August 30, 2022

Dear Mike and Peter,

It was great to see you both at ANTS XV. I apologize for the delay in getting this letter to you. I started writing it the weekend after ANTS but then realized I wanted to include computations that could not be completed until I returned to Cambridge, which happened just last week.

As I mentioned when we spoke, I'm writing in regard to a phenomenon that was recently observed in the paper *Murmurations of elliptic curves*, by Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver, and Alexey Pozdnyakov. Their paper discusses a number of topics, but the observation I am most interested in is the remarkable oscillation in average Frobenius traces $\bar{a}_p$ of elliptic curves over $\mathbb{Q}$ of fixed rank with conductor in a given interval. You can clearly see this oscillation in the figures on the following page, which plots blue/red points $(p, \bar{a}_p)$ for elliptic curves with a fixed root number and conductor $N \in [2^n, 2^{n+1})$.

In each plot the x-axis ranges up to 2^n (the bottom of the conductor interval) and with this scaling the crossover points in the oscillations occur at fixed fractions of the conductor as the conductor range varies. This scale invariance is not particular to dyadic intervals, as can be seen in the plots on pages 9–10.

I should note three differences in the plots I am presenting here relative to those in the murmurations paper. First, I am plotting points at coordinates $(p, \bar{a}_p)$ rather than $(\pi(p), \bar{a}_p)$, which I think makes more sense; the log factor is a small difference, but it is enough to shift the crossover points. Second, I am using geometric conductor intervals of the form $[\alpha, c\alpha]$ for some fixed $c > 0$; the arithmetic intervals of the form $(\alpha, \alpha + c)$ used in the murmurations paper are too narrow (for large α they may not contain any conductors). Third, I am only fixing the parity of the rank (root number); this should make no difference asymptotically.

The main observations I would like to make in this letter are the following:

- This phenomenon is robust and easy to reproduce (even with small conductor bounds). The plots on the next page all look similar, but each involves disjoint sets of elliptic curves. The first plot involves about 15,000 elliptic curves, while the last involves more than a million, but they all have the same shape. I am quite surprised that no one seems to have noticed this behavior before. But perhaps they have and I am just unaware of it – I trust you will enlighten me if that is the case.

- Ordering by conductor is crucial. If you order by naïve height, Faltings height, j-invariant, or pretty much anything that is not compatible with the conductor ordering, you will not see the oscillation (or at least not see it as clearly). There is a series of plots computed using the naive height ordering on page 3 and the oscillations are nowhere to be seen.

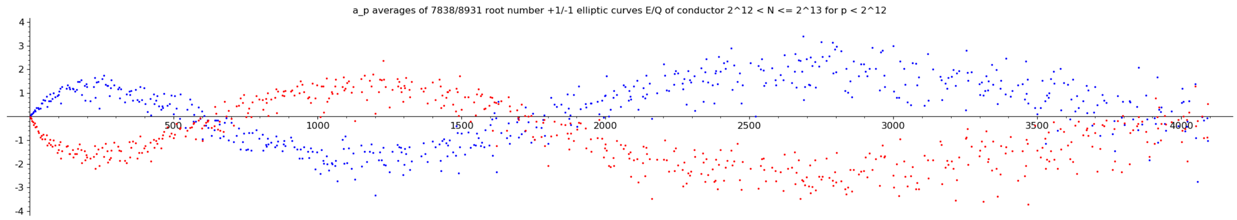
- This phenomenon is not specific to elliptic curves, I believe it applies to any family of primitive arithmetic L-functions ordered by conductor, so long as you normalize the a_p appropriately. You can see examples for modular forms and weights $w = 2, 4, 6$ on pages 4–6 (in each case the averages $\bar{a}_p$ are computed over $a_p/p^{(w-2)/2}$), and you can see examples for genus 2 L-functions on page 7. The shapes of the plots vary, but oscillations that scale with the conductor are clear in every case.

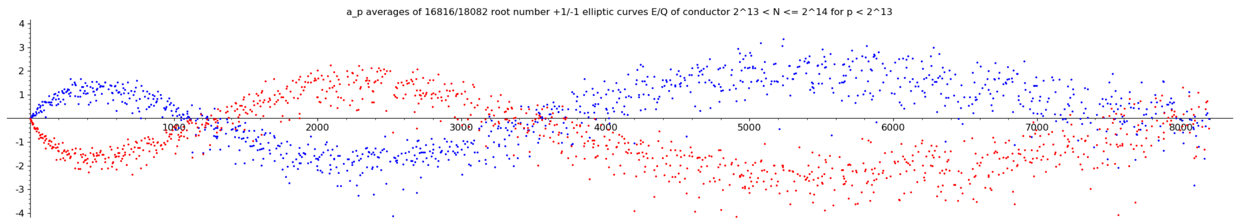
- Primitivity matters. On page 8 you can see plots for two special types of genus 2 L-functions. The plots on the top use L-functions of genus 2 curves with Sato-Tate group $\mathrm{SU}(2) \times \mathrm{SU}(2)$, almost all of which are products of elliptic curve L-functions. While there is a clear separation of even and odd rank for small p, no oscillations are visible. The plots on the bottom are for L-functions of genus 2 curves with Sato-Tate group $N(\mathrm{SU}(2) \times \mathrm{SU}(2))$, which all correspond to (primitive) L-functions of Hilbert or Bianchi modular forms, and here one sees the same oscillations as for the generic Sato-Tate group $\mathrm{USp}(4)$ (even though the Sato-Tate distributions are quite different).

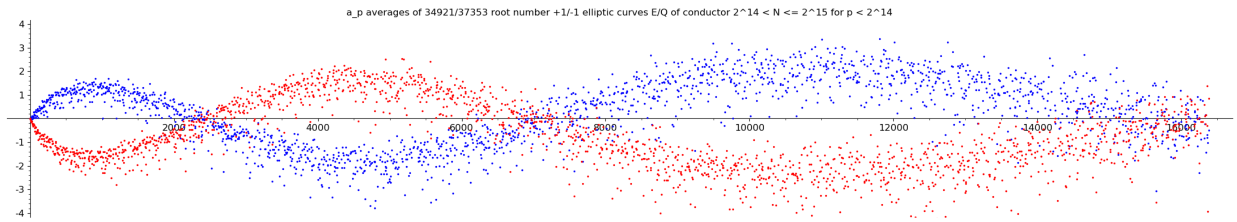
It was the last point that actually made me think of you. I believe you made a similar observation regarding the distinction between primitive and imprimitive L-functions when discussing generalizations to Artin L-functions in Section 5 of your *Chebyshev's Bias* paper. In any case, I'd love to hear what you think about this phenomenon. If there is a known explanation for it I expect you will know it, and if there is not, I expect you are more likely than anyone to be able to figure it out, although I should mention that I have also discussed this question with Andy Booker and Andrew Granville, and they may have some ideas.

