Abelian surfaces and their L-functions

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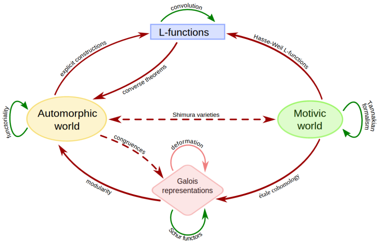
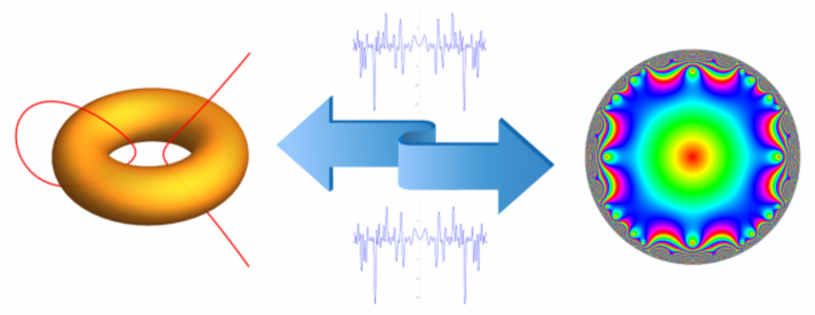


Simons Collaboration in Arithmetic Geometry, Number Theory, and Computation

Elliptic curves and their L-functions



Theorem (Eichler-Shimura, Langlands-Tunnel, Serre, Ribet, Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor)

For each positive integer N, the set of L-functions $L(E,s)$ of elliptic curves $E/\mathbb{Q}$ of conductor N is equal to the set of L-functions $L(f,s)$ of newforms $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ of weight 2 and level N with rational q-expansions.

Enumerating elliptic curves by conductor

To enumerate abelian varieties of dimension $g = 1$ over $\mathbb{Q}$ we may proceed as follows:

1. Prove the modularity theorem.
2. Enumerate rational modular forms $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ for $N = 1, 2, 3, \ldots$
3. Use Eichler-Shimura to get an isogeny class representative E_f for each f.
4. Fill out isogeny classes by finding all the elliptic curves $E/\mathbb{Q}$ isogenous to E_f.

For $N \leq 500000$ this yields 3064705 elliptic curves with 2164260 distinct L-functions that have been computed to high precision.

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Each one of these steps is substantially more difficult for $g > 1$, even for $g = 2$.

In particular, there is no obvious analog to step 3 (not even in principle).

Progress for $g = 2$

- Potential automorphy: abelian surfaces over totally real fields satisfy the Hasse-Weil conjecture [Boxer-Calegari-Gee-Piloni 2018].
- Paramodular conjecture: the set of L-functions of abelian surfaces $A/\mathbb{Q}$ with $\mathrm{End}(A) = \mathbb{Z}$ of conductor N coincides with the set of L-functions of suitable weight-2 paramodular forms of weight 2 and level N [Brumer-Kramer 2010, 2018].
- Examples of paramodularity: the isogeny classes `277.a`, `353.a`, `587.a` are paramodular [Brumer-Pacetti-Poor, Tornaría-Voight-Yuen 2018].
- Computing paramodular forms: all paramodular forms of level $N < 388$ are known (one each of levels $N = 249, 277, 295, 349, 353$), and we have heuristic tables of paramodular forms for levels $N \leq 1000$ [Poor-Yuen 2016, 2018, 2020, ...].
- L-functions from nothing: the possible conductors $N \leq 500$ of modular abelian surfaces over $\mathbb{Q}$ have been determined [Farmer-Koutsoliotas-Lemurell 2015]

