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Sato-Tate groups of $y^2 = x^8 + c$ and $y^2 = x^7 - cx$.

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Abstract. We consider the distribution of normalized Frobenius traces for two families of genus 3 hyperelliptic curves over $\mathbb{Q}$ that have large automorphism groups: $y^2 = x^8 + c$ and $y^2 = x^7 - cx$ with $c \in \mathbb{Q}^*$. We give efficient algorithms to compute the trace of Frobenius for curves in these families at primes of good reduction. Using data generated by these algorithms, we obtain a heuristic description of the Sato-Tate groups that arise, both generically and for particular values of c. We then prove that these heuristic descriptions are correct by explicitly computing the Sato-Tate groups via the correspondence between Sato-Tate groups and Galois endomorphism types.

Contents

1. Introduction

In this paper we consider two families of hyperelliptic curves over $\mathbb{Q}$:

$$C_1 \colon y^2 = x^8 + c, \qquad C_2 \colon y^2 = x^7 - cx.$$

For $c \in \mathbb{Q}^*$, these equations define hyperelliptic curves of genus 3 with good reduction at primes $p > 3$ for which $v_p(c) = 0$ (in fact, C_1 also has good reduction at 3). For each such p we have the *trace of Frobenius*

$$t_p(C_i) := p + 1 - \#\overline{C}_i(\mathbb{F}_p),$$

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where $\overline{C}_i$ denotes the reduction of C_i modulo p. From the Weil bounds, we know that t_p lies in the interval $[-6\sqrt{p}, 6\sqrt{p}]$. We wish to study the distribution of normalized Frobenius traces $t_p/\sqrt{p} \in [-6, 6]$, as p varies over primes of good reduction up to a bound N.

The generalized Sato-Tate conjecture predicts that as $N \to \infty$ this distribution converges to the distribution of traces in the *Sato-Tate group*, a compact subgroup of USp(6) associated to the Jacobian of the curve. For the two families considered here, the curves C_i have Jacobians that are $\mathbb{Q}$-isogenous to the product of an elliptic curve and an abelian surface.[1] This allows us to apply the classification of Sato-Tate groups for abelian surfaces obtained in [**FKRS12**] to determine the Sato-Tate groups that arise. This is achieved in §5.

After recalling the definition of the Sato-Tate group of an abelian variety in §2, we begin in §3 by deriving formulas for the Frobenius trace $t_p(C_i)$ in terms of the Hasse-Witt matrix of $\overline{C}_i$. These formulas allow us to design particularly efficient algorithms for computing $t_p(C_i)$. In §4, under the assumption of the Sato-Tate conjecture, we use the numerical data obtained by applying these algorithms to heuristically guess the isomorphism class of the Sato-Tate groups of C_1 and C_2. The explicit computation in §5 proves that, in fact, these guesses are correct, without appealing to the Sato-Tate conjecture.

Strictly speaking, §4 and §5 are independent of each other. However, we should emphasize that in the process of achieving our results, there was a constant and mutually beneficial interplay between the two distinct approaches.

Up to dimension 3, the Sato-Tate group of an abelian variety defined over a number field k is determined by its ring of endomorphisms over an algebraic closure of k. Although the Sato-Tate group does not capture the ring structure of the endomorphisms, it does codify the $\mathbb{R}$-algebra generated by the endomorphism ring, and the structure of this $\mathbb{R}$-algebra as a Galois module, what we refer to as the *Galois endomorphism type* of the abelian variety. As an example, in §6 we compute the Galois endomorphism type of the Jacobian of C_2.

The problem of analysing the Frobenius trace distributions and determining the Sato-Tate groups that arise in these two families was originally posed as part of a course given by the authors at the winter school *Frobenius Distributions on Curves* held in February, 2014, at the Centre International de Rencontres Mathématiques in Luminy. This problem turned out to be more challenging than we anticipated (the analogous question in genus 2 is quite straight-forward); this article represents a solution.

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2. Background

We start by briefly recalling the definition of the Sato-Tate group of an abelian variety A defined over a number field k, and set some notation. For a more detailed presentation we refer to [**Ser12**, Chap. 8] or [**FKRS12**, §2].

[1] As we shall see, this abelian surface may itself be $\mathbb{Q}$-isogenous to a product of elliptic curves and is in any case never simple over $\overline{\mathbb{Q}}$.

2.1. The Sato-Tate group of an abelian variety. Let $\overline{k}$ denote a fixed algebraic closure of k, and let g be the dimension of A. For each prime ℓ we have a continuous homomorphism

$$\varrho_{A,\ell}\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$$

arising from the action of $\mathrm{Gal}(\overline{k}/k)$ on the rational Tate module $(\varprojlim A[\ell^n]) \otimes \mathbb{Q}$. Here GSp denotes the group of symplectic similitudes, which preserve a symplectic form up to a scalar; in our setting the preserved symplectic form arises from the Weil pairing. Let G_ℓ be the Zariski closure of the image of $\varrho_{A,\ell}$, and let G_ℓ^1 be the kernel of the similitude character $G_\ell \to \mathbb{Q}_\ell^*$. We now choose an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, and for each prime ideal $\mathfrak{p}$ of the ring of integers of k, let $\mathrm{Frob}_\mathfrak{p}$ denote an arithmetic Frobenius at $\mathfrak{p}$ and let $N(\mathfrak{p})$ be the cardinality of its residue field.

Definition 2.1. The *Sato-Tate group of A*, denoted $\mathrm{ST}(A)$, is a maximal compact subgroup of $G_\ell^1 \otimes_\iota \mathbb{C}$. For each prime $\mathfrak{p}$ of good reduction for A, let $s(\mathfrak{p}) := \varrho_{A,\ell}(\mathrm{Frob}_\mathfrak{p}) \otimes_\iota N(\mathfrak{p})^{-1/2}$.

Let $\mathrm{USp}(2g)$ denote the group of $2g \times 2g$ complex matrices that are unitary and preserve a fixed symplectic form; this is a real Lie group of dimension $g(2g+1)$. One can show that $\mathrm{ST}(A)$ is well-defined up to conjugacy in $\mathrm{USp}(2g)$, and that $s(\mathfrak{p})$ determines a conjugacy class in $\mathrm{ST}(A)$.

Conjecture 2.2 (generalized Sato-Tate). *Let X denote the set of conjugacy classes of* $\mathrm{ST}(A)$*. Then:*

(i) The conjugacy class of $\mathrm{ST}(A)$ *in* $\mathrm{USp}(2g)$ *and the conjugacy classes* $s(\mathfrak{p})$ *in* $\mathrm{ST}(A)$ *are independent of the choice of the prime ℓ and the embedding ι.*

(ii) When the primes $\mathfrak{p}$ are ordered by norm, the $s(\mathfrak{p})$ are equidistributed on X with respect to the projection of the Haar measure of $\mathrm{ST}(A)$ *on X.*

It follows from [**BK15**] that part (*i*) of the above conjecture is true for $g \leq 3$. We next summarize some basic properties of the Sato-Tate group that we will need in our forthcoming discussion. If L/k is a field extension, we write A_L for the base change of A to L. We denote by K_A the minimal extension L/k over which all the endomorphisms of A are defined, that is, the minimal extension for which $\mathrm{End}(A_L) \simeq \mathrm{End}(A_{\bar{k}})$.

The Sato-Tate group $\mathrm{ST}(A)$ is a compact real Lie group, but it need not be connected. We use $\mathrm{ST}^0(A)$ to denote the connected component of the identity.

Proposition 2.3 (Prop. 2.17 of [**FKRS12**]). If $g \leq 3$, then the group of connected components $\mathrm{ST}(A)/\mathrm{ST}^0(A)$ is isomorphic to $\mathrm{Gal}(K_A/k)$.

