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Sato-Tate groups of some weight 3 motives

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Abstract. We establish the group-theoretic classification of Sato-Tate groups of self-dual motives of weight 3 with rational coefficients and Hodge numbers $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$. We then describe families of motives that realize some of these Sato-Tate groups, and provide numerical evidence supporting equidistribution. One of these families arises in the middle cohomology of certain Calabi-Yau threefolds appearing in the Dwork quintic pencil; for motives in this family, our evidence suggests that the Sato-Tate group is always equal to the full unitary symplectic group USp(4).

Contents

1. Introduction

For a fixed elliptic curve without complex multiplication defined over a number field, the *Sato-Tate conjecture* predicts the average distribution of the Frobenius trace at a variable prime. This conjecture may be naturally generalized to an arbitrary motive over a number field in terms of equidistribution of classes within a certain compact Lie group, the *Sato-Tate group*, as described in [**Ser95**, §13], [**Ser12**, Ch. 8], and [**FKRS12**, §2]. This equidistribution problem reduces naturally (as described in [**Ser68**, Appendix to Chapter 1]) to establishing analytic

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properties of certain motivic L-functions, but unfortunately this latter problem is generally quite difficult. Besides cases of complex multiplication, one of the few cases where equidistribution is known is elliptic curves over totally real number fields [**BLGG11**].

However, the problem of classifying the Sato-Tate groups that can arise from a given class of motives is more tractable. This problem splits naturally into two subproblems: the *group-theoretic classification* problem of identifying those groups consistent with certain group-theoretic restrictions known to apply to Sato-Tate groups in general, and the *arithmetic matching* problem of correlating the resulting groups with the arithmetic of motives in the family. In the case of 1-motives of abelian surfaces, both subproblems have been solved in [**FKRS12**]: there turn out to be exactly 52 groups that arise, up to conjugation within the unitary symplectic group $\mathrm{USp}(4)$.

In this paper, we consider a different family of motives for which we solve the group-theoretic classification problem, give some partial results towards the arithmetic matching problem, and present numerical evidence supporting the equidistribution conjecture. Before describing the family of motives in question, let us recall the general formulation of the group-theoretic classification problem for self-dual motives with rational coefficients of fixed weight w, dimension d, and Hodge numbers $h^{p,q}$. The problem is to identify groups obeying the *Sato-Tate axioms*, as formulated in [**FKRS12**] (modulo one missing condition; see Remark 2.3).

(ST1) The group G is a closed subgroup of $\mathrm{USp}(d)$ or $\mathrm{O}(d)$, depending on whether w is odd or even (respectively).

(ST2) (Hodge condition) There exists a subgroup H of G, called a *Hodge circle*, which is the image of a homomorphism $\theta\colon \mathrm{U}(1) \to G^0$ such that $\theta(u)$ has eigenvalues u^{p-q} with multiplicity $h^{p,q}$. Moreover, the Hodge circles generate a dense subgroup of the identity component G^0.

(ST3) (Rationality condition) For each component C of G and each irreducible character χ of $\mathrm{GL}_d(\mathbb{C})$, the expected value (under the Haar measure) of $\chi(\gamma)$ over $\gamma \in C$ is an integer.

For fixed $w, d, h^{p,q}$, there are only finitely many groups G satisfying (ST1), (ST2), and (ST3), up to conjugation within $\mathrm{USp}(d)$ or $\mathrm{O}(d)$; see Remark 3.3 in [**FKRS12**].

Since the group-theoretic classification is known for 1-motives of abelian varieties of dimensions 1 and 2, it is natural to next try the case of abelian threefolds. We are currently working on this classification, but it is likely to be rather complicated, involving many hundreds of groups. In this paper, we instead consider the case where $w = 3$, $d = 4$, and $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$. We have chosen this case because, on the one hand, it is similar enough to the case of abelian surfaces that much of the analysis of [**FKRS12**] carries over, and, on the other hand, it is of some arithmetic interest due to the multiple ways in which such motives arise. One of these ways is by taking the symmetric cube of the 1-motive associated to an elliptic curve. Another way is to consider a member of the *Dwork pencil* of Calabi-Yau projective threefolds defined by the equation

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = t x_0 x_1 x_2 x_3 x_4, \tag{1.1}$$

in which t represents a nonzero parameter, and then extract the 3-motive invariant under the action of the automorphism group $(\mathbb{Z}/5\mathbb{Z})^4$. These two constructions are closely related: for instance, the coincidence between certain mod ℓ Galois representations arising from the two constructions is exploited in [**HSBT10**] to yield one of the key ingredients in the proof of the Sato-Tate conjecture for elliptic curves. Additional constructions can be achieved using direct sums and tensor products of motives associated to elliptic curves and modular forms (the latter case was suggested to us by Serre).

The primary result of this paper is the resolution of the group-theoretic classification problem for motives of the shape we have just described. This turns out to be similar to the classification problem in [**FKRS12**] but substantially simpler due to the less symmetric shape of the Hodge circle: we end up with only 26 groups up to conjugation. These groups are described in §2 and summarized in Table 1. As in [**FKRS12**], we compute moment sequences associated to these groups in order to facilitate numerical experiments; these appear in §3.

As a partial result towards the arithmetic matching problem, we describe several constructions yielding motives of the given form and then match examples of these constructions to our list of Sato-Tate groups based on numerical experiments. For example, the symmetric cube construction gives rise to Sato-Tate groups with identity component U(1) or SU(2), depending on whether or not the original elliptic curve has complex multiplication (CM), and we can provisionally identify the exact Sato-Tate group (up to conjugation) by comparing experimentally derived moment statistics with the moment sequences computed in §3. In the CM case we are actually able to prove equidistribution using the techniques developed in [**FS12**]; this follows from Lemma 6.5. More generally, using the direct sum of a pair of motives arising from CM modular forms of weights 2 and 4, we obtain examples matching all 10 of the groups in our classification that have identity component U(1), and we are able to prove equidistribution in each of these cases (see Lemma 5.4). Additional cases arise from considering Hilbert modular forms and Hecke characters over CM fields. In total, we exhibit examples that appear to realize 25 of the 26 possible Sato-Tate groups obtained by our classification.

For the Dwork pencil construction, we are able to collect numerical evidence thanks to the work of Candelas, de la Ossa, and Rodriguez Villegas [**COR00**, **COR03**], who, motivated by the appearance of the Dwork pencil in the study of *mirror symmetry* in mathematical physics, described some p-adic analytic formulas for the L-function coefficients. The resulting evidence may be a bit surprising on first glance: one might expect (by analogy with abelian varieties) that the group USp(4) arises for most members of the pencil with a sparse but infinite set of exceptions, but in fact we found no exceptions at all other than $t = 0$ (the Fermat quintic). A Hodge-theoretic heuristic suggesting the existence of only finitely many exceptions in this family (and also applicable in many other cases) has been proposed by de Jong [**dJ02**].

For a gentle introduction to motives, we refer the reader to [**Mil13**].

2. Group-theoretic classification

In this section, we classify, up to conjugation, the groups $G \subseteq \mathrm{GL}_4(\mathbb{C})$ that satisfy the Sato-Tate axioms (ST1), (ST2), and (ST3); the list of possible groups (in notation introduced later in this section) can be found in Table 1. As in [**FKRS12**], we exhibit explicit representatives of each conjugacy class for the purposes of computing moments, which are needed for our numerical experiments (see §3). This forces us to give an explicit description of the matrix groups we are using.

Let M (resp. S) denote a matrix of $\mathrm{GL}_4(\mathbb{C})$ corresponding to a Hermitian (resp. symplectic) form, that is, a matrix satisfying $M^t = \overline{M}$ (resp. $S^t = -S$). The unitary symplectic group of degree 4 (relative to the forms M and S) is defined as

$$\mathrm{USp}(4) := \left\{ A \in \mathrm{GL}_4(\mathbb{C}) \,|\, A^t S A = S\,,\ \overline{A}^t M A = M \right\}.$$

For the purposes of the classification, it will be convenient to make different choices of S and M according to the different possibilities for the identity component G^0 of G. Unless otherwise specified, we will take M to be the identity matrix Id.

As in [**FKRS12**, Lemma 3.7], one shows that if G satisfies the Sato-Tate axioms, then G^0 is conjugate to one of

$$\mathrm{U}(1),\ \mathrm{SU}(2),\ \mathrm{U}(2),\ \mathrm{U}(1)\times\mathrm{U}(1),\ \mathrm{U}(1)\times\mathrm{SU}(2),\ \mathrm{SU}(2)\times\mathrm{SU}(2),\ \mathrm{USp}(4).$$

(The case $\mathrm{U}(2)$ does not occur in [**FKRS12**, Lemma 3.7]; see Remark 2.3 for the reason why.) We now proceed by considering each of these options in turn. Throughout the discussion, let Z and N denote the centralizer and normalizer, respectively, of G^0 in $\mathrm{USp}(4)$, so that $N/(ZG^0)$ is finite and $G \subseteq N$. (Beware that this convention is followed in [**FKRS12**, §3.4] but not in [**FKRS12**, §3.5].)

2.1. The case $G^0 = \mathrm{U}(1)$. To treat the case $G^0 = \mathrm{U}(1)$, we assume that the symplectic form preserved by $\mathrm{USp}(4)$ is given by the matrix

$$S := \begin{pmatrix} 0 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0 \end{pmatrix}.$$

In this case G^0 must be equal to a Hodge circle H, which we may take to be the image of the homomorphism

$$\theta\colon \mathrm{U}(1) \to \mathrm{USp}(4)\,, \qquad \theta(u) := \begin{pmatrix} U & 0 \\ 0 & \overline{U} \end{pmatrix}, \qquad U := \begin{pmatrix} u^3 & 0 \\ 0 & u \end{pmatrix}. \tag{2.1}$$

Note that the centralizer of G^0 within $\mathrm{GL}(4,\mathbb{C})$ consists of diagonal matrices. For such a matrix to be symplectic and unitary it must be of the form

$$\begin{pmatrix} V_2 & 0 \\ 0 & \overline{V}_2 \end{pmatrix}, \qquad V_2 := \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \tag{2.2}$$

where v_1 and v_2 are in $\mathrm{U}(1)$. We thus conclude that $Z \simeq \mathrm{U}(1)\times\mathrm{U}(1)$. The quotient N/Z injects into the continuous automorphisms $\mathrm{Aut}^{\mathrm{cont}}(G^0)$ of G^0. Since $\mathrm{Aut}^{\mathrm{cont}}(\mathrm{U}(1))$ consists just of the identity and complex conjugation, Z has index 2 in N. Thus N has the form

$$N = Z \cup JZ, \qquad J := \begin{pmatrix} 0 & J_2 \\ -J_2 & 0 \end{pmatrix}, \qquad J_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Conjugation on Z by J corresponds to complex conjugation, thus we have

$$N/G^0 \simeq \mathrm{U}(1) \rtimes \mathbb{Z}/2\mathbb{Z}\,,$$

where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{U}(1)$ by complex conjugation.

We first enumerate the options for G assuming that $G \subseteq Z$. Any finite subgroup of order n of $Z/G^0 \simeq \mathrm{U}(1)$ is cyclic. It lifts to a subgroup C_n of Z, for which we may choose the following presentation:

$$C_n := \langle G^0, \zeta_n \rangle\,, \qquad \zeta_n := \begin{pmatrix} \Theta_n & 0 \\ 0 & \overline{\Theta}_n \end{pmatrix}, \qquad \Theta_n := \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & 1 \end{pmatrix}.$$

LEMMA 2.1. *If the rationality condition (ST3) is satisfied for C_n, then n lies in* $\{1, 2, 3, 4, 6\}$.

PROOF. By the rationality condition, the average over $r \in [0, 1]$ of the fourth power of the trace of the matrix

$$\theta(e^{2\pi i r})\zeta_n$$

is an integer. It is an elementary but tedious computation to check that this average is equal to

$$36 + 8\cos\left(\frac{2\pi}{n}\right).$$

This implies $\cos\left(\frac{2\pi}{n}\right) = \frac{i}{2}$, for $i \in \{-2, -1, 0, 1, 2\}$, hence $n \in \{1, 2, 3, 4, 6\}$. □

We now consider the case $G \not\subseteq Z$. For $n \in \{1, 2, 3, 4, 6\}$, define

$$J(C_n) := \langle G^0, \zeta_n, J\rangle .$$

LEMMA 2.2. *Let G be a subgroup of N satisfying the rationality condition (ST3), and for which $\theta(\mathrm{U}(1)) \subseteq G \not\subseteq Z$. Then G is conjugate to $J(C_n)$ for some* $n \in \{1, 2, 3, 4, 6\}$.

PROOF. By hypothesis, G contains an element of JZ, which is of the form

$$JV = \begin{pmatrix} 0 & J_2V_2 \\ -J_2\overline{V}_2 & 0 \end{pmatrix}, \qquad \text{where} \qquad J_2V_2 = \begin{pmatrix} v_1 & 0 \\ 0 & -v_2 \end{pmatrix},$$

where v_1 and v_2 are in $\mathrm{U}(1)$. The conjugate of JV by the matrix

$$W := \begin{pmatrix} 0 & W_2 \\ -\overline{W}_2 & 0 \end{pmatrix}, \qquad W_2 := \begin{pmatrix} -\sqrt{v_1} & 0 \\ 0 & \sqrt{v_2} \end{pmatrix}$$

is J. Thus the conjugate of G by W is of the form $H \rtimes \langle J\rangle$, where H is a subgroup of Z satisfying the rationality condition. As we have already seen, H must be equal to C_n for some $n \in \{1, 2, 3, 4, 6\}$. □

2.2. The case $G^0 = \mathrm{SU}(2)$. To treat the case $G^0 = \mathrm{SU}(2)$, we consider the standard representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$ and take the embedding of $\mathrm{SU}(2)$ in $\mathrm{USp}(4)$ corresponding to the representation $\mathrm{Sym}^3(\mathbb{C}^2)$. More explicitly, if $a, b \in \mathbb{C}$ are such that $a\overline{a} + b\overline{b} = 1$, we consider the embedding of $\mathrm{SU}(2)$ in $\mathrm{USp}(4)$ given by

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mapsto \begin{pmatrix} a^3 & a^2b & ab^2 & b^3 \\ -3a^2\overline{b} & a^2\overline{a} - 2ab\overline{b} & 2a\overline{a}b - b^2\overline{b} & 3\overline{a}b^2 \\ 3a\overline{b}^2 & b\overline{b}^2 - 2a\overline{a}\overline{b} & a\overline{a}^2 - 2\overline{a}b\overline{b} & 3\overline{a}^2b \\ -\overline{b}^3 & \overline{a}\overline{b}^2 & -\overline{a}^2\overline{b} & \overline{a}^3 \end{pmatrix}. \tag{2.3}$$

In this section, the Hodge circle is the image of the homomorphism

$$\theta\colon \mathrm{U}(1) \to \mathrm{USp}(4), \qquad \theta(u) := \begin{pmatrix} U & 0 \\ 0 & \overline{u}^4U \end{pmatrix}, \qquad U := \begin{pmatrix} u^3 & 0 \\ 0 & u \end{pmatrix}, \tag{2.4}$$

and we assume that the symplectic and Hermitian forms preserved by $\mathrm{USp}(4)$ are respectively given by the matrices

$$S := \begin{pmatrix} 0 & 0 & 0 & z \\ 0 & 0 & -1/z & 0 \\ 0 & 1/z & 0 & 0 \\ -z & 0 & 0 & 0 \end{pmatrix}, \qquad M := \begin{pmatrix} 1/z & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1/z \end{pmatrix},$$

where $z = \sqrt{3}$. Since the embedded SU(2) contains the embedded U(1) of the previous section, the centralizer Z of G^0 in USp(4) consists of matrices of the form (2.2). Imposing the condition that conjugation by such a matrix preserves any element of the embedded SU(2), one finds that $v_1 = v_2 = \overline{v}_1 = \overline{v}_2$. Thus $Z = \{\pm\mathrm{Id}\} \subseteq G^0$. The group $N/G^0 = N/(ZG^0)$ embeds into the group of continuous outer automorphisms $\mathrm{Out}^{\mathrm{cont}}(\mathrm{SU}(2))$, which is trivial; consequently, this case yields only the single group $D := G^0$.

2.3. The case $G^0 = \mathrm{U}(2)$. To treat the case $G^0 = \mathrm{U}(2)$, we again assume that the symplectic form preserved by USp(4) is given by the matrix

$$S := \begin{pmatrix} 0 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0 \end{pmatrix}.$$

The group U(2) embeds into USp(4) via the map given in block form by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix},$$

as in [**FKRS12**, (3.1)]. As indicated in [**FKRS12**, §3], we have $Z = \{\pm\mathrm{Id}\} \subseteq G^0$ and $N = \mathrm{U}(2) \cup J(\mathrm{U}(2))$ for

$$J := \begin{pmatrix} 0 & J_2 \\ -J_2 & 0 \end{pmatrix}, \qquad J_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We thus obtain two groups: U(2) and $N(\mathrm{U}(2))$.

Remark 2.3. Note that U(2) is missing from [**FKRS12**, Theorem 3.4] even though it satisfies the Sato-Tate axioms as formulated in [**FKRS12**, Definition 3.1]. The reason is that axiom (ST2) is stated incorrectly there: it fails to include the condition that the Hodge circles generate a dense subgroup of G^0; see [**Ser12**, 8.2.3.6(i)].

Let us see this point more explicitly. Let $\theta\colon \mathrm{U}(1) \to \mathrm{U}(2)$ be a continuous homomorphism. The map $\mathrm{U}(1) \times \mathrm{SU}(2) \to \mathrm{U}(2)$ taking (u, A) to uA is an isogeny of degree 2 with kernel generated by $(-1, -\mathrm{Id}_2)$. We may thus identify $\mathrm{U}(2)/\mathrm{SU}(2)$ with $\mathrm{U}(1)/\{\pm 1\}$ and then with U(1) via the squaring map. There must then exist an integer a such that for all $u \in \mathrm{U}(1)$, the image of $\theta(u)$ in U(1) is u^a. The formula $u \mapsto u^{-a}\theta(u)^2$ defines a homomorphism $\mathrm{U}(1) \to \mathrm{SU}(2)$, so there must exist an integer b such that for all $u \in \mathrm{U}(1)$, the image of $u \in \mathrm{U}(1)$ in SU(2) has eigenvalues u^b and u^{-b}. The eigenvalues of $\theta(u^2)$ must then be u^{a+2b} and u^{a-2b}. If we then embed U(2) into USp(4), the image of $\theta(u^2)$ has eigenvalues $u^{a+2b}, u^{a-2b}, u^{-a+2b}, u^{-a-2b}$.

In this paper, we get a Hodge circle by taking θ as above with $a = 4, b = 1$. By contrast, in the setting of [**FKRS12**], the eigenvalues must be u^2, u^2, u^{-2}, u^{-2}, in some order. We may assume without loss of generality that $a + 2b = 2$; we must then have $a - 2b \in \{-2, 2\}$, implying that either $a = 0$ or $b = 0$. If $a = 0$, then the conjugates of the image of θ all lie inside SU(2), and if $b = 0$, then the conjugates all lie inside U(1). Thus no Hodge circle can exist.

2.4. The remaining cases for G^0. We now treat the remaining cases for G^0. These turn out to give exactly the same answers as in [**FKRS12**, §3.6], modulo the position of the Hodge circle, which we will ignore (see Remark 2.4); it thus suffices to recall these answers briefly. The case $G^0 = \mathrm{USp}(4)$ is trivial, so we focus on the split cases. As in [**FKRS12**, §3.6], we assume that the symplectic form preserved by $\mathrm{USp}(4)$ is defined by the block matrix

$$S := \begin{pmatrix} S_2 & 0 \\ 0 & S_2 \end{pmatrix}, \qquad S_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and that product groups are embedded compatibly with this decomposition of the symplectic form.

