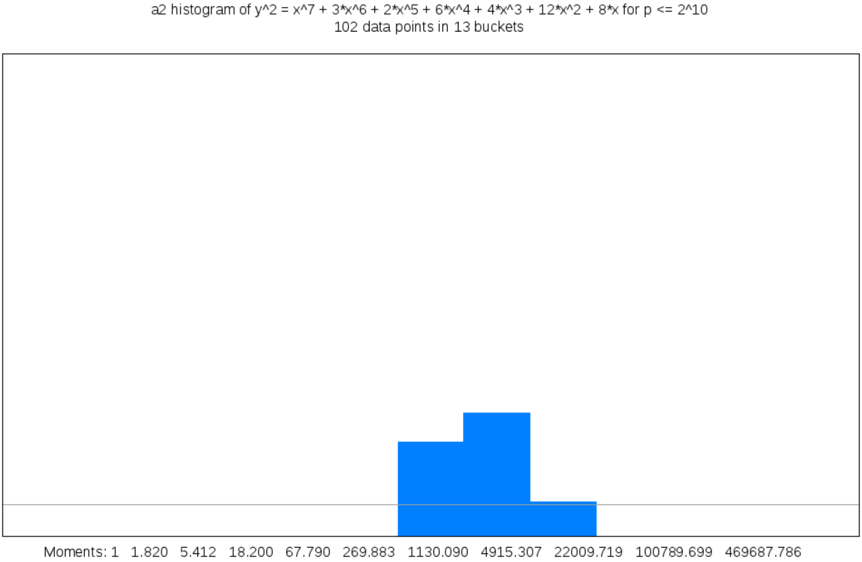
	$d=7$				$d=8$
N	**new**(3+)	**new**(2)	**new**(1)	**hypellfrob**	**new**(0)
2^{14}	0.2	0.3	0.3	6.8	0.5
2^{15}	0.5	0.7	1.0	15.6	1.5
2^{16}	1.2	1.9	2.7	37.6	4.3
2^{17}	3.0	5.1	7.0	95.0	11.1
2^{18}	7.5	12.3	16.3	250	28.1
2^{19}	18.1	29.6	38.7	681	67.3
2^{20}	43.4	70.4	91.7	1920	161
2^{21}	105	169	212	5460	385
2^{22}	246	405	489	16300	921
2^{23}	571	948	1123	49400	2140
2^{24}	1330	2220	2540	152000	4980
2^{25}	3090	5130	6510	467000	11500
2^{26}	7190	11900	16600	1490000	26500

Genus 3 comparison of **new** (# rat Weierstrass pts) with **hypellfrob**.

(Times in CPU seconds).



click histogram to animate (requires adobe reader)



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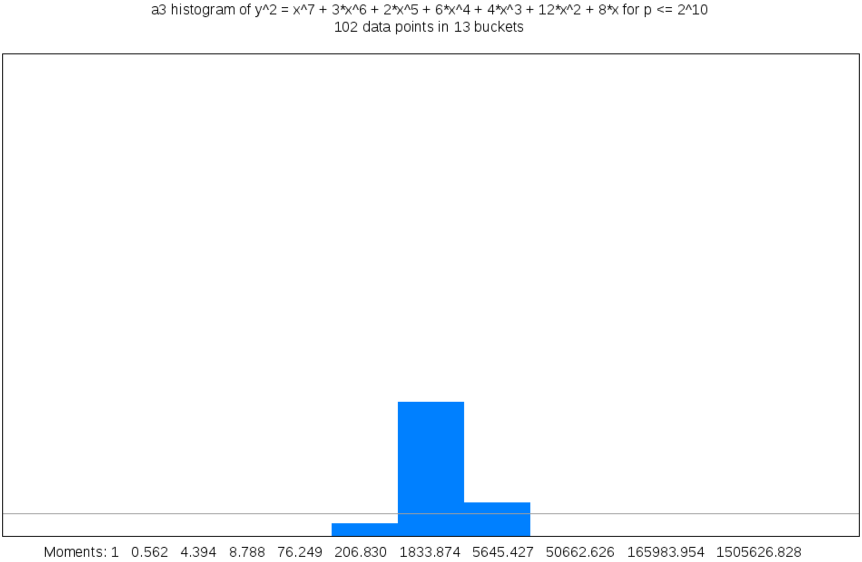

click histogram to animate (requires adobe reader)