This proposition implies, in particular, that a prime $\mathfrak{p}$ of good reduction for A splits completely in K_A if and only if $s(\mathfrak{p}) \in \mathrm{ST}^0(A)$. One can in fact show a little bit more: for any algebraic extension L/k, the Sato-Tate group $\mathrm{ST}(A_L)$ is a subgroup of $\mathrm{ST}(A)$ with $\mathrm{ST}^0(A_L) = \mathrm{ST}^0(A)$ and

$$\mathrm{ST}(A_L)/\mathrm{ST}^0(A_L) \simeq \mathrm{Gal}(K_A/(K_A \cap L)) \subseteq \mathrm{Gal}(K_A/k).$$

2.2. Galois endomorphism types. We now work in the category $\mathcal{C}$ of pairs (G, E), where G is a finite group and E is an $\mathbb{R}$-algebra equipped with an $\mathbb{R}$-linear action of G. A morphism $\Phi\colon (G,E) \to (G', E')$ of $\mathcal{C}$ consists of a pair $\Phi := (\phi_1, \phi_2)$, where $\phi_1\colon G \to G'$ is a morphism of groups and $\phi_2\colon E \to E'$ is an equivariant morphism of $\mathbb{R}$-algebras, that is,

$$\phi_2(\phi_1(g)e) = \phi_2(g)(\phi_1(e)) \qquad \text{for all } g \in G \text{ and } e \in E.$$

DEFINITION 2.4. The *Galois endomorphism type* of A is the isomorphism class in $\mathcal{C}$ of the pair $(\mathrm{Gal}(K_A/k), \mathrm{End}(A_{K_A}) \otimes_{\mathbb{Z}} \mathbb{R})$.

By [**FKRS12**, Prop. 2.19], for $g \leq 3$, the Galois endomorphism type is determined by the Sato-Tate group (in fact, the proof of this statement is effective, as we will illustrate in §6). This result admits a converse statement at least for $g \leq 2$.

THEOREM 2.5 (Thm. 4.3 of [**FKRS12**]). *For fixed $g \leq 2$, the Sato-Tate group and the Galois endomorphism type of an abelian variety A defined over a number field k uniquely determine each other. For $g = 1$ (resp. $g = 2$) there are* 3 *(resp.* 52*) possibilities for the Galois endomorphism type, all of which arise for some choice of A and k.*

For $g = 1$ the 3 possible Sato-Tate groups are $\mathrm{SU}(2) = \mathrm{USp}(2)$, a copy of the unitary group $\mathrm{U}(1)$ embedded in $\mathrm{SU}(2)$, and its normalizer in $\mathrm{SU}(2)$; these arise, respectively, for elliptic curves E/k without CM, with CM by a field contained in k, and with CM by a field not contained in k. For $g = 2$ a complete list of the 52 possible Sato-Tate groups can be found in [**FKRS12**].

In order to simplify the notation, when C is a smooth projective curve defined over the number field k, we may simply write

$$\mathrm{ST}(C) := \mathrm{ST}(\mathrm{Jac}(C)), \qquad \mathrm{ST}^0(C) := \mathrm{ST}^0(\mathrm{Jac}(C)), \qquad \text{and} \qquad K_C := K_{\mathrm{Jac}(C)}\,.$$

3. Trace formulas

Let p be an odd prime, and let $\overline{C}/\mathbb{F}_p$ be a smooth projective curve of genus $g \geq 1$ defined by an equation of the form $y^2 = f(x)$ with $f \in \mathbb{F}_p[x]$ squarefree. Let $n = (p-1)/2$ and let f_k^n denote the coefficient of x^k in the polynomial $f(x)^n$. The *Hasse–Witt* matrix of $\overline{C}$ is the $g \times g$ matrix $W_p := [w_{ij}]$ over $\mathbb{F}_p$, where

$$w_{ij} := f^n_{ip-j} \qquad (1 \leq i, j \leq g).$$

It is shown in [**Man61**, **Yui78**] that the characteristic polynomial $\chi(\lambda)$ of the Frobenius endomorphism of $\mathrm{Jac}(\overline{C})$ satisfies

$$\chi(\lambda) \equiv (-1)^g \lambda^g \det(W_p - \lambda I) \bmod p.$$

In particular,

$$\operatorname{tr} W_p \equiv t_p \bmod p,$$

where $t_p := p + 1 - \#\overline{C}(\mathbb{F}_p)$ is the trace of Frobenius. The Weil bounds imply $|t_p| \leq 2g\sqrt{p}$, which means that for all $p \geq 16g^2$, the trace of W_p uniquely determines the integer t_p.

Let us now specialize to the case where $f(x)$ has the form

$$f(x) = ax^d + bx^e,$$

with $d \in \{2g+1, 2g+2\}$, $e \in \{0, 1\}$, and $a, b \in \mathbb{F}_p^*$; this includes the families C_i defined in §1. Writing

$$f(x)^n = x^{en}(ax^{d-e} + b)^n$$

and applying the binomial theorem yields

$$f^n_{en+(d-e)r} = \binom{n}{r} a^r b^{n-r},$$

and we have $f^n_k = 0$ whenever k is not of the form $k = en + (d-e)r$. Setting $k = ip - j = i(2n+1) - j$ and solving for $r = r_{ij}$ yields

$$r_{ij} := \frac{(2i-e)n + i - j}{d-e} \qquad (1 \le i, j \le g).$$

The entries of the Hasse-Witt matrix for $y^2 = ax^d + bx^e$ are thus given by

$$w_{ij} = \begin{cases} \binom{n}{r_{ij}} a^{r_{ij}} b^{n-r_{ij}} & \text{if } r_{ij} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For any fixed integer $i \in [1, g]$, the quantity $(2i - e)n + i - j$ lies in an interval of width $g - 1 < (d - e)/2$, as j varies over integers in $[1, g]$. This implies that at most one entry w_{ij} in each row of W_p is nonzero, and for this entry r_{ij} is simply the nearest integer to $(2i - e)n/(d - e)$.

We now specialize to the two families of interest and assume $p > 3$. For $C_1\colon y^2 = x^8 + c$ we have $d = 8, e = 0, a = 1$, and $b = \overline{c}$, where $\overline{c}$ denotes the image of c in $\mathbb{F}_p$. We thus have

$$r_{ij} = \frac{2in + i - j}{8} = \frac{ip - j}{8}.$$

For integers $i, j \in [1, 3]$, the integral values of r_{ij} that arise are listed below:

$$\begin{aligned} p \equiv 1 \bmod 8: &\quad r_{11} = \tfrac{n}{4}, \quad r_{22} = \tfrac{n}{2}, \quad r_{33} = \tfrac{3n}{4}; \\ p \equiv 3 \bmod 8: &\quad r_{13} = \tfrac{n-1}{4}, \quad r_{31} = \tfrac{3n+1}{4}; \\ p \equiv 5 \bmod 8: &\quad r_{22} = \tfrac{n}{2}; \\ p \equiv 7 \bmod 8: &\quad \text{none.} \end{aligned}$$

This yields the following formulas for the trace of Frobenius:

$$t_p(C_1) \equiv_p \begin{cases} \binom{n}{n/2}\overline{c}^{n/2} + \binom{n}{n/4}\overline{c}^{n/4} + \binom{n}{n/4}\overline{c}^{3n/4} & \text{if } p \equiv 1 \bmod 8, \\ \binom{n}{n/2}\overline{c}^{n/2} & \text{if } p \equiv 5 \bmod 8, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

For $C_2\colon y^2 = x^7 - cx$ we have $d = 7, e = 1, a = 1$, and $b = -\overline{c}$. We thus have

$$r_{ij} = \frac{(2i-1)n + i - j}{6}.$$

For integers $i, j \in [1, 3]$, the integral values of r_{ij} that arise are listed below:

$$\begin{aligned} p \equiv 1 \bmod 12: &\quad r_{11} = \tfrac{n}{6}, \quad r_{22} = \tfrac{n}{2}, \quad r_{33} = \tfrac{5n}{6}; \\ p \equiv 5 \bmod 12: &\quad r_{13} = \tfrac{n-2}{6}, \quad r_{22} = \tfrac{n}{2}, \quad r_{31} = \tfrac{5n+2}{6}; \\ p \equiv 7 \bmod 12: &\quad \text{none;} \\ p \equiv 11 \bmod 12: &\quad \text{none.} \end{aligned}$$

This yields the following formulas for the trace of Frobenius:

$$t_p(C_2) \equiv_p \begin{cases} \binom{n}{n/2}(-\overline{c})^{n/2} + \binom{n}{n/6}(-\overline{c})^{n/6} + \binom{n}{n/6}(-\overline{c})^{5n/6} & \text{if } p \equiv 1 \bmod 12, \\ \binom{n}{n/2}(-\overline{c})^{n/2} & \text{if } p \equiv 5 \bmod 12, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

3.1. Algorithms. Computing the powers of $\overline{c}$ that appear in the formulas (2) and (3) for $t_p(C_i)$ is straight-forward; using binary exponentiation this requires just $O(\log p)$ multiplication in $\mathbb{F}_p$. The only potential difficulty is the computation of the binomial coefficients $\binom{n}{n/2}, \binom{n}{n/4}, \binom{n}{n/6}$ modulo p, where $n = (p-1)/2$ and p is known to lie in a suitable residue class. Fortunately, there are very efficient formulas for computing these particular binomial coefficients modulo suitable primes p. These are given by the lemmas below, in which $(\frac{2}{p}) \in \{\pm 1\}$ denotes the Legendre symbol, and m, x, and y denote integers.

