Sieve theory and small gaps between primes: Introduction

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(on behalf of D.H.J. Polymath)

Explicit Methods in Number Theory
MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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A quick historical overview

$$\Delta_m := \liminf_{n\to\infty} \frac{p_{n+m} - p_n}{\log p_n} \qquad H_m := \liminf_{n\to\infty} (p_{n+m} - p_n)$$

Twin Prime Conjecture: $H_1 = 2$ Prime Tuples Conjecture: $H_m \sim m \log m$

1896	Hadamard–Vallée Poussin	$\Delta_1 \leq 1$
1926	Hardy–Littlewood	$\Delta_1 \leq 2/3$ under GRH
1940	Rankin	$\Delta_1 \leq 3/5$ under GRH
1940	Erdős	$\Delta_1 < 1$
1956	Ricci	$\Delta_1 \leq 15/16$
1965	Bombieri–Davenport	$\Delta_1 \leq 1/2$, $\Delta_m \leq m - 1/2$
...	...	...
1988	Maier	$\Delta_1 < 0.2485.$
2005	Goldston-Pintz-Yıldırım	$\Delta_1 = 0$, $\Delta_m \leq m - 1$, EH $\Rightarrow H_1 \leq 16$
2013	Zhang	$H_1 < 70,000,000$
2013	Polymath 8a	$H_1 \leq 4680$
2013	Maynard-Tao	$H_1 \leq 600$, $H_m \ll m^3 e^{4m}$, EH $\Rightarrow H_1 \leq 12$
2014	Polymath 8b	$H_1 \leq 246$, $H_m \ll e^{3.815m}$, GEH $\Rightarrow H_1 \leq 6$
		$H_2 \leq 398,130$, $H_3 \leq 24,797,814$, ...

The prime number theorem in arithmetic progressions

Define the weighted prime counting functions[1]

$$\Theta(x) := \sum_{\text{prime } p \le x} \log p, \qquad \Theta(x;q,a) := \sum_{\substack{\text{prime } p \le x \\ p \equiv a \bmod q}} \log p.$$

Then $\Theta(x) \sim x$ (the prime number theorem), and for $a \perp q$,

$$\Theta(x;q,a) \sim \frac{x}{\phi(q)}.$$

We are interested in the discrepancy between these two quantities.
Clearly $\frac{-x}{\phi(q)} \le \Theta(x;q,a) - \frac{x}{\phi(q)} \le \left(\frac{x}{q} + 1\right) \log x$, and for any $Q < x$,

$$\sum_{q \le Q} \max_{a \perp q} \left| \Theta(x;q,a) - \frac{x}{\phi(q)} \right| \le \sum_{q \le Q} \left(\frac{2x \log x}{q} + \frac{x}{\phi(q)} \right) \ll x(\log x)^2.$$

[1] One can also use $\psi(x) := \sum_{n \le x} \Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function.

The Elliott-Halberstam conjecture

For any $0 < \theta < 1$, let EH$[\theta]$ denote the claim that for any $A \geq 1$,

$$\sum_{q \leq x^\theta} \max_{a \perp q} \left| \Theta(x; q, a) - \frac{x}{\phi(q)} \right| \ll \frac{x}{(\log x)^A}.$$

1965: Bombieri and Vinogradov prove EH$[\theta]$ for all $\theta < 1/2$.

Enrico Bombieri

Ivan Vinogradov

1968: Elliott and Halberstam conjecture EH$[\theta]$ for all $\theta < 1$.

Peter Elliot

Heini Halberstam

Prime tuples

Let $\mathcal{H} = \{h_1, \ldots h_k\}$ be a set of k integers. We call $\mathcal{H}$ *admissible* if it does not form a complete set of residues modulo any prime.

inadmissible: $\{0,1\}$, $\{0,2,4\}$, $\{0,2,6,8,12,14\}$.
admissible: $\{0\}$, $\{0,2\}$, $\{0,2,6\}$, $\{0,4,6\}$, $\{0,4,6,10,12,16\}$.

Let $\pi(n+\mathcal{H})$ count the primes in $n+\mathcal{H} := \{n+h_1, \ldots, n+h_k\}$.