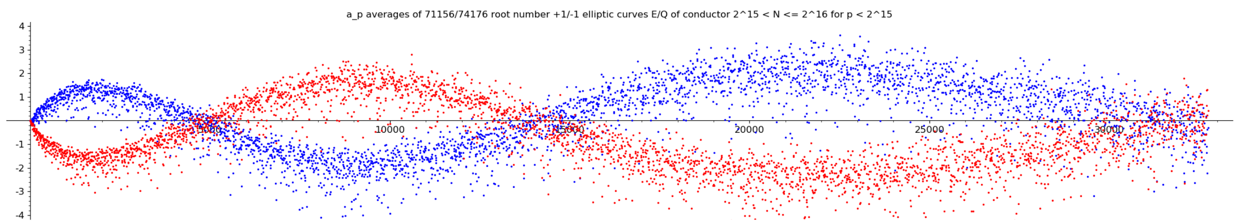
I wish you all the best – Drew

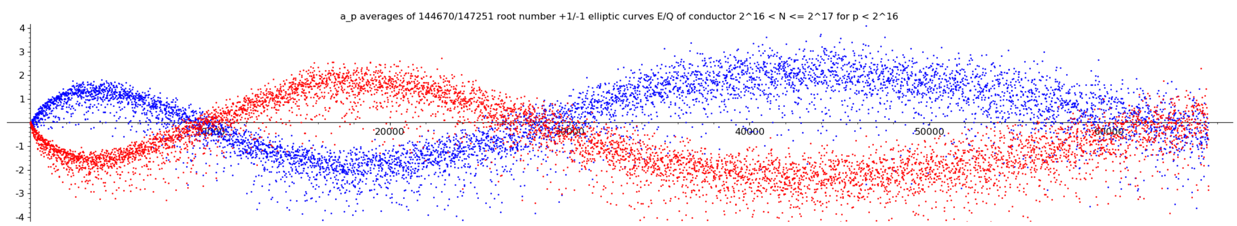
Plots for isogeny classes of non-CM elliptic curves of conductor $N \in [2^n, 2^{n+1})$ for $12 \leq n \leq 17$ and $N \in [250000, 500000)$. The x-axis ranges up to the bottom of the conductor interval in each case. A blue (red) dot at $(p, \bar{a}_p)$ means that $\bar{a}_p$ is the average of a_p over even (odd) rank isogeny classes.