Automorphic forms associated to abelian surfaces

Type	Conductor	Curve Equation	Motive	Modular form
$\mathbf{A}[C_1]_{(s)}$	$277 = 277^1$	$y^2+(x^3+x^2+x+1)y = -x^2-x$	typical surface	paramodular form
$\mathbf{B}[C_1]_s$	$529 = 23^2$	$y^2+(x^3+x+1)y = -x^5$	surface with RM by $\mathbb{Q}(\sqrt{5})$ over $\mathbb{Q}$	CMF 23.2.1.a
$\mathbf{B}[C_1]_{ns}$	$294 = 2^1 3^1 7^2$	$y^2+(x^3+1) = x^4+x^2$	product of ECs 14a4 and 21a4 over $\mathbb{Q}$	CMFs 14.2.1.a and 21.2.1.a
$\mathbf{B}[C_2]_s$	$10368 = 2^7 3^4$	$y^2+x^2y = 3x^5-4x^4+6x^3-3x^2+1$	surface with RM by $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}(\sqrt{2})$	HMF 162.1-a over $\mathbb{Q}(\sqrt{2})$
$\mathbf{B}[C_2]_{ngs}$	$1088 = 2^6 17^1$	$y^2+(x^3+x^2+x+1)y = x^4+x^3+2x^2+x+1$	Weil restriction of 17.1-a1 over $\mathbb{Q}(\sqrt{2})$	HMF 17.1-a over $\mathbb{Q}(\sqrt{2})$
$\mathbf{C}[C_2]_{(ns)}$	$448 = 2^6 7^1$	$y^2+(x^3+x)y = x^4-7$	product of PCM EC 32a3 and EC 14a6 over $\mathbb{Q}$	CMFs 32.2.1.a and 14.2.1.a
$\mathbf{D}[C_4]_{(s)}$	$3125 = 5^5$	$y^2+y = x^5$	surface with CM by $\mathbb{Q}(\zeta_5)$ over $\mathbb{Q}(\zeta_5)$	CM HMF 125.1-a over $\mathbb{Q}(\sqrt{5})$
$\mathbf{D}[D_2]_{(ns)}$	$8192 = 2^{13}$	$y^2 = x^6-9x^4+16x^2-8$	product of PCM ECs 32a3 and 256d1 over $\mathbb{Q}$	CMFs 32.2.1.a and 256.2.1.d
$\mathbf{E}[C_1]_{(ns)}$	$196 = 2^2 7^2$	$y^2+(x^2+x)y = x^6+3x^5+6x^4+7x^3+6x^2+3x+1$	square of EC 14a1 over $\mathbb{Q}$	CMF 14.2.1.a
$\mathbf{E}[C_2, \mathbb{C}]_{(ngs)}$	$576 = 2^6 3^2$	$y^2+(x^3+x^2+x+1)y = -x^3-x$	square of EC 9.1-a3 over $\mathbb{Q}(\sqrt{2})$	CMF 24.2.13.a
$\mathbf{E}[C_3]_{(ngs)}$	$324 = 2^2 3^4$	$y^2+(x^3+x+1)y = x^5+2x^4+2x^3+x^2$	square of EC 8.1-a1 over 3.3.81.1	CMF 18.2.13.a
$\mathbf{E}[C_4]_{(ngs)}$	$256 = 2^8$	$y^2+y = 2x^5-3x^4+x^3+x^2-x$	square of EC 1.1-a5 over 4.4.2048.1	CMF 16.2.5.a
$\mathbf{E}[C_6]_{(ngs)}$	$169 = 13^2$	$y^2+(x^3+x+1)y = x^5+x^4$	square of EC 1.1-a3 over 6.6.371293.1	CMF 13.2.4.a
$\mathbf{E}[C_2, \mathbb{R} \times \mathbb{R}]_s$	$455625 = 3^6 5^4$	$y^2+(x^3+x^2+x+1)y = x^5 - 3x^4-2x-1$	surface with QM $(D=6)$ over 2.0.3.1	BMF over 2.0.3.1 of level 50625
$\mathbf{E}[C_2, \mathbb{R} \times \mathbb{R}]_{ngs}$	$3969 = 3^4 7^2$	$y^2+(x^2+x+1)y = -3x^5+5x^4-4x^3+x$	Weil restriction of 441.2-a over 2.0.3.1	BMF 2.0.3.1-441.2-a
$\mathbf{E}[C_2, \mathbb{R} \times \mathbb{R}]_{ns}$	$675 = 3^3 5^2$	$y^2 = -x^6-14x^5-44x^4+28x^3-44x^2-14x-1$	product of ECs 15a2 and 45a2 over $\mathbb{Q}$	CMFs 15.2.1.a and 45.2.1.a
$\mathbf{E}[D_2]_s$	$20736 = 2^8 3^4$	$y^2 = -27x^6-54x^5-27x^4+18x^3+18x^2-2$	surface with QM $(D=6)$ over 4.0.576.2	HMF 324.1-b over $\mathbb{Q}(\sqrt{2})$
$\mathbf{E}[D_3]_s$	$34992 = 2^4 3^7$	$y^2 = -2x^6-6x^5+10x^3+9x^2-18x+6$	surface with QM $(D=6)$ over 6.0.2834352.2	BMF over 2.0.3.1 of level 3888
$\mathbf{E}[D_4]_s$	$20736 = 2^8 3^4$	$y^2+y = 6x^5+9x^4-x^3-3x^2$	surface with QM $(D=6)$ over 8.0.339738624.10	BMF over 2.0.3.1 of level 2304
$\mathbf{E}[D_6]_s$	$8100 = 2^2 3^4 5^2$	$y^2+x^3y = x^6+3x^5-42x^4+43x^3+21x^2-60x-28$	surface with QM $(D=6)$ over degree 12 field	BMF over 2.0.3.1 of level 900
$\mathbf{E}[D_2]_{ngs}$	$6400 = 2^8 5^2$	$y^2 = 2x^5+5x^4+8x^3+7x^2+6x+2$	square of EC 256.1-a1 over $\mathbb{Q}(\sqrt{5})$	HMF 2.2.5.1-256.1-a
$\mathbf{E}[D_3]_{ngs}$	$2187 = 3^7$	$y^2+(x^3+1)y = -1$	square of EC over 6.0.177147.2	BMF over 2..0.3.1 of level 243
$\mathbf{E}[D_4]_{ngs}$	$3600 = 2^4 3^2 5^2$	$y^2+x^2y = x^5-3x^4+11x^2-16x$	square of EC over 4.0.13500.2	BMF over $\mathbb{Q}(i)$ of level 225
$\mathbf{E}[D_6]_{ngs}$	$3600 = 2^4 3^2 5^2$	$y^2+x^3y = 14x^3 - 20$	square of EC over 6.0.7200000.1	BMF over 2.0.3.1 of level 400
$\mathbf{F}[D_2, C_2, \mathcal{H}]_{ngs}$	$576 = 2^6 3^2$	$y^2+x^3y = 5x^3-2$	square of PCM EC 1.1-a2 over $\mathbb{Q}(\sqrt{6})$	CM HMF 1.1-a over $\mathbb{Q}(\sqrt{6})$
$\mathbf{F}[C_2, C_1, \mathrm{M}_2(\mathbb{R})]_{ns}$	$729 = 3^6$	$y^2+y = -48x^6+15x^3-1$	square of PCM EC 27.a4 over $\mathbb{Q}$	CM CMF 27.2.1.a

