Sato-Tate distributions of abelian varieties

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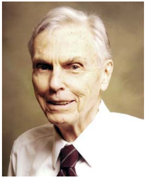
January 12, 2016

Mikio Sato



John Tate

Joint work with F. Fité, K.S. Kedlaya, and V. Rotger, and with D. Harvey.

Sato-Tate in dimension 1

Let $E/\mathbb{Q}$ be an elliptic curve, which we can write in the form

$$y^2 = x^3 + ax + b,$$

and let p be a prime of good reduction ($4a^3 + 27b^2 \not\equiv 0 \bmod p$).

The number of $\mathbb{F}_p$-points on the reduction E_p of E modulo p is

$$\#E_p(\mathbb{F}_p) = p + 1 - t_p,$$

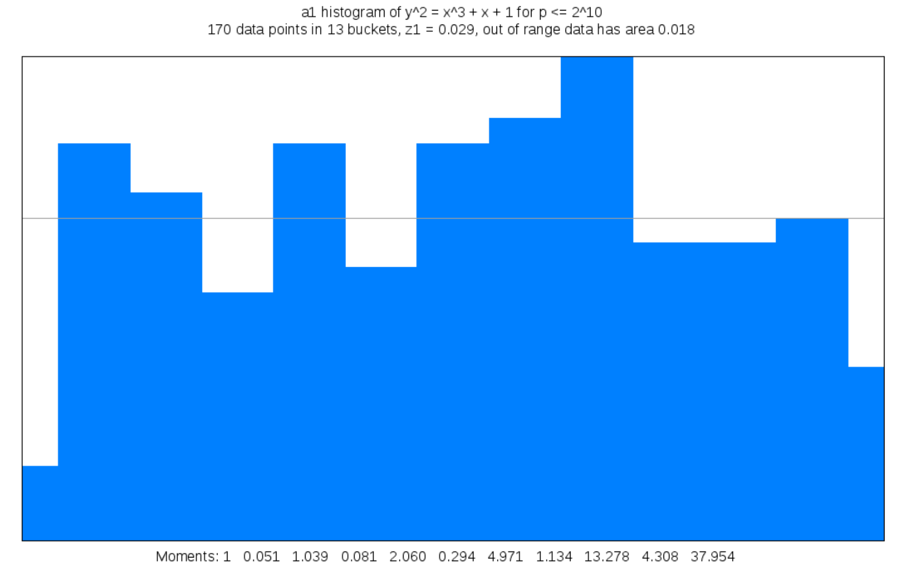
where the trace of Frobenius t_p is an integer in $[-2\sqrt{p}, 2\sqrt{p}]$.

We are interested in the limiting distribution of $x_p = -t_p/\sqrt{p} \in [-2, 2]$, as p varies over primes of good reduction up to N, as $N \to \infty$.

Example: $y^2 = x^3 + x + 1$

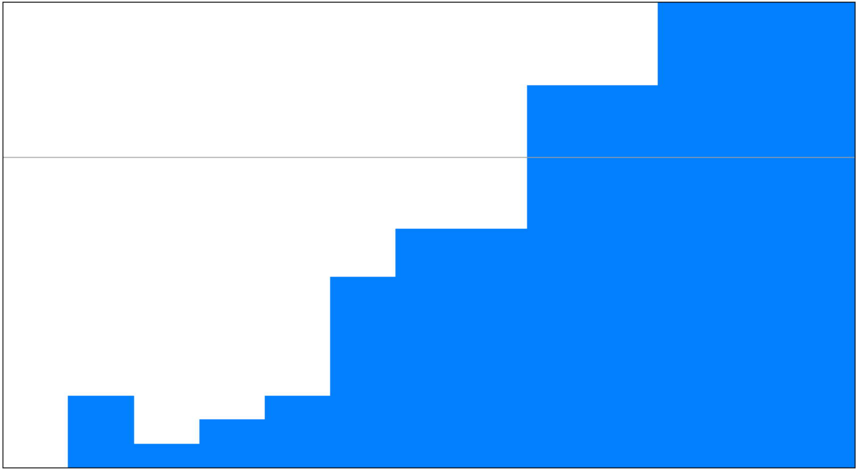
p	t_p	x_p	p	t_p	x_p	p	t_p	x_p
3	0	$\mathbf{0.000000}$	71	13	$\mathbf{-1.542816}$	157	-13	$\mathbf{1.037513}$
5	-3	$\mathbf{1.341641}$	73	2	$\mathbf{-0.234082}$	163	-25	$\mathbf{1.958151}$
7	3	$\mathbf{-1.133893}$	79	-6	$\mathbf{0.675053}$	167	24	$\mathbf{-1.857176}$
11	-2	$\mathbf{0.603023}$	83	-6	$\mathbf{0.658586}$	173	2	$\mathbf{-0.152057}$
13	-4	$\mathbf{1.109400}$	89	-10	$\mathbf{1.059998}$	179	0	$\mathbf{0.000000}$
17	0	$\mathbf{0.000000}$	97	1	$\mathbf{-0.101535}$	181	-8	$\mathbf{0.594635}$
19	-1	$\mathbf{0.229416}$	101	-3	$\mathbf{0.298511}$	191	-25	$\mathbf{1.808937}$
23	-4	$\mathbf{0.834058}$	103	17	$\mathbf{-1.675060}$	193	-7	$\mathbf{0.503871}$
29	-6	$\mathbf{1.114172}$	107	3	$\mathbf{-0.290021}$	197	-24	$\mathbf{1.709929}$
37	-10	$\mathbf{1.643990}$	109	-13	$\mathbf{1.245174}$	199	-18	$\mathbf{1.275986}$
41	7	$\mathbf{-1.093216}$	113	-11	$\mathbf{1.034793}$	211	-11	$\mathbf{0.757271}$
43	10	$\mathbf{-1.524986}$	127	2	$\mathbf{-0.177471}$	223	-20	$\mathbf{1.339299}$
47	-12	$\mathbf{1.750380}$	131	4	$\mathbf{-0.349482}$	227	0	$\mathbf{0.000000}$
53	-4	$\mathbf{0.549442}$	137	12	$\mathbf{-1.025229}$	229	-2	$\mathbf{0.132164}$
59	-3	$\mathbf{0.390567}$	139	14	$\mathbf{-1.187465}$	233	-3	$\mathbf{0.196537}$
61	12	$\mathbf{-1.536443}$	149	14	$\mathbf{-1.146925}$	239	-22	$\mathbf{1.423062}$
67	12	$\mathbf{-1.466033}$	151	-2	$\mathbf{0.162758}$	241	22	$\mathbf{-1.417145}$