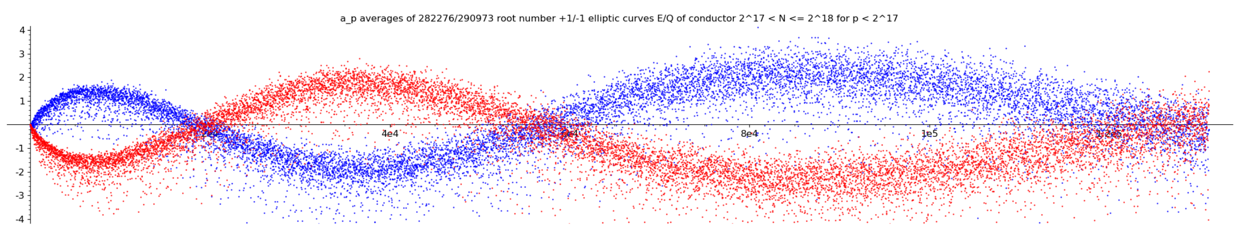


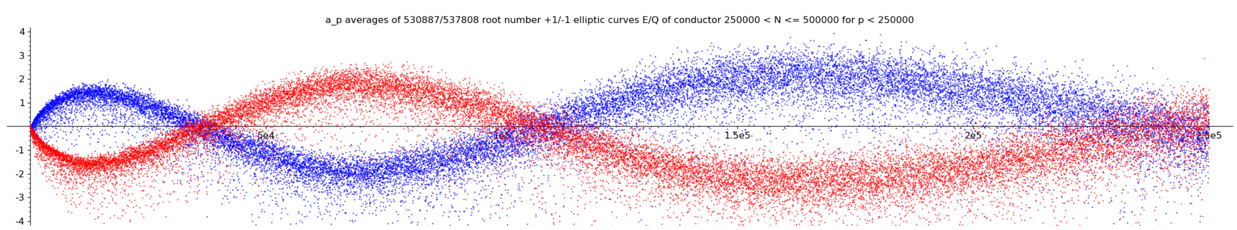




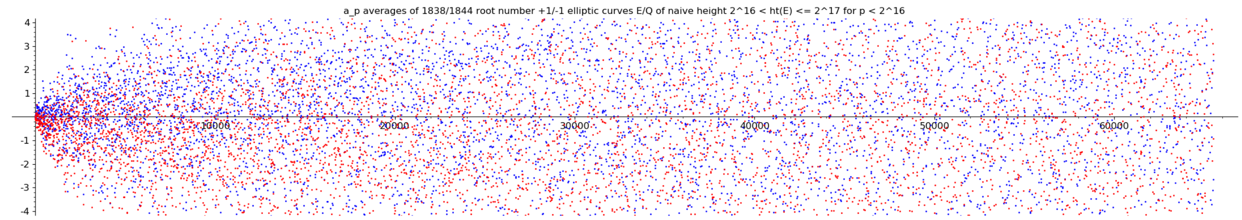


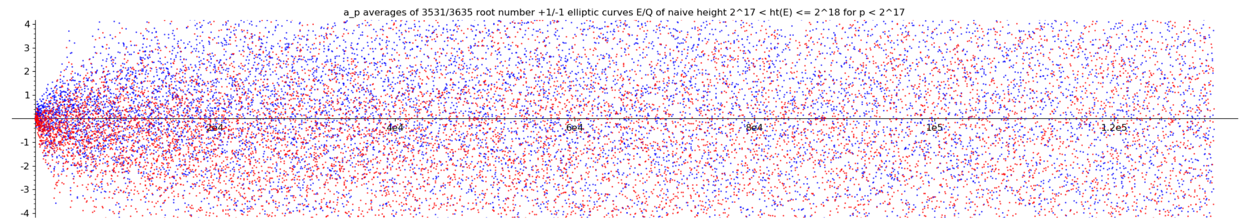


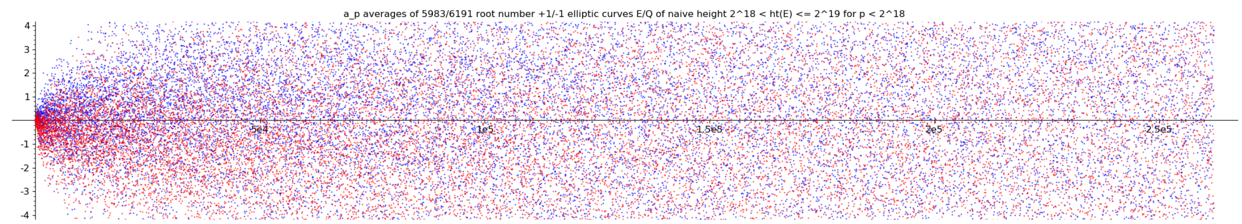


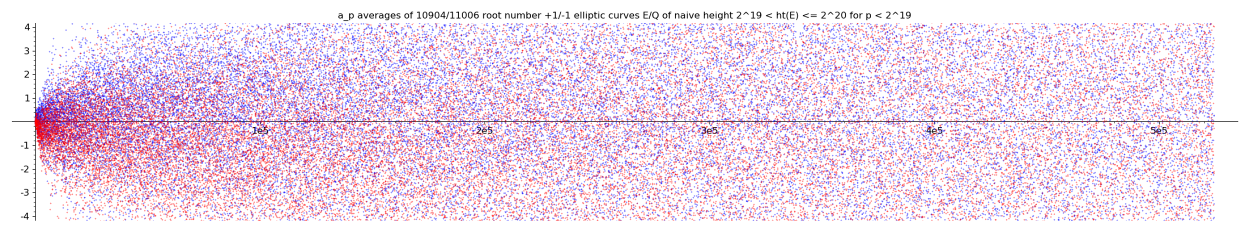


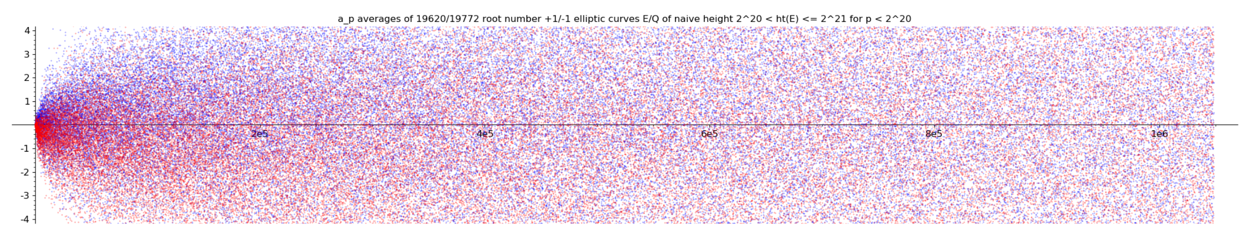
Plots for $E : y^2 = x^3 + Ax + B$ with naïve height $\mathrm{ht}(E) := \max(4|A|^3, 27B^2)$ in $[2^n, 2^{n+1})$ for $16 \leq n \leq 22$. The x-axis ranges up to 2^n in each case. A blue (red) dot at $(p, \bar{a}_p)$ means that $\bar{a}_p$ is the average of a_p over even (odd) rank elliptic curves E.