Conjecture (Hardy-Littlewood 1923)

Let $\mathcal{H}$ be an admissible k-tuple. There is an explicit $c_{\mathcal{H}} > 0$ for which

$$\pi_{\mathcal{H}}(x) := \#\{n \le x : \pi(n+\mathcal{H}) = k\} \sim c_{\mathcal{H}} \int_2^x \frac{dt}{(\log t)^k},$$

Godfrey Hardy

John Littlewood

The GPY Theorem

Let DHL$[k, r]$ denote the claim that for every admissible k-tuple $\mathcal{H}$, $\pi(n+\mathcal{H}) \geq r$ for infinitely many n. Put $\operatorname{diam}(\mathcal{H}) := \max(\mathcal{H}) - \min(\mathcal{H})$.

Then DHL$[k, m+1] \Rightarrow H_m \leq \operatorname{diam}(\mathcal{H})$ for any admissible k-tuple $\mathcal{H}$.

Theorem (Goldston-Pintz-Yıldırım 2005)

For $0 < \theta < 1$, if $k \geq 2$ and $\ell \geq 1$ are integers for which

$$2\theta > \left(1 + \frac{1}{2\ell+1}\right)\left(1 + \frac{2\ell+1}{k}\right),$$

then EH$[\theta] \Rightarrow$ DHL$[k, 2]$*. In particular,* EH$[\frac{1}{2}+\epsilon] \Rightarrow H_1 < \infty$*.*


Daniel Goldston


János Pintz


Cem Yıldırım

The GPY method

Let $\Theta(n+\mathcal{H}) := \sum_{n+h_i \text{ prime}} \log(n+h_i)$, where $\mathcal{H} = \{h_1, \ldots, h_k\}$.

To prove DHL$[k, m+1]$ it suffices to show that for any admissible k-tuple $\mathcal{H}$ there exist nonnegative weights w_n for which

$$\sum_{x<n\leq 2x} w_n \bigl(\Theta(n+\mathcal{H}) - m\log(3x)\bigr) > 0 \qquad (1)$$

holds for all sufficiently large x.

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holds for all sufficiently large x. GPY used weights $w(n)$ of the form

$$w_n := \Bigg(\sum_{\substack{d \mid \prod_i (n+h_i) \\ d<R}} \lambda_d\Bigg)^2, \qquad \lambda_d := \mu(d)f(d), \qquad R := x^{\theta/2}$$

to establish (1) with $m = 1^*$ and $\theta > \frac{1}{2}$, using $f(d) \approx \left(\log \frac{R}{d}\right)^{k+\ell}$.
(motivation: $\sum_{d|n} \mu(d) \left(\log \frac{n}{d}\right)^k$ vanishes when $\omega(n) > k$).

* As noted by GPY, their method cannot address $m > 1$, even under EH.

Zhang's Theorem

Let MPZ$[\varpi, \delta]$ denote the claim that for any $A \geq 1$ we have

$$\sum_q \left| \Theta(x; q, a) - \frac{x}{\phi(q)} \right| \ll \frac{x}{(\log x)^A},$$

where q varies over x^δ-smooth squarefree integers up to $x^{1/2+2\varpi}$ and a is a fixed x^δ-coarse integer (depending on x but not q).*

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Theorem (Zhang 2013)

1. *For any $\varpi, \delta > 0$ and all sufficiently large k,*

$$\text{MPZ}[\varpi, \delta] \Rightarrow \text{DHL}[k, 2].^\dagger$$

2. MPZ$[\varpi, \delta]$ *holds for all* $\varpi, \delta \leq 1/1168$.

*Zhang imposes an additional constraint on a that can be eliminated.
†A similar (weaker) implication was proved earlier by Motohashi and Pintz (2006).

Zhang's result

Using $\varpi = \delta = 1/1168$, Zhang proved that DHL$[k, 2]$ holds for all

$$k \geq 3.5 \times 10^6.$$

For $k = 3.5 \times 10^6$, taking the first k primes greater than k yields an admissible k-tuple of diameter less than 7×10^7.* It follows that

$$H_1 = \liminf_{n\to\infty} (p_{n+1} - p_n) < 7 \times 10^7.$$

Yitang Zhang

*In fact, less than 6×10^7.

The Polymath project

Goals of Polymath8a:

1. Improve Zhang's bound on H_1,
2. Attempt to better understand and refine Zhang's argument.

Natural sub-projects for addressing the first goal:

1. Minimizing $H(k)$ by constructing narrow admissible k-tuples.
2. Minimizing k for which MPZ(ϖ, δ) implies DHL$(k, 2)$.
3. Maximizing ϖ for which MPZ(ϖ, δ) holds.