`http://math.mit.edu/~drew/g1SatoTateDistributions.html`



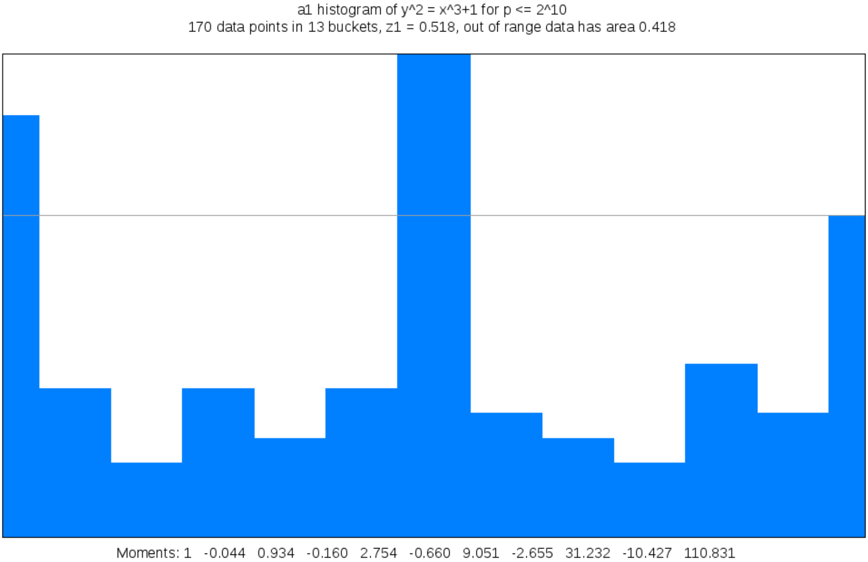
click histogram to animate (requires adobe reader)

a1 histogram of y^2 + xy + y = x^3 - x^2 - 20067762415575526585033208209338542750930230312178956502x
+ 34481611795030556467032985690390720374855944359319180361266008296291939448732243429 for p <= 2^10
172 data points in 13 buckets, z1 = 0.023, out of range data has area 0.250



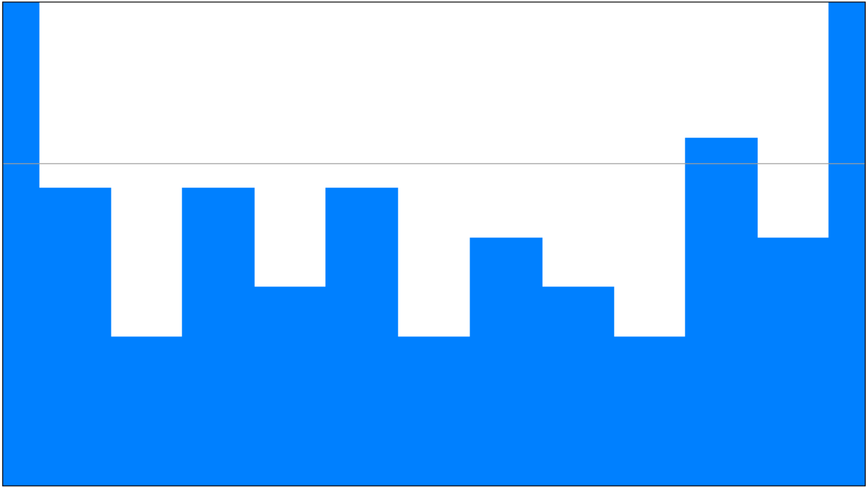
Moments: 1   1.034   1.716   2.532   4.446   7.203   13.024   22.220   40.854   72.100   133.961

click histogram to animate (requires adobe reader)



click histogram to animate (requires adobe reader)

a1 histogram of y^2 = x^3+1 over Q(sqrt(-3)) for split p <= 2^10
164 data points in 13 buckets, out of range data has area 0.122



Moments: 1 -0.092 1.935 -0.331 5.710 -1.368 18.765 -5.504 64.750 -21.616 229.771

click histogram to animate (requires adobe reader)

Sato-Tate distributions in dimension 1

1. Typical case (no CM)

Elliptic curves $E/\mathbb{Q}$ w/o CM have the semi-circular trace distribution. (This is also known for E/k, where k is a totally real number field).

[Barnet-Lamb, Clozel, Geraghty, Harris, Shepherd-Barron, Taylor]

2. Exceptional cases (CM)

Elliptic curves E/k with CM have one of two distinct trace distributions, depending on whether k contains the CM field or not.

[classical (Hecke, Deuring)]

Sato-Tate groups in dimension 1

The *Sato-Tate group* of E is a closed subgroup G of $\mathrm{SU}(2) = \mathrm{USp}(2)$ derived from the ℓ-adic Galois representation attached to E.

The refined Sato-Tate conjecture implies that the distribution of normalized traces of E_p converges to the distribution of traces in the Sato-Tate group of G, under the Haar measure.

G	G/G^0	E	k	$\mathrm{E}[a_1^0], \mathrm{E}[a_1^2], \mathrm{E}[a_1^4] \ldots$
$\mathrm{U}(1)$	C_1	$y^2 = x^3 + 1$	$\mathbb{Q}(\sqrt{-3})$	$1, 2, 6, 20, 70, 252, \ldots$
$N(\mathrm{U}(1))$	C_2	$y^2 = x^3 + 1$	$\mathbb{Q}$	$1, 1, 3, 10, 35, 126, \ldots$
$\mathrm{SU}(2)$	C_1	$y^2 = x^3 + x + 1$	$\mathbb{Q}$	$1, 1, 2, 5, 14, 42, \ldots$

In dimension 1 there are three possible Sato-Tate groups, two of which arise for elliptic curves defined over $\mathbb{Q}$.

Zeta functions and L-polynomials

For a smooth projective curve $C/\mathbb{Q}$ of genus g and each prime p of good redution for C we have the *zeta function*

$$Z(C_p/\mathbb{F}_p;T) := \exp\left(\sum_{k=1}^{\infty} N_k T^k/k\right),$$

where $N_k = \#C_p(\mathbb{F}_{p^k})$. This is a rational function of the form

$$Z(C_p/\mathbb{F}_p;T) = \frac{L_p(T)}{(1-T)(1-pT)},$$

where $L_p(T)$ is an integer polynomial of degree $2g$.

For $g=1$ we have $L_p(t) = pT^2 + c_1T + 1$, and for $g=2$,

$$L_p(T) = p^2T^4 + c_1pT^3 + c_2T^2 + c_1T + 1.$$

Normalized L-polynomials

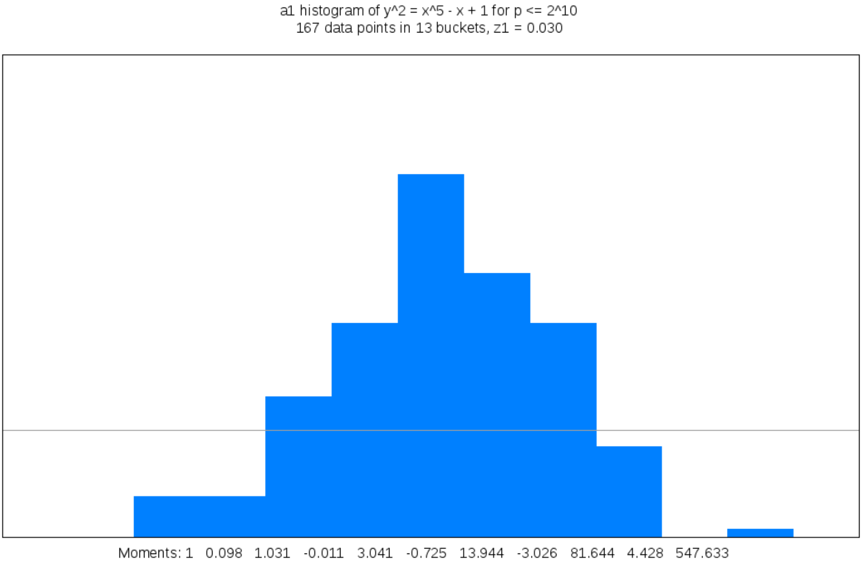
The normalized polynomial

$$\bar{L}_p(T) := L_p(T/\sqrt{p}) = \sum_{i=0}^{2g} a_i T^i \in \mathbb{R}[T]$$

is monic, reciprocal ($a_i = a_{2g-i}$), and unitary (roots on the unit circle). The coefficients a_i necessarily satisfy $|a_i| \leq \binom{2g}{i}$.