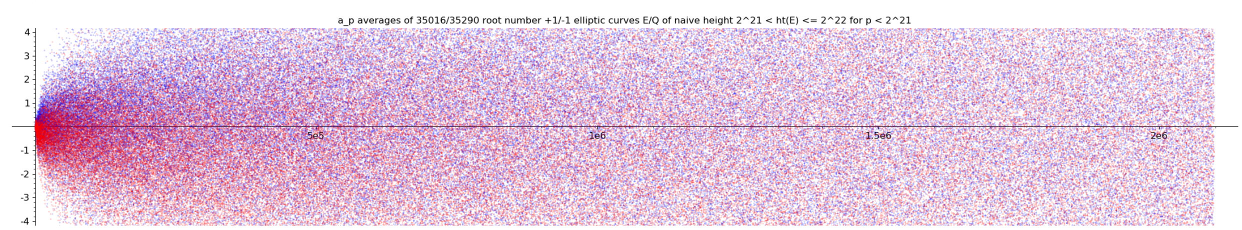


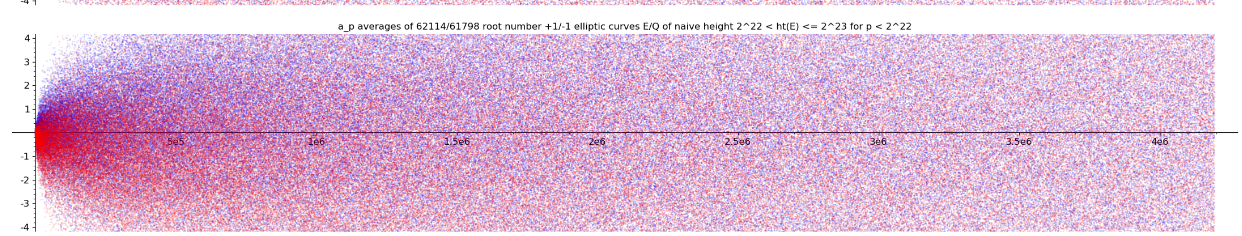




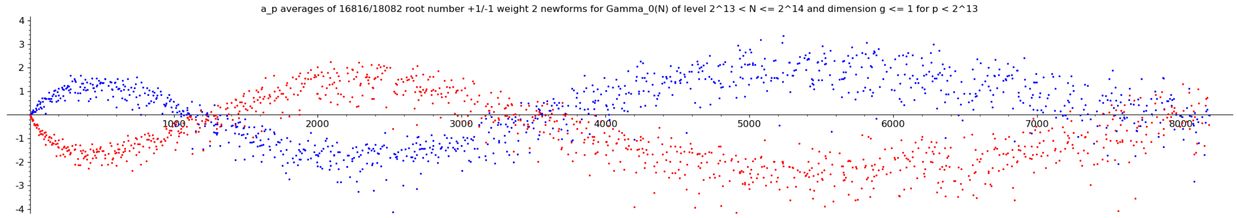


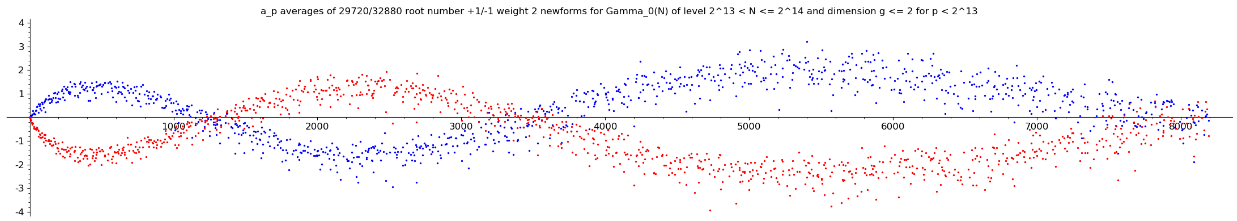


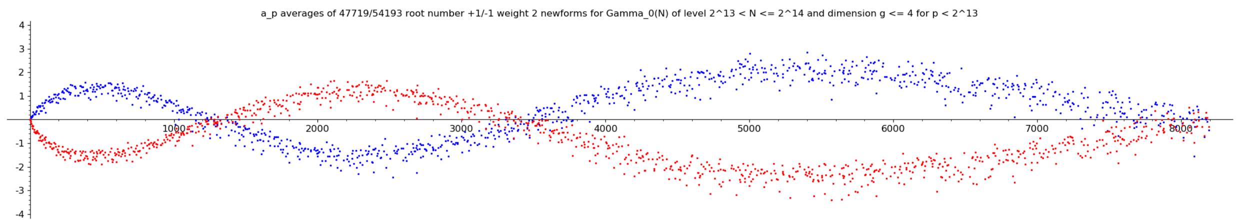


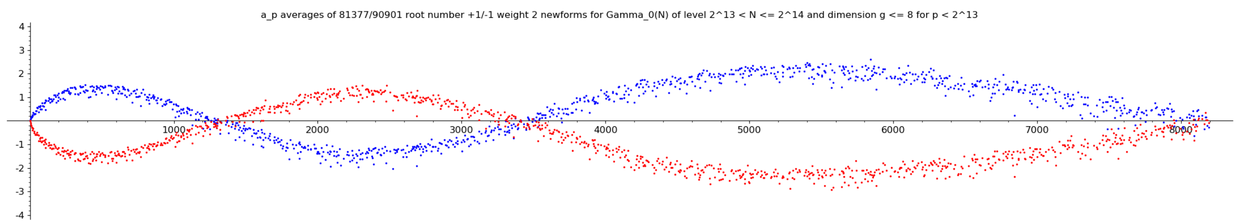


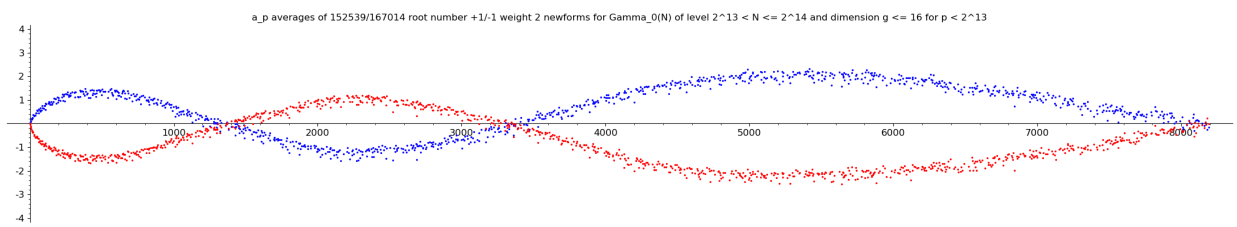
Plots for weight 2 newforms on $\Gamma_0(N)$ for $N \in [2^{13}, 2^{14})$ in Galois orbits of size at most 2^d for $d \in [0, 5]$ followed by a plot for all weight 2 newforms. The x-axis ranges up to 2^{13} in each plot. A blue (red) dot at $(p, \bar{a}_p)$ means that $\bar{a}_p$ is the average of a_p over even (odd) analytic rank newforms.