Questions relevant to the second goal:

1. What role do the Weil conjectures play?
2. Can the hypotheses in MPZ(ϖ, δ) be usefully modified?

Polymath8 web page.

Polymath 8a results

ϖ, δ	k	H	
$\varpi = \delta = 1/1168$	3 500 000	70 000 000	Zhang's paper
$\varpi = \delta = 1/1168$	3 500 000	55 233 504	Optimize $H = H(k)$
$\varpi = \delta = 1/1168$	341 640	4 597 926	Optimize $k = k(\varpi, \delta)$
$\varpi = \delta = 1/1168$	34 429	386 344	Make $k \propto \varpi^{-3/2}$
$828\varpi + 172\delta < 1$	22 949	248 816	Allow $\varpi \neq \delta$
$280\varpi + 80\delta < 3$	873	6712	Strengthen MPZ(ϖ, δ)
$280\varpi + 80\delta < 3$	720	5414	Make k less sensitive to δ
$600\varpi + 180\delta < 7$	632	4680	Further optimize ϖ, δ

Using only the Riemann hypothesis for curves:

$168\varpi + 48\delta < 1$	1783	14 950

A detailed timeline of improvements can be found here.

Optimized GPY Theorem

In the GPY Theorem (and Zhang's result), we have $k \propto \varpi^{-2}$.
This can be improved to $k \propto \varpi^{-3/2}$.

Theorem (D.H.J. Polymath 2013)

Let $k \geq 2$ *and* $0 < \varpi < 1/4$ *and* $0 < \delta < 1/4 + \varpi$ *satisfy*

$$(1 + 4\varpi)(1 - \kappa) \; > \; \frac{j_{k-2}^2}{k(k-1)},$$

where j_k *is the first positive zero of the Bessel function* J_k *of the first kind and* $\kappa = \kappa(\varpi, \delta, k)$ *is an explicit error term.*

Then $\mathrm{MPZ}[\varpi, \delta] \Rightarrow \mathrm{DHL}[k, 2]$.
Moreover, $\mathrm{EH}[1/2 + 2\varpi] \Rightarrow \mathrm{DHL}[k, 2]$ *with* $\kappa = 0$.*

We have $j_n = n + cn^{1/3} + O(n^{-1/3})$, so $\frac{j_{k-2}^2}{k(k-1)} \sim 1 + 2ck^{-2/3}$.

*The second statement was independently proved by Farkas, Pintz, and Revesz.

Dense divisibility

For each $i \in \mathbb{Z}_{\geq 0}$ and $y \in \mathbb{R}_{\geq 1}$, we define *$i$-tuply y-dense divisibility*:

1. Every natural number n is 0-tuply y-densely divisible.
2. n is i-tuply y-densely divisible if for all $j, k \geq 0$ with $j + k = i - 1$ and $1 \leq R \leq yn$ we can write $n = qr$ for some j-tuply y-densely divisible q and k-tuply y-densely divisible r with $\frac{1}{y}R \leq r \leq R$.

This can be viewed as a generalization of y-smoothness:

$$n \text{ is } y\text{-smooth} \iff n \text{ is } i\text{-tuply } y\text{-densely divisible for all } i.$$

But for any fixed i and y, the largest prime that may divide an i-tuply y-densely divisible integer n is unbounded.

Example (i-tuply 5-densely divisible but not 5-smooth $n \leq 100$)

i-tuply but not $(i+1)$-tuply 5-densely divisible non-5-smooth integers:

$i = 1$: $14, 21, 33, 35, 39, 44, 52, 55, 65, 66, 68, 76, 78, 85, 88, 95, 98.$
$i = 2$: $28, 42, 63, 70, 99.$
$i = 3$: $56, 84.$

A stronger form of MPZ$[\varpi, \delta]$

Let MPZ$^{(i)}[\varpi, \delta]$ denote MPZ$[\varpi, \delta]$ with the x^δ-smoothness constraint on the modulus q replaced by i-tuply x^δ-divisibility.

Then MPZ$^{(i)}[\varpi, \delta] \Rightarrow$ MPZ$[\varpi, \delta] \Rightarrow$ DHL$[k, 2]$ for each $i \geq 0$.