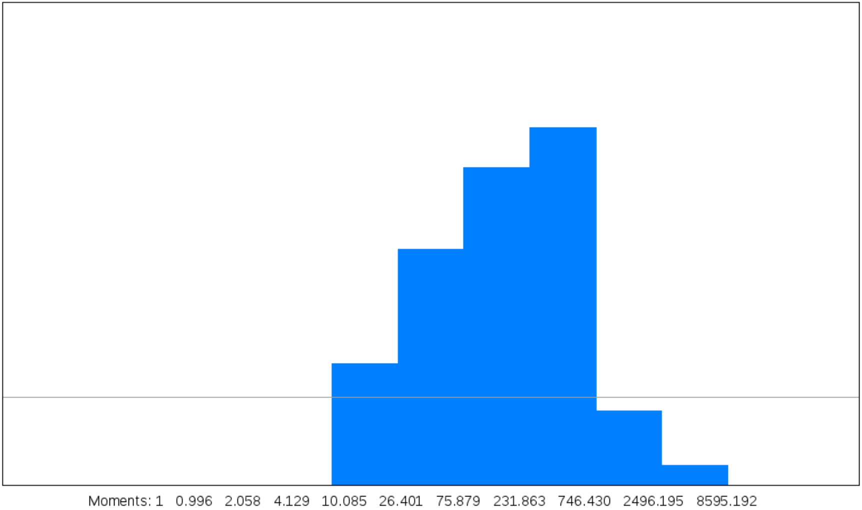
We now consider the limiting distribution of $a_1, a_2, \ldots, a_g$ over all primes $p \leq N$ of good reduction, as $N \to \infty$.

`http://math.mit.edu/~drew/g2SatoTateDistributions.html`

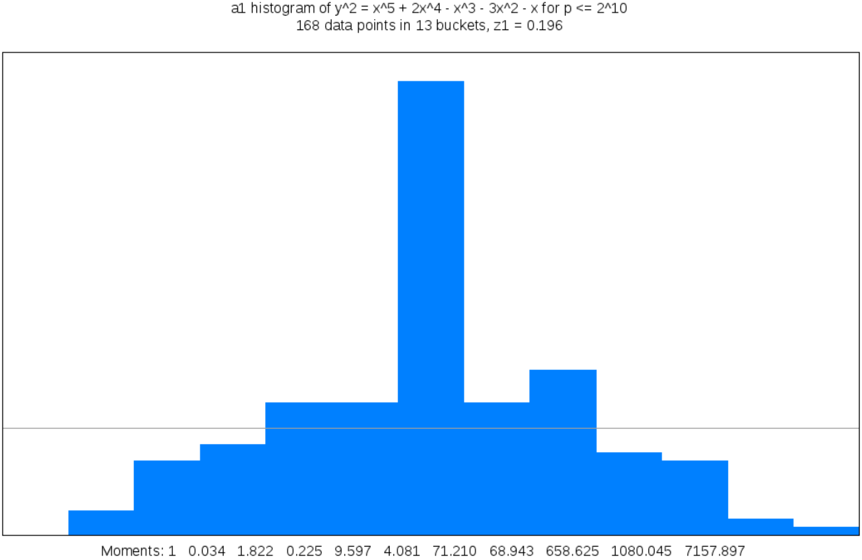


click histogram to animate (requires adobe reader)

a2 histogram of y^2 = x^5 - x + 1 for p <= 2^10
167 data points in 13 buckets

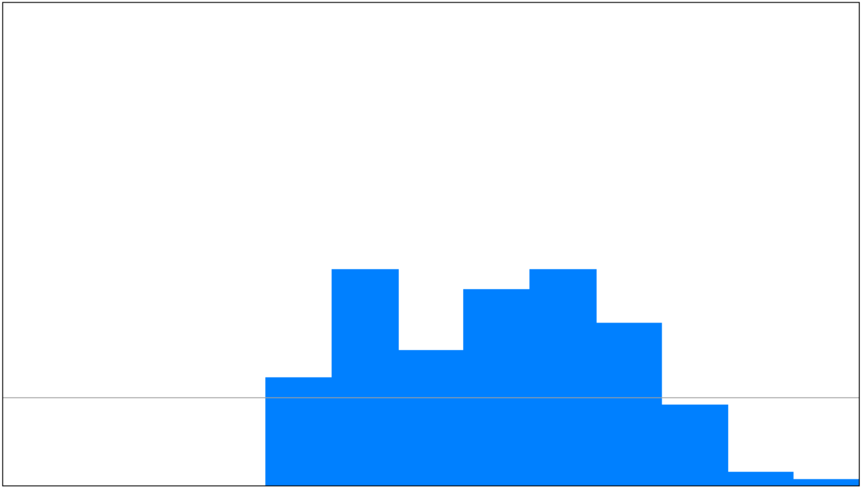


click histogram to animate (requires adobe reader)



click histogram to animate (requires adobe reader)

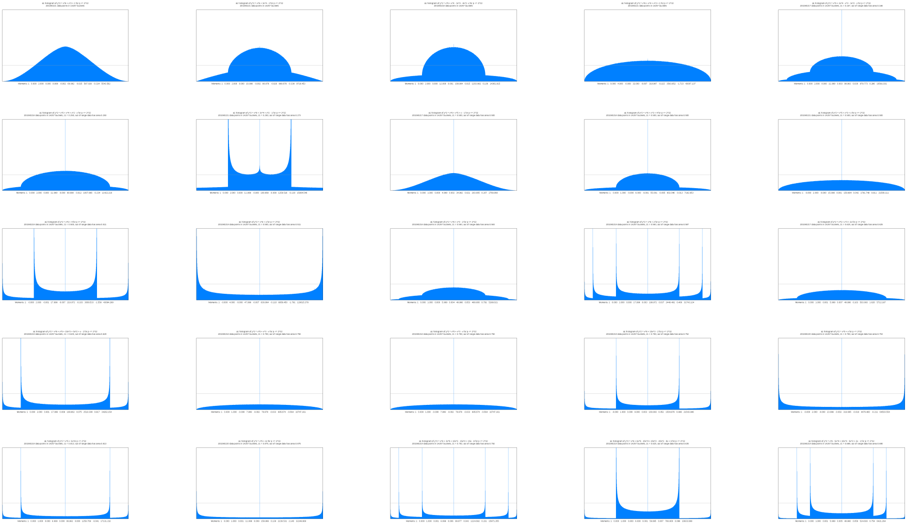
a2 histogram of y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x for p <= 2^10
168 data points in 13 buckets, z2 = [0.006 0.000 0.000 0.000 0.012]