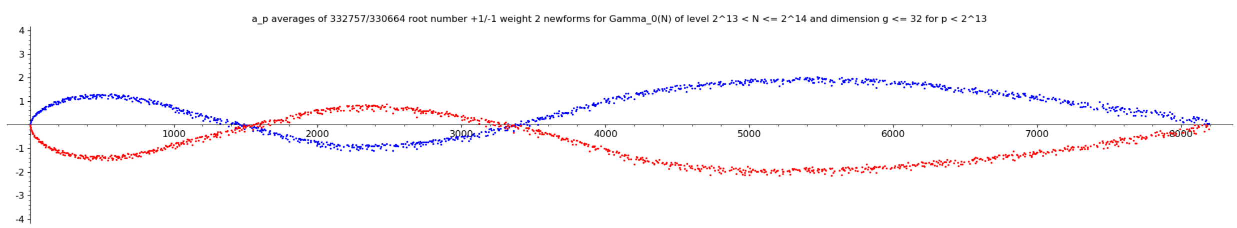


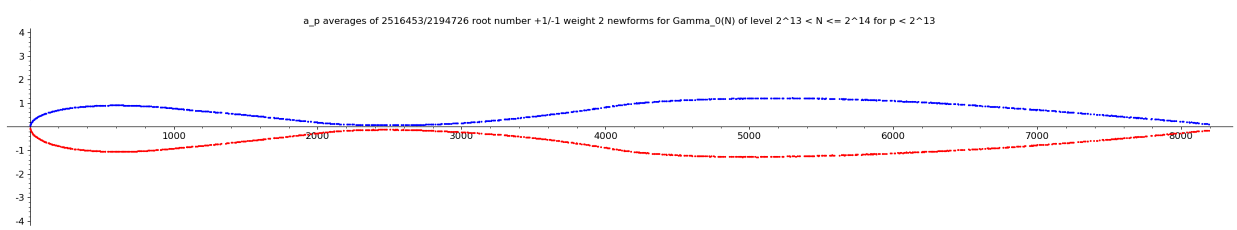




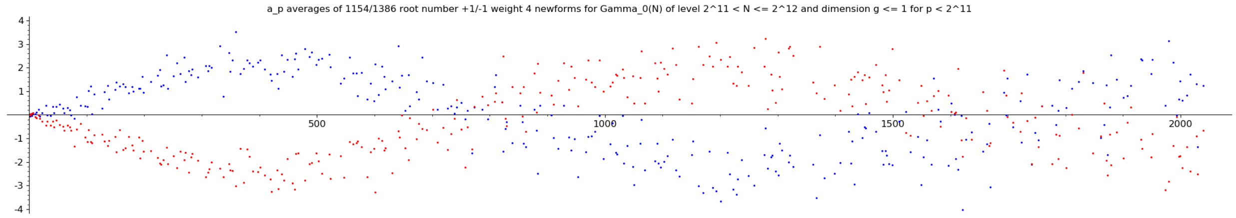


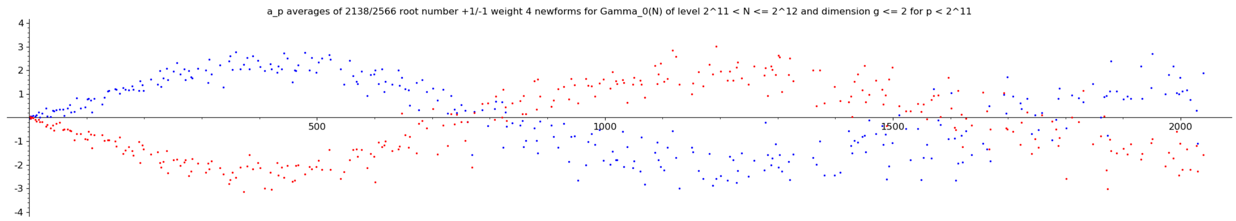


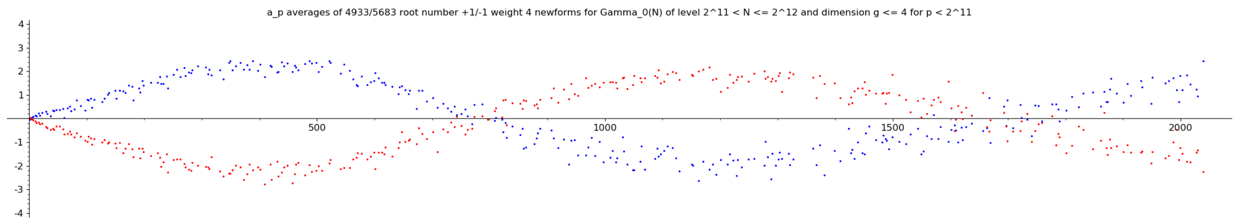


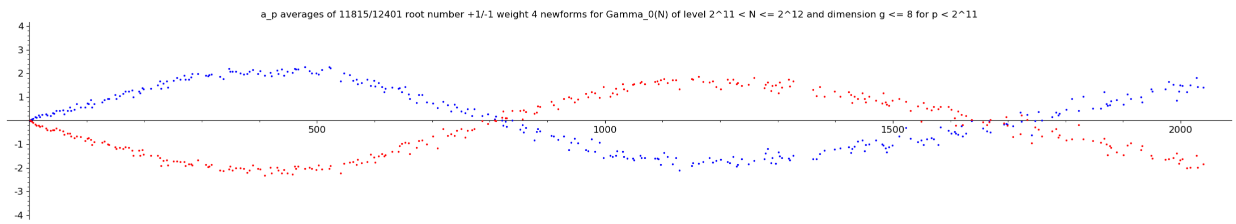


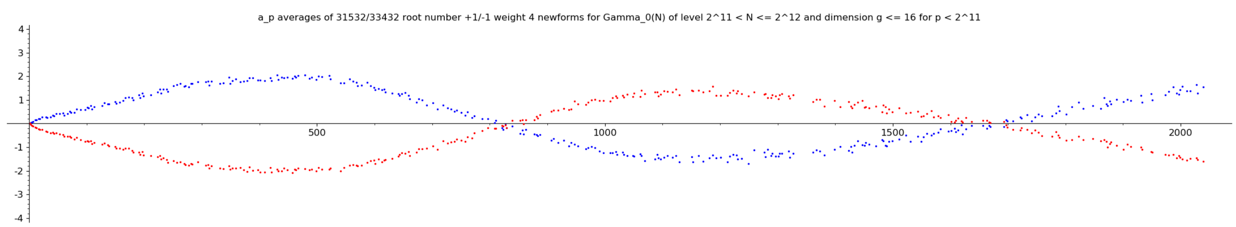
Plots for weight 4 newforms on $\Gamma_0(N)$ for $N \in [2^{10}, 2^{11})$ in Galois orbits of size at most 2^d for $d \in [0, 5]$ followed by a plot for all weight 4 newforms. The x-axis ranges up to 2^{11} in each plot. A blue (red) dot at $(p, \bar{a}_p)$ means that $\bar{a}_p$ is the average of a_p/p over even (odd) analytic rank newforms.