LEMMA 3.1. *Let* $p = 4m + 1 = x^2 + y^2$ *be prime, with* $x \equiv -\left(\frac{2}{p}\right) \bmod 4$. *Then*

$$\binom{2m}{m} \equiv 2(-1)^{m+1}x \bmod p.$$

PROOF. See [**BEW98**, Thm. 9.2.2]. □

LEMMA 3.2. *Let* $p = 8m+1 = x^2+2y^2$ *be prime, with* $x \equiv -\left(\frac{2}{p}\right) \bmod 4$. *Then*

$$\binom{4m}{m} \equiv 2(-1)^{m+1}x \bmod p.$$

PROOF. See [**BEW98**, Thm. 9.2.8]. □

LEMMA 3.3. *Let* $p = 12m + 1 = x^2 + y^2$ *be prime, with* $x \equiv -\left(\frac{2}{p}\right) \bmod 4$, *and define* ϵ *to be* 0 *if* $x \equiv 0 \bmod 3$ *and* 1 *otherwise. Then*

$$\binom{6m}{m} \equiv 2(-1)^{m+\epsilon}x \bmod p.$$

PROOF. See [**BEW98**, Thm. 9.2.10] (replace ρ_4^2 with $(-1)^{\epsilon-1}$). □

To apply these lemmas, one uses Cornacchia's algorithm to find a solution (x, y) to $p = x^2 + dy^2$, where $d = 1$ when computing $\binom{n}{n/2} \bmod p$ or $\binom{n}{n/6} \bmod p$, and $d = 2$ when computing $\binom{n}{n/4}$. Cornacchia's algorithm requires as input a square-root δ of $-d$ modulo p (if no such δ exists then $p = x^2 + dy^2$ has no solutions).

CORNACCHIA'S ALGORITHM

Given integers $1 \le d < m$ and an integer $\delta \in [0, m/2]$ such that $\delta^2 \equiv -d \bmod m$, find a solution (x, y) to $x^2 + dy^2 = m$ or determine that none exist as follows:

1. Set $x_0 := m$, $x_1 := \delta$, and $i = 1$.
2. While $x_i^2 \ge m$, set $x_{i+1} := x_{i-1} \bmod x_i$ with $x_{i+1} \in [0, x_i)$ and increment i.
3. If $(m - x_i^2)/d = y^2$ for some $y \in \mathbb{Z}$, output the solution (x_i, y).
   Otherwise, report that no solution exists.

See [**Bas04**] for a simple proof of the correctness of this algorithm. We now consider its computational complexity, using $\mathsf{M}(n)$ to denote the time to multiply two n-bit integers; we may take $\mathsf{M}(n) = O(n \log n \log\log n)$ via [**SS71**]. The first two steps correspond to half of the standard Euclidean algorithm for computing the GCD of m and δ, whose bit-complexity is bounded by $O(\log^2 m)$; see [**GG13**, Thm. 3.13]. The time required in step 3 to perform a division and check whether the result is a square integer is also $O(\mathsf{M}(\log m))$; see [**GG13**, Thm. 9.8, Thm. 9.28]). Thus the overall complexity is $O(\log^2 m)$, the same as the Euclidean algorithm.

REMARK 3.4. There is an asymptotically faster version of the Euclidean algorithm that allows one to compute any particular pair of remainders (x_{i-1}, x_i), including the unique pair for which $x_{i-1} \geq \sqrt{m} > x_i$, in $O(\mathsf{M}(\log m) \log\log m)$ time; see [**PW03**]. This yields a faster version of Cornacchia's algorithm that runs in quasi-linear time, but we will not use this.

We now turn to the problem of computing the square-root δ of $-d \bmod m$ that is required by Cornacchia's algorithm. There are two basic strategies for doing this:

1. (Cipolla-Lehmer) Use a probabilistic root-finding algorithm to factor $x^2 + d$ in $\mathbb{F}_p[x]$. This takes $O(\mathsf{M}(\log p) \log p)$ expected time.
2. (Tonelli-Shanks) Given a generator g for the 2-Sylow subgroup of $\mathbb{F}_p^*$, compute the discrete logarithm e of $(-d)^s \in \langle g \rangle$ and let $\delta = g^{-e/2}(-d)^{(s+1)/2}$, where $p = 2^v s + 1$ with s odd. This takes $O(\mathsf{M}(\log p)(\log p + v \log v / \log\log v))$ time if the algorithm in [**Sut11**] is used to compute the discrete logarithm.

We will exploit both approaches. To obtain a generator for the 2-Sylow subgroup of $\mathbb{F}_p^*$ one may take α^s for any quadratic non-residue α. Half the elements of $\mathbb{F}_p^*$ are non-residues, so randomly selecting elements and computing Legendre symbols will yield a non-residue after 2 attempts, on average, and each attempt takes $O(\mathsf{M}(\log p) \log\log p)$ time, via [**BZ10**]. Unfortunately, we know of no efficient way to deterministically obtain a quadratic non-residue modulo p without assuming the generalized Riemann hypothesis (GRH). Under the GRH the least non-residue is $O((\log p)^2)$ [**Bac90**], thus if we simply test increasing integers $2, 3, \ldots$ we can obtain a non-residue α for a total cost of $O(\mathsf{M}(\log p) \log^2 p \log\log p)$.

But we are actually interested in computing $t_p(C_i)$ for many primes $p \leq N$, for some large bound N; on average, this approach will find a non-residue very quickly. As $N \to \infty$ the average value of the least non-residue converges to

$$\sum_{k=1}^{\infty} \frac{p_k}{2^k} = 3.674643966\ldots,$$

where p_k denotes the kth prime, as shown by Erdös [**Erd61**].

Finally, we should mention an alternative approach to solving $p = x^2 + dy^2$ that is completely deterministic. Construct an elliptic curve $E/\mathbb{F}_p$ with complex multiplication by the imaginary quadratic order $\mathcal{O}$ with discriminant $D = -d$ (or $D = -4d$ if $-d \not\equiv 0, 1 \bmod 4$) and then use Schoof's algorithm [**Sch85**] to compute the trace of Frobenius t of E. We then have $4p = t^2 - v^2 D$, since the Frobenius endomorphism with trace t and norm p corresponds to $\frac{t \pm v\sqrt{D}}{2} \in \mathcal{O}$, and therefore $(t/v)^2 \equiv D \bmod p$. If $D = -d$, we have a square root of $-d$ modulo p and can use Cornacchia's algorithm to solve $p = x^2 + dy^2$. If $D = -4d$, then t is even and $(t/2, 2v)$ is already a solution to $p = x^2 + dy^2$. We are specifically interested in the cases $D = -4$ and $D = -8$. For $D = -4$ we can take $E\colon y^2 = x^3 - x$, and for $D = -8$ we can take $E\colon y^2 = x^4 + 4x^2 + 2x$; see §3 of Appendix A in [**Sil94**].

We collect all of these observations in the following theorem.

THEOREM 3.5. *Let $C_1\colon y^2 = x^8 + c$ and $C_2\colon y^2 = x^7 - cx$ be as above. Let $p > 3$ be a prime with $v_p(c) = 0$. We can compute $t_p(C_i)$:*

- *probabilistically in $O(\mathsf{M}(\log p) \log p)$ expected time;*
- *deterministically in $O(\mathsf{M}(\log p) \log^2 p \log\log p)$ time, assuming GRH;*
- *deterministically in $O(\mathsf{M}(\log^3 p) \log^2 p / \log\log p)$ time.*

For any positive integer N, we can compute $t_p(C_i)$ for all $3 < p \le N$ with $v_p(c) = 0$ deterministically in $O(N\,\mathsf{M}(\log N))$ time.

Proof. Since we are computing asymptotic bounds, we may assume $p \ge 144$ (if not, just count points naïvely). Then $t_p(C_i) \bmod p$ uniquely determines $t_p(C_i) \in \mathbb{Z}$.

For the first bound we use the Cipolla-Lehmer approach to probabilistically compute the square root required by Cornacchia's algorithm in $O(\mathsf{M}(\log p)\log p)$ time, matching the time required to apply any of Lemmas 2.1-4, and the time required by the exponentiations of $\overline{c}$ needed to compute $t_p(C_i)$.