For $G^0 = \mathrm{SU}(2) \times \mathrm{SU}(2)$, as in [**FKRS12**, §3.6] we have the group $G_{3,3} := G^0$ itself and its normalizer $N(G_{3,3})$, obtained by adjoining to G^0 the matrix

$$\begin{pmatrix} 0 & S_2 \\ -S_2 & 0 \end{pmatrix}.$$

For $G^0 = \mathrm{U}(1) \times \mathrm{U}(1)$, the normalizer in $\mathrm{USp}(4)$ contains $\mathrm{U}(1) \times \mathrm{U}(1)$ with index 8, and the quotient is isomorphic to the dihedral group D_4 and generated by matrices

$$a := \begin{pmatrix} S_2 & 0 \\ 0 & \mathrm{Id}_2 \end{pmatrix}, \qquad b := \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & S_2 \end{pmatrix}, \qquad c := \begin{pmatrix} 0 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0 \end{pmatrix},$$

each of which defines an involution on the component group. We write F_S for the group generated by G^0 and a subset S of $\langle a, b, c \rangle$. As in [**FKRS12**, §3.6], up to conjugation we obtain eight groups

$$F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c}.$$

For $G^0 = \mathrm{U}(1) \times \mathrm{SU}(2)$, we obtain the group $G_{1,3} := \mathrm{U}(1) \times \mathrm{SU}(2)$ and its normalizer $N(G_{1,3}) = \langle G_{1,3}, a \rangle$.

REMARK 2.4. Note that in some of the cases with $G^0 = \mathrm{U}(1) \times \mathrm{U}(1)$, there is more than one way to embed the Hodge circle H into G up to conjugation. This is irrelevant for questions of equidistribution, but it does matter when one attempts to relate the Sato-Tate group of a motive with the real endomorphism algebra of its Hodge structure (as in [**FKRS12**, §4]). Since we will not attempt that step in this paper at more than a heuristic level, we have chosen to ignore this ambiguity.

3. Testing the generalized Sato-Tate conjecture

In the sections that follow, we describe various explicit constructions that give rise to self-dual 3-motives with Hodge numbers $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$ and rational coefficients. For each of these motives M, we then perform numerical tests of the generalized Sato-Tate conjecture by comparing the distribution of the normalized L-polynomials of M with the distribution of characteristic polynomials in one of the candidate Sato-Tate groups G found by the classification in §2. More precisely, we ask whether the normalized L-polynomials of M appear to be equidistributed with respect to the image of the Haar measure under the map $G \to \mathrm{Conj}(\mathrm{USp}(4))$, where Conj denotes the space of conjugacy classes. To make this determination, we compare *moment statistics* of the motive M to *moment sequences* associated to G, as described below.

Table 1 lists invariants that allow us to distinguish the groups G. As in [**FKRS12**], d denotes the real dimension of G; c is the number $|G/G^0|$ of connected components of G; and z_1 and z_2 are defined by

$$z_1 := z_{1,0}, \qquad z_2 := [z_{2,-2}, z_{2,-1}, z_{2,0}, z_{2,1}, z_{2,2}],$$

TABLE 1. Candidate Sato-Tate groups of self-dual motives of weight 3 with Hodge numbers $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$ and rational coefficients. The final column indicates where within the article to find explicit constructions that yield matching moment statistics.

d	c	G	$[G/G^0]$	z_1	z_2	Examples
1	1	C_1	C_1	0	$0,0,0,0,0$	5.5, 6.7, 6.10
1	2	C_2	C_2	0	$0,0,0,0,0$	5.6
1	3	C_3	C_3	0	$0,0,0,0,0$	5.7, 6.4
1	4	C_4	C_4	0	$0,0,0,0,0$	5.8
1	6	C_6	C_6	0	$0,0,0,0,0$	5.9
1	2	$J(C_1)$	D_1	1	$0,0,0,0,1$	5.5, 6.7, 6.10
1	4	$J(C_2)$	D_2	2	$0,0,0,0,2$	5.6
1	6	$J(C_3)$	D_3	3	$0,0,0,0,3$	5.7
1	8	$J(C_4)$	D_4	4	$0,0,0,0,4$	5.8
1	12	$J(C_6)$	D_6	6	$0,0,0,0,6$	5.9
3	1	D	C_1	0	$0,0,0,0,0$	6.7
4	1	$\mathrm{U}(2)$	C_1	0	$0,0,0,0,0$	6.1, 6.16
4	2	$N(\mathrm{U}(2))$	C_2	1	$0,0,0,0,0$	6.16
2	1	F	C_1	0	$0,0,0,0,0$	5.1, 6.2, 6.15
2	2	F_a	C_2	0	$0,0,0,0,1$	5.1
2	2	F_c	C_2	1	$0,0,0,0,0$	6.15, 6.2
2	2	F_{ab}	C_2	1	$0,0,0,0,1$	6.2
2	4	F_{ac}	C_4	3	$0,0,2,0,1$	7.2
2	4	$F_{a,b}$	D_2	1	$0,0,0,0,3$	6.15, 6.2
2	4	$F_{ab,c}$	D_2	3	$0,0,0,0,1$	6.15
2	8	$F_{a,b,c}$	D_4	5	$0,0,2,0,3$	None (but see 8.3)
4	1	$G_{1,3}$	C_1	0	$0,0,0,0,0$	5.2
4	2	$N(G_{1,3})$	C_2	0	$0,0,0,0,1$	5.2
6	1	$G_{3,3}$	C_1	0	$0,0,0,0,0$	5.3
6	2	$N(G_{3,3})$	C_2	1	$0,0,0,0,0$	8.1
10	1	$\mathrm{USp}(4)$	C_1	0	$0,0,0,0,0$	7.3

where $z_{i,j}$ is the number of connected components of G for which the ith coefficient a_i of the characteristic polynomial of each of its elements is equal to the integer j. We use $[G/G^0]$ to denote the isomorphism class of the component group of G, and the notations C_n and D_n indicate the cyclic group of n elements and the dihedral group of $2n$ elements, respectively. For each of the motives M constructed in the sections that follow, the nature of the construction allows us to predict the type of identity component and the number of components, as well as the values of the invariants z_1 and z_2, which is enough to uniquely determine a candidate Sato-Tate group G. The last column of Table 1 references the example motives M whose candidate Sato-Tate group is G. For all but one group ($F_{a,b,c}$) there is at least one such example, and in many cases there are multiple constructions that lead to the same candidate Sato-Tate group.

3.1. Experimental methodology — moment statistics. All of the motives M/K that we consider have L-polynomials of the form

$$L_{\mathfrak{p}}(T) = p^6T^4 + c_1p^3T^3 + c_2pT^2 + c_1T + 1, \tag{3.1}$$

where $\mathfrak{p}$ is a prime of K of good reduction for M, $p = N(\mathfrak{p})$ is its absolute norm, and c_1 and c_2 are integers satisfying the Weil bounds $|c_1| \le 4p^{3/2}$ and $|c_2| \le 6p^2$ (in fact $c_2 \ge -2p^2$). For the purpose of computing moment statistics we may restrict our attention to primes $\mathfrak{p}$ of degree 1, so we assume that p is prime. Note that c_1 is the *negation* of the trace of Frobenius, and c_2 is obtained by *removing a factor* of p from the coefficient of T^2 in $L_{\mathfrak{p}}(T)$.

The normalized L-polynomial coefficients of M/K are then defined by

$$a_1(\mathfrak{p}) := c_1/N(\mathfrak{p})^{3/2} \qquad \text{and} \qquad a_2(\mathfrak{p}) := c_2/N(\mathfrak{p})^2, \tag{3.2}$$

which are real numbers in the intervals $[-4, 4]$ and $[-2, 6]$, respectively.

Given a norm bound B, we let $S(B)$ denote the set of degree 1 primes of K with norm at most B, and for $i = 1, 2$ we define the nth *moment statistic* of a_i for the motive M (with respect to B) by

$$M_n[a_i] := \frac{1}{\#S(B)} \sum_{\mathfrak{p} \in S(B)} a_i(\mathfrak{p})^n.$$

Similarly, given a candidate Sato-Tate group G, we let $a_i := a_i(g)$ denote the ith coefficient of the characteristic polynomial of a random element g of G (according to the Haar measure). We then let $M_n[a_i]$ denote the expected value of a_i^n; this is the nth *moment* of a_i for the group G, which is always an integer (see axiom (ST3) in [**FKRS12**, Def. 3.1]). In what follows it will be clear from context whether $M_n[a_i]$ refers to a moment statistic of M (with respect to a norm bound B) or a moment of G.

To test for equidistribution with respect to a candidate Sato-Tate group G, for increasing values of B we compare moment statistics $M_n[a_i]$ for the motive M to the corresponding moments $M_n[a_i]$ of the group G and ask whether the former appear to converge to the latter as B increases. As may be seen in the tables of moment statistics listed in §9, in cases where it is computationally feasible to make B sufficiently large (up to 2^{40}), we see very strong evidence for convergence; the moment statistics of M generally agree with the moments of G to within one part in ten thousand.

It should be noted that the correct statement of the generalized Sato-Tate conjecture is somewhat more precise than what we are testing here. It includes both a defined group G attached to the motive (the *Sato-Tate group*) and a sequence of elements of $\mathrm{Conj}(G)$ that should be equidistributed for the image of the Haar measure, even before projecting to $\mathrm{Conj}(\mathrm{USp}(4))$. The formulation in [**FKRS12**, §2] is only valid for motives of weight 1; for a reformulation in terms of absolute Hodge cycles that applies to motives of any odd weight, see [**BK15a**, **BK15b**].

Since we do not introduce the definition of the Sato-Tate group here, we do not attempt to verify in our examples that the candidate Sato-Tate group we identify actually coincides with the Sato-Tate group of the motive. It is unclear how difficult this is to achieve, especially for the motives appearing in the Dwork pencil. Moreover, we do not claim that our list of constructions is exhaustive. It may (or may not) be that the group $N(G_{3,3})$, which we are unable to match with an explicit construction, can be realized by other methods (compare Remark 8.4).

3.2. Moment sequences of candidate Sato-Tate groups. In this section we compute moment sequences associated to each of the subgroups G of $\mathrm{USp}(4)$ encountered in §2; these are listed in Tables 2 and 3. Let G be a compact subgroup of $\mathrm{USp}(4)$. For $i = 1, 2$, let $a_i := a_i(g)$ denote the ith coefficient of the characteristic polynomial of a random element g of G (according to the Haar measure). For a nonnegative integer n, the nth moment $M_n[a_i]$ is the expected value of a_i^n.

We note that 13 of the 26 groups encountered in §2 already appeared in the classification of [**FKRS12**], and we do not need to compute their moments again. We proceed to the computation of the moment sequences for the restriction of a_i to every connected component of each of the remaining groups. Let t (resp. s) denote the trace of a random element in $\mathrm{U}(1)$ (resp. $\mathrm{SU}(2)$). Recall that

$$M_{2n}[t] = \binom{2n}{n}, \qquad M_{2n}[s] = \frac{1}{n+1}\binom{2n}{n}, \tag{3.3}$$

whereas the odd moments are all zero in both cases.

The group D. In this case we have a single connected component, whose moments can be computed by noting that

$$M_n[a_1(g) \,|\, g \in D] = \mathrm{E}[(-s^3 + 2s)^n],$$
$$M_n[a_2(g) \,|\, g \in D] = \mathrm{E}[(s^4 - 3s^2 + 2)^n],$$

and then applying the second equality in (3.3).

The groups $\mathrm{U}(2)$ and $N(\mathrm{U}(2))$. We can use the isomorphism $\mathrm{U}(2) \simeq \mathrm{U}(1) \times \mathrm{SU}(2)/\langle -1 \rangle$ to deduce that

$$M_n[a_1(g) \,|\, g \in \mathrm{U}(2)] = \mathrm{E}[(-t \cdot s)^n],$$
$$M_n[a_2(g) \,|\, g \in \mathrm{U}(2)] = \mathrm{E}[(s^2 + t^2 - 2)^n],$$

and, if J is as in §2.3, that

$$M_n[a_1(g) \,|\, g \in J\mathrm{U}(2)] = 0,$$
$$M_n[a_2(g) \,|\, g \in J\mathrm{U}(2)] = \mathrm{E}[(-s^2 + 2)^n].$$

The groups C_n and $J(C_n)$. We have $a_1(g) = 0$ and $a_2(g) = 2$ for any element g in the connected component of $\zeta_m^k J$ (where ζ_m and J are as in §2.1). Let $C(\zeta_m^k)$ denote the connected component of the matrix ζ_m^k. Then

$$M_n[a_1(g) \,|\, g \in C(\zeta_m^k)] = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \left(\cos\left(3r + \frac{2\pi k}{m}\right) + \cos(r) \right)^n dr,$$
$$M_n[a_2(g) \,|\, g \in C(\zeta_m^k)] = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \left(1 + \cos\left(4r + \frac{2\pi k}{m}\right) + \cos\left(2r + \frac{2\pi k}{m}\right) \right)^n dr.$$

TABLE 2. Moments $M_n = M_n[a_1]$ for the groups listed in Table 1.

G	M_2	M_4	M_6	M_8	M_{10}	M_{12}	M_{14}	M_{16}
C_1	4	44	580	8092	116304	1703636	25288120	379061020
C_2	4	36	400	4956	65904	919116	13236080	194789660
C_3	4	36	400	4900	63504	854216	11806652	166685220
C_4	4	36	400	4900	63504	853776	11778624	165640540
C_6	4	36	400	4900	63504	853776	11778624	165636900
$J(C_1)$	2	22	290	4046	58152	851818	12644060	189530510
$J(C_2)$	2	18	200	2478	32952	459558	6618040	97394830
$J(C_3)$	2	18	200	2450	31752	427108	5903326	83342610
$J(C_4)$	2	18	200	2450	31752	426888	5889312	82820270
$J(C_6)$	2	18	200	2450	31752	426888	5889312	82818450
D	1	4	34	364	4269	52844	679172	8976188
$\mathrm{U}(2)$	2	12	100	980	10584	121968	1472328	18404100
$N(\mathrm{U}(2))$	1	6	50	490	5292	60984	736164	9202050
F	4	36	400	4900	63504	853776	11778624	165636900
F_a	3	21	210	2485	31878	427350	5891028	82824885
F_c	2	18	200	2450	31752	426888	5889312	82818450
F_{ab}	2	18	200	2450	31752	426888	5889312	82818450
F_{ac}	1	9	100	1225	15876	213444	2944656	41409225
$F_{a,b}$	2	12	110	1260	16002	213906	2946372	41415660
$F_{ab,c}$	1	9	100	1225	15876	213444	2944656	41409225
$F_{a,b,c}$	1	6	55	630	8001	106953	1473186	20707830
$G_{1,3}$	3	20	175	1764	19404	226512	2760615	34763300
$N(G_{1,3})$	2	11	90	889	9723	113322	1380522	17382365
$G_{3,3}$	2	10	70	588	5544	56628	613470	6952660
$N(G_{3,3})$	1	5	35	294	2772	28314	306735	3476330
$\mathrm{USp}(4)$	1	3	14	84	594	4719	40898	379236

TABLE 3. Moments of $M_n = M_n[a_2]$ for the groups listed in Table 1.

G	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9
C_1	2	8	38	196	1052	5774	32146	180772	1024256
C_2	2	8	32	148	712	3614	18916	101700	557384
C_3	2	8	32	148	712	3584	18496	97444	521264
C_4	2	8	32	148	712	3584	18496	97444	521096
C_6	2	8	32	148	712	3584	18496	97444	521096
$J(C_1)$	2	6	23	106	542	2919	16137	90514	512384
$J(C_2)$	2	6	20	82	372	1839	9522	50978	278948
$J(C_3)$	2	6	20	82	372	1824	9312	48850	260888
$J(C_4)$	2	6	20	82	372	1824	9312	48850	260804
$J(C_6)$	2	6	20	82	372	1824	9312	48850	260804
D	1	2	5	16	62	272	1283	6316	31952
$\mathrm{U}(2)$	1	4	11	44	172	752	3383	15892	76532
$N(\mathrm{U}(2))$	1	3	7	25	91	386	1709	7981	38329
F	2	8	32	148	712	3584	18496	97444	521096
F_a	2	6	20	82	372	1824	9312	48850	260804
F_c	1	5	16	77	356	1802	9248	48757	260548
F_{ab}	2	6	20	82	372	1824	9312	48850	260804
F_{ac}	1	3	10	41	186	912	4656	24425	130402
$F_{a,b}$	2	5	14	49	202	944	4720	24553	130658
$F_{ab,c}$	1	4	10	44	186	922	4656	24460	130402
$F_{a,b,c}$	1	3	7	26	101	477	2360	12294	65329
$G_{1,3}$	2	6	20	76	312	1364	6232	29460	142952
$N(G_{1,3})$	2	5	14	46	172	714	3180	14858	71732
$G_{3,3}$	2	5	14	44	152	569	2270	9524	41576
$N(G_{3,3})$	1	3	7	23	76	287	1135	4769	20788
$\mathrm{USp}(4)$	1	2	4	10	27	82	268	940	3476

4. Modular forms and Hecke characters

Modular forms and Hecke characters play a key role in many of our motive constructions. Before giving explicit examples, we first recall some theoretical facts concerning modular forms with complex multiplication (CM), following the exposition given in [**Sch06**, Chap. II]. These facts allow us to actually prove equidistribution in several cases (see Lemma 5.4), and they facilitate our numerical computations (via Lemma 4.2).

Notation: To avoid potential confusion with the normalized L-polynomial coefficients a_1 and a_2 (and the integer L-polynomial coefficients c_1 and c_2), we generally use b_n (or d_n or e_n) to denote the Fourier coefficients of a modular form $f = f(z) = \sum b_n q^n$, where $q = \exp(2\pi i z)$. Unless otherwise indicated, the symbols ω and i denote, respectively, the third and fourth roots of unity in the upper half plane.

When possible, we identify specific modular forms by their labels in the LMFDB database of L-functions, modular forms, and related objects [**LMFDB**]. These identifiers are formatted as $N.k.cs$, where N is the level, k is the weight, c is an index indicating the character, and s is an alphabetic string that distinguishes the form from others of the same weight, level, and character. The trivial character is always indexed by the label $c = 1$.

4.1. Newforms with complex multiplication. Let $S_k(\Gamma_1(N))$ denote the complex space of weight k cusp forms for $\Gamma_1(N)$. There is a decomposition

$$S_k(\Gamma_1(N)) = \bigoplus_{\varepsilon} S_k(\Gamma_0(N), \varepsilon),$$

where $\varepsilon\colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ runs over the characters of $(\mathbb{Z}/N\mathbb{Z})^*$ and $S_k(\Gamma_0(N), \varepsilon)$ denotes the space of weight k cusp forms for $\Gamma_0(N)$ with nebentypus ε. We denote by $S_k^{\mathrm{new}}(\Gamma_1(N))$ the complex subspace generated by the newforms. We say that $f = \sum_{n\geq 1} b_n q^n \in S_k(\Gamma_1(N))$ is a *newform* if it is an eigenform for all the Hecke operators, it is new at level N, and it is normalized so that $b_1 = 1$.

The newform $f \in S_k(\Gamma_1(N))$ is said to have complex multiplication (CM) by a (quadratic) Dirichlet character χ if $b_p = \chi(p) b_p$ for a set of primes of density 1.