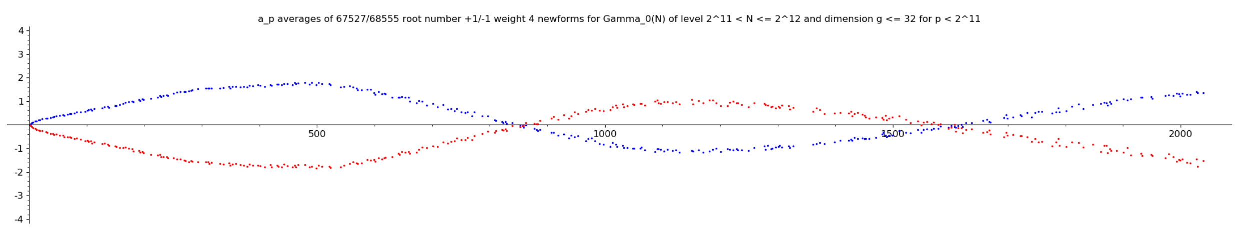


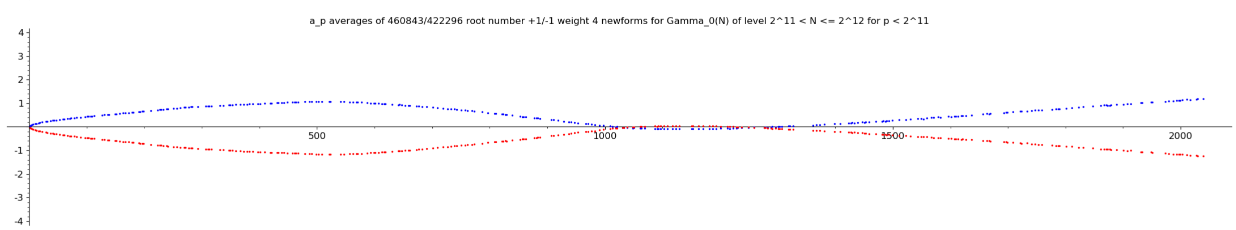




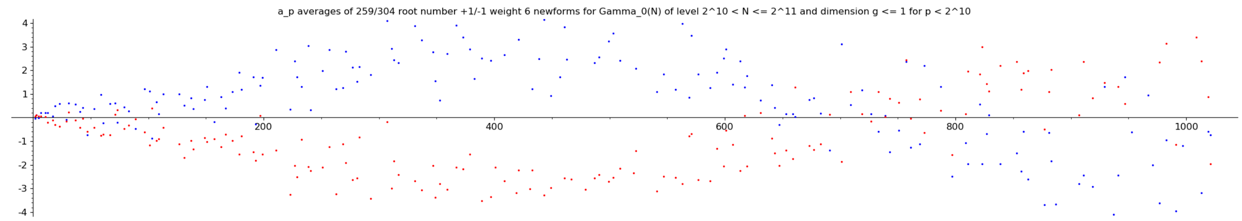


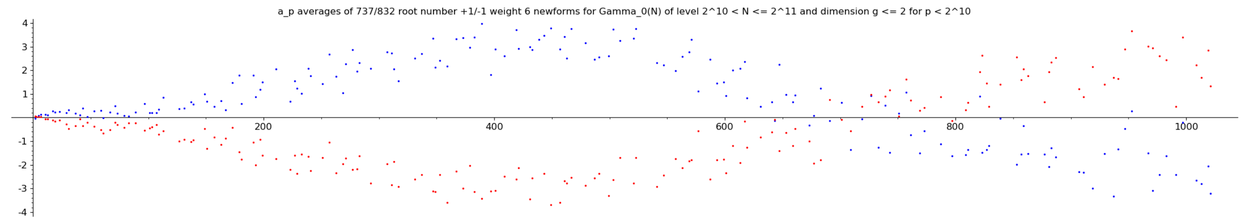


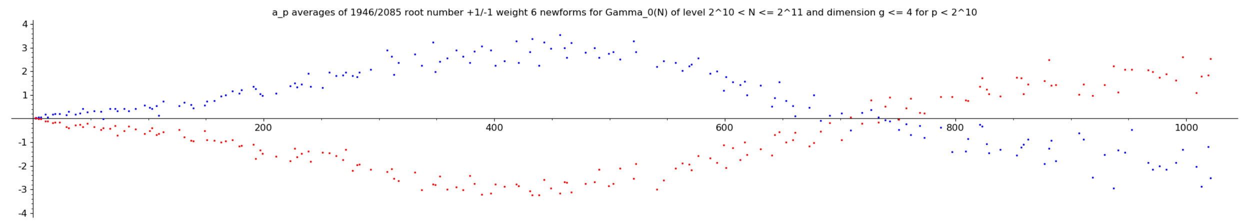


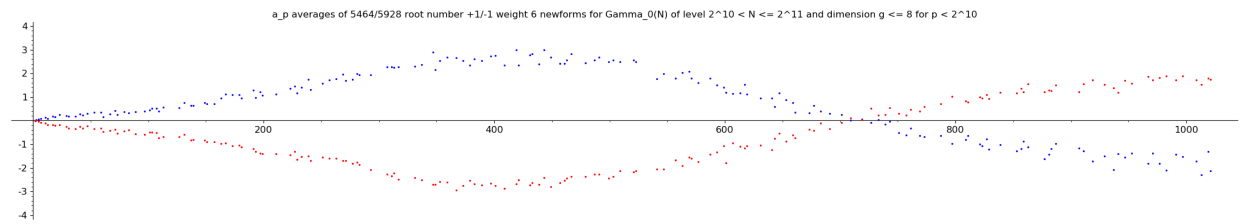


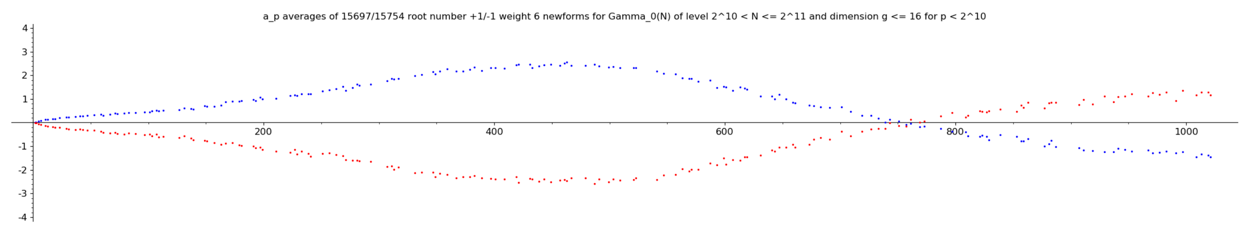
Plots for weight 6 newforms on $\Gamma_0(N)$ for $N \in [2^8, 2^9)$ in Galois orbits of size at most 2^d for $d \in [0, 5]$ followed by a plot for all weight 6 newforms. The x-axis ranges up to 2^{10} in each plot. A blue (red) dot dot at $(p, \bar{a}_p)$ means that $\bar{a}_p$ is the average of a_p/p^2 over even (odd) analytic rank newforms.