For the second bound we instead use the Tonelli-Shanks approach to computing square roots, relying on iteratively testing increasing integers to find a non-residue. Under the GRH this takes $O(\mathsf{M}(\log p)\log^2 p \log\log p)$ time, which dominates everything else.

For the third bound, we instead use Schoof's approach to solve $p = x^2 + dy^2$. The analysis in [**SS14**, Cor. 11] shows that Schoof's algorithm can be implemented to run in $O(\mathsf{M}(\log^3 p)\log^2 p/\log\log p)$ time.

For the final bound we proceed as in the GRH bound but instead rely on the Erdös bound for the least non-residue modulo $p \le N$, on average. By the prime number theorem there are $O(N/\log N)$ primes $p \le N$; the total number of quadratic residue tests is thus $O(N/\log N)$. It takes $O(\mathsf{M}(\log N)\log\log N)$ time for each test, so the total time spent finding non-residues is $O(N\,\mathsf{M}(\log N)\log\log N/\log N)$. The average 2-adic valuation of $p-1$ over primes $p \le N$ is $O(1)$, so the total time spent computing square roots modulo primes $p \le N$ using the Tonelli-Shanks approach is $O((N/\log N)\,\mathsf{M}(\log N)\log N)) = O(N\,\mathsf{M}(\log N))$ which dominates the time spent finding non-residues and matches the time spent on everything else. ☐

We note that the average time per prime $p \le N$ using a deterministic algorithm is $O(\mathsf{M}(\log p)\log p)$, which matches the expected time when applying our probabilistic approach for any particular prime $p \le N$; both bounds are quasi-quadratic $O((\log p)^{2+o(1)})$. For comparison, the average time per prime $p \le N$ achieved using the average polynomial time algorithm in [**HS14a**, **HS16**] is $O((\log p)^{4+o(1)})$.

Remark 3.6. Although Theorem 3.5 only addresses the computation of $t_p(C_i)$, for $p \not\equiv 3 \bmod 8$ (resp. $p \not\equiv 5 \bmod 12$) we can readily compute the entire Hasse-Witt matrix W_p for C_1 (resp. C_2) using the same approach and within the same complexity bounds.

4. Guessing Sato-Tate groups

In this section we analyze the Sato-Tate distributions of the curves C_i and arrive at a heuristic characterization of their Sato-Tate groups up to isomorphism, based on statistics collected using the algorithm described in §3.1. In §5 we will unconditionally prove that our heuristic characterizations are correct.

4.1. The Sato-Tate distribution of C_1. Before applying any heuristics we can derive some information about the structure of the Sato-Tate group directly from the formulas developed in the previous section. The possible shapes of the Hasse-Witt matrix for C_1 at a primes $p \equiv 1, 3, 5, 7 \bmod 8$ are depicted below, with the residue class of $p \bmod 8$ in parentheses:

$$\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} (1), \qquad \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ * & 0 & 0 \end{bmatrix} (3), \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{bmatrix} (5), \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (7).$$

From this we can (unconditionally) conclude the following:

(a) the component group $\mathrm{ST}(C_1)/\mathrm{ST}^0(C_1)$ has order divisible by 4;
(b) we have $s(p)$ in $\mathrm{ST}^0(C_1)$ only if $p \equiv 1 \bmod 8$;
(c) the field K_{C_1} contains $\mathbb{Q}(i, \sqrt{2})$.

We note that (c) follows immediately from (b): a prime $p > 2$ splits completely in $\mathbb{Q}(i, \sqrt{2})$ if and only if $p \equiv 1 \bmod 8$.

Table 1 lists moment statistics M_n for the curve $C_1 \colon y^2 = x^8 + c$ for selected values of c, where M_n is the average value of the nth power of the normalized L-polynomial coefficient

$$a_1 := -t_p/\sqrt{p},$$

over odd primes $p \leq 2^{40}$ not dividing c. The moment statistics M_n for odd n are all close to zero, so we list M_n only for even n.

c	M_2	M_4	M_6	M_8	M_{10}
1	3.000	50.999	1229.971	33634.058	978107.050
2	2.000	27.000	619.987	16834.560	489116.939
3	2.000	24.000	469.984	11234.520	297593.517
4	3.000	51.000	1229.990	33634.650	978125.742
5	2.000	23.999	469.976	11234.211	297585.653
6	2.000	23.999	469.979	11234.275	297587.173
7	2.000	23.999	469.968	11234.007	297579.866
8	2.000	27.000	619.987	16834.560	498116.939
9	2.000	27.000	619.991	16834.654	498118.664
2^4	3.000	50.999	1229.971	33634.058	978107.050
3^3	2.000	24.000	469.987	11234.520	297594.971
2^5	2.000	27.000	619.987	16834.560	498116.939
2^6	3.000	51.000	1229.990	33634.650	978125.742
3^4	3.000	51.000	1229.990	33634.593	978121.494

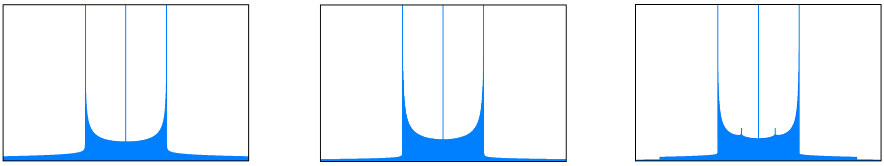
TABLE 1. Trace moment statistics for $C_1 \colon y^2 = x^8 + c$ for $p \leq 2^{40}$.

There appear to be three distinct trace distributions that arise, depending on whether the integer c is in

$$\mathbb{Q}(i, \sqrt{2})^{*4}, \quad \mathbb{Q}(i, \sqrt{2})^{*2} \setminus \mathbb{Q}(i, \sqrt{2})^{*4}, \quad \text{or} \quad \mathbb{Q}(i, \sqrt{2})^{*} \setminus \mathbb{Q}(i, \sqrt{2})^{*2};$$

these can be distinguished by whether the nearest integer to M_4 is 51, 27, or 24, respectively. Histogram plots of representative examples are shown with $c = 1, 2, 3$ in Figure 1. We note that in each histogram the central spike at 0 has area 1/2, while the spikes at -2 and 2 have area zero.

Based on the data in Table 1, we expect K_{C_1} to contain $\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{c})$. If we now require c to be a fourth-power and restrict to primes $p \equiv 1 \bmod 8$, we can



$y^2 = x^8 + 1$ $y^2 = x^8 + 2$ $y^2 = x^8 + 3$

FIGURE 1. a_1-histograms for three representative curves C_1.

investigate the Sato-Tate distribution of C_1 over the number field $\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{c})$. For $c = 1$ we obtain the moments listed below:

c	M_2	M_4	M_6	M_8	M_{10}
1	10.000	197.997	4899.892	134466.452	3912182.569

The corresponding histogram is shown in Figure 2.

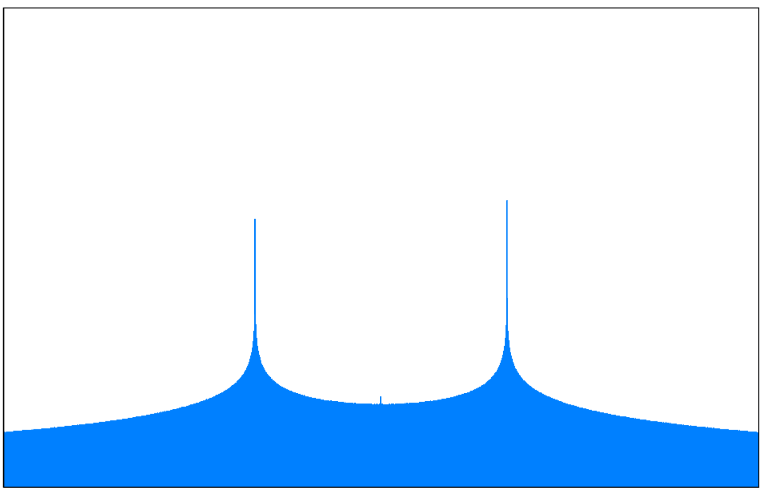


FIGURE 2. a_1-histogram for $y^2 = x^8 + 1$ over $\mathbb{Q}(i, \sqrt{2})$.