One page of the “giant table” [Booker-Sijsling-S-Voight-Yasaki 2022?]

Geometric sources of abelian surfaces

Some geometric sources of abelian surfaces that are computationally accessible:

- The Jacobian of a genus 2 curve over $\mathbb{Q}$.
- Products of elliptic curves over $\mathbb{Q}$.
- The Weil-restriction of an elliptic curve over a quadratic field.
- Isogeny factors of Jacobians of higher genus curves (classical Prym varieties and generalized Pryms that are not necessarily PPAVs).
- Isogeny twists of elliptic curves over $\mathbb{Q}$ (typically not PPAVs).
- $(1, d)$-polarized abelian surfaces (not PPAVs for $d > 1$, see Elkies talk for $d = 3$).

Prior results from enumerating curves

There have been several large scale enumerations of curves of genus 2 and 3 that have provided explicit examples of abelian surfaces:

- An enumeration of $\approx 10^{17}$ genus 2 curves yielded the 66158 curves with $|D| \leq 10^6$ in the LMFDB [Booker-Sijsling-S-Voight-Yasaki 2016].
- A similar enumeration of $\approx 10^{17}$ hyperelliptic genus 3 curves yielded 67879 curves with $|D| \leq 10^7$. Abelian surface isogeny factors with $N = 924, 945$ match previously unmatched paramodular forms [S 2018].
- An enumeration of $\approx 10^{17.5}$ nonhyperelliptic genus 3 curves yielded 82240 curves with $|D| \leq 10^7$. Abelian surface isogeny factors with $N =$ 550, 702, 760, 969 match previously unmatched paramodular forms [S 2018].