Let K be a quadratic imaginary field, $\mathfrak{M}$ an ideal of K, and $l \in \mathbb{N}$. Let $I_{\mathfrak{M}}$ stand for the group of fractional ideals of K coprime to $\mathfrak{M}$. A Hecke character of K of modulus $\mathfrak{M}$ and infinite type $(l, 0)$, or simply l, is a homomorphism

$$\psi\colon I_{\mathfrak{M}} \to \mathbb{C}^*$$

such that $\psi(\alpha\mathcal{O}_K) = \alpha^l$ for all $\alpha \in K^*$ with[1] $\alpha \equiv 1 \pmod{\mathfrak{M}}$. We extend ψ by defining it to be 0 for all fractional ideals of K that are not coprime to $\mathfrak{M}$. We say that $\mathfrak{M}$ is the conductor of ψ if the following holds: if ψ is defined modulo $\mathfrak{M}'$, then $\mathfrak{M} | \mathfrak{M}'$. The L-function of ψ is then defined by

$$L(\psi, s) := \prod_{\mathfrak{p}} (1 - \psi(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1},$$

[1]To simplify notation, we will simply write $\equiv$, but the reader should be aware that in this context we are alluding to multiplicative congruence by this sign.

where the product runs over all prime ideals of K. Let Δ_K denote the absolute value of the discriminant of K and let χ_K denote the Dirichlet character associated to K. By results of Hecke and Shimura, the inverse Mellin transform

$$f_\psi := \sum_{\mathfrak{a}\subseteq\mathcal{O}_K} \psi(\mathfrak{a})q^{N(\mathfrak{a})} =: \sum_{n\geq 1} b_n q^n$$

of $L(\psi, s)$ is an eigenform of weight $l+1$, level $\Delta_K N(\mathfrak{M})$, and nebentypus $\chi_K\eta$, where

$$\eta(n) = \frac{\psi(n\mathcal{O}_K)}{n^l} \quad \text{if } (n, N(\mathfrak{M})) = 1\,,$$

and $\eta(n) = 0$, otherwise. Moreover, f_ψ is new at this level if and only if $\mathfrak{M}$ is the conductor of ψ and, by construction, we have $\overline{b_n} = \chi_K(n)b_n$. Thus the modular form f_ψ has CM by χ_K (we also say that f_ψ has CM by K). It follows from results of Ribet that every CM newform in $S_k(\Gamma_1(N))$ arises in this way; see Proposition 4.4 and Theorem 4.5 in [**Rib77**].

In this article we only consider newforms with rational coefficients. The following result describes the nebentypus in this case.

PROPOSITION 4.1 ([**Sch06**], Cor. II.1.2). *Let $f \in S_k(\Gamma_1(N))$ be a newform with real coefficients.*

i) If k is even then the nebentypus ε is trivial.
ii) If k is odd then the nebentypus ε is quadratic and f has CM by ε.

To ease notation, when the nebentypus is trivial, we simply write $S_k(N)$ in place of $S_k(\Gamma_0(N), \varepsilon_{\mathrm{triv}})$ and we use $S_k^{\mathrm{new}}(N)$ to denote the subspace of $S_k(N)$ generated by newforms.

We now describe two constructions that play a key role in what follows. These involve certain weight 4 newforms with CM by $K = \mathbb{Q}(i)$ or $K = \mathbb{Q}(\omega)$, and twists of these forms by a quartic or sextic character (respectively). We first recall two definitions.

Let $K = \mathbb{Q}(i)$. The biquadratic residue symbol of $\alpha \in \mathcal{O}_K = \mathbb{Z}[i]$ is the homomorphism

$$\left(\frac{\alpha}{\cdot}\right)_4 : I_{((1+i)\alpha)} \to \mathcal{O}_K^* = \langle i\rangle$$

uniquely characterized by the property that

$$\alpha^{(N(\mathfrak{p})-1)/4} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)_4 \pmod{\mathfrak{p}}\,.$$

Using biquadratic reciprocity, one can show that this is a Hecke character of infinite type 0. We define $\left(\frac{\alpha}{\cdot}\right)_4$ to be zero at fractional ideals of K that are not coprime to $(i+1)\alpha$.

Now let $K = \mathbb{Q}(\omega)$. The sextic residue symbol of $\alpha \in \mathcal{O}_K = \mathbb{Z}\oplus\omega\mathbb{Z}$ is the homomorphism

$$\left(\frac{\alpha}{\cdot}\right)_6 : I_{(2\sqrt{-3}\alpha)} \to \mathcal{O}_K^* = \langle \omega\rangle$$

uniquely characterized by the property that

$$\alpha^{(N(\mathfrak{p})-1)/6} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)_6 \pmod{\mathfrak{p}}\,.$$

Using cubic reciprocity, one can show that it is also a Hecke character of infinite type 0. We define $\left(\frac{\alpha}{\cdot}\right)_6$ to be zero at fractional ideals of K that are not coprime to $2\sqrt{-3}\alpha$.

4.2. CM newforms of weights 3 and 4 with a quartic twist. Let $K = \mathbb{Q}(i)$. For any prime ideal $\mathfrak{p}$ of K there exists $\alpha_{\mathfrak{p}} \in \mathcal{O}_K$ such that $\mathfrak{p} = (\alpha_{\mathfrak{p}})$, and if $\mathfrak{p}$ is coprime to $1+i$, then by multiplying $\alpha_{\mathfrak{p}}$ by an element of $\mathcal{O}_K^* = \langle i \rangle$, we may assume that $\alpha_{\mathfrak{p}} \equiv 1 \bmod (1+i)^3$. Moreover, this uniquely determines $\alpha_{\mathfrak{p}}$ (see [**IR82**, Chap. 9, Lemma 7]). Now define

$$\psi(\mathfrak{p}) := \alpha_{\mathfrak{p}}\,.$$

This is a Hecke character of infinite type 1 and conductor $\mathfrak{M} = (1+i)^3$. By [**IR82**, Chap. 18, §4], this is the Hecke character attached to the elliptic curve $y^2 = x^3 - x$. The newform $f_\psi \in S_2^{\mathrm{new}}(32)$ has rational coefficients and LMFDB identifier 32.2.1a.

The Hecke character ψ^3 has infinite type 3 and conductor $\mathfrak{M} = (1+i)^3$. Thus f_{ψ^3} is a newform in $S_4^{\mathrm{new}}(32)$, and its identifier is 32.4.1b.

Let $\phi := \left(\frac{3}{\cdot}\right)_4$. The Hecke character $\psi^3 \otimes \phi$ has infinite type 3, but we do not necessarily know its conductor *a priori*. However, we may use the above recipe to compute ψ and the first several Fourier coefficients of $f_{\psi^3\otimes\phi} = \sum_{n\geq 1} b_n q^n$; for primes $p \equiv 1 \bmod 4$ with $3 < p \leq 97$, we obtain

p:	5	13	17	29	37	41	53	61	73	89	97
b_p:	4	-18	-104	284	-214	-472	572	-830	-1098	176	-594

Let $\chi\colon (\mathbb{Z}/24\mathbb{Z})^* \to \mathbb{C}^*$ denote the quadratic Dirichlet character defined by

$$\chi(n) := \begin{cases} 1 & \text{if } n \equiv 1,\,7,\,17,\,23 \bmod 24; \\ -1 & \text{if } n \equiv 5,\,11,\,13,\,19 \bmod 24. \end{cases}$$

One may verify that that the Fourier coefficients of $f_{\psi^3\otimes\phi} \otimes \chi$ coincide with those of a new form of weight 4 and level 288. Moreover, we have

$$f_{\psi^3\otimes\left(\frac{3}{\cdot}\right)_4} \otimes \chi = f_{\psi^3\otimes\left(\frac{-3}{\cdot}\right)_4},$$

thus it is a quartic twist of 32.4.1b.[2]

The Hecke character ψ^2 has infinite type 2 and conductor $(1+i)^2$. Indeed, observe that for $\alpha \in K^*$ we have

$$\psi^2(\alpha\mathcal{O}_K) = \psi(\alpha^2\mathcal{O}_K)$$

and $\alpha^2 \equiv 1 \bmod (1+i)^3$ if $\alpha \equiv 1 \bmod (1+i)^2$. Thus f_{ψ^2} is a newform in $S_3^{\mathrm{new}}(\Gamma_1(16))$. Let $\phi := \left(\frac{27}{\cdot}\right)_4$. Proceeding as in the previous case, one may show that $f_{\psi^2\otimes\phi} = \sum_{n\geq 1} b_n q^n$ is new at level 576 and that its first Fourier coefficients, for primes $p \equiv 1 \bmod 4$ with $3 < p \leq 97$, are

p:	5	13	17	29	37	41	53	61	73	89	97
b_p:	-8	-10	16	40	-70	-80	-56	-22	110	160	-130

[2] If $f \in S_k^{\mathrm{new}}(N)$ is an eigenform and $\chi : (\mathbb{Z}/M\mathbb{Z})^* \to \mathbb{C}^*$ is a Dirichlet character, then $f\otimes\chi$ is a (not necessarily new) eigenform of $S_k(\mathrm{lcm}(N, M^2))$. The minimal level of $f_{\psi^3\otimes\phi}$ should thus be a divisor of 576. Data for this level is not yet available in [**LMFDB**], but one may use [**Magma**] or [**Sage**] to identify $f_{\psi^3\otimes\phi} = f \otimes \chi$ as a newform at level 576.

4.3. A weight 4 CM newform with cubic and sextic twists. Let $K = \mathbb{Q}(\omega)$. Since K has class number 1, for any prime ideal $\mathfrak{p}$ of K there exists $\alpha_\mathfrak{p} \in \mathcal{O}_K$ such that $\mathfrak{p} = (\alpha_\mathfrak{p})$. For $\mathfrak{p}$ coprime to $2\sqrt{-3}$, by multiplying $\alpha_\mathfrak{p}$ by an element of $\mathcal{O}_K^* = \langle\omega\rangle$, we may assume that $\alpha_\mathfrak{p} \equiv 1 \bmod 3$, and this uniquely determines $\alpha_\mathfrak{p}$ (see [**IR82**, Prop. 9.3.5]). We now define

$$\psi(\mathfrak{p}) := \alpha_\mathfrak{p}\,.$$

This is the Hecke character of infinite type 1 and conductor $\mathfrak{M} = (3)$ attached to the elliptic curve $y^2 + y = x^3$. The newform $f_\psi \in S_2^{\mathrm{new}}(27)$ has rational coefficients and identifier 27.2.1a.

The Hecke character ψ^3 has infinite type 3 and conductor $\mathfrak{M} = \left(\sqrt{-3}\right)$. Indeed, observe that for $\alpha \in K^*$ we have

$$\psi^3(\alpha\mathcal{O}_K) = \psi(\alpha^3\mathcal{O}_K)$$

and $\alpha^3 \equiv 1 \bmod 3$ if $\alpha \equiv 1 \bmod \sqrt{-3}$. Thus f_{ψ^3} is a newform in $S_4^{\mathrm{new}}(9)$, and its identifier is 9.4.1a.

Let $\phi := \left(\frac{2}{\cdot}\right)_6$. The Hecke character $\psi^3 \otimes \phi$ has infinite type 3. As before we compute ψ and the first several Fourier coefficients of $f_{\psi^3\otimes\phi} = \sum_{n\geq 1} b_n q^n$; for primes $p \equiv 1 \bmod 6$ with $3 < p \leq 97$, we obtain

$p:$	7	13	19	31	37	43	61	67	73	79	97
$b_p:$	17	−89	−107	308	433	520	901	−1007	−271	503	1853

Let $\chi\colon (\mathbb{Z}/24\mathbb{Z})^* \to \mathbb{C}^*$ denote the quadratic Dirichlet character defined by

$$\chi(n) := \begin{cases} 1 & \text{if } n \equiv 1,\ 5,\ 7,\ 11 \bmod 24; \\ -1 & \text{if } n \equiv 13,\ 17,\ 19,\ 23 \bmod 24. \end{cases}$$

One may verify that the Fourier coefficients of $f_{\Psi\otimes\phi} \otimes \chi$ coincide with those of a newform of weight 4 and level 108. Moreover, we have

$$f_{\psi^3\otimes\left(\frac{2}{\cdot}\right)_6} \otimes \chi = f_{\psi^3\otimes\left(\frac{2}{\cdot}\right)_6\otimes\left(\frac{2}{N(\cdot)}\right)} = f_{\psi^3\otimes\left(\frac{4}{\cdot}\right)_3}\,.$$

Thus $f_{\Psi\otimes\phi} \otimes \chi$ (resp. $f_{\psi^3\otimes\phi}$) is a cubic (resp. sextic) twist of 9.4.1a.[3]

In §6.3 we also consider the newform $f_{\psi^2} \in S_3^{\mathrm{new}}(\Gamma_1(27))$.

4.4. Computing Fourier coefficients of newforms. One of the key advantages of working with CM newforms f_{ψ^2} or f_{ψ^3} is that we can derive their Fourier coefficients from the corresponding coefficients of the weight 2 CM newform f_ψ, which we can compute very quickly.

LEMMA 4.2. *Let ψ be a Hecke character of an imaginary quadratic field K and suppose that f_ψ has trivial nebentypus. Suppose that we have Fourier q-expansions $f_\psi = \sum b_n q^n$, $f_{\psi^2} = \sum d_n q^n$, and $f_{\psi^3} = \sum e_n q^n$. Then*

$$d_p = b_p^2 - 2p \qquad \text{and} \qquad e_p = b_p^3 - 3pb_p \tag{4.1}$$

for primes p that split in K. For primes p inert in K, we have $d_p = e_p = 0$.

[3] Although we will not need it in what follows, we might ask about the minimal level of $f_{\psi^3\otimes\phi}$. It must be a divisor of $\mathrm{lcm}(108, 24^2) = 1728$. This is again out of the range of [**LMFDB**], but one may use [**Magma**] or [**Sage**] to determine that $f_{\psi^3\otimes\phi}$ is new at level 1728.

PROOF. If $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, then $b_p = \psi(\mathfrak{p}) + \psi(\overline{\mathfrak{p}})$, and the pth Fourier coefficients of f_{ψ^2} and f_{ψ^3} are given by

$$d_p = \psi(\mathfrak{p})^2 + \psi(\overline{\mathfrak{p}})^2 = (\psi(\mathfrak{p}) + \psi(\overline{\mathfrak{p}}))^2 - 2\psi(\mathfrak{p})\psi(\overline{\mathfrak{p}}) = b_p^2 - 2p\,,$$
$$e_p = \psi(\mathfrak{p})^3 + \psi(\overline{\mathfrak{p}})^3 = (\psi(\mathfrak{p}) + \psi(\overline{\mathfrak{p}}))^3 - 3\psi(\mathfrak{p})\psi(\overline{\mathfrak{p}})(\psi(\mathfrak{p}) + \psi(\mathfrak{p})) = b_p^3 - 3pb_p\,.$$

If p is inert in K, then $d_p = e_p = 0$, because f_{ψ^2} and f_{ψ^3} have CM by K. $\square$

We note that, in particular, the Fourier coefficients b_p of 27.2.1a (resp. 32.2.1a) and the Fourier coefficients d_p of 9.4.1a (resp. 32.4.1b) satisfy (4.1).

Efficiently computing the Fourier coefficients of a general[4] weight 4 newform is more difficult. Here we use the modular symbols approach implemented in [**Magma**] and [**Sage**], with a running time of $\tilde{O}(N^2)$. To improve the constant factors in the running time, we use some specialized C code to handle the most computationally intensive steps, a strategy suggested to us by William Stein. This optimization speeds up the computation by more than a factor of 100, making it easy to handle norm bounds as large as $B = 2^{24}$.

5. Direct sum constructions

Following a suggestion of Serre, in this section we construct $M = M_1 \oplus M_2$ as the direct sum of the Tate twist M_1 of the motive associated to a weight 2 newform f_1 (with Hodge numbers $h^{2,1} = h^{1,2} = 1$) and the motive M_2 associated to a weight 4 newform f_2 (with Hodge numbers $h^{3,0} = h^{0,3} = 1$). We require both f_1 and f_2 to have rational Fourier coefficients.

The motive M is defined over $\mathbb{Q}$, but we may also consider its base change to a number field K. Let $f_1 = \sum b_n q^n$ and $f_2 = \sum d_n q^n$ be the q-expansions of f_1 and f_2. Since f_1 and f_2 both have trivial nebentypus (by Proposition 4.1), the coefficients of the L-polynomial $L_{\mathfrak{p}}(T)$ of the motive M at a prime $\mathfrak{p}$ of K are given by

$$c_1 = -(pb_p + d_p) \qquad \text{and} \qquad c_2 = b_p d_p + 2p^2, \tag{5.1}$$

where $p = N(\mathfrak{p})$ and the integer coefficients c_1 and c_2 are as defined in (3.1). The normalized coefficients are then $a_1(\mathfrak{p}) = c_1/p^{3/2}$ and $a_2(\mathfrak{p}) = c_2/p^2$.

5.1. Direct sums of uncorrelated newforms. We first consider the case where f_1 and f_2 have no special relationship; the case where f_1 and f_2 are related (for example, by having the same CM field) is addressed in the next section. When f_1 and f_2 are unrelated, we expect the identity component of the Sato-Tate group of M to be one of the three product groups $\mathrm{U}(1) \times \mathrm{U}(1)$, $\mathrm{U}(1) \times \mathrm{SU}(2)$, or $\mathrm{SU}(2) \times \mathrm{SU}(2)$, depending on whether both, one, or neither of the newforms has complex multiplication.

In the case where exactly one of the forms has complex multiplication, we expect to see the same distribution regardless of which form has CM, and this is confirmed by our numerical experiments. Thus to facilitate the computations, in both of the first two cases we take f_2 to be a CM newform to which Lemma 4.2 applies, allowing us to use norm bounds $B = 2^n$ ranging from 2^{12} to 2^{40}. In the third case we cannot choose f_2 to have CM, which makes the computations more

[4]Mark Watkins points out that a few examples can be generated using η products, whose Fourier coefficients can be computed efficiently using the power series expansion of η. For example, the form 5.4a used in Example 5.3 can be realized as $\eta(q)^4\eta(q^5)^4$.

difficult; here we only let B range from 2^{12} to 2^{24}. Fortunately there are only two possible Sato-Tate groups in this case and their moments are easily distinguished.

EXAMPLE 5.1 ($\boldsymbol{F, F_a, F_{a,b}}$). Let f_1 be the weight 2 newform 32.2.1a, corresponding to (the isogeny class of) the elliptic curve $y^2 = x^3 - x$, which has CM by $\mathbb{Q}(i)$, and let f_2 by the weight 4 newform 9.4.1a, which has CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \oplus M_2$ over the fields $K = \mathbb{Q}(i,\omega), \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for the groups $G = F, F_a, F_{a,b}$. With $K = \mathbb{Q}(i)$ one obtains essentially the same moment statistics as with $K = \mathbb{Q}(\omega)$; this is as expected, since the groups F_a and F_b are conjugate.

EXAMPLE 5.2 ($\boldsymbol{G_{1,3}, N(G_{1,3})}$). Let f_1 be the weight 2 newform 11.2.1a, corresponding to the elliptic curve $y^2 + y = x^3 - x^2 - 10x - 20$, which does not have CM, and let f_2 by the weight 4 newform 9.4.1a, which has CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = G_{1,3}, N(G_{1,3})$.

EXAMPLE 5.3 ($\boldsymbol{G_{3,3}}$). Let f_1 be the weight 2 newform 11.2.1a, and let f_2 be the weight 4 newform 5.4.1a, neither of which has complex multiplication. Moment statistics for the motive $M = M_1 \oplus M_2$ over $K = \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = G_{3,3}$.