We claim that this distribution corresponds to a connected Sato-Tate group, namely, the group

$$\mathrm{U}(1)_2 \times \mathrm{U}(1) := \left\langle \begin{bmatrix} U(u) & 0 & 0 \\ 0 & U(u) & 0 \\ 0 & 0 & U(v) \end{bmatrix} : u, v \in \mathrm{U}(1) \right\rangle,$$

where for $u \in \mathrm{U}(1) := \{e^{i\theta} : \theta \in [0, 2\pi)\}$ the matrix $U(u)$ is defined by

$$U(u) := \begin{bmatrix} u & 0 \\ 0 & \overline{u} \end{bmatrix}. \tag{4}$$

The a_1-moment sequence for $\mathrm{U}(1)_2 \times \mathrm{U}(1)$ can be computed as the binomial convolution of the a_1-moment sequences for $\mathrm{U}(1)_2$ and $\mathrm{U}(1)$ given in [**FKRS12**]. Explicitly, if $M_n(G)$ denotes the nth moment of a_1 (or any class function), for $G = H_1 \times H_2$, we have

$$M_n(G) = \sum_{k=0}^{n} \binom{n}{k} M_k(H_1) M_{n-k}(H_2). \tag{5}$$

Applying this to $G = \mathrm{U}(1)_2 \times \mathrm{U}(1)$ yields:

	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
$\mathrm{U}(1)_2$	1	0	8	0	96	0	1280	0	17920	0	258048
$\mathrm{U}(1)$	1	0	2	0	6	0	20	0	70	0	252
$\mathrm{U}(1)_2 \times \mathrm{U}(1)$	1	0	10	0	198	0	4900	0	1344700	0	3912300

This is in close agreement (within 0.1%) with the moment statistics for $y^2 = x^8 + 1$ over $\mathbb{Q}(i, \sqrt{2})$. We thus conjecture that the identity component is

$$\mathrm{ST}^0(C_1) = \mathrm{U}(1)_2 \times \mathrm{U}(1),$$

up to conjugacy in $\mathrm{USp}(6)$, and

$$K_{C_1} = \mathbb{Q}(i, \sqrt{2}, \sqrt[4]{c}).$$

For generic c the component group of $\mathrm{ST}(C_1)$ is then isomorphic to

$$\mathrm{Gal}(K_{C_1}/\mathbb{Q}) \simeq \mathrm{D}_4 \times \mathrm{C}_2,$$

where D_4 is the dihedral group of order 8 and C_2 is the cyclic group of order 2.

4.2. The Sato-Tate distribution of C_2. The possible shapes of the Hasse-Witt matrix for C_2 at a primes $p \equiv 1, 5, 7, 11 \bmod 12$ are depicted below, with the residue class of p mod 12 in parentheses:

$$\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} (1), \quad \begin{bmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{bmatrix} (5), \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (7), \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (11).$$

From this information we can conclude that:

(a) the order of the component group $\mathrm{ST}(C_2)/\mathrm{ST}^0(C_2)$ is a multiple of 4;
(b) we have $s(p) \in \mathrm{ST}^0(C_2)$ only if $p \equiv 1 \bmod 12$;
(c) the field K_{C_2} contains $\mathbb{Q}(i, \sqrt{3})$.

We note that (c) follows immediately from (b): a prime $p > 3$ splits completely in $\mathbb{Q}(i, \sqrt{3})$ if and only if $p \equiv 1 \bmod 12$.

Table 2 lists moment statistics M_n for the curve $C_2 \colon y^2 = x^7 - cx$ for various values of c. There now appear to be just two distinct trace distributions that arise, depending on whether the integer c is a cube or not; these can be distinguished by whether the nearest integer to M_2 is 2 or 3, respectively. Histogram plots of three representative examples are shown for $c = 1, 2$ in Figure 3. In the histogram for $c = 1$ the central spike at 0 has area 1/2 and the spikes at -2 and 2 have area zero, but in the histogram for $c = 2$ the central spike has area 7/12, while the spikes at $-4, -2, 2, 4$ have area zero. This gives us a further piece of information: the order of the component group $\mathrm{ST}(C_2)/\mathrm{ST}^0(C_2)$ should be divisible by 12.

Based on the data in Table 1, we expect K_{C_2} to contain $\mathbb{Q}(i, \sqrt{3}, \sqrt[3]{c})$. We now require c to be a cube and restrict to primes $p \equiv 1 \bmod 12$ in order to investigate

c	M_2	M_4	M_6	M_8	M_{10}
1	3.000	62.999	1829.927	57434.041	1860104.868
2	2.000	29.999	719.982	20649.366	641569.043
3	2.000	29.999	719.972	20649.083	641561.180
4	2.000	30.000	719.985	20649.447	641572.217
5	2.000	30.000	719.988	20649.586	641578.161
6	2.000	30.000	720.004	20650.090	641593.419
7	2.000	30.000	719.991	20649.656	641579.324
8	3.000	62.999	1829.978	57434.221	1860110.123
9	2.000	29.999	719.973	20649.084	641561.181
2^4	2.000	30.000	719.985	20649.447	641572.217
3^3	3.000	62.999	1829.972	57434.041	1860104.867
2^5	2.000	29.999	719.982	20649.366	641569.043
2^6	3.000	62.999	1829.972	57434.041	1860104.868
3^4	2.000	29.999	719.973	20649.084	641561.181

TABLE 2. Trace moment statistics for $C_2\colon y^2 = x^7 - cx$ for $p \le 2^{40}$.

$y^2 = x^7 - x$ $\qquad$ $y^2 = x^7 - 2x$

FIGURE 3. a_1-histograms for two representative curves C_2.

the Sato-Tate distribution of C_2 over the number field $\mathbb{Q}(i, \sqrt{3}, \sqrt[3]{c})$. For $c = 1$ we obtain the moments listed below:

c	M_2	M_4	M_6	M_8	M_{10}
1	10.000	245.997	7299.909	229666.846	7440189.620

The corresponding histogram is shown in Figure 4, and is clearly *not* the distribution of the identity component; one can see directly that there are (at least) two components.

This suggests that we should try computing the Sato-Tate distribution over a quadratic extension of $\mathbb{Q}(i, \sqrt{3})$. After a bit of experimentation, one finds that $\mathbb{Q}(i, \sqrt[4]{-3})$ works. With $c = 1$ we obtain the moments statistics:

c	M_2	M_4	M_6	M_8	M_{10}
1	18.000	485.994	14579.770	459261.673	14880044.545

The corresponding histogram is shown in Figure 5.

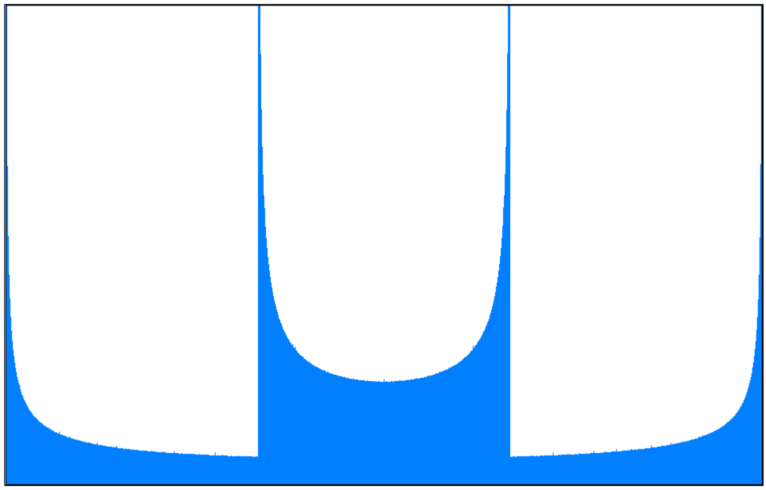


FIGURE 4. a_1-histogram for $y^2 = x^7 - x$ over $\mathbb{Q}(i, \sqrt{3})$.

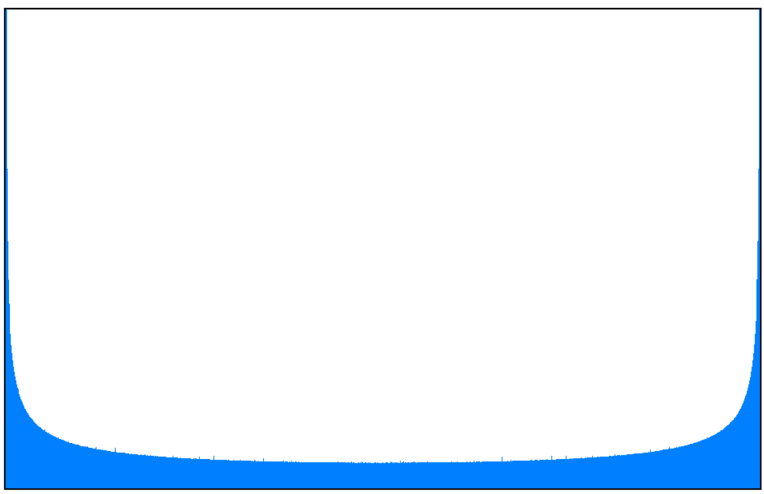


FIGURE 5. a_1-histogram for $y^2 = x^7 - x$ over $\mathbb{Q}(i, \sqrt[4]{-3})$.