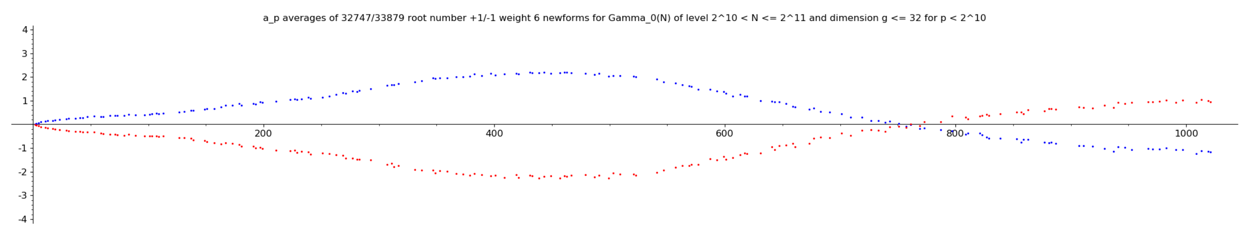


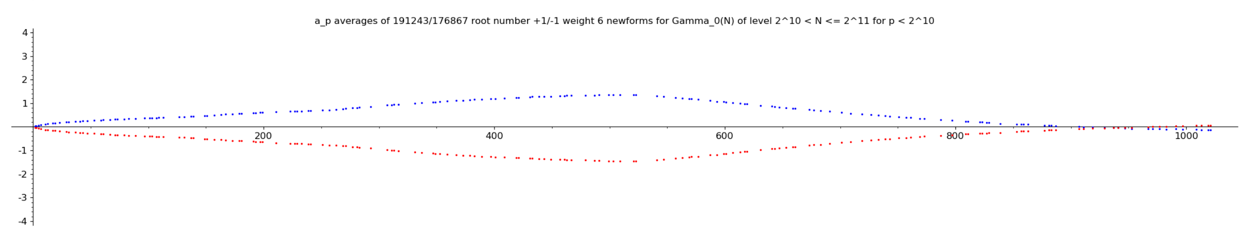






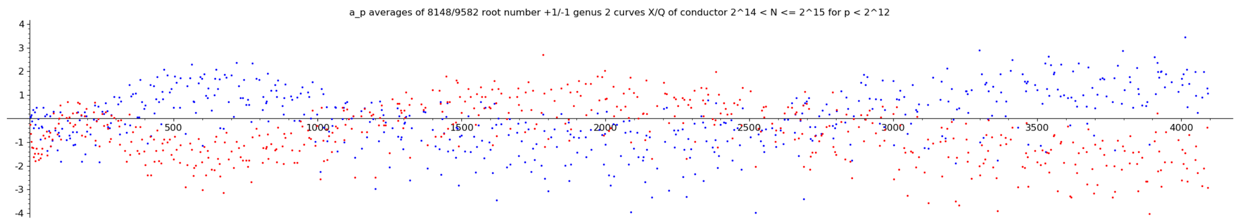


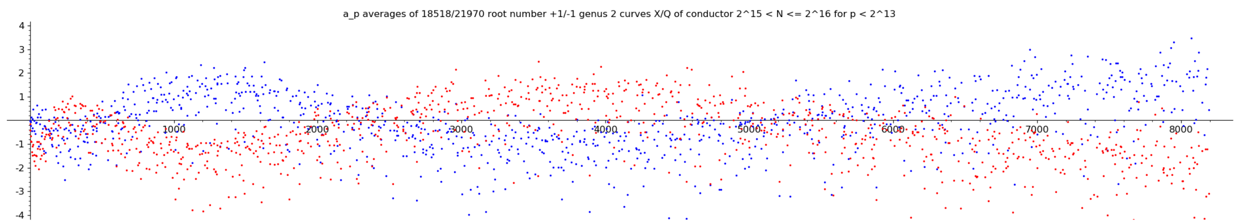


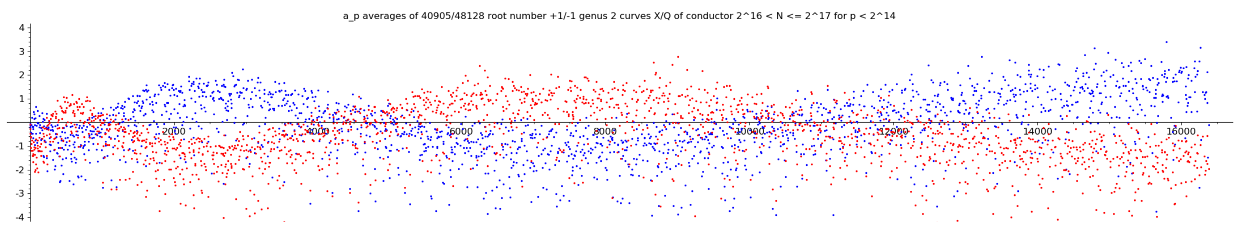


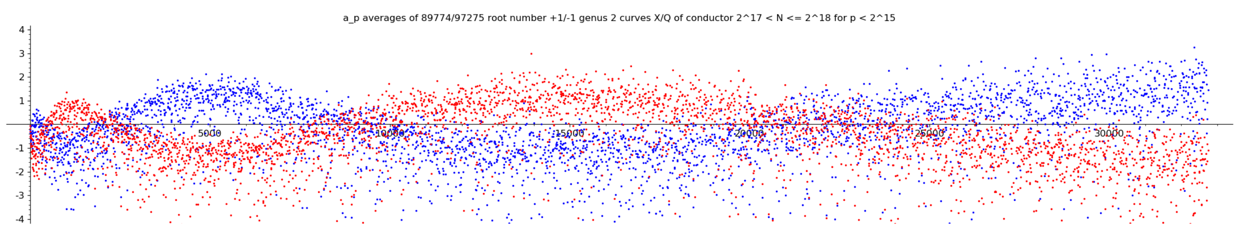
Plots for genus 2 L-functions with Sato-Tate group USp(4) of conductor $N \in [2^n, 2^{n+1})$ for $14 \le n \le 19$; here the x-axis ranges up to 2^{n-2} in each plot, 1/4 the scale of the elliptic curve and modular form plots (so the period of the oscillations is much shorter in genus 2 than genus 1, relative to the conductor). A blue (red) dot at $(p, \bar{a}_p)$ means that $\bar{a}_p$ is the average of a_p over even (odd) analytic rank L-functions.

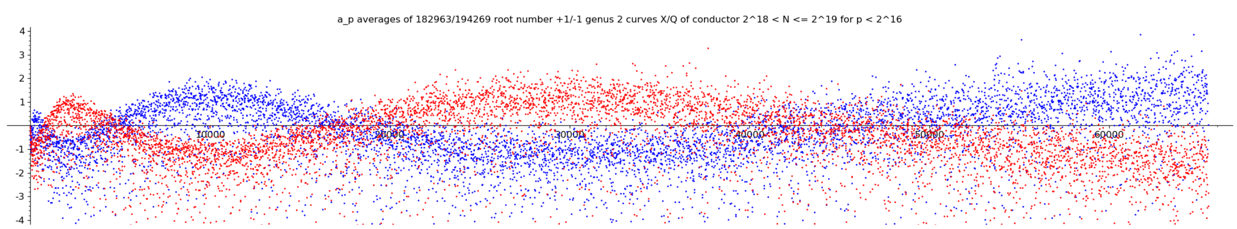
These plots were generated using a recently constructed dataset that includes more than five million genus 2 curves of conductor up to 2^{20}. This vastly exceeds the scope of the genus 2 database in the LMFDB (which imposes constraints on the discriminant as well as the conductor) but is surely still incomplete. However, it seems unlikely that the shape of the graphs below would change even if more data was available.