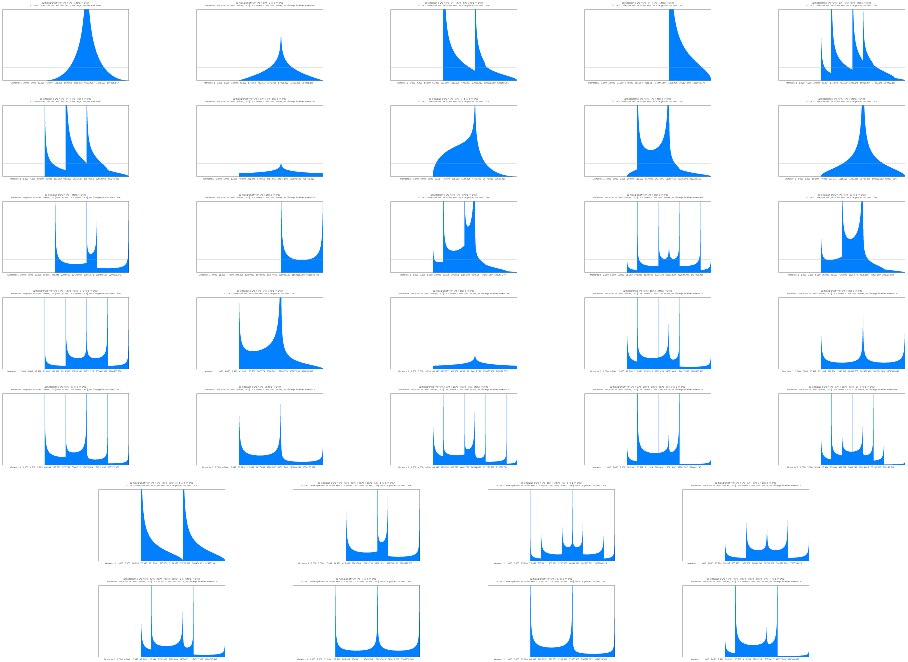


Moments: 1   0.914   3.679   8.930   33.618   120.114   506.202   2236.335   10692.989   53523.391   278878.343

click histogram to animate (requires adobe reader)

Exceptional distributions for abelian surfaces over $\mathbb{Q}$:





L-polynomials of Abelian varieties

Let A be an abelian variety of dimension $g \geq 1$ over a number field k and fix a prime ℓ.

Let $\rho_\ell \colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ be the Galois representation arising from the action of $G_k := \mathrm{Gal}(\bar{k}/k)$ on the ℓ-adic Tate module

$$V_\ell(A) := \varprojlim A[\ell^n] \otimes \mathbb{Q}.$$

For each prime $\mathfrak{p}$ of good reduction for A we have the *L-polynomial*

$$L_\mathfrak{p}(T) := \det(1 - \rho_\ell(\mathrm{Frob}_\mathfrak{p})T),$$
$$\bar{L}_\mathfrak{p}(T) := L_\mathfrak{p}(T/\sqrt{\|\mathfrak{p}\|}) = \sum a_i T^i.$$

In the case that A is the Jacobian of a genus g curve C, this agrees with our earlier definition of $L_\mathfrak{p}(T)$ as the numerator of the zeta function of C.

The Sato-Tate problem for an abelian variety

The polynomials $\bar{L}_\mathfrak{p} \in \mathbb{R}[T]$ are monic, symmetric, unitary, and have degree $2g$.

Every such polynomial arises as the characteristic polynomial of a conjugacy class in the unitary symplectic group $\mathrm{USp}(2g)$.

Each probability measure on $\mathrm{USp}(2g)$ determines a distribution of conjugacy classes (hence a distribution of characteristic polynomials).

The *Sato-Tate problem*, in its simplest form, is to find a measure for which these classes are equidistributed.

Conjecturally, such a measure arises as the Haar measure of a compact subgroup ST_A of $\mathrm{USp}(2g)$.

The Sato-Tate group

Recall that the action of G_k on $V_\ell(A)$ induces the representation

$$\rho_\ell\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell).$$

Fixing an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we now apply

$$\ker(G_k \xrightarrow{\chi_\ell} \mathbb{Q}_\ell^\times) \xrightarrow{\overline{\rho_\ell}} \mathrm{Sp}_{2g}(\mathbb{Q}_\ell) \xrightarrow{\otimes_\iota \mathbb{C}} \mathrm{Sp}_{2g}(\mathbb{C}),$$

and define ST_A to be a maximal compact subgroup of the image.

Conjecturally, ST_A does not depend on ℓ or ι; this is known for $g \leq 3$.

Definition [Serre]

$\mathrm{ST}_A \subseteq \mathrm{USp}(2g)$ is the *Sato-Tate group* of A.

The refined Sato-Tate conjecture

Let $s(\mathfrak{p})$ denote the conjugacy class of the image of $\mathrm{Frob}_{\mathfrak{p}}$ in ST_A.

Let μ_{ST_A} denote the image of the Haar measure on $\mathrm{Conj}(\mathrm{ST}_A)$, which does not depend on the choice of ℓ or ι.

Conjecture

The conjugacy classes $s(\mathfrak{p})$ are equidistributed with respect to μ_{ST_A}.

In particular, the distribution of $\bar{L}_{\mathfrak{p}}(T)$ matches the distribution of characteristic polynomials of random matrices in ST_A.

We can test this numerically by comparing statistics of the coefficients $a_1, \ldots, a_g$ of $\bar{L}_{\mathfrak{p}}(T)$ over $\|\mathfrak{p}\| \le N$ to the predictions given by μ_{ST_A}.

`https://hensel.mit.edu:8000/home/pub/6`

The Sato-Tate axioms

The Sato-Tate axioms for abelian varieties (weight-1 motives):

1. G is closed subgroup of $\mathrm{USp}(2g)$.
2. **Hodge condition**: G contains a Hodge circle[1] whose conjugates generate a dense subset of G.
3. **Rationality condition**: for each component H of G and each irreducible character χ of $\mathrm{GL}_{2g}(\mathbb{C})$ we have $\mathrm{E}[\chi(\gamma) : \gamma \in H] \in \mathbb{Z}$.

For any fixed g, the set of subgroups $G \subseteq \mathrm{USp}(2g)$ that satisfy the *Sato-Tate axioms* is **finite** up to conjugacy (3 for $g = 1$, 55 for $g = 2$).

[1] An embedding $\theta\colon \mathrm{U}(1) \to G^0$ where $\theta(u)$ has eigenvalues u and u^{-1} each with multiplicity g.

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Theorem

For $g \leq 3$, the group ST_A satisfies the Sato-Tate axioms.

This is expected to hold for all g.

[1] An embedding $\theta\colon \mathrm{U}(1) \to G^0$ where $\theta(u)$ has eigenvalues u and u^{-1} each with multiplicity g.

Galois endomorphism modules

Let A be an abelian variety defined over a number field k.
Let K be the minimal extension of k for which $\mathrm{End}(A_K) = \mathrm{End}(A_{\bar{\mathbb{Q}}})$.
$\mathrm{Gal}(K/k)$ acts on the $\mathbb{R}$-algebra $\mathrm{End}(A_K)_{\mathbb{R}} = \mathrm{End}(A_K) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition

The *Galois endomorphism type* of A is the isomorphism class of $[\mathrm{Gal}(K/k), \mathrm{End}(A_K)_{\mathbb{R}}]$, where $[G, E] \simeq [G', E']$ iff there are isomorphisms $G \simeq G'$ and $E \simeq E'$ that are compatible with the Galois action.

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Theorem [FKRS 2012]

For abelian varieties A/k of dimension $g \leq 3$ there is a one-to-one correspondence between Sato-Tate groups and Galois types.

More precisely, the identity component G^0 is uniquely determined by $\mathrm{End}(A_k)_{\mathbb{R}}$ and $G/G^0 \simeq \mathrm{Gal}(K/k)$ (with corresponding actions).