5.2. Direct sums of correlated newforms. We now consider motives $M = M_1 \oplus M_2$ where M_1 and M_2 are associated to newforms f_1 and f_2 that bear a special relationship. More specifically, we shall take f_1 to be a weight 2 newform f_ψ with CM by K, where ψ is a Hecke character of K (of infinite type 1), and then take f_2 to be a weight 4 newform $f_{\psi^3\otimes\phi}$, where ϕ is a finite order Hecke character (of infinite type 0). Using variations of the two constructions given in §4.2 and §4.3 we are able to construct motives whose L-polynomial distributions match all ten of the candidate Sato-Tate groups $G = C_n, J(C_n)$ with identity component U(1), where $n = 1, 2, 3, 4, 6$. Moreover, via arguments analogous to those used in [**FS12**], we are able to prove equidistribution in each of these cases (alternatively, as we are concerned with a CM construction, equidistribution could be directly deduced from the work of Johansson [**Joh14**]).

LEMMA 5.4. *Let ψ be a Hecke character of K of infinite type 1 such that f_ψ has rational coefficients. Let M_1 be the Tate twist of the motive associated to the weight 2 newform f_ψ. Let M_2 be the motive associated to the weight 4 newform $f_{\psi^3\otimes\phi}$, where ϕ is a finite order Hecke character (of infinite type 0) such that $f_{\psi^3\otimes\phi}$ has rational coefficients. The distribution of the normalized Frobenius eigenvalues of $M_1\oplus M_2$ (resp. the extension of scalars $(M_1\oplus M_2)_K$) coincides with the distribution of the eigenvalues of a random element in the group $J(C_{\mathrm{ord}(\phi)})$ (resp. $C_{\mathrm{ord}(\phi)}$).*

PROOF. Since f_ψ has rational coefficients, its nebentypus is trivial. Thus, if p is inert in K, then the normalized Frobenius eigenvalues of $M_1 \oplus M_2$ are $i, -i, i, -i$. It is straightforward to check that, for any $n \in \mathbb{N}$, these are precisely the eigenvalues of the matrix

$$\begin{pmatrix} \Theta_n & 0 \\ 0 & \overline{\Theta}_n \end{pmatrix} J\,,$$

where Θ_n and J are as in §2.1. If $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, then the Frobenius eigenvalues of $M_1 \oplus M_2$ are

$$N(\mathfrak{p})\psi(\mathfrak{p}),\ N(\mathfrak{p})\overline{\psi(\mathfrak{p})},\ \psi(\mathfrak{p})^3\phi(\mathfrak{p}),\ \overline{\psi(\mathfrak{p})}^3\overline{\phi(\mathfrak{p})}\,.$$

Setting $x_\mathfrak{p} := \frac{\psi(\mathfrak{p})}{N(\mathfrak{p})^{1/2}}$, we find that the normalized Frobenius eigenvalues are

$$x_\mathfrak{p},\ \overline{x}_\mathfrak{p},\ (x_\mathfrak{p})^3\,\phi(\mathfrak{p}),\ (\overline{x}_\mathfrak{p})^3\,\overline{\phi(\mathfrak{p})}\,.$$

Now let $\mathfrak{M}$ be the conductor of ϕ, let $K_\mathfrak{M}$ be the ray class field of K of modulus $\mathfrak{M}$, and let $(\cdot, K_\mathfrak{M}/K)\colon I_\mathfrak{M} \to \mathrm{Gal}(K_\mathfrak{M}/K)$ denote the Artin map. Since $(\cdot, K_\mathfrak{M}/K)$ is surjective, for any $\mathfrak{a} \in I_\mathfrak{M}$ the equality

$$\tilde{\phi}((\mathfrak{a}, K_\mathfrak{M}/K)) = \phi(\mathfrak{a})$$

uniquely determines a character $\tilde{\phi}\colon \mathrm{Gal}(K_\mathfrak{M}/K) \to \mathbb{C}^*$. Since the kernel of ϕ coincides with the kernel of $(\cdot, K_\mathfrak{M}/K)$ (consisting of those $\alpha\mathcal{O}_K$ with $\alpha \in K^*$ for which $\alpha \equiv 1 \,(\mathrm{mod}\,\mathfrak{M})$), the map $\tilde{\phi}$ is well defined. We thus have a commutative diagram

$$\begin{tikzcd} I_\mathfrak{M} \arrow[rr, "\phi"] \arrow[dr, two heads, "{(\cdot, K_\mathfrak{M}/K)}"'] & & \mathbb{C}^* \\ & \mathrm{Gal}(K_\mathfrak{M}/K)\,, \arrow[ur, "\tilde{\phi}"] & \end{tikzcd}$$

with $\tilde{\phi}$ satisfying $\tilde{\phi}(\mathrm{Frob}_\mathfrak{p}) = \phi(\mathfrak{p})$ for every prime $\mathfrak{p}$ coprime to $\mathfrak{M}$. The lemma then follows from Proposition 3.6 of [**FS12**], which asserts that the $x_\mathfrak{p}$'s are equidistributed on U(1), even when $\mathfrak{p}$ is subject to the condition that $\mathrm{Frob}_\mathfrak{p} = c$ for some fixed conjugacy class c of $\mathrm{Gal}(K_\mathfrak{M}/K)$. □

Example 5.5 ($\boldsymbol{C_1, J(C_1)}$). Let $f_1 = f_\psi$ be the weight 2 newform 27.2.1a, corresponding to the elliptic curve $y^2 + y = x^3$, and let $f_2 = f_{\psi^3}$ be the weight 4 newform 9.4.1a; both f_1 and f_2 have CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = C_1, J(C_1)$.

Example 5.6 ($\boldsymbol{C_2, J(C_2)}$). Let $f_1 = f_\psi$ be the weight 2 newform 27.2.1a, and let $f_2 = f_{\psi^3} \otimes \chi_4$ be the weight 4 newform, which is the quadratic twist of 9.4.1a by the nontrivial Dirichlet character χ_4 of modulus 4; both f_1 and f_2 have CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = C_2, J(C_2)$.

Example 5.7 ($\boldsymbol{C_3, J(C_3)}$). Let $f_1 = f_\psi$ be the weight 2 newform 27.2.1a, and let $f_2 = f_{\psi^3 \otimes \left(\frac{2}{\cdot}\right)_6} \otimes \chi$ be the weight 4 newform, which is a cubic twist of 9.4.1a, as shown in §4.3 where χ is defined; both f_1 and f_2 have CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = C_3, J(C_3)$.

Example 5.8 ($\boldsymbol{C_4, J(C_4)}$). In this case we may apply a quartic twist to either f_ψ or f_{ψ^3}, and it is computationally more convenient to do the former. So let f_1 be the weight 2 newform, corresponding to the elliptic curve $y^2 = x^3 - 2x$, which is a quartic twist of the form $f_\psi = 32.2.1\mathrm{a}$. Let $f_2 = f_{\psi^3}$ be the weight 4 newform 32.4.1b; both f_1 and f_2 have CM by $\mathbb{Q}(i)$. Moment statistics for the motive $M = M_1 \oplus M_2$ over $K = \mathbb{Q}(i), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = C_4, J(C_4)$.

Example 5.9 ($\boldsymbol{C_6, J(C_6)}$). Let $f_1 = f_\psi$ be the weight 2 newform 27.2.1a, and let $f_2 = f_{\psi^3 \otimes \left(\frac{2}{\cdot}\right)_6}$ be the weight 4 newform of level 576 constructed in §4.3, which is a sextic twist of 9.4.1a; both f_1 and f_2 have CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = C_6, J(C_6)$.

6. Tensor product constructions

We now consider motives of the form $M = M_1 \otimes M_2$, in which M_1 is a 1-motive with Hodge numbers $h^{1,0} = h^{0,1} = 1$, and M_2 is a 2-motive with Hodge numbers $h^{2,0} = h^{0,2} = 1$. We also consider the related construction in which M is the symmetric cube of M_1.

6.1. Tensor product constructions using elliptic curves. We first consider examples in which M_1 is the 1-motive of an elliptic curve E_1 and M_2 is the complement of the Tate motive in the symmetric square of an elliptic curve E_2 with complex multiplication defined over K. When E_1 does not have complex multiplication, the Sato-Tate group should be $\mathrm{U}(2)$. If E_1 has complex multiplication and is not $\overline{K}$-isogenous to E_2, the Sato-Tate group should be F or F_c depending on whether its complex multiplication is defined over K or not.[5] In the case that E_1 and E_2 are $\overline{K}$-isogenous, the Sato-Tate group should have identity component $\mathrm{U}(1)$; this case is discussed in further detail below.

For any prime $\mathfrak{p}$ of K where both E_1 and E_2 have good reduction, the coefficients of the L-polynomial $L_{\mathfrak{p}}(T)$ of the motive $M_1 \otimes M_2$ can be derived directly from the Frobenius traces t_1 and t_2 of E_1 and E_2 at $\mathfrak{p}$. If $\alpha_1, \overline{\alpha}_1$ and $\alpha_2, \overline{\alpha}_2$ are the Frobenius eigenvalues of the reductions of E_1 and E_2 modulo $\mathfrak{p}$ respectively, then the Frobenius eigenvalues of $M_1 \otimes M_2$ at $\mathfrak{p}$ are $\alpha_1\alpha_2^2, \alpha_1\overline{\alpha}_2^2, \overline{\alpha}_1\alpha_2^2$, and $\overline{\alpha}_1\overline{\alpha}_2^2$. The L-polynomial coefficients c_1 and c_2 of (3.1) may be computed via

$$c_1 = -t_1(t_2^2 - 2p) \qquad \text{and} \qquad c_2 = pt_1^2 + (t_2^2 - 2p)^2 - 2p^2, \tag{6.1}$$

where $p = N(\mathfrak{p})$, and the normalized coefficients are then $a_1(\mathfrak{p}) = c_1/p^{3/2}$ and $a_2(\mathfrak{p}) = c_2/p^2$.

By using the `smalljac` software [**KS08**] to compute the sequences of Frobenius traces of E_1 and E_2 and applying (6.1) to the results, we can very efficiently compute the moment statistics of a_1 and a_2, which allows us to use norm bounds $B = 2^n$ ranging from 2^{12} to 2^{40}.

Example 6.1 (**$\mathbf{U(2)}$**). Let E_1 be the elliptic curve $y^2 = x^3 + x + 1$, which does not have CM, and let E_2 be the elliptic curve $y^2 = x^3 + 1$, which has CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega)$ are listed in Table 9, along with the corresponding moments for the group $G = \mathrm{U}(2)$. (One can also achieve $N(\mathrm{U}(2))$ by considering this example over $\mathbb{Q}$; compare Example 6.16.)

Example 6.2 (**$\boldsymbol{F, F_c}$**). Let E_1 be the elliptic curve $y^2 = x^3 + x$ with CM by $\mathbb{Q}(i)$, and let E_2 be the elliptic curve $y^2 = x^3 + 1$ with CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \otimes M_2$ over $K = \mathbb{Q}(i, \omega), \mathbb{Q}(\omega)$ are listed in Table 9, along with the corresponding moments for $G = F, F_c$. (One can also achieve F_{ab} and $F_{ab,c}$ by considering this example over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{3})$; compare Example 6.15.)

[5] To see why it must be F_c, as opposed to F_a or F_{ab}, which also have component groups of order 2, note that (6.1) implies that G must have invariants $z_1 = 1$ and $z_2 = [0, 0, 0, 0, 0]$.

REMARK 6.3. Here we appear to be able to realize the Sato-Tate group F_c, the first of the three groups ruled out in [**FKRS12**] for weight 1 motives arising from abelian surfaces, and we also appear to realize the second such group, $F_{ab,c}$; see Example 6.15. It is unclear whether the remaining group $F_{a,b,c}$ ruled out in [**FKRS12**] can be realized by a weight 3 motive with rational coefficients (but see Example 8.3).

We now consider the case where E_1 and E_2 are $\overline{K}$-isogenous. Without loss of generality (for the purpose of realizing groups) we may suppose that E_1 and E_2 are actually $\overline{K}$-isomorphic, that is, twists. The case where E_1 and E_2 are K-isomorphic corresponds to taking the symmetric cube of an elliptic curve, which we consider in the next section; here we assume that E_1 and E_2 are twists that are not isomorphic over K.

If E_1 and E_2 are quadratic twists, the Sato-Tate group of $M_1 \otimes M_2$ will be the same as that of $\mathrm{Sym}^3 M_1$; this is evident from (6.1), since multiplying either t_1 or t_2 by $\chi(p) \in \{\pm 1\}$ for some quadratic character χ will not change any of the a_1 and a_2 moments, and these moments determine the Sato-Tate group (as can be seen in Tables 2 and 3). However, the situation changes if we take a cubic twist.

EXAMPLE 6.4 ($\boldsymbol{C_3}$). Consider the elliptic curves $E_1\colon y^2 = x^3+4$ and $E_2\colon y^2 = x^3+1$, both of which have CM by $K = \mathbb{Q}(\omega)$. Moment statistics for $M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega)$ are listed in Table 9 along with the corresponding moments for $G = C_3$. Note that the moment $M_{12}[a_1] = 854216$ distinguishes C_3, and the moment statistics for $M_{12}[a_1]$ obtained by this example are much closer to this value than any of the other $M_{12}[a_1]$ values in Table 2. (One can also achieve $J(C_3)$ by considering this example over $\mathbb{Q}$; compare Example 6.13.)

We also get C_3 if we use a sextic twist, for the same reason that using a quadratic twist yields C_1. One might hope that taking E_1 to be a quartic twist of $E_2\colon y^2 = x^3 - x$ would yield C_2, but we actually get C_1 instead. All of this behavior is explained by the following lemma and remark.

LEMMA 6.5. *For $A, B \in K^*$, where $K = \mathbb{Q}(\omega)$, let M_1 be the 1-motive of the elliptic curve $E_A\colon y^2 = x^3 + A$ over K, and let M_2 be the complement of the Tate motive in the symmetric square of the elliptic curve $E_B\colon y^2 = x^3 + B$ over K. Let $L = K((A/B)^{1/6})$, let $d = [L : K]$, and let $n = d/(d,4)$. Then the distribution of the normalized Frobenius eigenvalues of $M_1 \otimes M_2$ coincides with the distribution of the eigenvalues of a random element of the group C_n.*

PROOF. Let $\mathrm{End}_{\overline{\mathbb{Q}}}(E_A, E_B)$ denote the ring of endomorphisms from E_A to E_B defined over $\overline{\mathbb{Q}}$. Since E_A and E_B have complex multiplication by K and are isogenous over L, the vector space $\mathrm{End}_{\overline{\mathbb{Q}}}(E_A, E_B) \otimes \mathbb{Q}$ is endowed with the structure of a $K[\mathrm{Gal}(L/K)]$-module; let χ denote its character. Then for a prime ℓ, as in [**FS12**, §3.3], we have the following isomorphism of $\overline{\mathbb{Q}}_\ell[G_K]$-modules

$$V_\ell(E_A) \otimes \overline{\mathbb{Q}}_\ell \simeq V_\sigma(E_B) \otimes \chi \oplus V_\sigma(E_B) \otimes \overline{\chi}\,. \tag{6.2}$$

Here $V_\ell(E_A)$ denotes the ℓ-adic Tate module of E_A, σ and $\overline{\sigma}$ stand for the two embeddings of M into $\overline{\mathbb{Q}}_\ell$, and

$$V_\sigma(E_B) := V_\ell(E_B) \otimes_{M,\sigma} \overline{\mathbb{Q}}_\ell\,, \qquad V_{\overline{\sigma}}(E_B) := V_\ell(E_B) \otimes_{M,\overline{\sigma}} \overline{\mathbb{Q}}_\ell\,.$$

Let $\mathfrak{p}$ be a prime of K of good reduction for E_A and E_B such that $\mathrm{Frob}_\mathfrak{p}$ has order f in $\mathrm{Gal}(L/K)$. Since χ is injective, it follows from (6.2) that if $\{\alpha_\mathfrak{p}, \overline{\alpha}_\mathfrak{p}\}$ are the normalized eigenvalues of the action of $\mathrm{Frob}_{\mathfrak{p}}$ on $V_\ell(E_B)$, then $\{\zeta\alpha_{\mathfrak{p}}, \overline{\zeta}\overline{\alpha}_{\mathfrak{p}}\}$ are the normalized eigenvalues of the action of $\mathrm{Frob}_{\mathfrak{p}}$ on $V_\ell(E_A)$, where ζ is a primitive fth root of unity. Thus the normalized eigenvalues of the action of $\mathrm{Frob}_{\mathfrak{p}}$ relative to $M_1 \otimes M_2$ are

$$\{\zeta\alpha_{\mathfrak{p}}^3, \overline{\zeta}\overline{\alpha}_{\mathfrak{p}}^3, \overline{\zeta}\alpha_{\mathfrak{p}}, \zeta\overline{\alpha}_{\mathfrak{p}}\}\,. \tag{6.3}$$

By [**FS12**, Proposition 3.6], the sequence of $\alpha_{\mathfrak{p}}$'s with $\mathrm{ord}(\mathrm{Frob}_{\mathfrak{p}}) = f$ is equidistributed on $\mathrm{U}(1)$ with respect to the Haar measure. By the translation invariance of the Haar measure, the sequence of $\beta_{\mathfrak{p}}$'s with $\mathrm{ord}(\mathrm{Frob}_{\mathfrak{p}}) = f$ is also equidistributed, where $\beta_{\mathfrak{p}} := \overline{\zeta}\alpha_{\mathfrak{p}}$. Now (6.3) reads

$$\{\zeta^4\beta_{\mathfrak{p}}^3, \overline{\zeta}^4\overline{\beta}_{\mathfrak{p}}^3, \beta_{\mathfrak{p}}, \overline{\beta}_{\mathfrak{p}}\}\,.$$

Let $s = f/(f,4)$. We deduce that the normalized eigenvalues of the action of $\mathrm{Frob}_{\mathfrak{p}}$ relative to $M_1 \otimes M_2$ with $\mathrm{ord}(\mathrm{Frob}_{\mathfrak{p}}) = f$ are equidistributed as the eigenvalues of a random matrix in the connected component of the matrix

$$\begin{pmatrix} \Theta_s U & 0 \\ 0 & \overline{\Theta_s U} \end{pmatrix}$$

(in the notation of §2.1). The extension L/K is cyclic of order dividing 6, which implies that the normalized Frobenius eigenvalues of $M_1 \otimes M_2$ have the same distribution as the eigenvalues of a random matrix in the group C_n. $\square$

Remark 6.6. The same proof works when $K = \mathbb{Q}(i)$, $L = K((A/B)^{1/4})$, $E_A\colon y^2 = x^3 + Ax$, and $E_B\colon y^2 = x^3 + Bx$. In this case, $n = d/(4,d) = 1$, and the distribution of the normalized Frobenius eigenvalues of $M_1 \otimes M_2$ is thus always governed by C_1.

6.2. Symmetric cubes of elliptic curves. We next consider motives of the form $M = \mathrm{Sym}^3 M_1$ over a field K in which M_1 is the 1-motive of an elliptic curve E_1. The Sato-Tate group in this case should be $C_1, J(C_1)$, or D, depending on whether E has complex multiplication defined over K, complex multiplication that is not defined over K, or no complex multiplication at all. This is effectively a special case of the product construction $M_1 \otimes M_2$ with $E_1 = E_2$, except that we do not necessarily require $E_1 = E_2$ to have complex multiplication. To compute the L-polynomial coefficients of M we simply apply the equations in (6.1) with $t_1 = t_2$.