But this implication can be proved directly in a way that makes k essentially independent of δ; this lets us increase ϖ and decrease k.

Theorem (D.H.J. Polymath 2013)

(i) MPZ$^{(4)}[\varpi, \delta]$ *holds for all* $\varpi, \delta > 0$ *satisfying* $600\varpi + 180\delta < 7$.
(ii) MPZ$^{(2)}[\varpi, \delta]$ *holds for all* $\varpi, \delta > 0$ *satisfying* $168\varpi + 48\delta < 1$.
The proof of (ii) does not require any of Deligne's results.

The Maynard-Tao approach

Recall that in the GPY method we require weights $w_n \geq 0$ that satisfy

$$\sum_{x<n\leq 2x} w_n\big(\Theta(n+\mathcal{H}) - m\log(3x)\big) > 0$$

for sufficiently large x. GPY achieved this using weights of the form

$$w_n := \Bigg(\sum_{\substack{d\mid\prod_i(n+h_i)\\ d<x^{\theta/2}}} \lambda_d\Bigg)^2, \qquad \lambda_d := \mu(d)f(d).$$

Maynard (and independently, Tao) instead use weights of the form

$$w_n := \Bigg(\sum_{\substack{d_i\mid n+h_i\\ \prod d_i<x^{\theta/2}}} \lambda_{d_1,\ldots,d_k}\Bigg)^2, \qquad \lambda_{d_1,\ldots,d_k} :\approx \Big(\prod \mu(d_i)\Big)f(d_1,\ldots,d_k).$$

where f is defined in terms of a smooth F that we are free to choose.

A variational problem

Let $F\colon [0,1]^k \to \mathbb{R}$ denote a nonzero square-integrable function with support in the simplex $\mathcal{R}_k := \{(x_1,\dots,x_k) \in [0,1]^k : \sum_i x_i \le 1\}$.

$$I(F) := \int_0^1 \cdots \int_0^1 F(t_1,\dots,t_k)^2 dt_1 \dots dt_k,$$

$$J(F) := \sum_{i=1}^{k} \int_0^1 \cdots \int_0^1 \left(\int_0^1 F(t_1,\dots,t_k) dt_i \right)^2 dt_1 \dots dt_{i-1} dt_{i+1} \dots dt_k,$$

$$\rho(F) := \frac{J(F)}{I(F)}, \qquad M_k := \sup_F \rho(F)$$

Theorem (Maynard 2013)

For any $0 < \theta < 1$, if EH$[\theta]$ *and $M_k > \frac{2m}{\theta}$, then* DHL$[k, m+1]$.

Explicitly bounding M_k

To prove $M_k > \frac{2m}{\theta}$, it suffices to exhibit an F with $\rho(F) > \frac{2m}{\theta}$.

Maynard considers functions F defined by a polynomial of the form

$$P := \sum_{a+2b \leq d} c_{a,b}(1 - P_1)^a P_2^b,$$

where $P_1 := \sum_i t_i,\ P_2 := \sum_i t_i^2$, with support restricted to $\mathcal{R}_k$.

The function $\rho(F)$ is then a ratio of quadratic forms in the $c_{a,b}$ that are completely determined by our choice of k and d.

If $\mathbf{I}$ and $\mathbf{J}$ are the matrices of these forms (which are real, symmetric, and positive definite), we want to choose k and d so that $\mathbf{I}^{-1}\mathbf{J}$ has an eigenvalue $\lambda > 4m$.

The corresponding eigenvector then determines the coefficients $c_{a,b}$.

Example: With $k = 5$ and $d = 3$ we can explicitly compute

$$\mathbf{I} = \frac{1}{1995840}\begin{bmatrix} 16632 & 3960 & 2772 & 495 & 792 & 297 \\ 3960 & 1100 & 495 & 110 & 110 & 33 \\ 2772 & 495 & 792 & 110 & 297 & 132 \\ 495 & 110 & 110 & 20 & 33 & 12 \\ 792 & 110 & 297 & 33 & 132 & 66 \\ 297 & 33 & 132 & 12 & 66 & 36 \end{bmatrix},$$