Real endomorphism algebras of abelian surfaces

abelian surface	$\mathbf{End}(A_K)_{\mathbb{R}}$	$\mathbf{ST}_A^0$
square of CM elliptic curve	$\mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1)_2$
• QM abelian surface • square of non-CM elliptic curve	$\mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2)_2$
• CM abelian surface • product of CM elliptic curves	$\mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1)$
product of CM and non-CM elliptic curves	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2)$
• RM abelian surface • product of non-CM elliptic curves	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
generic abelian surface	$\mathbb{R}$	$\mathrm{USp}(4)$

(factors in products are assumed to be non-isogenous)

Sato-Tate groups in dimension 2

Theorem [FKRS 2012]

Up to conjugacy, 55 subgroups of $\mathrm{USp}(4)$ satisfy the Sato-Tate axioms:

$$\begin{array}{rl}
\mathrm{U}(1)_2: & C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O, \\
& J(C_1), J(C_2), J(C_3), J(C_4), J(C_6), \\
& J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O), \\
& C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1 \\
\mathrm{SU}(2)_2: & E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6) \\
\mathrm{U}(1)\times\mathrm{U}(1): & F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c} \\
\mathrm{U}(1)\times\mathrm{SU}(2): & \mathrm{U}(1)\times\mathrm{SU}(2),\ N(\mathrm{U}(1)\times\mathrm{SU}(2)) \\
\mathrm{SU}(2)\times\mathrm{SU}(2): & \mathrm{SU}(2)\times\mathrm{SU}(2),\ N(\mathrm{SU}(2)\times\mathrm{SU}(2)) \\
\mathrm{USp}(4): & \mathrm{USp}(4)
\end{array}$$

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 & J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O), \\
 & C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1 \\
\mathrm{SU}(2): & E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6) \\
\mathrm{U}(1)\times\mathrm{U}(1): & F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c} \\
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Of these, exactly 52 arise as ST_A for an abelian surface A (34 over $\mathbb{Q}$).

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& J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O),\\
& C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1\\
\mathrm{SU}(2): & E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6)\\
\mathrm{U}(1)\times\mathrm{U}(1): & F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c}\\
\mathrm{U}(1)\times\mathrm{SU}(2): & \mathrm{U}(1)\times\mathrm{SU}(2), N(\mathrm{U}(1)\times\mathrm{SU}(2))\\
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\end{array}$$

Of these, exactly 52 arise as ST_A for an abelian surface A (34 over $\mathbb{Q}$).

This theorem says nothing about equidistribution, however this is now known in many special cases [FS 2012, Johansson 2013].

Sato-Tate groups in dimension 2 with $G^0 = \mathrm{U}(1)_2$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
1	1	C_1	C_1	0	0, 0, 0, 0, 0	8, 96, 1280, 17920	4, 18, 88, 454
1	2	C_2	C_2	1	0, 0, 0, 0, 0	4, 48, 640, 8960	2, 10, 44, 230
1	3	C_3	C_3	0	0, 0, 0, 0, 0	4, 36, 440, 6020	2, 8, 34, 164
1	4	C_4	C_4	1	0, 0, 0, 0, 0	4, 36, 400, 5040	2, 8, 32, 150
1	6	C_6	C_6	1	0, 0, 0, 0, 0	4, 36, 400, 4900	2, 8, 32, 148
1	4	D_2	D_2	3	0, 0, 0, 0, 0	2, 24, 320, 4480	1, 6, 22, 118
1	6	D_3	D_3	3	0, 0, 0, 0, 0	2, 18, 220, 3010	1, 5, 17, 85
1	8	D_4	D_4	5	0, 0, 0, 0, 0	2, 18, 200, 2520	1, 5, 16, 78
1	12	D_6	D_6	7	0, 0, 0, 0, 0	2, 18, 200, 2450	1, 5, 16, 77
1	2	$J(C_1)$	C_2	1	1, 0, 0, 0, 0	4, 48, 640, 8960	1, 11, 40, 235
1	4	$J(C_2)$	D_2	3	1, 0, 0, 0, 1	2, 24, 320, 4480	1, 7, 22, 123
1	6	$J(C_3)$	C_6	3	1, 0, 0, 2, 0	2, 18, 220, 3010	1, 5, 16, 85
1	8	$J(C_4)$	$\mathrm{C}_4 \times \mathrm{C}_2$	5	1, 0, 2, 0, 1	2, 18, 200, 2520	1, 5, 16, 79
1	12	$J(C_6)$	$\mathrm{C}_6 \times \mathrm{C}_2$	7	1, 2, 0, 2, 1	2, 18, 200, 2450	1, 5, 16, 77
1	8	$J(D_2)$	$\mathrm{D}_2 \times \mathrm{C}_2$	7	1, 0, 0, 0, 3	1, 12, 160, 2240	1, 5, 13, 67
1	12	$J(D_3)$	D_6	9	1, 0, 0, 2, 3	1, 9, 110, 1505	1, 4, 10, 48
1	16	$J(D_4)$	$\mathrm{D}_4 \times \mathrm{C}_2$	13	1, 0, 2, 0, 5	1, 9, 100, 1260	1, 4, 10, 45
1	24	$J(D_6)$	$\mathrm{D}_6 \times \mathrm{C}_2$	19	1, 2, 0, 2, 7	1, 9, 100, 1225	1, 4, 10, 44
1	2	$C_{2,1}$	C_2	1	0, 0, 0, 0, 1	4, 48, 640, 8960	3, 11, 48, 235
1	4	$C_{4,1}$	C_4	3	0, 0, 2, 0, 0	2, 24, 320, 4480	1, 5, 22, 115
1	6	$C_{6,1}$	C_6	3	0, 2, 0, 0, 1	2, 18, 220, 3010	1, 5, 18, 85
1	4	$D_{2,1}$	D_2	3	0, 0, 0, 0, 2	2, 24, 320, 4480	2, 7, 26, 123
1	8	$D_{4,1}$	D_4	7	0, 0, 2, 0, 2	1, 12, 160, 2240	1, 4, 13, 63
1	12	$D_{6,1}$	D_6	9	0, 2, 0, 0, 4	1, 9, 110, 1505	1, 4, 11, 48
1	6	$D_{3,2}$	D_3	3	0, 0, 0, 0, 3	2, 18, 220, 3010	2, 6, 21, 90
1	8	$D_{4,2}$	D_4	5	0, 0, 0, 0, 4	2, 18, 200, 2520	2, 6, 20, 83
1	12	$D_{6,2}$	D_6	7	0, 0, 0, 0, 6	2, 18, 200, 2450	2, 6, 20, 82
1	12	T	A_4	3	0, 0, 0, 0, 0	2, 12, 120, 1540	1, 4, 12, 52
1	24	O	S_4	9	0, 0, 0, 0, 0	2, 12, 100, 1050	1, 4, 11, 45
1	24	O_1	S_4	15	0, 0, 6, 0, 6	1, 6, 60, 770	1, 3, 8, 30
1	24	$J(T)$	$\mathrm{A}_4 \times \mathrm{C}_2$	15	1, 0, 0, 8, 3	1, 6, 60, 770	1, 3, 7, 29
1	48	$J(O)$	$\mathrm{S}_4 \times \mathrm{C}_2$	33	1, 0, 6, 8, 9	1, 6, 50, 525	1, 3, 7, 26