As of 2018, these computations combined with examples of genus 2 curves previously obtained by Brumer addressed all but one of the paramodular forms of level $N \leq 1000$ computed by Poor-Yuen. There is a single unmatched paramodular form of level 903!

Recent computations

- Aug 29: we enumerated $\approx 10^{18.5}$ genus 2 curves $y^2 = f(x)$ with $\|f\| \le 270$ finding $\approx 6 \times 10^7$ curves with $\mathrm{rad}(D_{odd}) \le 1000$.

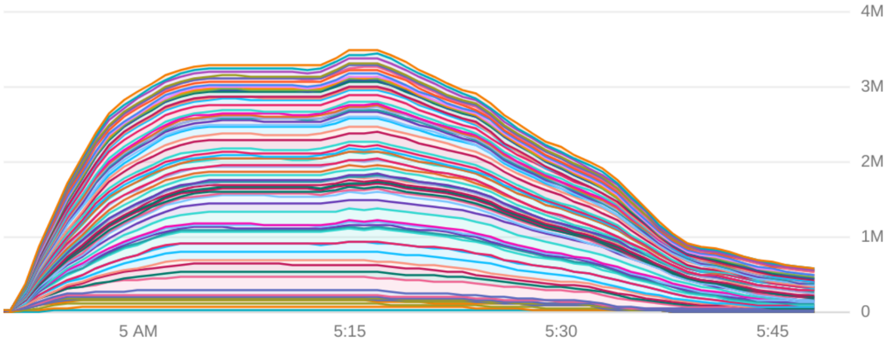
  Twisting yields $\approx 3 \times 10^8$ with $\mathrm{rad}(D_{odd}) \le 1000$, of which $\approx 10^6$ have $N \le 10^6$.

- Jan 9: we enumerated $\approx 10^{17.5}$ genus 2 curves $y^2 + h(x)y = f(x)$ with $\|f\| \le 127$ finding $\approx 5 \times 10^9$ with $|D_{\mathrm{odd}}| \le 2^{63}$, $|D_{p^{12}\text{-}free}| \le 2^{36}$, $\mathrm{rad}(D_{odd}) \le 10^6$.

  Twisting yields $\approx 1.7 \times 10^6$ isomorphism classes with $N \le 10^6$.

Combining the two computations yields more than 2 million isomorphism classes of genus 2 curves with conductor $N \le 10^6$ in at least 1.7×10^6 distinct isogeny classes.

Computation of Jan 9, 2022



We used 126080 32-vCPU instances in 73 data centers in 23 different locations, including Taiwan, Hong Kong, Tokyo, Osaka, Mumbai, Singapore, Sydney, Finland, Belgium, London, Frankfurt, the Netherlands, Zurich, Montréal, São Paulo, Iowa, South Carolina, Virginia, Oregon, Los Angeles, Salt Lake City, and Las Vegas.

Results

We found more than 500 genus 2 curves with conductors $N \leq 1000$, including the curve

$$C_{903} : y^2 + (x^2+1)y = x^5 + 3x^4 - 13x^3 - 25x^2 + 61x - 28$$

of conductor 903 and whose L-function coefficients match the a_n of the paramodular form of conductor 903 computed by Poor-Yuen and extended by Mellit to $n \leq 100$.

We also found curves of conductor 657, 760, 775, 924 not previously known to occur for Jacobians, examples of arithmetic phenomena that do not arise among the curves in the LMFDB, and many more curves of small conductor:

conductor bound	1000	10000	100000	1000000
curves in LMFDB	159	3069	20265	66158
curves found	569	16583	243498	2069955
L-functions in LMFDB	109	2807	19775	65534
L-functions found	182	8434	173193	1679113