Example 6.7 ($\boldsymbol{C_1, J(C_1), D}$). See Table 9 for moment statistics of the motive $M = \mathrm{Sym}^3 M_1$ in three cases: (1) E_1 is the elliptic curve $y^2 = x^3 + 1$ over $\mathbb{Q}(\omega)$; (2) E_1 is the elliptic curve $y^2 = x^3 + 1$ over $\mathbb{Q}$; and (3) E_1 is the elliptic curve $y^2 = x^3 + x + 1$; along with the corresponding moments for $G = C_1, J(C_1), D$.

6.3. Tensor product constructions using modular forms. We now consider motives $M = M_1 \otimes M_2$ that arise as the tensor product of the motive M_1 associated to a weight 2 newform f_1 (with Hodge numbers $h^{1,0} = h^{0,1} = 1$) and the motive M_2 associated to a weight 3 newform f_2 (with Hodge numbers $h^{2,0} = h^{0,2} = 1$). We assume that both f_1 and f_2 have rational Fourier coefficients.

By Proposition 4.1, f_1 has trivial nebentypus and f_2 has CM by its (quadratic) nebentypus χ. The motive M is defined over $\mathbb{Q}$, and we consider its base change to a number field K. If the q-expansions of f_1 and f_2 are given by $f_1 = \sum b_n q^n$ and

$f_2 = \sum d_n q^n$, then the coefficients of the L-polynomial $L_{\mathfrak{p}}(T)$ at a prime $\mathfrak{p}$ of K of good reduction for M are given by

$$c_1 = -b_p d_p \qquad \text{and} \qquad c_2 = \chi(p) p b_p^2 + d_p^2 - 2\chi(p)p^2, \tag{6.4}$$

where $p = N(\mathfrak{p})$ and the integer coefficients c_1 and c_2 are as defined in (3.1). The normalized coefficients are then $a_1(\mathfrak{p}) = c_1/p^{3/2}$ and $a_2(\mathfrak{p}) = c_2/p^2$.

LEMMA 6.8. *Let M_1 be the motive associated to a weight 2 newform f_1 and let M_2 be the motive associated to a weight 3 newform f_2. Assume that both f_1 and f_2 have rational Fourier coefficients. Then $M = M_1 \otimes M_2$ is self-dual.*

PROOF. It is enough to show that the (normalized) Frobenius eigenvalues of M at prime of good reduction p come in conjugate pairs. Let α_p and $\overline{\alpha}_p$ denote the normalized Frobenius eigenvalues of M_1. We have $\alpha_p\overline{\alpha}_p = 1$, since the nebentypus of f_1 is trivial. For the normalized Frobenius eigenvalues of M_2 we have two cases according to the value of the nebentypus χ of f_2 at p: (1) if $\chi(p) = -1$, then they are 1 and -1, since f_2 has CM by χ, and (2) if $\chi(p) = 1$, then they are β_p and $\overline{\beta}_p$ with $\beta_p\overline{\beta}_p = 1$. In any of the two cases, we readily check that the normalized Frobenius eigenvalues of M come in conjugate pairs

$$\begin{aligned} (1):&\quad \{\alpha_p\beta_p,\, \alpha_p\overline{\beta}_p,\, \overline{\alpha}_p\beta_p,\, \overline{\alpha}_p\overline{\beta}_p\}\,,\\ (2):&\quad \{\alpha_p,\, -\alpha_p,\, \overline{\alpha}_p,\, -\overline{\alpha}_p\}\,. \end{aligned}$$

Consequently, M is self-dual. □

REMARK 6.9. With the hypothesis of the lemma, M_2 is not self-dual, since the nebentypus of f_2 is not trivial (and note therefore that this is not an obstruction for $M_1 \otimes M_2$ being self-dual). In particular, seen as a motive over $\mathbb{Q}$, the Sato-Tate group

$$\left\langle \begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : u \in \mathbb{C},\, u\cdot\overline{u} = 1 \right\rangle$$

of M_2 is a subgroup of $\mathrm{U}(2)$ not contained in $\mathrm{SU}(2)$.

EXAMPLE 6.10 (**$C_1, J(C_1)$**). Let $f_1 = f_\psi$ be the weight 2 newform 27.2.1a, corresponding to the elliptic curve $y^2 + y = x^3$, and let $f_2 = f_{\psi^2}$, which is a weight 3 newform of level 27; both f_1 and f_2 have CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = C_1, J(C_1)$.

REMARK 6.11. The sequence of L-polynomials of the motive constructed as a tensor product $M_1 \otimes M_2$ in Example 6.10, using $f_1 = 27.2.1\text{a}$ and $f_2 = f_{\psi^2}$, is identical to the sequence of L-polynomials of the motive constructed as a direct sum $M_1 \oplus M_2$ in Example 5.5, using $f_1 = 27.2.1\text{a}$ and $f_2 = 9.4.1\text{a}$.

EXAMPLE 6.12 (**$C_2, J(C_2)$**). Let $f_1 = f_\psi$ be the weight 2 newform 32.2.1a, corresponding to the elliptic curve $y^2 = x^3 - x$, and let $f_2 = f_{\psi^2 \otimes \phi}$ be the weight 3 newform of level 576 constructed in §4.2, which is a quartic twist of f_{ψ^2}; both f_1 and f_2 have CM by $\mathbb{Q}(i)$. Moment statistics for the motive $M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = C_2, J(C_2)$.

Example 6.13 ($\boldsymbol{C_3, J(C_3)}$). Let f_1 be the weight 2 newform 36.2.1a, which is the cubic twist of f_ψ =27.2.1a corresponding to the elliptic curve $y^2 = x^3 + 1$, and let $f_2 = f_{\psi^2}$, a weight 3 newform of level 27; both f_1 and f_2 CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = C_3, J(C_3)$.

Remark 6.14. The behavior observed in the above examples can be explained by means of arguments completely analogous to those of Lemma 5.4. Let ψ be a Hecke character of a quadratic imaginary field K of infinite type 1 such that f_ψ has rational coefficients. Let ϕ_1 (resp. ϕ_2) be a Hecke character of finite order n such that $f_{\psi^2\otimes\phi_1}$ (resp. $f_{\psi\otimes\phi_2}$) has rational coefficients. We then have:

(i) If $f_1 := f_{\psi^2}$ and $f_2 := f_{\psi\otimes\phi_2}$, then the distribution of the normalized Frobenius eigenvalues of $M_1 \otimes M_2$ (resp. of the base change $(M_1 \otimes M_2)_K$) coincides with the distribution of the eigenvalues of a random element in $J(C_t)$ (resp. C_t), where $t = n/(n, 2)$.
(ii) If $f_1 := f_{\psi^2\otimes\phi_1}$ and $f_2 := f_\psi$, then the distribution of the normalized Frobenius eigenvalues of $M_1 \otimes M_2$ (resp. of the base change $(M_1 \otimes M_2)_K$) coincides with the distribution of the eigenvalues of a random element in $J(C_t)$ (resp. C_t), where $t = n/(n, 4)$.

Example 6.15 ($\boldsymbol{F, F_{ab}, F_c, F_{ab,c}}$). Let f_1 be the weight 2 newform 32.2.1.a, which has CM by $\mathbb{Q}(i)$, and let $f_2 := f_{\psi^2}$, a weight 3 newform of level 27, which has CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \otimes M_2$ over the fields $K = \mathbb{Q}(i, \omega)$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(i)$, $\mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = F, F_{ab}, F_c, F_{ab,c}$. With $K = \mathbb{Q}(\omega)$ one obtains essentially the same moment statistics as with $K = \mathbb{Q}(i)$; this is as expected, since the groups F_{abc} and F_c are conjugate.

Example 6.16 ($\mathbf{U(2)}, \boldsymbol{N}(\mathbf{U(2)})$). Let f_1 be the weight 2 newform 11.2.1a, corresponding to the elliptic curve $y^2 + y = x^3 - x^2 - 10x - 20$, which does not have CM, and let $f_2 := f_{\psi^2}$, a weight 3 newform of level 27, which has CM by $\mathbb{Q}(\omega)$. Moment statistics for the motive $M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega), \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = \mathrm{U}(2), N(\mathrm{U}(2))$.

7. The Dwork pencil

We next describe a construction that gives rise to motives whose L-polynomial distributions match the group $\mathrm{USp}(4)$, something that cannot be achieved using any of the preceding methods. To facilitate explicit computations with the Dwork pencil of threefolds, we work with a family of hypergeometric motives defined by fixed parameters $\alpha = (1/5, 2/5, 3/5, 4/5)$ and $\beta = (0, 0, 0, 0)$, and a varying parameter $z = (5/t)^5$, where t is the Dwork pencil parameter, as described in [**CR12**]. We first summarize the general setup in [**CR12**] and then specialize to the case of interest.

7.1. Trace formulas and algorithms. For a prime p, let $\mathbb{Q}_{(p)}$ denote the ring of rational numbers with denominators prime to p. For $z \in \mathbb{Q}_{(p)}$, we write $\mathrm{Teich}(z)$ to denote the Teichmüller lift of the reduction of z modulo p. Letting $\Gamma_p(x)$ denote the p-adic gamma function, for each prime power $q = p^f$, we define $\Gamma_q^*(x) := \prod_{v=0}^{f-1} \Gamma_p(\{p^v x\})$, where $\{\cdot\}$ denotes the fractional part of a rational number, and then define a p-adic analogue of the Pochhammer symbol by setting

$$(x)_m^* := \frac{\Gamma_q^*\left(x + \frac{m}{1-q}\right)}{\Gamma_q^*(x)}.$$

Given vectors $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ in $\mathbb{Q}_{(p)}^r$ and $z \in \mathbb{Q}_{(p)}$, we define

$$H_q\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| z\right) := \frac{1}{1-q}\sum_{m=0}^{q-2}(-p)^{\eta_m(\alpha)-\eta_m(\beta)}q^{\xi_m(\beta)}\left(\prod_{j=1}^{r}\frac{(\alpha_j)_m^*}{(\beta_j)_m^*}\right)\mathrm{Teich}(z)^m, \tag{7.1}$$

using the notations

$$\eta_m(x_1, \ldots, x_r) := \sum_{j=1}^{r}\sum_{v=0}^{f-1}\left\{p^v\left(x_j + \frac{m}{1-q}\right)\right\} - \{p^v x_j\},$$

and

$$\xi_m(\beta) := \#\{j : \beta_j = 0\} - \#\left\{j : \beta_j + \frac{m}{1-q} = 0\right\}.$$

(with $\beta = (0,0,0,0)$ we have $\xi_m(\beta) = 4$ for all nonzero m and $\xi_0(\beta) = 0$).

Now let X_ψ be the quintic threefold given in (1.1),

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = t x_0 x_1 x_2 x_3 x_4,$$

with the parameter $t = 5\psi$. Let V_ψ be the subspace of $H^3(X_\psi, \mathbb{C})$ fixed by the automorphism group

$$\{(\zeta_1, \ldots, \zeta_5) | \zeta_i^5 = 1, \zeta_1 \cdots \zeta_5 = 1\},$$

acting by $x_i \mapsto \zeta_i x_i$. For primes $p \neq 5$ for which we have $\psi^5 \not\equiv 1 \bmod p$ and $\psi \neq \infty \bmod p$, the Euler factor of the L-function of V_ψ at p has the form (3.1),

$$L_p(T) = p^6T^4 + c_1p^3T^3 + c_2pT^2 + c_1T + 1,$$

where c_1 and c_2 are integers. For $\psi \not\equiv 0 \bmod p$, the trace of the geometric Frobenius on V_ψ is given by

$$\mathrm{Trace}\left(\mathrm{Frob}_q\big|_{V_\psi}\right) = H_q\left(\begin{matrix}\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5}\\ 0 & 0 & 0 & 0\end{matrix}\middle|\psi^{-5}\right).$$

Abbreviating the right-hand side as H_q, we have

$$c_1 = -H_p, \qquad \text{and} \qquad c_2 = \frac{1}{2p}\left(H_p^2 - H_{p^2}\right).$$

The Weil bounds imply that $|c_1| \leq 4p^{3/2}$, so for $p > 64$ it suffices to compute $H_p \bmod p^2$. Computing c_2 requires more precision: we have $-4p^3 < 2pc_2 \leq 12p^3$, so for $p > 16$ it is enough to compute H_p and H_{p^2} modulo p^4.

Specializing $\alpha = (1/5, 2/5, 3/5, 4/5)$ and $\beta = (0,0,0,0)$ in (7.1) allows us to simplify the formulas. For the sake of brevity (and ease of computation), we focus on the problem of computing $H_p \bmod p^2$, so $q = p$ and $f = 1$. We have $\eta_0(x) =$

$\xi_0(x) = 0$ and $(\alpha_j)_0^* = (\beta_j)_0^* = 1$, thus the $m = 0$ term in (7.1) is equal to 1. For $m > 0$ we have $\xi_m(\beta) = 4$, and one finds that

$$\eta_m(\alpha) - \eta_m(\beta) = \begin{cases} -4 & \text{if } 0 < m < \lfloor \frac{p+4}{5} \rfloor, \\ -3 & \text{if } \lfloor \frac{p+4}{5} \rfloor \le m < \lfloor \frac{2p+3}{5} \rfloor, \\ -2 & \text{if } \lfloor \frac{2p+3}{5} \rfloor \le m < \lfloor \frac{3p+2}{5} \rfloor, \\ -1 & \text{if } \lfloor \frac{3p+2}{5} \rfloor \le m < \lfloor \frac{4p+1}{5} \rfloor, \\ 0 & \text{if } m \ge \lfloor \frac{4p+1}{5} \rfloor. \end{cases}$$

When working modulo p^2, only the first two ranges of m are relevant (the other terms in the sum are all divisible by p^2), and we may write

$$H_p\left(\begin{smallmatrix} \frac{1}{5} & \frac{2}{4} & \frac{3}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 \end{smallmatrix} \middle| z \right) \equiv \frac{1 + S_1 - pS_2}{1-p} \bmod p^2,$$

where

$$S_1 = \sum_{m=1}^{m_1 - 1} \left(\prod_{j=1}^{4} \frac{(j/5)_m^*}{(0)_m^*} \right) \mathrm{Teich}(z)^m, \quad S_2 = \sum_{m=m_1}^{m_2 - 1} \left(\prod_{j=1}^{4} \frac{(j/5)_m^*}{(0)_m^*} \right) \mathrm{Teich}(z)^m,$$

with $m_1 = \lfloor \frac{p+4}{5} \rfloor$ and $m_2 = \lfloor \frac{2p+3}{5} \rfloor$.

To compute $H_p \bmod p^2$, it suffices to compute $S_1 \bmod p^2$ and $S_2 \bmod p$. Evaluating the Pochhammer symbols $(\cdot)_m^*$ that appear in the formulas for S_1 and S_2 thus reduces to computing $\Gamma_p(x)$ modulo p^2, or modulo p. To compute $\Gamma_p(x) \bmod p^2$ for $x \in \mathbb{Q}_{(p)}$, we first reduce x modulo p^2 and use

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{for } x \in \mathbb{Z}_p^*, \\ -\Gamma_p(x) & \text{for } x \in p\mathbb{Z}_p, \end{cases} \tag{7.2}$$

to shift the argument down so that it is divisible by p. We then apply

$$\Gamma_p(py) \equiv 1 + \left(1 + \frac{1}{(p-1)!}\right) y \bmod p^2.$$

For $x = x_0 + px_1$ with $0 < x_0 < p$, we have

$$\begin{aligned} \Gamma_p(x) &\equiv (-1)^{x_0}(px_1 + 1) \cdots (px_1 + x_0 - 1) \left(1 + \left(1 + \frac{1}{(p-1)!}\right) x_1\right) \bmod p^2 \\ &\equiv (-1)^{x_0} \left(px_1 \sum_{k=1}^{x_0 - 1} \frac{(x_0 - 1)!}{k} + (x_0 - 1)!\right) \left(1 + \left(1 + \frac{1}{(p-1)!}\right) x_1\right) \bmod p^2. \end{aligned}$$

To compute $\Gamma_p(x) \bmod p$, simply apply the above formula with $x_1 = 0$.

Now let $F_n := n!$ and $T_n := \sum_{k=1}^{n} \frac{n!}{k}$. We may compute F_n and T_n modulo p^2 for $0 \le n < p$ via the recurrences $F_{n+1} = (n+1)F_n$ and $T_{n+1} = (n+1)T_n + F_n$, with $F_0 = 1$ and $T_0 = 0$. Having computed the F_n and T_n using $O(p)$ operations in $\mathbb{Z}/p^2\mathbb{Z}$, we can use the above formulas to compute $\Gamma_p(x)$ for any $x \in \mathbb{Z}/p^2\mathbb{Z}$ using $O(1)$ operations in $\mathbb{Z}/p^2\mathbb{Z}$. Noting that $\mathrm{Teich}(z) \equiv z^p \bmod p^2$, we can compute $H_p \bmod p^2$ using a total of $O(p)$ operations in $\mathbb{Z}/p^2\mathbb{Z}$.

To efficiently compute the moment statistics of a_1 for a large set S of parameter values z in parallel, for each p up to a given bound N we compute $H_p(z)$ as a polynomial in $\mathrm{Teich}(z)$ with coefficients in $\mathbb{Z}/p^2\mathbb{Z}$. For $p < \min(\#S, N)$, we then compute $H_p(z^p) \bmod p^2$ for every nonzero $z \in \mathbb{Z}/p\mathbb{Z}$ using fast algorithms for multipoint polynomial evaluation [**GG03**, Alg. 10.8], and construct a lookup table that maps values of z in $\mathbb{Z}/p\mathbb{Z}$ to values of a. If $M = \#S$, then we can compute $H_p(z) \bmod p^2$ for all primes $p \le N$ and all $z \in S$ in time

$$O\big(\pi(N)\mathsf{M}(N)\mathsf{M}(\log N)\log N + M\pi(N)\log N\big),$$

where $\mathsf{M}(n)$ denotes the cost of multiplication. For $M \ge N$, this corresponds to an average cost of $O((\log N)^{3+o(1)})$ per $H_p(z)$ computation.

Computing the moment statistics of a_2 is substantially more work, since we then also need to compute $H_{p^2}(z)$ (modulo p^4), which involves $O(p^2)$ arithmetic operations, compared to the $O(p)$ operations needed to compute $H_p(z)$. To compute $\Gamma_p(x) \bmod p^4$ for $x \in \mathbb{Q}_{(p)}$, we first reduce x modulo p^4 and use (7.2) to shift the argument down so that it is divisible by p. We then apply the formula

$$\Gamma_p(py) \equiv 1 + a_1 y + a_2 y^2 + a_3 y^3 \bmod p^4,$$

with

$$\begin{aligned}
a_2 &\equiv -\big((p-1)! + 1/(p-1)! + 2\big)/2 \bmod p^4,\\
a_1 &\equiv -\big(8(p-1)! + (2p)!/(2p^2) + 4a_2 + 7\big)/6 \bmod p^4,\\
a_3 &\equiv -\big((p-1)! + 1 + a_1 + a_2\big) \bmod p^4.
\end{aligned}$$

After computing $H_p(z)$ and $H_{p^2}(z)$, one then computes $H_p(z)^2 - H_{p^2} \bmod p^4$, lifts this value to an integer that is known to lie in the interval $[-4p^3, 12p^3]$, and then divides by $2p$ to obtain the L-polynomial coefficient c_2, and $a_2 = c_2/p^2$.