Sato-Tate groups in dimension 2 with $G^0 \neq \mathrm{U}(1)_2$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
3	1	E_1	C_1	0	0, 0, 0, 0, 0	4, 32, 320, 3584	3, 10, 37, 150
3	2	E_2	C_2	1	0, 0, 0, 0, 0	2, 16, 160, 1792	1, 6, 17, 78
3	3	E_3	C_3	0	0, 0, 0, 0, 0	2, 12, 110, 1204	1, 4, 13, 52
3	4	E_4	C_4	1	0, 0, 0, 0, 0	2, 12, 100, 1008	1, 4, 11, 46
3	6	E_6	C_6	1	0, 0, 0, 0, 0	2, 12, 100, 980	1, 4, 11, 44
3	2	$J(E_1)$	C_2	1	0, 0, 0, 0, 0	2, 16, 160, 1792	2, 6, 20, 78
3	4	$J(E_2)$	D_2	3	0, 0, 0, 0, 0	1, 8, 80, 896	1, 4, 10, 42
3	6	$J(E_3)$	D_3	3	0, 0, 0, 0, 0	1, 6, 55, 602	1, 3, 8, 29
3	8	$J(E_4)$	D_4	5	0, 0, 0, 0, 0	1, 6, 50, 504	1, 3, 7, 26
3	12	$J(E_6)$	D_6	7	0, 0, 0, 0, 0	1, 6, 50, 490	1, 3, 7, 25
2	1	F	C_1	0	0, 0, 0, 0, 0	4, 36, 400, 4900	2, 8, 32, 148
2	2	F_a	C_2	0	0, 0, 0, 0, 1	3, 21, 210, 2485	2, 6, 20, 82
2	2	F_c	C_2	1	0, 0, 0, 0, 0	2, 18, 200, 2450	1, 5, 16, 77
2	2	F_{ab}	C_2	1	0, 0, 0, 0, 1	2, 18, 200, 2450	2, 6, 20, 82
2	4	F_{ac}	C_4	3	0, 0, 2, 0, 1	1, 9, 100, 1225	1, 3, 10, 41
2	4	$F_{a,b}$	D_2	1	0, 0, 0, 0, 3	2, 12, 110, 1260	2, 5, 14, 49
2	4	$F_{ab,c}$	D_2	3	0, 0, 0, 0, 1	1, 9, 100, 1225	1, 4, 10, 44
2	8	$F_{a,b,c}$	D_4	5	0, 0, 2, 0, 3	1, 6, 55, 630	1, 3, 7, 26
4	1	G_4	C_1	0	0, 0, 0, 0, 0	3, 20, 175, 1764	2, 6, 20, 76
4	2	$N(G_4)$	C_2	0	0, 0, 0, 0, 1	2, 11, 90, 889	2, 5, 14, 46
6	1	G_6	C_1	0	0, 0, 0, 0, 0	2, 10, 70, 588	2, 5, 14, 44
6	2	$N(G_6)$	C_2	1	0, 0, 0, 0, 0	1, 5, 35, 294	1, 3, 7, 23
10	1	$\mathrm{USp}(4)$	C_1	0	0, 0, 0, 0, 0	1, 3, 14, 84	1, 2, 4, 10

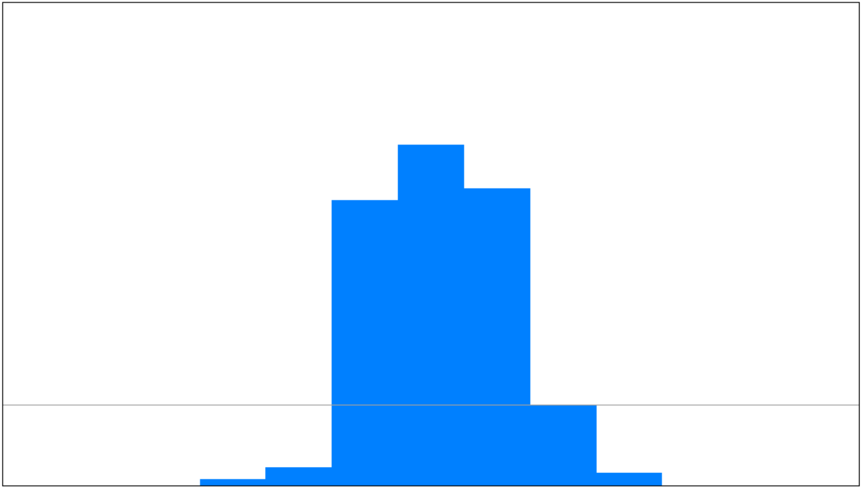
Genus 2 curves realizing Sato-Tate groups with $G^0 = \mathrm{U}(1)$

Group	Curve $y^2 = f(x)$	k	K
C_1	$x^6 + 1$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3})$
C_2	$x^5 - x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2})$
C_3	$x^6 + 4$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
C_4	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a); a^4 + 17a^2 + 68 = 0$
C_6	$x^6 + 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
D_2	$x^5 + 9x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
D_3	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
D_4	$x^5 + 3x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
D_6	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3}, a); a^3 + 3a - 2 = 0$
T	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a, b);$
			$a^3 - 7a + 7 = b^4 + 4b^2 + 8b + 8 = 0$
O	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-11}, a, b);$
			$a^3 - 4a + 4 = b^4 + 22b + 22 = 0$
$J(C_1)$	$x^5 - x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_2)$	$x^5 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(C_4)$	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}$	see entry for C_4
$J(C_6)$	$x^6 - 15x^4 - 20x^3 + 6x + 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{3}, a); a^3 + 3a^2 - 1 = 0$
$J(D_2)$	$x^5 + 9x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
$J(D_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(D_4)$	$x^5 + 3x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
$J(D_6)$	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}$	see entry for D_6
$J(T)$	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}$	see entry for T
$J(O)$	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}$	see entry for O
$C_{2,1}$	$x^6 + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3})$
$C_{4,1}$	$x^5 + 2x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt[4]{2})$
$C_{6,1}$	$x^6 + 6x^5 - 30x^4 + 20x^3 + 15x^2 - 12x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, a); a^3 - 3a + 1 = 0$
$D_{2,1}$	$x^5 + x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$D_{4,1}$	$x^5 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$D_{6,1}$	$x^6 + 6x^5 - 30x^4 - 40x^3 + 60x^2 + 24x - 8$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-3}, a); a^3 - 9a + 6 = 0$
$D_{3,2}$	$x^6 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$D_{4,2}$	$x^6 + x^5 + 10x^3 + 5x^2 + x - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a); a^4 - 14a^2 + 28a - 14 = 0$
$D_{6,2}$	$x^6 + 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
O_1	$x^6 + 7x^5 + 10x^4 + 10x^3 + 15x^2 + 17x + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a, b);$
			$a^3 + 5a + 10 = b^4 + 4b^2 + 8b + 2 = 0$

Genus 2 curves realizing Sato-Tate groups with $G^0 \neq \mathrm{U}(1)$

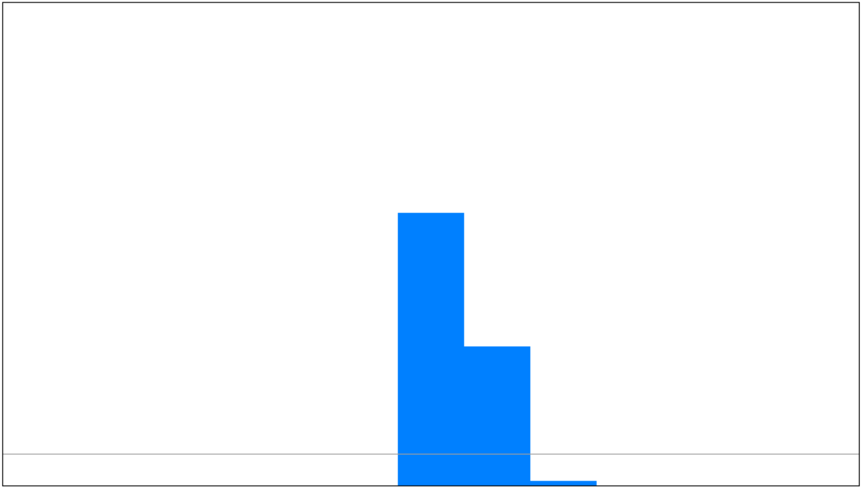
Group	Curve $y^2 = f(x)$	k	K
F	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i, \sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_a	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
F_{ab}	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_{ac}	$x^5 + 1$	$\mathbb{Q}$	$\mathbb{Q}(a); a^4 + 5a^2 + 5 = 0$
$F_{a,b}$	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
E_1	$x^6 + x^4 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
E_2	$x^6 + x^5 + 3x^4 + 3x^2 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$
E_3	$x^5 + x^4 - 3x^3 - 4x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a); a^3 - 3a + 1 = 0$
E_4	$x^5 + x^4 + x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a); a^4 - 5a^2 + 5 = 0$
E_6	$x^5 + 2x^4 - x^3 - 3x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{7}, a); a^3 - 7a - 7 = 0$
$J(E_1)$	$x^5 + x^3 + x$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$J(E_2)$	$x^5 + x^3 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(E_3)$	$x^6 + x^3 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$J(E_4)$	$x^5 + x^3 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$J(E_6)$	$x^6 + x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$G_{1,3}$	$x^6 + 3x^4 - 2$	$\mathbb{Q}(i)$	$\mathbb{Q}(i)$
$N(G_{1,3})$	$x^6 + 3x^4 - 2$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$G_{3,3}$	$x^6 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
$N(G_{3,3})$	$x^6 + x^5 + x - 1$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$\mathrm{USp}(4)$	$x^5 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}$