We claim that this distribution corresponds to a connected Sato-Tate group, namely, the group

$$\mathrm{U}(1)_3 := \left\langle \begin{bmatrix} U(u) & 0 & 0 \\ 0 & U(u) & 0 \\ 0 & 0 & U(u) \end{bmatrix} : u \in \mathrm{U}(1) \right\rangle.$$

The a_1-moment sequence for $\mathrm{U}(1)_3$ can be computed as the $3a_1$-moment sequence for $\mathrm{U}(1)$, which simply scales the nth moment by 3^n. This yields the moments:

	M_2	M_4	M_6	M_8	M_{10}
$\mathrm{U}(1)_3$	18	486	14580	459270	14880348

which are in close agreement (better than 0.1%) with the moment statistics for $y^2 = x^7 - x$ over the field $\mathbb{Q}(i, \sqrt[4]{-3})$.

A complication arises if we repeat the experiment using a cube $c \neq 1$; we no longer get a connected Sato-Tate group! Taking c to be a sixth-power works, but we now need to ask whether, generically, the degree 48 extension $\mathbb{Q}(i, \sqrt[4]{-3}, \sqrt[6]{c})$ is the minimal extension required to get a connected Sato-Tate group. We have good reason to believe that a degree 24 extension *is* necessary, since K_{C_2} appears to properly contain the degree 12 field $\mathbb{Q}(i, \sqrt{3}, \sqrt[3]{c})$, but it is not clear that a degree 48 extension is required. We thus check various quadratic subextensions of $\mathbb{Q}(i, \sqrt[4]{-3}, \sqrt[6]{c})$ and find that $\mathbb{Q}(i, \sqrt[3]{c}, \sqrt{c\sqrt{-3}})$ works consistently.

We thus conjecture that

$$K_{C_2} = \mathbb{Q}(i, \sqrt[3]{c}, \sqrt{c\sqrt{-3}}).$$

This implies that for generic c, the component group of $\mathrm{ST}(C_2)$ is isomorphic to

$$\mathrm{Gal}(K_{C_2}/\mathbb{Q}) \simeq \mathrm{C}_3 \rtimes \mathrm{D}_4 \qquad (\text{GAP id} : \langle 24, 8 \rangle).$$

As noted above, we conjecture that the identity component is

$$\mathrm{ST}^0(C_2) = \mathrm{U}(1)_3,$$

up to conjugacy in $\mathrm{USp}(6)$.

REMARK 4.1. While we are able to give a general description of the Sato-Tate group in both cases just by looking at the a_1-distribution of the curves C_i, it should be noted that our characterization of the Sato-Tate group in terms of its identity component and the isomorphism type of its component group is far from sufficient to determine the Sato-Tate distribution. For this we need an explicit description of the Sato-Tate group as a subgroup (up to conjugacy) of $\mathrm{USp}(6)$; this is addressed in the next section.

5. Determining Sato-Tate groups

In this section we compute the Sato-Tate groups of the curves $C_1\colon y^2 = x^8 + c$ and $C_2\colon y^2 = x^7 - cx$ for *generic* values of $c \in \mathbb{Q}^*$. The meaning of generic will be specified in each case, but it ensures that the order of the group of components of the Sato-Tate group is as large as possible. The Sato-Tate groups for the non-generic cases can then be obtained as subgroups.

The description of the Sato-Tate group in terms of the *twisted Lefschetz group* introduced by Banaszak and Kedlaya [**BK15**] is a useful tool for explicitly determining Sato-Tate groups (see [**FGL16**], for example, where this is exploited), but here we take a different approach that is better suited to our special situation. Our strategy is to identify an elliptic quotient of each of the curves C_1 and C_2 and then use the classification results of [**FKRS12**] to identify the Sato-Tate group of the complement abelian surface. We then reconstruct the Sato-Tate group of the curves C_1 and C_2 from this data.

To determine the splitting of the Jacobians of C_1 and C_2 we benefit from the fact that these are curves with large automorphism groups. For generic c, the automorphism group of C_1 over K_{C_1} has order 32 (GAP id $\langle 32, 9 \rangle$), and the automorphism group of C_2 over K_{C_2} has order 24 (GAP id $\langle 24, 5 \rangle$).

We start by fixing the following matrix notations:

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; K := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \; Z_n := \begin{bmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{bmatrix}.$$

Also, for $u \in \mathrm{U}(1)$, recall the notation $U(u)$ introduced in (4). Whenever we consider matrices of the unitary symplectic group $\mathrm{USp}(6)$, we do it with respect to the symplectic form given by the matrix

$$H := \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}. \tag{6}$$

If A and A' are two abelian varieties defined over k, we write $A \sim A'$ to indicate that A and A' are related by an isogeny defined over k. Finally, we let ζ_3 denote a primitive third root of unity in $\overline{\mathbb{Q}}$.

5.1. Sato-Tate group of $C_1\colon y^2 = x^8 + c$.

LEMMA 5.1. *Let $c \in \mathbb{Q}^*$ and $C_1\colon y^2 = x^8 + c$. Then*

$$\mathrm{Jac}(C_1) \sim E \times \mathrm{Jac}(C)\,,$$

where $E\colon y^2 = x^4 + c$ and $C\colon y^2 = x^5 + cx$ over $\mathbb{Q}$. Thus $K_{C_1} = \mathbb{Q}(i, \sqrt{-2}, c^{1/4})$.

PROOF. First note that we can write nonconstant morphisms defined over $\mathbb{Q}$:

$$\begin{aligned} &\phi_E : C_1 \to E\,, \qquad \phi_E(x,y) = (x^2, y)\,, \\ &\phi_C : C_1 \to C\,, \qquad \phi_C(x,y) = (x^2, xy)\,. \end{aligned} \tag{7}$$

We clearly have that $K_E = \mathbb{Q}(i)$. To see that $K_C = \mathbb{Q}(i, \sqrt{-2}, c^{1/4})$, first set $F = \mathbb{Q}(c^{1/4})$, and consider the automorphism

$$\alpha\colon C_F \to C_F, \qquad \alpha(x,y) = \left(\frac{c^{1/2}}{x}, \frac{c^{3/4}}{x^3}\, y \right).$$

Since α has order 2 and is nonhyperelliptic, $C_F/\langle\alpha\rangle$ is an elliptic curve E' defined over F. Poincaré's decomposition theorem implies that $\mathrm{Jac}(C)_F \sim E' \times E''$, where E'' is an elliptic curve defined over F. Observe that we also have the automorphism

$$\gamma\colon C_{F(i)} \to C_{F(i)}, \qquad \gamma(x,y) = (-x, iy)\,.$$

Since α and γ do not commute, we deduce that $\mathrm{End}(\mathrm{Jac}(C)_{F(i)})$ is nonabelian. It follows that $E'_{F(i)}$ and $E''_{F(i)}$ are $F(i)$-isogenous and that $\mathrm{Jac}(C)_{F(i)} \sim E'^2_{F(i)}$. One may readily find an equation for the quotient curve $E' = C_F/\langle\alpha\rangle$, and, by computing its j-invariant, determine that E' has complex multiplication by $\mathbb{Q}(\sqrt{-2})$. From this we may conclude that $K_C = F(i, \sqrt{-2})$. The asserted splitting of the Jacobian $\mathrm{Jac}(C_1)$ follows from the existence of the morphisms of equation (7) and the fact that E and E' are not $\overline{\mathbb{Q}}$-isogenous. This latter fact also implies that K_{C_1} is the compositum of K_E and K_C. □

DEFINITION 5.2. In this subsection, we say that $c \in \mathbb{Q}^*$ is generic if $[K_{C_1} : \mathbb{Q}]$ is maximal, that is, $[K_{C_1} : \mathbb{Q}] = 16$. Equivalently, $c \notin \mathbb{Q}(i, \sqrt{-2})^{*2}$.