REMARK 7.1. Given the higher cost of computing moment statistics for a_2, for the purposes of comparison with USp(4), we choose to mainly focus on a_1. This is reasonable because the a_1 moments of USp(4) easily distinguish it from any of the other candidate Sato-Tate groups, as can be seen in Table 2.

On the other hand, an ongoing project of the second author with Edgar Costa and David Harvey is expected to yield a computation of a_2 using only $O(p)$ arithmetic operations. The strategy is to view the members of the Dwork pencil as nondegenerate toric hypersurfaces, then make a careful computation in p-adic cohomology in the style of the work of the second author [**Ked01**] on hyperelliptic curves.

Note that the algorithms described above cannot be used when $t = 0$, because then the condition $\psi \not\equiv 0 \pmod p$ is never satisfied. For completeness, we describe this case separately.

EXAMPLE 7.2 ($\boldsymbol{F_{ac}}$). Let M be the motive arising from the quintic threefold (1.1) with parameter $t = 0$. The L-polynomials in this case where computed by Weil in terms of Jacobi sums; they coincide with the L-polynomials of the unique algebraic Hecke character over $\mathbb{Q}(\zeta_5)$ of conductor $(1-\zeta_5)^2$ and infinite type $(3,0),(2,1)$. The latter can be computed efficiently using [**Magma**], as demonstrated to us by Mark Watkins. Moment statistics for the motive M over $K = \mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = F_{ac}$.

7.2. Experimental results. Using the algorithms described in the previous section, we computed a_1 moment statistics for the family of hypergeometric motives with rational parameter z of height at most 10^3; the set S of such z has cardinality greater than 10^6. We computed c_1 values for all $z \in S$ and all $p \le 2^{14}$, and for a subset of the $z \in S$ we continued the computation over $p \le 2^{20}$. For each value of z we computed the moment statistic $M_n[a_1]$ for $1 \leq n \leq 12$. In every case the moment statistics appeared to match the a_1 moment sequence of USp(4) listed in Table 2. We note that USp(4) is the only group with $M_4[a_1] = 3$, and its sixth moment $M_6[a_1] = 14$ is less than half any of the other values for $M_6[a_1]$ listed in Table 2; these differences are clearly evident in the moment statistics, even when using a norm bound as small as $B = 2^{14}$.

We then conducted similar experiments for each of the following families:

- $z = (5/t)^5$ for rational t of height at most 1000;
- $z = 1 + 1/n$ for integers n of absolute value at most 10^5.
- $z = (z_3\zeta^3 + z_2\zeta^2 + z_1\zeta + z_0)^{-1}$ for a primitive fifth root of unity ζ and integers z_0, z_1, z_2, and z_3 of absolute value at most 10.

In every case the moment statistics again appeared to match the USp(4) moment sequence; we found no exceptional cases aside from the excluded case $t = 0$ (see Example 7.2).

Example 7.3 (**USp(4)**). Let M be the motive arising from the quintic threefold (1.1) with parameter $t = -5$ (that is, $z = -1$), as described in §7.1, over the field $K = \mathbb{Q}$. Table 9 lists moment statistics of a_1 as the norm bound $B = 2^n$ varies from 2^{10} to 2^{24}, and moment statistics of a_2 with $B = 2^n$ varying from 2^{10} to 2^{13}. The corresponding moments for the group $G = \mathrm{USp}(4)$ are shown in the last line for comparison.

Remark 7.4. It is worth contrasting the behavior of the Dwork pencil of threefolds with the behavior of a universal family of elliptic curves, in which one always sees infinitely many curves with complex multiplication. It has been suggested by de Jong that the scarcity of special members of the Dwork family may be explained by Hodge-theoretic considerations (unpublished, but see [**dJ02**]). However, such considerations do not give any indication about the *number* of exceptional cases. It is entirely possible that there are some unobserved exceptional cases arising at large height and/or over a number field other than $\mathbb{Q}$.

Remark 7.5. The Dwork pencil is a family of *hypergeometric motives*, i.e., a family whose Picard-Fuchs equation is a hypergeometric differential equation. One can classify such families for fixed weight and Hodge numbers; for the values we are considering, there are 47 such families (as verified by the [**Magma**] command `PossibleHypergeometricData`). The computation of L-polynomials in these families has recently been implemented by Mark Watkins in [**Magma**], and leads to some other exceptional cases (e.g., example H126E5 in the *Magma Handbook*).

8. More modular constructions

At this point, all of the groups listed in Table 1 are accounted for except for $F_{a,b,c}$ and $N(G_{3,3})$. We conclude with some more exotic uses of modular forms, leading to a realization of $N(G_{3,3})$ and a tantalizing near-miss for $F_{a,b,c}$. Thanks to Mark Watkins for suggesting these examples and providing assistance with computations in [**Magma**].

8.1. Hilbert modular forms.

Example 8.1 ($\boldsymbol{N(G_{3,3})}$). There is a unique normalized Hilbert modular eigenform over $K = \mathbb{Q}(\sqrt{5})$ of level $\Gamma_0(2\sqrt{5})$ and weight $(2, 4)$. This gives rise to a motive

M of the desired form by a procedure described in [**BR93**] (which gives a motive over K) followed by a base change from K to $\mathbb{Q}$. Moment statistics for the motive M over $\mathbb{Q}$ are listed in Table 9, along with the corresponding moments for $G = N(G_{3,3})$. Due to computational limitations of [**Magma**], we were only able to compute a_1, and we were forced to limit the prime bound to 2^{14}, limiting the quality of the numerical evidence. However, note that $M_4[a_1]$ appears to be converging quite rapidly to 5, and that this value occurs for no groups in Table 1 other than $N(G_{3,3})$.

The motive in Example 8.1 is somewhat hard to write down explicitly. However, one expects that a generic example of this form should give the same Sato-Tate group, and there exist other examples where the motive appears much more explicitly.

EXAMPLE 8.2. Define the two-variable Chebyshev polynomial

$$P(x, y) = x^5 + y^5 - 5xy(x^2 + y^2) + 5xy(x + y) + 5(x^2 + y^2) - 5(x + y).$$

Form the affine threefold

$$\operatorname{Spec} \mathbb{Q}[x_1, x_2, x_3, x_4]/(P(x_1, x_2) - P(x_3, x_4)),$$

then take the Zariski closure in $\mathbb{P}^4_{\mathbb{Q}}$. It was observed by Consani-Scholten [**CS01**] that the resulting threefold has 120 ordinary double points and no other singularities. Blow up these double points to obtain a smooth threefold, then take middle cohomology to obtain a motive M.

It was conjectured by Consani-Scholten and proved by Dieulefait-Pacetti-Schütt [**DPS12**] that this is an example of a nonrigid modular Calabi-Yau threefold. More precisely, the L-function of M coincides with that of a certain Hilbert newform over $K = \mathbb{Q}(\sqrt{5})$ of level $\Gamma_0(30)$ (or rather its base change from K to $\mathbb{Q}$).

8.2. Other Hecke characters. So far we have only considered Hecke characters over quadratic fields. However, algebraic Hecke characters over larger fields also correspond to motives, as described in [**Sch88**]. We have seen one instance of this in another guise in Example 7.2. It is tempting to try to realize $F_{a,b,c}$ using a variant of that example; this turns out to be possible for motives with coefficients in a real quadratic field, but it remains unclear whether rational coefficients can be achieved.

EXAMPLE 8.3. Consider the number field $K = \mathbb{Q}[\alpha]/(\alpha^4 - 2\alpha^3 + 5\alpha^2 - 4\alpha + 2)$, labeled 4.0.1088.2 in [**LMFDB**]; this is a CM field of class number 1 whose Galois group is the dihedral group of order 8. Let $\mathfrak{p}$ be the unique (ramified) prime of norm 17. There is then a unique algebraic Hecke character ψ of conductor $\mathfrak{p}$ and infinite type $(3, 0), (1, 2)$. The resulting motive M is defined over $\mathbb{Q}$ but has coefficients in $\mathbb{Q}(\sqrt{17})$; it is thus not covered by our classification. Nonetheless, one can compute L-polynomial coefficients in [**Magma**] and observe good agreement with moment statistics for the group $F_{a,b,c}$.

REMARK 8.4. One can construct similar examples of infinite type $(1, 0), (1, 0)$. One thus obtains motives with the Hodge numbers of an abelian surface, but having Sato-Tate group $F_{a,b,c}$ which is shown not to occur for abelian surfaces in [**FKRS12**]. In particular, the three groups appearing in the group-theoretic classification of [**FKRS12**] which are not realized by abelian surfaces appear to be realized by motives with nonrational coefficients.

REMARK 8.5. For any example constructed from Hecke characters as above, the connected part of the Sato-Tate group should be a torus. If so, one can prove equidistribution using the work of Johansson [**Joh14**].

9. Moment statistics

This section lists moment statistics for the various motives constructed in the previous three sections. In each of the tables that follow, the column n indicates the norm bound $B = 2^n$ on the degree 1 primes $\mathfrak{p}$ of K for which L-polynomials $L_{\mathfrak{p}}(T)$ were computed. The remaining columns list various moment statistics $M_n[a_i]$ of the normalized L-polynomial coefficients a_1 and a_2. Following each example, the corresponding moments of the candidate Sato-Tate group G are listed for comparison.

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}(i,\omega)$ with $f_1 = 32.2.1\mathrm{a}$ and $f_2 = 9.4.1\mathrm{a}$ (Example 5.1)												
12	3.848	32.096	329.646	3772.44	46139.8	589946	2.025	7.562	28.615	125.379	573.450	2761.95	13699.7
16	4.068	36.349	399.331	4828.99	61695.2	816810	2.043	8.062	32.259	148.124	707.888	3533.35	18071.1
20	3.977	35.643	394.090	4803.74	61964.3	829450	1.991	7.958	31.691	146.135	700.526	3514.70	18079.7
24	3.984	35.765	396.577	4849.68	62756.6	842562	1.994	7.966	31.802	146.893	705.735	3548.37	18292.4
28	3.999	35.978	399.623	4893.66	63399.1	852060	2.000	7.997	31.982	147.884	711.270	3579.48	18468.2
32	4.000	36.005	400.061	4900.72	63512.0	853854	2.000	8.001	32.004	148.022	712.107	3584.53	18498.5
36	4.000	35.998	399.973	4899.58	63497.5	853676	2.000	8.000	31.999	147.991	711.949	3583.70	18494.2
40	4.000	35.999	399.988	4899.82	63501.3	853735	2.000	8.000	31.999	147.996	711.978	3583.87	18495.3
F	4	36	400	4900	63504	853776	2	8	32	148	712	3584	18496
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 = 32.2.1\mathrm{a}$ and $f_2 = 9.4.1\mathrm{a}$ (Example 5.1)												
12	2.902	18.647	169.722	1858.66	22413.6	285365	2.012	5.720	17.955	68.817	293.457	1366.79	6681.5
16	3.022	21.088	208.818	2439.48	30845.9	407177	2.021	6.023	20.079	81.788	368.544	1791.49	9062.4
20	2.987	20.792	206.682	2432.41	31050.9	414422	1.996	5.975	19.824	80.948	365.647	1786.17	9087.3
24	2.991	20.872	208.156	2458.18	31482.7	421452	1.997	5.981	19.893	81.401	368.634	1804.98	9203.9
28	2.999	20.988	209.800	2481.69	31823.8	426468	2.000	5.999	19.990	81.938	371.616	1821.64	9297.6
32	3.000	21.002	210.026	2485.31	31881.3	427380	2.000	6.000	20.002	82.009	372.046	1824.22	9313.1
36	3.000	20.999	209.985	2484.77	31874.5	427297	2.000	6.000	19.999	81.995	371.972	1823.84	9311.1
40	3.000	21.000	209.994	2484.91	31876.6	427328	2.000	6.000	20.000	81.998	371.988	1823.93	9311.6
F_a	3	21	210	2485	31878	472350	2	6	20	82	372	1824	9312
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}$ with $f_1 = 32.2.1\mathrm{a}$ and $f_2 = 9.4.1\mathrm{a}$ (Example 5.1)												
12	1.936	10.706	88.681	933.61	11110.1	140883	2.006	4.848	12.907	42.033	160.871	706.14	3358.2
16	2.003	12.000	109.030	1233.04	15433.8	203133	2.011	5.008	14.019	48.784	199.706	924.84	4580.2
20	1.991	11.884	108.258	1232.87	15578.2	207307	1.998	4.987	13.908	48.453	198.716	924.53	4604.8
24	1.996	11.934	109.048	1246.20	15799.2	210887	1.999	4.990	13.945	48.690	200.261	934.20	4664.5
28	2.000	11.994	109.897	1258.30	15974.3	213457	2.000	4.999	13.995	48.968	201.802	942.79	4712.6
32	2.000	12.001	110.011	1260.13	16003.3	213917	2.000	5.000	14.001	49.004	202.020	944.10	4720.4
36	2.000	12.000	109.992	1259.88	16000.2	213879	2.000	5.000	14.000	48.997	201.985	943.92	4719.5
40	2.000	12.000	109.997	1259.95	16001.3	213895	2.000	5.000	14.000	48.999	201.994	943.97	4719.8
$F_{a,b}$	2	12	110	1260	16002	213906	2	5	14	49	202	944	4720

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 = 11.2.1\text{a}$ and $f_2 = 9.4.1\text{a}$ (Example 5.2)												
12	2.831	18.700	162.998	1653.35	18513.5	222275	1.955	5.772	18.845	70.897	289.489	1269.72	5856.2
16	2.912	18.900	162.111	1606.02	17364.6	199045	1.967	5.823	19.073	71.426	289.045	1247.86	5632.4
20	2.990	19.881	173.341	1739.61	19040.2	221046	1.997	5.982	19.902	75.459	308.989	1346.94	6134.4
24	2.999	19.981	174.808	1762.12	19385.8	226341	2.000	5.997	19.986	75.935	311.725	1362.83	6227.0
28	2.999	19.991	174.872	1761.94	19371.8	226024	2.000	5.999	19.992	75.955	311.735	1362.45	6223.0
32	3.000	19.998	174.956	1763.31	19393.8	226369	2.000	6.000	19.998	75.987	311.919	1363.52	6229.3
36	3.000	20.000	174.993	1763.88	19402.0	226480	2.000	6.000	20.000	75.998	311.988	1363.92	6231.5
40	3.000	20.000	174.999	1763.99	19403.8	226510	2.000	6.000	20.000	76.000	311.998	1363.99	6232.0
$G_{1,3}$	3	20	175	1764	19404	226512	2	6	20	76	312	1364	6232
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}$ with $f_1 = 11.2.1\text{a}$ and $f_2 = 9.4.1\text{a}$ (Example 5.2)												
12	1.890	10.162	82.694	822.34	9158.8	109806	1.978	4.875	13.355	43.107	159.141	659.35	2956.4
16	1.953	10.421	83.282	807.34	8675.0	99267	1.984	4.908	13.519	43.624	160.110	654.03	2871.4
20	1.994	10.934	89.117	876.26	9535.1	110519	1.998	4.990	13.947	45.710	170.407	705.07	3129.3
24	1.999	10.988	89.877	887.78	9710.7	113199	2.000	4.998	13.991	45.958	171.816	713.20	3176.5
28	1.999	10.995	89.933	887.94	9706.5	113074	2.000	4.999	13.996	45.976	171.863	713.20	3175.4
32	2.000	10.999	89.976	888.64	9717.7	113248	2.000	5.000	13.999	45.993	171.957	713.75	3178.6
36	2.000	11.000	89.996	888.94	9721.9	113305	2.000	5.000	14.000	45.999	171.994	713.96	3179.7
40	2.000	11.000	89.999	888.99	9722.9	113321	2.000	5.000	14.000	46.000	171.999	714.00	3180.0
$N(G_{1,3})$	2	11	90	889	9723	113322	2	5	14	46	172	714	3180
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}$ with $f_1 = 11.2.1\text{a}$ and $f_2 = 5.4.1\text{a}$ (Example 5.3)												
12	2.044	9.914	65.414	507.34	4354.5	40032	2.055	5.121	14.257	43.862	146.697	525.70	1990.5
14	2.001	10.005	70.915	613.85	6062.0	65576	2.010	5.003	14.045	44.357	154.995	590.70	2415.8
16	2.004	10.034	70.308	591.59	5604.0	57723	2.008	5.011	14.048	44.208	153.040	574.31	2298.4
18	2.003	10.007	69.991	587.09	5530.5	56512	2.005	5.005	14.016	44.025	152.053	569.03	2270.0
20	2.001	9.986	69.679	583.37	5486.3	55954	2.002	5.003	14.001	43.943	151.537	566.12	2254.0
22	1.999	10.003	69.991	586.98	5522.8	56293	2.000	5.001	14.003	44.012	151.998	568.64	2266.5
24	2.000	10.001	69.991	587.39	5531.0	56416	2.000	5.001	14.002	44.006	151.988	568.71	2267.4
$G_{3,3}$	2	10	70	588	5544	56628	2	5	14	44	152	569	2270

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 = 27.2.1\mathrm{a}$ and $f_2 = 9.4.1\mathrm{a}$ (Example 5.5)												
12	3.724	39.781	517.581	7140.47	101446.7	1467990	1.908	7.388	34.459	175.801	935.697	5096.06	28152.9
16	3.968	43.330	567.235	7865.66	112441.4	1639111	1.991	7.926	37.448	192.185	1026.865	5613.14	31134.8
20	3.991	43.809	576.712	8037.78	115425.6	1689536	1.997	7.976	37.838	194.976	1045.677	5735.41	31912.2
24	3.995	43.912	578.580	8069.13	115936.1	1697732	1.998	7.988	37.926	195.553	1049.301	5757.72	32047.9
28	4.000	43.991	579.834	8089.14	116256.1	1702843	2.000	7.999	37.993	195.950	1051.676	5771.97	32133.4
32	4.000	43.998	579.954	8091.18	116289.9	1703399	2.000	8.000	37.998	195.986	1051.909	5773.42	32142.3
36	4.000	43.999	579.990	8091.83	116301.1	1703588	2.000	8.000	38.000	195.997	1051.980	5773.88	32145.2
40	4.000	44.000	579.999	8091.98	116303.6	1703628	2.000	8.000	38.000	196.000	1051.997	5773.98	32145.9
C_1	4	44	580	8092	116304	1703636	2	8	38	196	1052	5774	32146
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}$ with $f_1 = 27.2.1\mathrm{a}$ and $f_2 = 9.4.1\mathrm{a}$ (Example 5.5)												
12	1.835	19.607	255.106	3519.41	50001.3	723547	1.955	5.670	21.041	94.763	477.416	2544.22	13941.0
16	1.977	21.592	282.664	3919.60	56031.6	816799	1.995	5.956	22.675	103.796	527.759	2829.24	15579.2
20	1.994	21.890	288.170	4016.29	57675.5	844222	1.999	5.987	22.909	105.430	538.511	2897.87	16009.8
24	1.997	21.949	289.194	4033.23	57948.8	848585	1.999	5.993	22.958	105.747	540.482	2909.92	16082.7
28	2.000	21.995	289.906	4044.43	58126.0	851391	2.000	5.999	22.996	105.972	541.820	2917.88	16130.1
32	2.000	21.998	289.971	4045.51	58143.8	851683	2.000	6.000	22.999	105.991	541.944	2918.65	16134.8
36	2.000	22.000	289.994	4045.90	58150.3	851791	2.000	6.000	23.000	105.998	541.988	2918.93	16136.6
40	2.000	22.000	289.999	4045.98	58151.7	851813	2.000	6.000	23.000	106.000	541.998	2918.99	16136.9
$J(C_1)$	2	22	290	4046	58152	851818	2	6	23	106	542	2919	16137
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 = 27.2.1\mathrm{a}$ and $f_2 = f_{\psi^3} \otimes \chi_4$ (Example 5.6)												
12	3.958	33.913	358.055	4234.21	54093.7	729652	2.025	7.857	30.376	135.639	630.635	3101.81	15779.8
16	3.902	34.652	379.674	4634.68	60772.9	837263	1.958	7.794	30.841	141.438	673.384	3384.33	17542.5
20	3.999	35.925	398.551	4930.94	65486.8	912313	2.001	7.991	31.943	147.570	709.241	3596.40	18806.9
24	3.998	35.946	398.928	4937.19	65589.1	913955	2.000	7.995	31.957	147.687	709.930	3600.80	18833.7
28	4.000	35.989	399.773	4952.00	65836.4	917992	2.000	7.999	31.991	147.935	711.560	3611.19	18898.4
32	4.000	35.996	399.935	4954.99	65888.1	918864	2.000	7.999	31.996	147.979	711.877	3613.28	18911.7
36	4.000	35.999	399.990	4955.83	65901.2	919069	2.000	8.000	32.000	147.997	711.981	3613.88	18915.3
40	4.000	36.000	399.999	4955.97	65903.5	919108	2.000	8.000	32.000	148.000	711.997	3613.98	18915.9
C_2	4	36	400	4956	65904	919116	2	8	32	148	712	3614	18916