$$\mathbf{J} = \frac{1}{11975040}\begin{bmatrix} 166320 & 35640 & 35640 & 5610 & 11880 & 4950 \\ 35640 & 8184 & 7260 & 1218 & 2376 & 990 \\ 35640 & 7260 & 8910 & 1287 & 3300 & 1485 \\ 5610 & 1218 & 1287 & 203 & 450 & 195 \\ 11880 & 2376 & 3300 & 450 & 1320 & 630 \\ 4950 & 990 & 1485 & 195 & 630 & 315 \end{bmatrix},$$

with row/column indexes ordered as $c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{3,0}$.
The largest eigenvalue of $\mathbf{I}^{-1}\mathbf{J}$ is ≈ 2.0027. An approximate eigenvector is

$$\mathbf{a} = [0, 5, -15, 70, 49, 0], \qquad \text{for which} \qquad \frac{\mathbf{a}^\intercal \mathbf{J}\mathbf{a}}{\mathbf{a}^\intercal \mathbf{I}\mathbf{a}} = \frac{1417255}{708216} > 2.$$

It follows that $\rho(F) = \frac{J(F)}{I(F)} = \frac{\mathbf{a}^\intercal \mathbf{J}\mathbf{a}}{\mathbf{a}^\intercal \mathbf{I}\mathbf{a}} = \frac{1417255}{708216} > 2$ for the function F defined by

$$P = 5P_2 - 15(1 - P_1) + 70(1 - P_1)P_2 + 49(1 - P_1)^2.$$

Thus $M_5 > 2$; under EH we get DHL$[5, 2]$ and $H_1 \leq 12$, using $\{0, 4, 6, 10, 12\}$.

Taking $k = 105$ and $d = 11$ gives $M_{105} > 4$, DHL$[105, 2]$, $H_1 \leq 600$ (unconditionally).

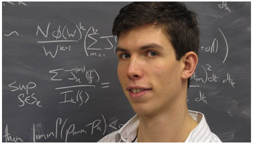
Maynard's results

Theorem (Maynard 2013)

*We have $M_5 > 2$, $M_{105} > 4$, and $M_k > \log k - 2\log\log k - 2$ for all sufficiently large k.** *These bounds imply*

1. $H_1 \leq 12$ *under* EH,
2. $H_1 \leq 600$,
3. $H_m \leq m^3 e^{4m+5}$ *for all* $m \geq 1$.

The bound $H_1 \leq 600$ relies only on Bombieri-Vinogradov.
In fact, for any $\theta > 0$ we have EH$[\theta] \Rightarrow H_1 < \infty$.



James Maynard

*$k \geq 200$ is sufficiently large (D.H.J. Polymath).

Dickson's conjecture

Conjecture (Dickson 1904)

Every admissible k-tuple has infinitely many translates composed entirely of primes (Dickson k-tuples).

The Maynard-Tao theorem implies that for each $k \geq 2$ a positive proportion of admissible k-tuples are Dickson k-tuples.

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The Maynard-Tao theorem implies that for each $k \geq 2$ a positive proportion of admissible k-tuples are Dickson k-tuples.

More precisely, there is a constant $c = c(k) > 0$ such that for all sufficiently large x the proportion of k-tuples in $[1, x]$ that are Dickson k-tuples is greater than c. Proof:

1. Let $m = k - 1$ and $K = m^3 e^{4m+5}$.
   Then every admissible K-tuple contains at least one Dickson k-tuple.
2. Let $S = \{n \in [1, x] : n \perp p \text{ for } p \leq K\}$. Every K-tuple in S is admissible.
3. There are at least $\binom{\#S}{K} / \binom{\#S-k}{K-k} = \binom{\#S}{k} / \binom{K}{k}$ Dickson k-tuples in S.
   The proportion of k-tuples in $[0, x]$ that lie in S is $\gg (\log K)^{-k}$.

The Polymath project (Polymath8b)

Goals:

1. Improve Maynard's bounds on H_1 and asymptotics for H_m.
2. Get explicit bounds on H_m for $m = 2, 3, 4, \ldots$

Natural sub-projects:

1. Constructing narrow admissible k-tuples for large k.
2. Explicit lower bounds on M_k.
3. Figuring out how to replace EH$[\frac{1}{2} + 2\varpi]$ with MPZ$[\varpi, \delta]$.

Key questions:

1. Maynard uses F of a specific form to bound M_k, would more complicated choices for F work better?
2. To what extent can Zhang's work and the Polymath8a results be combined with the Maynard-Tao approach?