Corollary 5.3. *For generic $c \in \mathbb{Q}^*$, the Sato-Tate group of* $\mathrm{Jac}(C_1)$ *is*

$$\left\langle \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}, \begin{bmatrix} 0 & J & 0 \\ -J & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \begin{bmatrix} Z_8 & 0 & 0 \\ 0 & \overline{Z}_8 & 0 \\ 0 & 0 & I \end{bmatrix}, \begin{bmatrix} U(u) & 0 & 0 \\ 0 & U(u) & 0 \\ 0 & 0 & U(v) \end{bmatrix} : u, v \in \mathrm{U}(1) \right\rangle .$$

Proof. Recall the notations of Lemma 5.1. It follows from the description of $\mathrm{Jac}(C)$ given in the proof of the lemma and the results of [**FKRS12**] that $\mathrm{ST}(C)$ can be presented as

$$\left\langle R := \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, S := \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}, T := \begin{bmatrix} Z_8 & 0 \\ 0 & \overline{Z}_8 \end{bmatrix}, \begin{bmatrix} U(u) & 0 \\ 0 & U(u) \end{bmatrix} : u \in \mathrm{U}(1) \right\rangle .$$

This is the group named $J(D_4)$ in [**FKRS12**]. Since $E/\mathbb{Q}$ has CM, we also have

$$\mathrm{ST}(E) = \langle J, U(u) : u \in \mathrm{U}(1) \rangle .$$

Since E is not a $\overline{\mathbb{Q}}$-isogeny factor of $\mathrm{Jac}(C)$, we have $\mathrm{ST}^0(C_1) \simeq \mathrm{ST}^0(E) \oplus \mathrm{ST}^0(C)$, which proves the part of the corollary concerning the identity component. By Proposition 2.3, we have isomorphisms

$$\begin{aligned} \psi_E \colon\ & \mathrm{ST}(E)/\,\mathrm{ST}^0(E) \xrightarrow{\sim} \mathrm{Gal}(K_E/\mathbb{Q}) , \\ \psi_C \colon\ & \mathrm{ST}(C)/\,\mathrm{ST}^0(C) \xrightarrow{\sim} \mathrm{Gal}(K_C/\mathbb{Q}) , \\ \psi_{C_1} \colon\ & \mathrm{ST}(C_1)/\,\mathrm{ST}^0(C_1) \xrightarrow{\sim} \mathrm{Gal}(K_{C_1}/\mathbb{Q}) . \end{aligned} \tag{8}$$

The isomorphism ψ_E identifies J with the nontrivial automorphism of K_E, whereas the isomorphism ψ_C identifies the images of the generators $g = R, S, T$ in

$$\mathrm{ST}(C)/\,\mathrm{ST}^0(C) \simeq \langle R, S, T \rangle / \langle -1 \rangle$$

with automorphisms $\sigma = r, s, t \in \mathrm{Gal}(K_{C_1}/\mathbb{Q}) = \mathrm{Gal}(K_C/\mathbb{Q})$ as indicated below:

g	$\sigma = \psi_C(g)$	$\sigma(i)$	$\sigma(\sqrt{-2})$	$\sigma(c^{1/4})$
R	r	$-i$	$\sqrt{-2}$	$c^{1/4}$
S	s	i	$-\sqrt{-2}$	$c^{1/4}$
T	t	i	$\sqrt{-2}$	$ic^{1/4}$

Let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be the three first generators of $\mathrm{ST}(C_1)$. To check the part of the theorem concerning the group of components of $\mathrm{ST}(C_1)$, one only needs to verify that $\mathcal{R}, \mathcal{S}, \mathcal{T}$ generate a group of components isomorphic to

$$\mathrm{Gal}(K_{C_1}/\mathbb{Q}) \simeq \langle \mathcal{R}, \mathcal{S}, \mathcal{T} \,|\, \mathcal{R}^2, \mathcal{S}^2, \mathcal{T}^4, \mathcal{RSRS}, \mathcal{RTRT}, \mathcal{STST}^3 \rangle,$$

and that their natural projections onto $\mathrm{ST}(E)/\,\mathrm{ST}^0(E)$ and onto $\mathrm{ST}(C)/\,\mathrm{ST}^0(C)$ are compatible with the isomorphisms of (8). In this case, this amounts to noting that $\mathcal{R}, \mathcal{S}, \mathcal{T}$ project onto R, S, T in $\mathrm{ST}(C)$; that the automorphism r restricts to the non-trivial element of $\mathrm{Gal}(K_E/\mathbb{Q})$, while $\mathcal{R}$ projects down to J in $\mathrm{ST}(E)$; and that the restrictions of s and t to K_E are trivial, as are the projections of $\mathcal{S}$ and $\mathcal{T}$ to $\mathrm{ST}(E)$. □

Remark 5.4. We note that even though $\mathrm{Jac}(C_1) \sim E \times \mathrm{Jac}(C)$, in the generic case the Sato-Tate group $\mathrm{ST}(C_1)$ is *not* isomorphic to the direct sum of $\mathrm{ST}(E)$ and $\mathrm{ST}(C)$, because $\mathrm{Gal}(K_{C_1}/\mathbb{Q})$ is not isomorphic to the direct product of $\mathrm{Gal}(K_E/\mathbb{Q})$ and $\mathrm{Gal}(K_C/\mathbb{Q})$. This highlights the importance of being able to write down an explicit description for $\mathrm{ST}(C_1)$ in terms of generators.

REMARK 5.5. To treat non-generic values of c, one replaces Z_8 in the third generator for $\mathrm{ST}(C_1)$ in Corollary 5.3 with Z_4 or Z_2 when $c \in \mathbb{Q}(i, \sqrt{2})^{*2} \backslash \mathbb{Q}(i, \sqrt{2})^{*4}$ or $c \in \mathbb{Q}(i, \sqrt{2})^{*4}$, respectively (in the latter case one can simply remove $\mathcal{T}$ since it is already realized by $u = -1$ and $v = 1$).

Using the explicit representation of $\mathrm{ST}(C_1)$ given in Corollary 5.3 one may compute moment sequences using the techniques described in §3.2 of [**FKS16**]. The table below lists moments not only for a_1, but also for a_2 and a_3, where a_i denotes the coefficient of T^i in the characteristic polynomial of a random element of $\mathrm{ST}(C_1)$ distributed according to the Haar measure (these correspond to normalized L-polynomial coefficients of $\mathrm{Jac}(C_1)$):

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
a_1:	0	2	0	24	0	470	0	11235
a_2:	2	9	56	492	5172	59691	726945	9178434
a_3:	0	9	0	1245	0	284880	0	79208745

The a_1 moments closely match the corresponding moment statistics listed in Table 1 in the cases where c is generic, as expected. For a further comparison, we computed moment statistics for a_1, a_2 a_3 by applying the algorithm of [**HS16**] to the curve $y^2 = x^8 + 3$ over primes $p \leq 2^{30}$. The a_1-moment statistics listed below have less resolution than those in Table 1, which covers $p \leq 2^{40}$ (with this higher bound we get $M_8 \approx 11234$, an even better match to the value 11235 predicted by the Sato-Tate group $\mathrm{ST}(C_1)$).

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
a_1:	0.00	2.00	0.00	23.98	0.04	469.26	1	11210
a_2:	2.00	9.00	55.95	491.22	5160.77	59527.55	724556	9143413
a_3:	0.00	8.99	0.04	1242.59	10.30	283980.23	2972	78866094

5.2. Sato-Tate group of $C_2\colon y^2 = x^7 - cx$.

LEMMA 5.6. *Let $c \in \mathbb{Q}^*$ and $C_2\colon y^2 = x^7 - cx$. Set $F := \mathbb{Q}(\zeta_3, c^{1/3})$. Then*

$$\mathrm{Jac}(C_2) \sim E \times A\,,$$

where $E\colon y^2 = x^3 - cx$ and A is an abelian surface defined over $\mathbb{Q}$ for which $A_F \sim E' \times E''$, where E' and E'' are elliptic curves defined over F by the equations

$$E'\colon y^2 = x^3 + 3c^{1/3}x\,, \qquad E''\colon y^2 = x^3 + 3\zeta_3 c^{1/3}x\,.$$

Thus $K_{C_2} = \mathbb{Q}(i, c^{1/3}, \sqrt{c\sqrt{-3}})$.