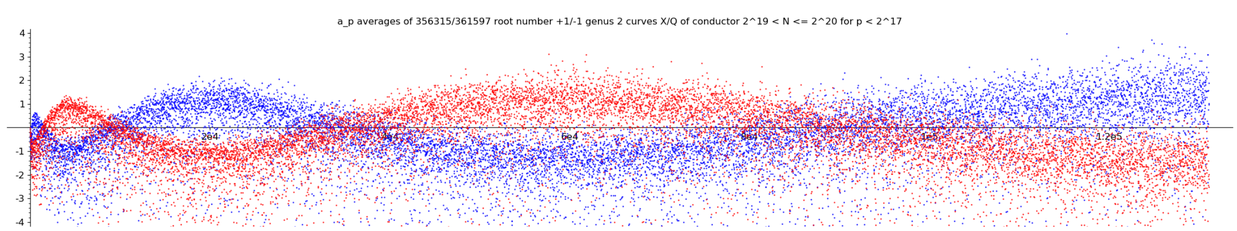




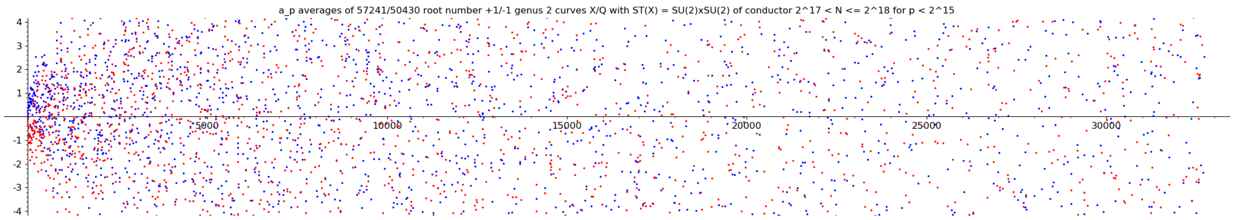


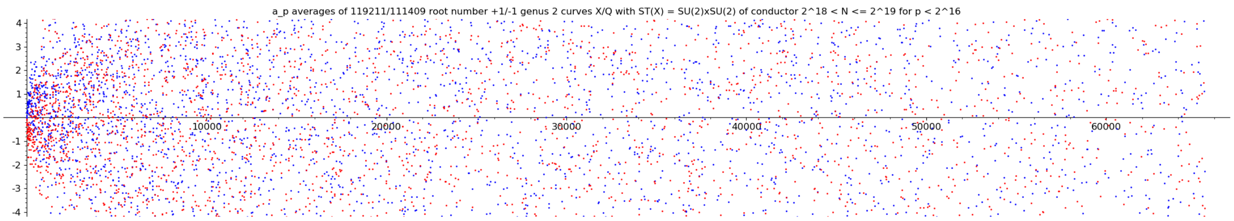


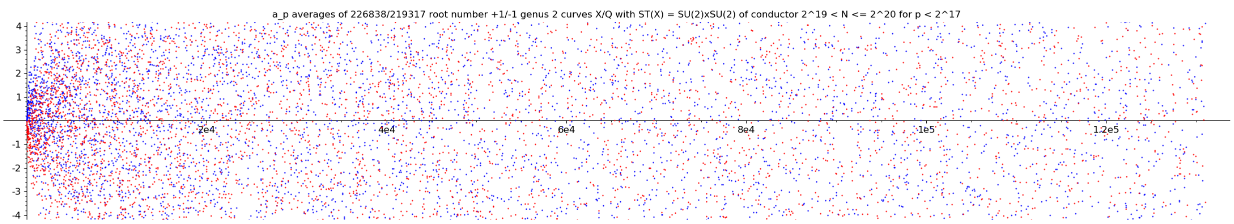




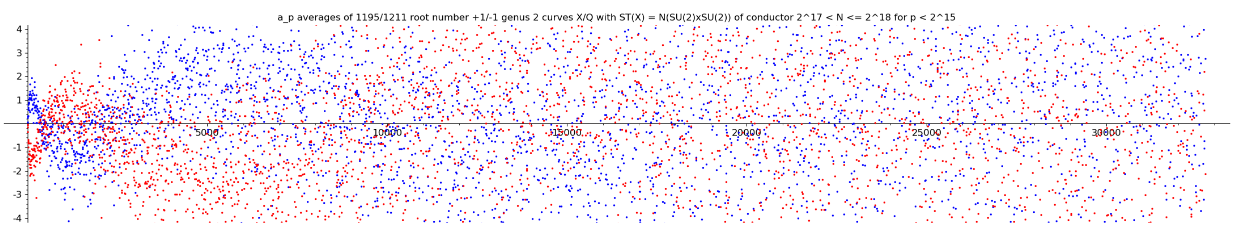
Plots for genus 2 L-functions with Sato-Tate group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of conductor $N \in [2^n, 2^{n+1})$ for $17 \leq n \leq 19$; the x-axis ranges up to 2^{n-2}. All of these L-functions are imprimitive: most are products of elliptic curve L-functions but a few are RM abelian surfaces (with the RM defined over $\mathbb{Q}$) that correspond to weight-2 modular forms with quadratic Hecke field. The murmuration patter present in the generic case is not visible here (there is some separation, but much less than in the generic case and the shape is different).

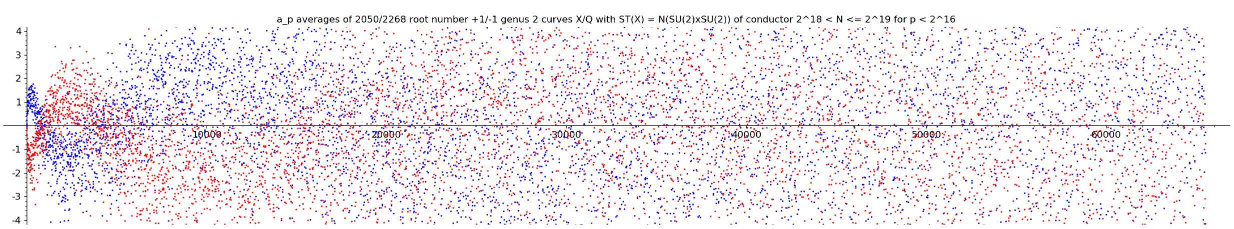


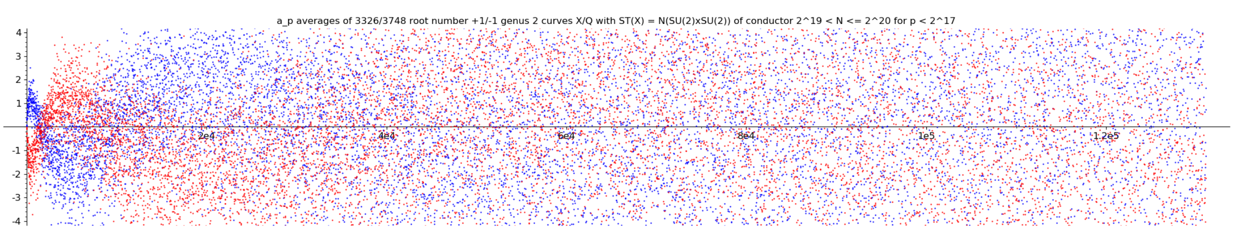




Plots for genus 2 L-functions with Sato-Tate group $N(\mathrm{SU}(2) \times \mathrm{SU}(2))$ of conductor $N \in [2^n, 2^{n+1})$ for $17 \leq n \leq 19$; the x-axis ranges up to 2^{n-2}. The L-functions of these curves are primitive L-functions arising from Hilbert or Bianchi modular forms. While the murmuration pattern is not as clear as it is for generic genus 2 L-functions (but it is a much smaller dataset), it is still visible and appears to have the same shape.







Plots for isogeny classes of elliptic curves of conductor $N \in [(3/2)^n, (3/2)^{n+1})$ for $25 \leq n \leq 31$. The x-axis ranges up to $(3/2)^n$. A blue red (dot) at $(p, \bar{a}_p)$ means that $\bar{a}_p$ is the average of a_p over even (odd) rank isogeny classes.