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}$ with $f_1 = 27.2.1\text{a}$ and $f_2 = f_{\psi^3} \otimes \chi_4$ (Example 5.6)												
12	1.951	16.715	176.479	2086.97	26661.8	359633	2.012	5.901	19.029	74.968	327.057	1561.28	7842.5
16	1.944	17.268	189.198	2309.55	30284.2	417223	1.979	5.891	19.382	78.508	351.613	1718.58	8806.0
20	1.998	17.951	199.147	2463.88	32722.2	455862	2.001	5.994	19.964	81.743	370.402	1829.06	9461.4
24	1.998	17.967	199.398	2467.78	32783.7	456826	2.000	5.997	19.975	81.822	370.853	1831.81	9477.8
28	2.000	17.994	199.879	2475.91	32917.0	458980	2.000	5.999	19.995	81.965	371.768	1837.53	9512.8
32	2.000	17.997	199.963	2477.45	32943.4	459423	2.000	6.000	19.998	81.988	371.932	1838.60	9519.7
36	2.000	18.000	199.994	2477.91	32950.5	459533	2.000	6.000	20.000	81.998	371.989	1838.93	9521.6
40	2.000	18.000	199.999	2477.98	32951.7	459553	2.000	6.000	20.000	82.000	371.998	1838.99	9521.9
$J(C_2)$	2	18	200	2478	32952	459558	2	6	20	82	372	1839	9522
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 = 27.2.1\text{a}$ and $f_2 = f_{\psi^3 \otimes (\frac{2}{\cdot})_6} \otimes \chi$ (Example 5.7)												
12	3.639	30.654	325.827	3854.12	48328.9	628104	1.879	7.240	27.488	123.398	576.740	2836.28	14308.3
16	3.957	35.065	382.912	4605.55	58579.1	773226	1.982	7.886	31.201	142.825	678.826	3375.60	17201.8
20	3.988	35.776	396.152	4836.03	62459.6	837365	1.997	7.974	31.819	146.821	704.618	3538.42	18217.8
24	3.999	35.962	399.222	4886.11	63268.3	850306	2.000	7.995	31.966	147.759	710.428	3574.03	18433.9
28	3.999	35.988	399.801	4896.72	63449.7	853322	2.000	7.998	31.989	147.936	711.613	3581.65	18481.6
32	4.000	35.997	399.937	4898.92	63486.2	853926	2.000	8.000	31.997	147.981	711.881	3583.25	18491.4
36	4.000	35.999	399.988	4899.80	63500.7	854161	2.000	8.000	31.999	147.996	711.977	3583.86	18495.1
40	4.000	36.000	399.998	4899.96	63503.3	854204	2.000	8.000	32.000	147.999	711.995	3583.97	18495.8
C_3	4	36	400	4900	63504	854216	2	8	32	148	712	3584	18496
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}$ with $f_1 = 27.2.1\text{a}$ and $f_2 = f_{\psi^3 \otimes (\frac{2}{\cdot})_6} \otimes \chi$ (Example 5.7)												
12	1.794	15.109	160.594	1899.63	23820.5	309582	1.941	5.597	17.605	68.935	300.493	1430.41	7117.2
16	1.972	17.474	190.812	2295.03	29191.0	385313	1.991	5.936	19.561	79.199	354.325	1714.23	8636.2
20	1.993	17.876	197.948	2416.45	31209.6	418412	1.999	5.986	19.902	81.368	368.092	1800.09	9167.0
24	1.999	17.975	199.545	2442.25	31623.7	425012	2.000	5.997	19.979	81.858	371.102	1818.43	9277.9
28	2.000	17.993	199.893	2448.27	31723.7	426646	2.000	5.999	19.994	81.966	371.794	1822.76	9304.5
32	2.000	17.998	199.965	2449.41	31742.5	426955	2.000	6.000	19.998	81.989	371.934	1823.59	9309.5
36	2.000	18.000	199.993	2449.89	31750.2	427079	2.000	6.000	20.000	81.998	371.987	1823.92	9311.5
40	2.000	18.000	199.999	2449.98	31751.6	427102	2.000	6.000	20.000	82.000	371.997	1823.98	9311.9
$J(C_3)$	2	18	200	2450	31752	427108	2	6	20	82	372	1824	9312

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}(i)$ with $f_1 =$ level 256 quartic twist of 32.2.1a and $f_2 =$ 32.4.1b (Example 5.8)												
12	3.956	35.366	385.901	4597.51	57507.0	741277	1.973	7.859	31.351	143.344	679.870	3357.91	16951.8
16	3.931	34.902	382.691	4623.07	59036.2	781489	1.968	7.838	31.057	142.474	678.992	3385.60	17299.0
20	3.983	35.704	395.127	4820.48	62212.4	832884	1.994	7.959	31.744	146.459	702.647	3527.38	18152.8
24	3.999	35.966	399.280	4887.18	63288.1	850214	2.000	7.998	31.975	147.796	710.614	3575.05	18439.9
28	3.999	35.980	399.691	4895.17	63428.5	852595	2.000	7.998	31.984	147.903	711.419	3580.56	18475.6
32	4.000	35.995	399.924	4898.78	63484.7	853470	2.000	7.999	31.996	147.976	711.855	3583.14	18490.8
36	4.000	35.999	399.990	4899.83	63501.1	853729	2.000	8.000	32.000	147.997	711.981	3583.88	18495.3
40	4.000	36.000	399.997	4899.95	63503.2	853763	2.000	8.000	32.000	147.999	711.994	3583.97	18495.8
C_4	4	36	400	4900	63504	853776	2	8	32	148	712	3584	18496
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}$ with $f_1 =$ level 256 quartic twist of 32.2.1a and $f_2 =$ 32.4.1b (Example 5.8)												
12	1.939	17.338	189.180	2253.84	28191.7	363397	1.987	5.892	19.447	78.428	349.606	1678.78	8375.5
16	1.957	17.379	190.556	2301.99	29396.2	389131	1.984	5.911	19.481	78.976	354.160	1717.95	8678.1
20	1.989	17.826	197.274	2406.71	31060.7	415833	1.997	5.977	19.855	81.134	366.833	1793.16	9127.2
24	1.999	17.977	199.576	2442.81	31633.9	424971	2.000	5.998	19.983	81.877	371.199	1818.96	9281.0
28	1.999	17.990	199.840	2447.52	31713.4	426286	2.000	5.999	19.991	81.950	371.700	1822.23	9301.6
32	2.000	17.997	199.958	2449.35	31741.8	426728	2.000	6.000	19.998	81.987	371.922	1823.54	9309.3
36	2.000	18.000	199.994	2449.91	31750.5	426863	2.000	6.000	20.000	81.998	371.989	1823.94	9311.6
40	2.000	18.000	199.998	2449.97	31751.6	426881	2.000	6.000	20.000	81.999	371.997	1823.98	9311.9
$J(C_4)$	2	18	200	2450	31752	426888	2	6	20	82	372	1824	9312
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 =$ 27.2.1a and $f_2 =$ level 576 sextic twist of 9.4.1a (§4.3) (Example 5.9)												
12	3.935	34.662	381.188	4653.42	60215.9	807506	2.027	7.831	30.936	141.517	678.383	3406.94	17565.0
16	3.945	35.020	384.306	4653.09	59626.5	792794	1.976	7.863	31.184	143.053	682.618	3409.08	17457.8
20	3.983	35.731	395.513	4825.30	62269.7	833562	1.995	7.965	31.770	146.569	703.148	3529.74	18164.4
24	3.999	35.953	399.062	4883.78	63239.0	849553	2.000	7.995	31.961	147.717	710.163	3572.51	18425.9
28	4.000	35.999	399.978	4899.61	63496.0	853613	2.000	8.000	31.998	147.992	711.953	3583.69	18493.8
32	3.999	35.992	399.876	4898.15	63476.2	853354	2.000	7.999	31.993	147.960	711.769	3582.68	18488.4
36	4.000	35.999	399.980	4899.66	63498.5	853688	2.000	8.000	31.999	147.994	711.962	3583.76	18494.5
40	4.000	36.000	399.997	4899.94	63503.0	853758	2.000	8.000	32.000	147.999	711.994	3583.96	18495.7
C_6	4	36	400	4900	63504	853776	2	8	32	148	712	3584	18496

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \oplus M_2$ over $K = \mathbb{Q}$ with $f_1 = 27.2.1\text{a}$ and $f_2 =$ level 576 sextic twist of 9.4.1a (§4.3) (Example 5.9)												
12	1.939	17.084	187.881	2293.59	29679.4	398006	2.013	5.888	19.305	77.865	350.591	1711.68	8722.4
16	1.966	17.451	191.507	2318.72	29713.0	395064	1.988	5.925	19.553	79.313	356.215	1730.91	8763.8
20	1.990	17.854	197.629	2411.09	31114.7	416512	1.998	5.981	19.877	81.242	367.357	1795.75	9140.4
24	1.999	17.971	199.465	2441.08	31609.0	424636	2.000	5.997	19.976	81.837	370.969	1817.67	9273.9
28	2.000	17.999	199.982	2449.72	31746.9	426791	2.000	6.000	19.999	81.993	371.964	1823.78	9310.6
32	2.000	17.996	199.934	2449.03	31737.4	426669	2.000	5.999	19.996	81.979	371.878	1823.30	9308.0
36	2.000	17.999	199.989	2449.82	31749.1	426842	2.000	6.000	20.000	81.997	371.980	1823.88	9311.2
40	2.000	18.000	199.998	2449.97	31751.5	426879	2.000	6.000	20.000	81.999	371.997	1823.98	9311.9
$J(C_6)$	2	18	200	2450	31752	426888	2	6	20	82	372	1824	9312
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega)$ with $E_1 \colon y^2 = x^3 + x + 1$ and $E_2 \colon y^2 = x^3 + 1$ (Example 6.1)												
12	2.111	13.965	128.743	1400.03	16758.8	212823	0.988	4.190	12.480	53.310	226.485	1062.80	5139.9
16	1.939	11.499	95.025	924.42	9930.1	114111	0.963	3.889	10.519	42.169	163.640	713.49	3195.5
20	1.984	11.832	98.258	960.44	10348.8	118999	0.995	3.966	10.844	43.318	168.953	737.94	3315.5
24	2.002	12.031	100.371	984.35	10635.6	122592	1.000	4.005	11.022	44.135	172.659	755.38	3399.8
28	2.000	11.996	99.964	979.61	10578.9	121894	1.000	3.999	10.997	43.986	171.939	751.71	3381.5
32	2.000	12.001	100.015	980.18	10585.9	121989	1.000	4.000	11.001	44.005	172.022	752.11	3383.5
36	2.000	11.999	99.983	979.78	10581.2	121931	1.000	4.000	10.999	43.994	171.970	751.85	3382.2
40	2.000	12.000	99.996	979.95	10583.3	121959	1.000	4.000	11.000	43.999	171.993	751.96	3382.8
U(2)	2	12	100	980	10584	121968	1	4	11	44	172	752	3383
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(i, \omega)$ with $E_1 \colon y^2 = x^3 - x$ and $E_2 \colon y^2 = x^3 + 1$ (Example 6.2)												
12	3.645	30.632	327.151	3892.89	49243.6	648190	1.899	7.140	27.528	123.913	582.112	2871.94	14551.7
16	3.937	34.821	381.322	4615.95	59193.8	787809	1.966	7.815	30.868	141.644	675.476	3375.44	17295.9
20	3.974	35.548	393.139	4796.22	61918.6	829320	1.992	7.941	31.621	145.818	699.274	3510.35	18067.4
24	3.997	35.945	399.071	4884.77	63256.3	849765	1.998	7.990	31.946	147.681	710.124	3572.81	18428.8
28	3.999	35.978	399.659	4894.77	63423.9	852548	1.999	7.996	31.980	147.885	711.343	3580.20	18474.0
32	4.000	35.999	399.958	4899.15	63488.5	853507	2.000	8.000	31.999	147.988	711.913	3583.40	18492.0
36	4.000	35.998	399.971	4899.51	63495.9	853646	2.000	8.000	31.999	147.991	711.942	3583.65	18493.8
40	4.000	36.000	399.996	4899.93	63502.8	853756	2.000	8.000	32.000	147.999	711.991	3583.95	18495.7
F	4	36	400	4900	63504	853776	2	8	32	148	712	3584	18496

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega)$ with $E_1\colon y^2 = x^3 - x$ and $E_2\colon y^2 = x^3 + 1$ (Example 6.2)												
12	1.760	14.792	157.976	1879.81	23778.9	313000	0.889	4.483	13.217	62.888	280.877	1396.83	7026.1
16	1.960	17.338	189.871	2298.41	29474.3	392273	0.979	4.904	15.378	73.561	336.382	1690.81	8612.3
20	1.983	17.741	196.207	2393.69	30902.3	413896	0.995	4.964	15.781	75.780	348.983	1761.96	9017.0
24	1.997	17.960	199.397	2440.70	31606.3	424589	0.998	4.993	15.961	76.791	354.814	1795.18	9208.0
28	1.999	17.988	199.818	2447.25	31710.2	426250	1.000	4.998	15.989	76.939	355.653	1800.00	9236.5
32	2.000	17.999	199.975	2449.52	31743.6	426744	1.000	5.000	15.999	76.993	355.948	1801.66	9245.8
36	2.000	17.999	199.984	2449.73	31747.7	426819	1.000	5.000	15.999	76.995	355.968	1801.81	9246.9
40	2.000	18.000	199.997	2449.96	31751.3	426877	1.000	5.000	16.000	76.999	355.995	1801.97	9247.8
F_c	2	18	200	2450	31752	426888	1	5	16	77	356	1802	9248
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega)$ with $E_1\colon y^2 = x^3 + 4$ and $E_2\colon y^2 = x^3 + 1$ (Example 6.4)												
12	3.639	30.654	325.827	3854.12	48328.9	628104	1.879	7.240	27.488	123.398	576.740	2836.28	14308.3
16	3.957	35.065	382.912	4605.55	58579.1	773226	1.982	7.886	31.201	142.825	678.826	3375.60	17201.8
20	3.988	35.776	396.152	4836.03	62459.6	837365	1.997	7.974	31.819	146.821	704.618	3538.42	18217.8
24	3.999	35.962	399.222	4886.11	63268.3	850306	2.000	7.995	31.966	147.759	710.428	3574.03	18433.9
28	3.999	35.988	399.801	4896.72	63449.7	853322	2.000	7.998	31.989	147.936	711.613	3581.65	18481.6
32	4.000	35.997	399.937	4898.92	63486.2	853926	2.000	8.000	31.997	147.981	711.881	3583.25	18491.4
36	4.000	35.999	399.988	4899.80	63500.7	854161	2.000	8.000	31.999	147.996	711.977	3583.86	18495.1
40	4.000	36.000	399.998	4899.96	63503.3	854204	2.000	8.000	32.000	147.999	711.995	3583.97	18495.8
C_3	4	36	400	4900	63504	854216	2	8	32	148	712	3584	18496
	$M = \mathrm{Sym}^3 M_1$ over $K = \mathbb{Q}(\omega)$ with $E_1\colon y^2 = x^3 + 1$ (Example 6.7)												
12	3.860	41.526	538.869	7414.00	105214.4	1523370	1.955	7.666	35.913	183.144	973.102	5291.48	29205.5
16	3.946	43.013	563.168	7811.99	111715.8	1629196	1.981	7.873	37.174	190.784	1019.598	5574.78	30929.7
20	3.985	43.768	576.467	8037.57	115456.1	1690342	1.995	7.967	37.804	194.863	1045.382	5735.23	31917.7
24	3.996	43.929	578.736	8070.58	115948.5	1697802	1.999	7.992	37.941	195.613	1049.552	5758.76	32052.1
28	3.999	43.982	579.715	8087.48	116232.8	1702512	2.000	7.998	37.985	195.910	1051.460	5770.79	32126.9
32	4.000	43.997	579.944	8091.09	116289.2	1703396	2.000	8.000	37.997	195.983	1051.893	5773.35	32142.1
36	4.000	43.999	579.980	8091.67	116298.6	1703548	2.000	8.000	37.999	195.994	1051.962	5773.77	32144.6
40	4.000	44.000	579.997	8091.95	116303.2	1703623	2.000	8.000	38.000	195.999	1051.995	5773.97	32145.8
C_1	4	44	580	8092	116304	1703636	2	8	38	196	1052	5774	32146