a1 histogram of y^2 = x^7-x+1 for p <= 2^10
168 data points in 13 buckets, z1 = 0.030

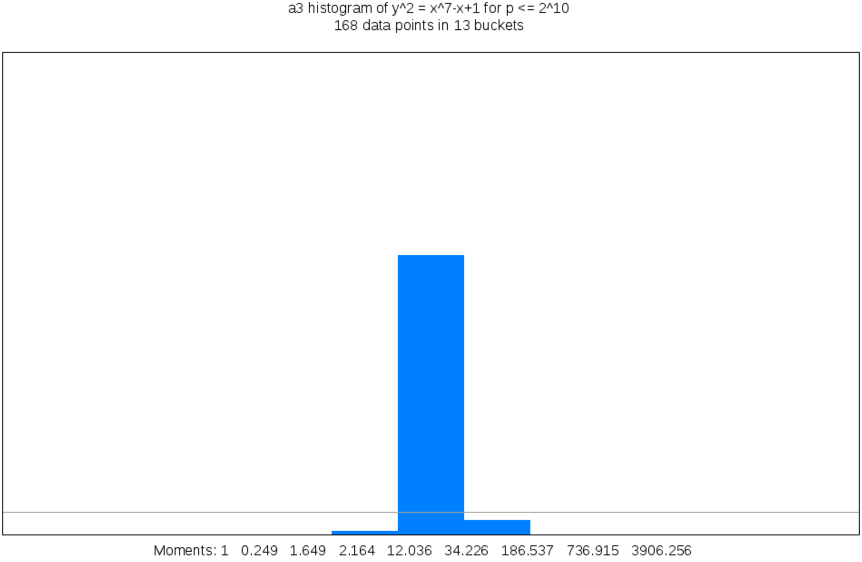


click histogram to animate (requires adobe reader)

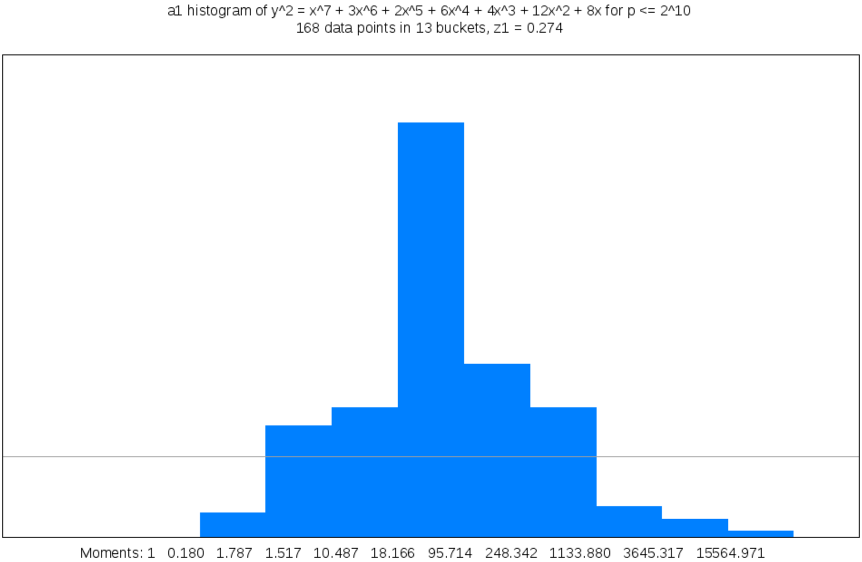
a2 histogram of y^2 = x^7-x+1 for p <= 2^10
168 data points in 13 buckets



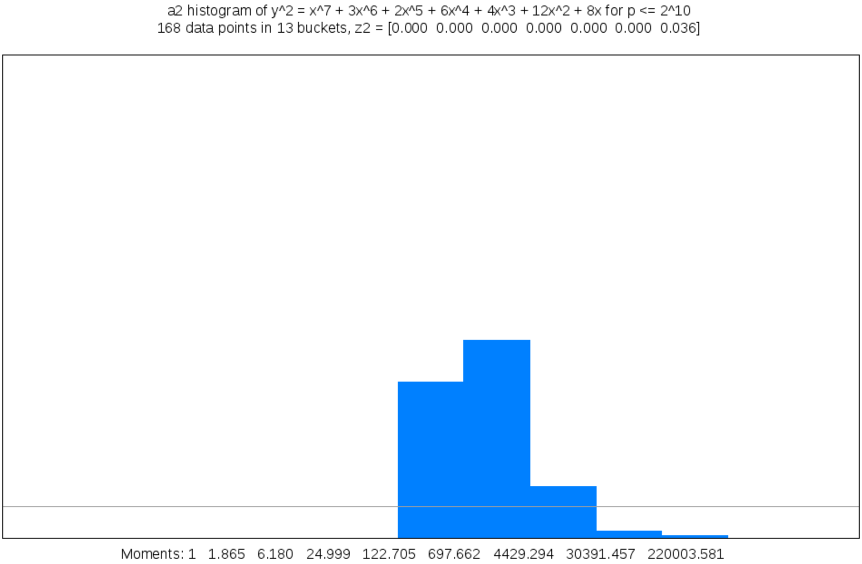
click histogram to animate (requires adobe reader)



click histogram to animate (requires adobe reader)

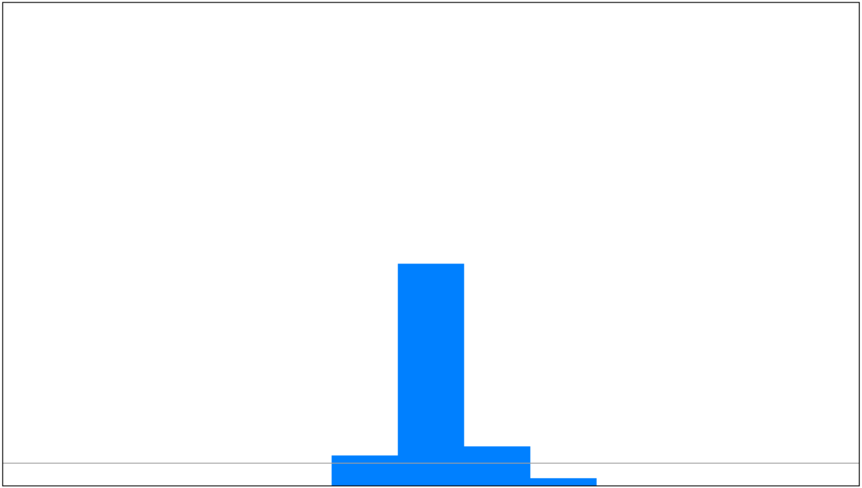


click histogram to animate (requires adobe reader)



click histogram to animate (requires adobe reader)

a3 histogram of y^2 = x^7 + 3x^6 + 2x^5 + 6x^4 + 4x^3 + 12x^2 + 8x for p <= 2^10
168 data points in 13 buckets, z3 = 0.274