Polymath8b results

Bounds that do not use Zhang or Polymath8a results:

k	M_k	m	H_m	
105	4	1	600	Maynard's paper
102	4	1	576	Optimized Maynard
54	4	1	270	More general F
50	4	1	246	ϵ-enlarged simplex
3	2	1	6	under GEH
51	4	2	252	under GEH
5511	6	3	52 116	under EH
41 588	8	4	474 266	under EH
309 661	10	5	4 137 854	under EH

We also prove

$$\frac{k}{k-1}\log k - c \ < \ M_k \ \leq \ \frac{k}{k-1}\log k,$$

for an effective constant $c \approx 3$.

Sieving an ϵ-enlarged simplex

Fix $\epsilon \in (0,1)$ and let $F\colon [0, 1+\epsilon)^k \to \mathbb{R}$ denote a square-integrable function with support in $(1+\epsilon)\mathcal{R}_k$. Define

$$J_{1-\epsilon}(F) := \sum_{i=1}^{k} \int_{(1-\epsilon)\mathcal{R}_{k-1}} \left(\int_0^{1+\epsilon} F(t_1, \dots, t_k) dt_i \right)^2 dt_1 \dots dt_{i-1} dt_{i+1} \dots dt_k,$$

$$M_{k,\epsilon} := \sup_F \frac{J_{1-\epsilon}(F)}{I(F)}.$$

Theorem (D.H.J. Polymath 2014)

Assume either EH$[\theta]$ *with* $1+\epsilon < \frac{1}{\theta}$ *or* GEH$[\theta]$ *with* $\epsilon < \frac{1}{k-1}$*.*
Then $M_{k,\epsilon} > \frac{2m}{\theta}$ *implies* DHL$[k, m+1]$*.*

$M_{50,1/25} > 4$ proves $H_1 \leq 246$ and $M_{4,0.168} > 2$ gives $H_1 \leq 8$ under GEH.
To get $H_1 \leq 6$ under GEH we prove a specific result for $k = 3$.

Polymath8b results

Bounds that incorporate Polymath8a results:

ϖ, δ	k	m	H_m
$600\varpi + 180\delta < 7$	35 410	2	398 130
$600\varpi + 180\delta < 7$	1 649 821	3	24 797 814
$600\varpi + 180\delta < 7$	75 845 707	4	1 431 556 072
$600\varpi + 180\delta < 7$	3 473 995 908	5	80 550 202 480

For comparison, if we only use Bombieri-Vinogradov we get $H_2 < 474\,266$ rather than $H_2 < 398\,130$ (but H_1 is not improved).

We also prove $H_m \ll m e^{(4-\frac{28}{157})m}$.

*Sharper bounds on H_m are listed on the Polymath8 web page, that use $1080\varpi + 330\delta < 13$, but this constraint has not been rigorously verified.

Sieving a truncated simplex

Fix positive $\varpi, \delta < 1/4$ such that MPZ$[\varpi, \delta]$ holds.
For $\alpha > 0$, define $M_k^{[\alpha]} := \sup_F \rho(F)$, with the supremum over nonzero square-integrable functions with support in $[0, \alpha]^k \cap \mathcal{R}_k$.

Theorem (D.H.J. Polymath 2014)

Assume MPZ$[\varpi, \delta]$. *Then* DHL$[k, m+1]$ *holds whenever*

$$M_k^{[\frac{\delta}{1/4+\varpi}]} > \frac{m}{1/4 + \varpi}.$$

Example

$H_2 \leq 398\,130$ is proved using an admissible 35410-tuple and showing

$$M_{35410}^{[\frac{\delta}{1/4+\varpi}]} > \frac{2}{1/4 + \varpi},$$

for some $\delta, \varpi > 0$ with $600\varpi + 180\delta < 7$ (which implies MPZ$[\varpi, \delta]$).

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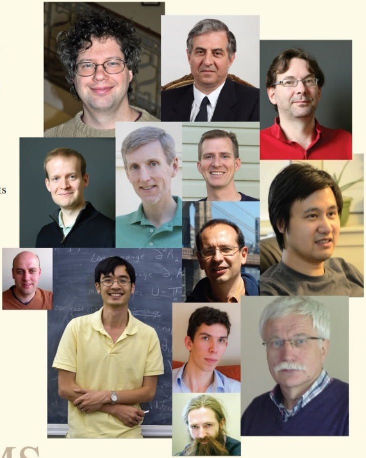

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