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = \mathrm{Sym}^3 M_1$ over $K = \mathbb{Q}$ with $E_1\colon y^2 = x^3 + 1$ (Example 6.7)												
12	1.903	20.468	265.599	3654.23	51858.3	750843	1.978	5.807	21.758	98.382	495.853	2640.53	14459.8
16	1.966	21.434	280.637	3892.86	55670.0	811858	1.991	5.930	22.538	103.098	524.138	2810.12	15477.0
20	1.991	21.870	288.047	4016.19	57690.7	844625	1.997	5.982	22.892	105.374	538.364	2897.78	16012.6
24	1.998	21.957	289.272	4033.95	57955.0	848620	2.000	5.995	22.966	105.777	540.607	2910.44	16084.8
28	1.999	21.990	289.847	4043.60	58114.3	851225	2.000	5.999	22.992	105.952	541.712	2917.29	16126.9
32	2.000	21.998	289.966	4045.46	58143.4	851681	2.000	6.000	22.998	105.990	541.937	2918.62	16134.7
36	2.000	21.999	289.989	4045.82	58149.1	851771	2.000	6.000	22.999	105.997	541.979	2918.87	16136.2
40	2.000	22.000	289.998	4045.97	58151.5	851811	2.000	6.000	23.000	105.999	541.997	2918.98	16136.9
$J(C_1)$	2	22	290	4046	58152	851818	2	6	23	106	542	2919	16137
	$M = \mathrm{Sym}^3 M_1$ over $K = \mathbb{Q}$ with $E_1\colon y^2 = x^3 + 1$ (Example 6.7)												
12	0.954	4.122	38.892	447.50	5499.8	70135	1.000	2.006	5.173	17.490	71.929	331.62	1623.4
16	0.979	3.741	30.989	328.20	3829.5	47298	1.000	1.965	4.802	14.987	56.932	246.65	1155.3
20	0.995	3.917	32.831	347.75	4041.6	49643	0.996	1.983	4.920	15.594	59.849	260.36	1219.4
24	1.001	4.005	34.076	365.31	4290.4	53178	1.000	2.000	5.003	16.020	62.140	272.91	1288.6
28	1.000	4.000	34.011	364.28	4274.8	52951	1.000	2.000	5.000	16.002	62.023	272.20	1284.4
32	1.000	4.000	34.001	364.01	4269.1	52847	1.000	2.000	5.000	16.001	62.002	272.01	1283.0
36	1.000	4.000	33.997	363.97	4268.7	52841	1.000	2.000	5.000	15.999	61.996	271.98	1282.9
40	1.000	4.000	33.999	363.99	4268.8	52842	1.000	2.000	5.000	16.000	61.998	271.99	1282.9
D	1	4	34	364	4269	52844	1	2	5	16	62	272	1283
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 = 27.2.1\mathrm{a}$ and $f_2 = f_{\psi^2}$ (Example 6.10)												
12	3.724	39.781	517.581	7140.47	101446.7	1467990	1.908	7.388	34.459	175.801	935.697	5096.06	28152.9
16	3.968	43.330	567.235	7865.66	112441.4	1639111	1.991	7.926	37.448	192.185	1026.865	5613.14	31134.8
20	3.991	43.809	576.712	8037.78	115425.6	1689536	1.997	7.976	37.838	194.976	1045.677	5735.41	31912.2
24	3.995	43.912	578.580	8069.13	115936.1	1697732	1.998	7.988	37.926	195.553	1049.301	5757.72	32047.9
28	4.000	43.991	579.834	8089.14	116256.1	1702843	2.000	7.999	37.993	195.950	1051.676	5771.97	32133.4
32	4.000	43.998	579.954	8091.18	116289.9	1703399	2.000	8.000	37.998	195.986	1051.909	5773.42	32142.3
36	4.000	43.999	579.990	8091.83	116301.1	1703588	2.000	8.000	38.000	195.997	1051.980	5773.88	32145.2
40	4.000	44.000	579.999	8091.98	116303.6	1703628	2.000	8.000	38.000	196.000	1051.997	5773.98	32145.9
C_1	4	44	580	8092	116304	1703636	2	8	38	196	1052	5774	32146

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}$ with $f_1 = 27.2.1\text{a}$ and $f_2 = f_{\psi^2}$ (Example 6.10)												
12	1.835	19.607	255.106	3519.41	50001.3	723547	1.955	5.670	21.041	94.763	477.416	2544.22	13941.0
16	1.977	21.592	282.664	3919.60	56031.6	816799	1.995	5.956	22.675	103.796	527.759	2829.24	15579.2
20	1.994	21.890	288.170	4016.29	57675.5	844222	1.999	5.987	22.909	105.430	538.511	2897.87	16009.8
24	1.997	21.949	289.194	4033.23	57948.8	848585	1.999	5.993	22.958	105.747	540.482	2909.92	16082.7
28	2.000	21.995	289.906	4044.43	58126.0	851391	2.000	5.999	22.996	105.972	541.820	2917.88	16130.1
32	2.000	21.998	289.971	4045.51	58143.8	851683	2.000	6.000	22.999	105.991	541.944	2918.65	16134.8
36	2.000	22.000	289.994	4045.90	58150.3	851791	2.000	6.000	23.000	105.998	541.988	2918.93	16136.6
40	2.000	22.000	289.999	4045.98	58151.7	851813	2.000	6.000	23.000	106.000	541.998	2918.99	16136.9
$J(C_1)$	2	22	290	4046	58152	851818	2	6	23	106	542	2919	16137
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(i)$ with $f_1 = 32.2.1\text{a}$ and $f_2 =$ level 576 quartic twist of f_{ψ^2} (§4.2) (Example 6.12)												
12	3.828	32.989	350.939	4159.81	53066.5	713569	1.908	7.546	29.322	132.113	616.295	3037.54	15451.9
16	3.930	34.768	380.034	4639.36	60922.4	840944	1.974	7.837	30.966	141.722	674.205	3388.58	17577.0
20	3.979	35.680	395.497	4891.89	64970.0	905221	1.993	7.954	31.725	146.497	703.659	3567.69	18655.8
24	3.995	35.911	398.564	4933.20	65544.7	913473	1.998	7.988	31.927	147.556	709.327	3598.01	18820.7
28	4.000	35.992	399.824	4952.59	65843.2	918079	2.000	7.999	31.993	147.948	711.635	3611.56	18900.2
32	4.000	35.997	399.944	4955.07	65889.1	918880	2.000	8.000	31.997	147.983	711.894	3613.35	18912.1
36	4.000	35.999	399.981	4955.69	65899.0	919035	2.000	8.000	31.999	147.994	711.964	3613.78	18914.7
40	4.000	36.000	399.995	4955.92	65902.7	919095	2.000	8.000	32.000	147.998	711.991	3613.94	18915.7
C_2	4	36	400	4956	65904	919116	2	8	32	148	712	3614	18916
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}$ with $f_1 = 32.2.1\text{a}$ and $f_2 =$ level 576 quartic twist of f_{ψ^2} (§4.2) (Example 6.12)												
12	1.877	16.172	172.041	2039.27	26014.9	349814	1.955	5.738	18.452	72.922	318.440	1521.72	7640.3
16	1.957	17.312	189.233	2310.10	30335.5	418736	1.987	5.911	19.436	78.601	351.777	1719.43	8816.5
20	1.987	17.814	197.459	2442.37	32437.5	451948	1.996	5.974	19.845	81.153	367.338	1813.28	9378.3
24	1.997	17.950	199.219	2465.81	32761.9	456591	1.999	5.993	19.960	81.757	370.555	1830.44	9471.3
28	2.000	17.996	199.907	2476.23	32920.7	459027	2.000	6.000	19.996	81.972	371.808	1837.73	9513.8
32	2.000	17.998	199.968	2477.50	32944.0	459432	2.000	6.000	19.998	81.990	371.941	1838.65	9519.9
36	2.000	17.999	199.990	2477.84	32949.4	459516	2.000	6.000	20.000	81.997	371.981	1838.89	9521.3
40	2.000	18.000	199.997	2477.96	32951.3	459547	2.000	6.000	20.000	81.999	371.995	1838.97	9521.8
$J(C_2)$	2	18	200	2478	32952	459558	2	6	20	82	372	1839	9522

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 = 36.2.1\text{a}$ and $f_2 = f_{\psi^2}$ (Example 6.13)												
12	3.749	31.983	340.480	4006.11	49839.4	642488	1.951	7.443	28.658	128.351	599.744	2936.99	14745.9
16	3.964	35.185	384.363	4624.51	58836.7	776721	1.987	7.913	31.337	143.436	681.908	3391.67	17288.5
20	3.987	35.740	395.657	4830.42	62399.9	836748	1.996	7.968	31.781	146.639	703.736	3534.46	18200.1
24	3.997	35.955	399.202	4886.09	63267.7	850284	1.999	7.994	31.959	147.748	710.396	3573.93	18433.2
28	3.999	35.988	399.792	4896.57	63447.9	853300	2.000	7.998	31.990	147.936	711.604	3581.58	18481.2
32	4.000	35.996	399.937	4898.94	63486.6	853932	2.000	8.000	31.997	147.981	711.880	3583.25	18491.4
36	4.000	36.000	399.998	4899.93	63502.3	854182	2.000	8.000	32.000	148.000	711.995	3583.95	18495.6
40	4.000	36.000	399.997	4899.95	63503.2	854202	2.000	8.000	32.000	147.999	711.994	3583.97	18495.8
C_3	4	36	400	4900	63504	854216	2	8	32	148	712	3584	18496
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 = 36.2.1\text{a}$ and $f_2 = f_{\psi^2}$ (Example 6.13)												
12	1.848	15.764	167.817	1974.54	24565.0	316671	1.976	5.697	18.182	71.376	311.831	1480.05	7332.9
16	1.975	17.534	191.535	2304.48	29319.4	387054	1.994	5.950	19.629	79.504	355.861	1722.24	8679.4
20	1.992	17.858	197.701	2413.65	31179.8	418104	1.998	5.983	19.883	81.277	367.651	1798.11	9158.2
24	1.998	17.971	199.535	2442.23	31623.4	425001	1.999	5.996	19.976	81.852	371.085	1818.38	9277.6
28	2.000	17.993	199.889	2448.20	31722.8	426634	2.000	5.999	19.994	81.966	371.790	1822.73	9304.3
32	2.000	17.998	199.964	2449.42	31742.7	426958	2.000	6.000	19.998	81.989	371.933	1823.59	9309.5
36	2.000	18.000	199.998	2449.96	31751.1	427089	2.000	6.000	20.000	82.000	371.996	1823.97	9311.8
40	2.000	18.000	199.998	2449.97	31751.6	427101	2.000	6.000	20.000	82.000	371.997	1823.98	9311.9
$J(C_3)$	2	18	200	2450	31752	427108	2	6	20	82	372	1824	9312
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(i, \omega)$ with $f_1 = 32.2.1\text{a}$ and $f_2 = f_{\psi^2}$ (Example 6.15)												
12	3.869	34.463	381.774	4669.32	60454.8	812582	1.863	7.519	30.279	140.449	677.403	3406.86	17563.1
16	3.964	35.792	398.231	4874.93	63064.3	845872	1.953	7.872	31.576	146.529	705.854	3554.26	18333.7
20	4.016	36.184	401.802	4916.79	63651.3	854877	1.997	8.028	32.108	148.611	714.639	3595.94	18547.2
24	3.997	35.935	398.962	4884.38	63269.9	850236	1.999	7.990	31.936	147.639	709.968	3572.63	18431.8
28	3.998	35.962	399.371	4890.17	63352.2	851433	2.000	7.995	31.968	147.800	710.801	3576.93	18454.5
32	4.000	35.993	399.900	4898.44	63478.9	853370	2.000	7.999	31.994	147.967	711.809	3582.87	18489.2
36	4.000	36.000	399.994	4899.89	63501.8	853735	2.000	8.000	32.000	147.998	711.988	3583.92	18495.4
40	4.000	36.000	399.996	4899.93	63502.8	853756	2.000	8.000	32.000	147.999	711.991	3583.95	18495.7
F	4	36	400	4900	63504	853776	2	8	32	148	712	3584	18496

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\sqrt{3})$ with $f_1 = 32.2.1\text{a}$ and $f_2 = f_{\psi^2}$ (Example 6.15)												
12	1.868	16.641	184.352	2254.74	29192.6	392382	1.930	5.692	18.744	76.066	343.596	1678.10	8546.9
16	1.967	17.762	197.624	2419.20	31295.9	419767	1.976	5.921	19.699	80.773	366.398	1796.05	9162.7
20	2.004	18.054	200.477	2453.21	31758.5	426537	1.999	6.010	20.028	82.165	372.599	1826.24	9318.1
24	1.997	17.956	199.351	2440.60	31614.3	424841	1.999	5.994	19.960	81.776	370.763	1817.17	9274.0
28	1.999	17.980	199.673	2444.93	31674.1	425690	2.000	5.998	19.983	81.896	371.379	1820.35	9290.7
32	2.000	17.996	199.943	2449.13	31738.3	426669	2.000	5.999	19.997	81.981	371.892	1823.37	9308.3
36	2.000	18.000	199.996	2449.93	31750.7	426865	2.000	6.000	20.000	81.999	371.992	1823.95	9311.7
40	2.000	18.000	199.997	2449.96	31751.3	426877	2.000	6.000	20.000	81.999	371.995	1823.97	9311.8
F_{ab}	2	18	200	2450	31752	426888	2	6	20	82	372	1824	9312
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(i)$ with $f_1 = 32.2.1\text{a}$ and $f_2 = f_{\psi^2}$ (Example 6.15)												
12	1.878	16.732	185.354	2266.99	29351.3	394514	0.902	4.672	14.724	71.213	329.033	1664.04	8527.8
16	1.975	17.836	198.443	2429.23	31425.7	421507	0.977	4.929	15.747	76.039	351.768	1781.20	9136.0
20	2.006	18.074	200.695	2455.87	31793.0	427000	1.000	5.012	16.042	77.235	356.966	1806.14	9264.1
24	1.997	17.955	199.341	2440.47	31612.7	424819	0.998	4.993	15.956	76.769	354.732	1795.06	9209.4
28	1.999	17.980	199.673	2444.93	31674.1	425689	1.000	4.997	15.983	76.895	355.378	1798.35	9226.7
32	2.000	17.996	199.945	2449.16	31738.7	426674	1.000	4.999	15.997	76.982	355.896	1801.39	9244.4
36	2.000	18.000	199.995	2449.92	31750.6	426864	1.000	5.000	16.000	76.998	355.991	1801.94	9247.6
40	2.000	18.000	199.997	2449.96	31751.3	426877	1.000	5.000	16.000	76.999	355.995	1801.97	9247.8
F_c	2	18	200	2450	31752	426888	1	5	16	77	356	1802	9248
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}$ with $f_1 = 32.2.1\text{a}$ and $f_2 = f_{\psi^2}$ (Example 6.15)												
12	0.921	8.202	90.866	1111.35	14388.9	193403	0.941	3.829	9.225	40.535	169.330	837.18	4212.7
16	0.984	8.881	98.812	1209.60	15648.0	209884	0.996	3.967	9.877	43.417	183.280	908.14	4581.6
20	1.002	9.024	100.201	1226.14	15873.3	213188	1.000	4.002	10.010	44.059	186.224	922.74	4657.3
24	0.998	8.974	99.639	1219.85	15801.3	212342	0.999	3.996	9.975	43.874	185.307	918.25	4635.2
28	0.999	8.990	99.834	1222.43	15836.6	212839	1.000	3.999	9.991	43.947	185.685	920.15	4645.2
32	1.000	8.998	99.971	1224.56	15869.1	213334	1.000	4.000	9.998	43.990	185.945	921.68	4654.1
36	1.000	9.000	99.997	1224.96	15875.3	213431	1.000	4.000	10.000	43.999	185.995	921.97	4655.8
40	1.000	9.000	99.999	1224.98	15875.6	213438	1.000	4.000	10.000	44.000	185.997	921.98	4655.9
$F_{ab,c}$	1	9	100	1225	15876	213444	1	4	10	44	186	922	4656

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(\omega)$ with $f_1 = 11.2.1\mathrm{a}$ and $f_2 = f_{\psi^2}$ (Example 6.16)												
12	2.050	12.696	106.126	1044.08	11462.6	135998	0.924	4.159	11.354	46.222	181.157	799.21	3637.5
16	1.962	11.714	98.200	972.33	10623.3	123788	0.982	3.930	10.743	43.127	169.178	744.94	3376.7
20	1.991	11.920	99.255	972.75	10517.0	121449	0.997	3.972	10.915	43.646	170.614	746.31	3360.0
24	1.997	11.962	99.543	974.22	10507.4	120917	0.999	3.993	10.968	43.835	171.159	747.59	3359.6
28	1.999	11.995	99.997	980.52	10596.9	122213	1.000	3.998	10.996	43.989	172.006	752.29	3385.8
32	2.000	11.999	99.995	979.98	10584.6	121987	1.000	4.000	10.999	43.997	171.988	751.97	3383.0
36	2.000	12.000	99.999	979.98	10583.7	121964	1.000	4.000	11.000	44.000	171.996	751.98	3382.9
40	2.000	12.000	99.998	979.97	10583.7	121964	1.000	4.000	11.000	43.999	171.995	751.98	3382.9
U(2)	2	12	100	980	10584	121968	1	4	11	44	172	752	3383
	$M = M_1 \otimes M_2$ over $K = \mathbb{Q}(i)$ with $f_1 = 32.2.1\mathrm{a}$ and $f_2 = f_{\psi^2}$ (Example 6.16)												
12	1.012	6.269	52.402	515.54	5659.9	67152	0.976	3.037	7.109	25.726	94.375	404.25	1813.2
16	0.978	5.838	48.942	484.60	5294.6	61695	0.991	2.958	6.862	24.490	89.347	381.27	1700.5
20	0.995	5.956	49.596	486.07	5255.2	60686	0.999	2.987	6.958	24.814	90.266	382.93	1696.5
24	0.998	5.979	49.755	486.95	5252.0	60439	0.999	2.996	6.982	24.911	90.552	383.68	1696.8
28	1.000	5.998	49.997	490.24	5298.3	61104	1.000	2.999	6.998	24.994	91.000	386.13	1710.3
32	1.000	6.000	49.996	489.98	5292.2	60992	1.000	3.000	7.000	24.998	90.992	385.98	1709.0
36	1.000	6.000	49.999	489.99	5291.8	60982	1.000	3.000	7.000	25.000	90.998	385.99	1708.9
40	1.000	6.000	49.999	489.99	5291.8	60982	1.000	3.000	7.000	25.000	90.998	385.99	1708.9
N(U(2))	1	6	50	490	5292	60984	1	3	7	25	91	386	1709
	M is the motive arising from the quintic threefold (1.1) with $t = 0$ (Example 7.2)												
12	0.937	8.579	96.545	1193.7	15578.9	210469	0.979	2.919	9.712	39.874	181.358	892.118	4568.07
16	0.991	8.881	97.986	1187.7	15199.3	201621	0.990	2.978	9.899	40.424	182.208	886.627	4488.21
20	0.997	8.944	99.076	1210.2	15640.6	209721	0.999	2.991	9.953	40.704	184.232	901.372	4592.70
24	0.998	8.961	99.440	1216.9	15756.9	211669	0.999	2.994	9.966	40.816	184.972	906.254	4623.50
26	1.000	8.997	99.945	1224.0	15858.9	213141	1.000	2.999	9.996	40.979	185.88	911.264	4651.34
28	1.000	8.996	99.920	1223.7	15855.2	213118	1.000	2.999	9.996	40.974	185.84	911.056	4650.38
F_{ac}	1	9	100	1225	15876	213444	1	3	10	41	186	912	4656

n	$M_2[a_1]$	$M_4[a_1]$	$M_6[a_1]$	$M_8[a_1]$	$M_{10}[a_1]$	$M_{12}[a_1]$	$M_1[a_2]$	$M_2[a_2]$	$M_3[a_2]$	$M_4[a_2]$	$M_5[a_2]$	$M_6[a_2]$	$M_7[a_2]$
	M is the motive arising from the quintic threefold (1.1) with $t = -5$ (Example 7.3)												
10	1.038	2.956	11.783	56.21	304.9	1800	0.999	2.002	3.826	9.221	22.507	61.02	170.2
13	0.974	2.833	12.281	65.88	404.0	2717	0.989	1.969	3.871	9.498	24.295	69.34	207.7
16	0.985	2.984	14.371	89.80	659.0	5372							
18	0.986	2.916	13.465	80.05	560.1	4384							
20	1.001	3.021	14.205	85.54	603.1	4740							
22	0.999	2.996	13.968	83.68	590.3	4673							
24	0.999	2.997	13.989	83.91	592.5	4693							
USp(4)	1	3	14	84	594	4719	1	2	4	10	27	82	268
	M corresponds to a Hilbert modular form over $\mathbb{Q}(\sqrt{5})$ of level $\Gamma_0(2\sqrt{5})$ and weight $(2,4)$ (Example 8.1)												
10	0.919	4.923	40.085	405.22	4599.98	55704.4							
11	0.935	5.067	40.982	404.54	4435.56	51570.9							
12	0.975	5.177	39.851	372.63	3898.24	43705.4							
13	0.985	5.143	38.553	348.98	3528.31	38283.2							
14	0.967	4.907	35.917	318.67	3171.62	34025.7							
15													
16													
$N(G_{3,3})$	1	5	35	294	2772	28314	1	3	7	23	76	287	1135

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