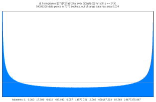
Moments: 1 0.395 5.812 16.472 208.554 1226.780 13225.147 105527.791 1072037.628

click histogram to animate (requires adobe reader)

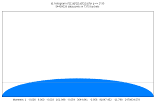
Real endomorphism algebras of abelian threefolds

abelian threefold	$\mathbf{End}(A_K)_{\mathbb{R}}$	$\mathbf{ST}_A^0$
cube of a CM elliptic curve	$\mathrm{M}_3(\mathbb{C})$	$\mathrm{U}(1)_3$
cube of a non-CM elliptic curve	$\mathrm{M}_3(\mathbb{R})$	$\mathrm{SU}(2)_3$
product of CM elliptic curve and square of CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1) \times \mathrm{U}(1)_2$
• product of CM elliptic curve and QM abelian surface • product of CM elliptic curve and square of non-CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{U}(1) \times \mathrm{SU}(2)_2$
product of non-CM elliptic curve and square of CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{SU}(2) \times \mathrm{U}(1)_2$
• product of non-CM elliptic curve and QM abelian surface • product of non-CM elliptic curve and square of non-CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$
• CM abelian threefold • product of CM elliptic curve and CM abelian surface • product of three CM elliptic curves	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$
• product of non-CM elliptic curve and CM abelian surface • product of non-CM elliptic curve and two CM elliptic curves	$\mathbb{C} \times \mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$
• product of CM elliptic curve and RM abelian surface • product of CM elliptic curve and two non-CM elliptic curves	$\mathbb{C} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$
• RM abelian threefold • product of non-CM elliptic curve and RM abelian surface • product of 3 non-CM elliptic curves	$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$
product of CM elliptic curve and abelian surface	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{USp}(4)$
product of non-CM elliptic curve and abelian surface	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{USp}(4)$
quadratic CM abelian threefold	$\mathbb{C}$	$\mathrm{U}(3)$
generic abelian threefold	$\mathbb{R}$	$\mathrm{USp}(6)$

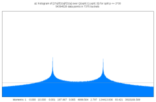
Connected Sato-Tate groups of abelian threefolds:



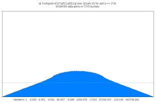
$\mathrm{U}(1)_3$



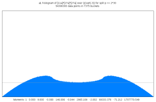
$\mathrm{SU}(2)_3$



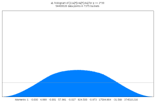
$\mathrm{U}(1) \times \mathrm{U}(1)_2$



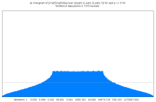
$\mathrm{U}(1) \times \mathrm{SU}(2)_2$



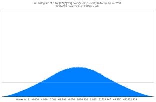
$\mathrm{SU}(2) \times \mathrm{U}(1)_2$



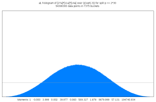
$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$



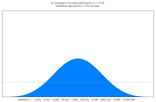
$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$



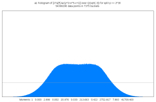
$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$



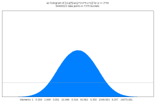
$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$



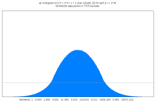
$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$



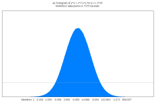
$\mathrm{U}(1) \times \mathrm{USp}(4)$



$\mathrm{SU}(2) \times \mathrm{USp}(4)$



$\mathrm{U}(3)$



$\mathrm{USp}(6)$

Partial classification of component groups

G_0	$G/G_0 \hookrightarrow$	$\lvert G/G_0\rvert$ divides
$\mathrm{USp}(6)$	C_1	1
$\mathrm{U}(3)$	C_2	2
$\mathrm{SU}(2)\times\mathrm{USp}(4)$	C_1	1
$\mathrm{U}(1)\times\mathrm{USp}(4)$	C_2	2
$\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$	S_3	6
$\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$	D_2	4
$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)$	D_4	8
$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$	$\mathrm{C}_2\wr\mathrm{S}_3$	48
$\mathrm{SU}(2)\times\mathrm{SU}(2)_2$	$\mathrm{D}_4,\ \ \mathrm{D}_6$	$8,\ 12$
$\mathrm{SU}(2)\times\mathrm{U}(1)_2$	$\mathrm{D}_6\times\mathrm{C}_2,\ \ \mathrm{S}_4\times\mathrm{C}_2$	48
$\mathrm{U}(1)\times\mathrm{SU}(2)_2$	$\mathrm{D}_4\times\mathrm{C}_2,\ \ \mathrm{D}_6\times\mathrm{C}_2$	$16,\ 24$
$\mathrm{U}(1)\times\mathrm{U}(1)_2$	$\mathrm{D}_6\times\mathrm{C}_2\times\mathrm{C}_2,\ \ \mathrm{S}_4\times\mathrm{C}_2\times\mathrm{C}_2$	96
$\mathrm{SU}(2)_3$	$\mathrm{D}_6,\ \ \mathrm{S}_4$	24
$\mathrm{U}(1)_3$	$\cdots$	$336,\ 1728$

(disclaimer: this is work in progress subject to verification)

Algorithms to compute zeta functions

Given a curve $C/\mathbb{Q}$, we want to compute its normalized L-polynomials $\overline{L}_p(T)$ at all good primes $p \le N$.

algorithm	complexity per prime (ignoring factors of $O(\log\log p)$) $g=1$	$g=2$	$g=3$
point enumeration	$p\log p$	$p^2\log p$	$p^3\log p$
group computation	$p^{1/4}\log p$	$p^{3/4}\log p$	$p^{5/4}\log p$
p-adic cohomology	$p^{1/2}\log^2 p$	$p^{1/2}\log^2 p$	$p^{1/2}\log^2 p$
CRT (Schoof-Pila)	$\log^5 p$	$\log^8 p$	$\log^{12} p$
average polytime	$\log^4 p$	$\log^4 p$	$\log^4 p$

	genus 2		genus 3	
N	`smalljac`	`hwlpoly`	`hypellfrob`	`hwlpoly`
2^{14}	0.2	0.1	7.2	0.4
2^{15}	0.6	0.3	16.3	1.0
2^{16}	1.7	0.9	39.1	2.9
2^{17}	5.5	2.2	98.3	7.8
2^{18}	19.2	5.3	255	18.3
2^{19}	78.4	12.5	695	43.2
2^{20}	271	27.8	1950	98.8
2^{21}	1120	64.5	5600	229
2^{22}	2820	155	16700	537
2^{23}	9840	357	51200	1240
2^{24}	31900	823	158000	2800
2^{25}	105000	1890	501000	6280
2^{26}	349000	4250	1480000	13900
2^{27}	1210000	9590	4360000	31100
2^{28}	4010000	21200	12500000	69700
2^{29}	13200000	48300	39500000	155000
2^{30}	45500000	108000	120000000	344000

(Intel Xeon E5-2697v2 2.7 GHz CPU seconds).