PROOF. We can write nonconstant morphisms:

$$\phi_E\colon (C_2)_F \to E_F\,, \qquad \phi_E(x,y) = (x^3, xy)\,,$$

$$\phi_{E'}\colon (C_2)_F \to E'\,, \qquad \phi_{E'}(x,y) = \left(\frac{x^2 - c^{1/3}}{x}, \frac{y}{x^2}\right),$$

$$\phi_{E''}\colon (C_2)_F \to E''\,, \qquad \phi_{E''}(x,y) = \left(\frac{x^2 - \zeta_3 c^{1/3}}{x}, \frac{y}{x^2}\right).$$

Note that the morphisms ϕ_E, $\phi_{E'}$, and $\phi_{E''}$ are quotient maps given by automorphisms α_E, $\alpha_{E'}$, and $\alpha_{E''}$ of $(C_2)_F$:

$$\alpha_E(x,y) := (\zeta_3 x, \zeta_3^2 y)\,, \quad \alpha_{E'}(x,y) := \left(\frac{-c^{1/3}}{x}, \frac{c^{2/3}y}{x^4}\right)\,, \quad \alpha_{E''} := \alpha_E \circ \alpha_{E'}\,.$$

To see that $\mathrm{Jac}(C_2)_F \sim E_F \times E' \times E''$, it is enough to check that we have an isomorphism of F-vector spaces of regular differential forms

$$\Omega_{(C_2)_F} = \phi_E^*(\Omega_{E_F}) \oplus \phi_{E'}^*(\Omega_{E'}) \oplus \phi_{E''}^*(\Omega_{E''})\,,$$

But this follows from the fact that $\omega_1 = dx/y$, $\omega_2 = x \cdot dx/y$, and $\omega_3 = x^2 \cdot dx/y$ constitute a basis for $\Omega_{(C_2)_F}$, together with the easy computation

$$\phi_E^*\left(\frac{dx}{y}\right) = 3\omega_2, \quad \phi_{E'}^*\left(\frac{dx}{y}\right) = c^{1/3}\omega_1 + \omega_3, \quad \phi_{E''}^*\left(\frac{dx}{y}\right) = \zeta_3 c^{1/3}\omega_1 + \omega_3.$$

Since E is defined over $\mathbb{Q}$, there exists an abelian surface A defined over $\mathbb{Q}$ such that $A_F \sim E' \times E''$. To see that $K_{C_2} = \mathbb{Q}(i, c^{1/3}, \sqrt{c\sqrt{-3}})$, first note that $F(i) \subseteq K_{C_2}$ and that $E' \sim E''$. Therefore, K_{C_2} is the minimal extension of $F(i)$ over which E and E' become isomorphic. Now observe that we have an isomorphism

$$(9) \quad \psi\colon E_{F(i,\sqrt{c\sqrt{-3}})} \to E'_{F(i,\sqrt{c\sqrt{-3}})}\,, \qquad \psi(x,y) = \left(\frac{\sqrt{-3}}{c^{1/3}}x, \frac{\sqrt{-3}\sqrt{c\sqrt{-3}}}{c}y\right),$$

from which we see that K_{C_2} is the extension of $F(i)$ obtained by adjoining the element $\sqrt{c\sqrt{-3}}$ to $F(i)$ (note: one needs to write formula in (9) carefully, otherwise one may be tempted to make K_{C_2} too large). □

DEFINITION 5.7. In this subsection, we say that $c \in \mathbb{Q}^*$ is generic if $[K_{C_2} : \mathbb{Q}]$ is maximal, that is, $[K_{C_2} : \mathbb{Q}] = 24$. Equivalently, c is not a cube in $\mathbb{Q}^*$.

COROLLARY 5.8. *For generic $c \in \mathbb{Q}^*$, the Sato-Tate group of* $\mathrm{Jac}(C_2)$ *is*

$$\left\langle \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}, \begin{bmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & J \end{bmatrix}, \begin{bmatrix} Z_3 & 0 & 0 \\ 0 & \overline{Z}_3 & 0 \\ 0 & 0 & I \end{bmatrix}, \begin{bmatrix} U(u) & 0 & 0 \\ 0 & U(u) & 0 \\ 0 & 0 & U(u) \end{bmatrix} : u \in \mathrm{U}(1) \right\rangle.$$

PROOF. We assume the notations of Lemma 5.6. Since E, E', and E'' are K_{C_2}-isogenous, we have $\mathrm{ST}^0(C_2) \simeq \mathrm{U}(1)$. Note that $K_A = \mathbb{Q}(i, \zeta_3, c^{1/3})$. We claim that $\mathrm{ST}(A)$ is the group named $D_{6,1}$ in [**FKRS12**]. As may be seen in [**FKRS12**, Table 8], there are three Sato-Tate groups with identity component $\mathrm{U}(1)$ and group of components isomorphic to $\mathrm{Gal}(K_A/\mathbb{Q}) \simeq D_6$, namely, $J(D_3)$, $D_{6,1}$, and $D_{6,2}$. We can rule out the latter option, since by [**FKRS12**, Table 2] this would imply that $\mathrm{Gal}(K_A/\mathbb{Q}(i))$ is a cyclic group of order 6, which is false. To rule out $J(D_3)$, we need to argue along the lines of [**FKRS12**, §4.6]: Let F denote $\mathbb{Q}(\zeta_3, c^{1/3})$ as in Lemma 5.6; if $\mathrm{ST}(A) = J(D_3)$, then $\mathrm{ST}(A_F) = J(C_1)$, whereas if $\mathrm{ST}(A) = D_{6,1}$, then $\mathrm{ST}(A_F) = C_{2,1}$. By the dictionary between Sato-Tate groups and Galois endomorphism types in dimension 2 given by Theorem 2.5 (see [**FKRS12**, Table 8]), the first option would imply that $\mathrm{End}(A_F) \otimes_{\mathbb{Z}} \mathbb{R}$ is isomorphic to the Hamilton quaternion algebra $\mathbb{H}$, whereas the second option would yield $\mathrm{End}(A_F) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$. Since Lemma 5.6, ensures that we are in the latter case, we must have $\mathrm{ST}(A) = D_{6,1}$.

For convenience, we take the following presentation of $D_{6,1}$, which is conjugate to the one given in [**FKRS12**]:

$$\left\langle R := \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, S := \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}, T := \begin{bmatrix} Z_3 & 0 \\ 0 & \overline{Z}_3 \end{bmatrix}, \begin{bmatrix} U(u) & 0 \\ 0 & U(u) \end{bmatrix} : u \in \mathrm{U}(1) \right\rangle .$$

Since $E/\mathbb{Q}$ has CM, we have

$$\mathrm{ST}(E) = \langle J, U(u) : u \in \mathrm{U}(1) \rangle .$$

By Proposition 2.3, we have isomorphisms

$$\psi_E\colon \mathrm{ST}(E)/\mathrm{ST}^0(E) \xrightarrow{\sim} \mathrm{Gal}(K_E/\mathbb{Q}),$$
$$\psi_A\colon \mathrm{ST}(A)/\mathrm{ST}^0(A) \xrightarrow{\sim} \mathrm{Gal}(K_A/\mathbb{Q}),$$
$$\psi_{C_2}\colon \mathrm{ST}(C_2)/\mathrm{ST}^0(C_2) \xrightarrow{\sim} \mathrm{Gal}(K_{C_2}/\mathbb{Q}).$$

To prove the corollary it suffices to make these isomorphisms explicit and show that they are compatible with the projections from $\mathrm{ST}(C_2)$ to $\mathrm{ST}(E)$ and $\mathrm{ST}(A)$, and with the restriction maps from $\mathrm{Gal}(K_{C_2}/\mathbb{Q})$ to $\mathrm{Gal}(K_E/\mathbb{Q})$ and $\mathrm{Gal}(K_A/\mathbb{Q})$.

The isomorphism ψ_E identifies the image of J in $\mathrm{ST}(E)/\mathrm{ST}^0(E)$ with the non-trivial element of $\mathrm{Gal}(K_E/\mathbb{Q})$, while the isomorphism ψ_A identifies the images of the generators $g = R, S, T$ in

$$\mathrm{ST}(A)/\mathrm{ST}^0(A) \simeq \langle R, S, T\rangle/\langle -1\rangle$$

with automorphisms $\sigma = r, s, t \in \mathrm{Gal}(K_A/\mathbb{Q})$ as indicated below:

g	$\sigma = \psi_A(g)$	$\sigma(i)$	$\sigma(\zeta_3)$	$\sigma(c^{1/3})$
R	r	$-i$	ζ_3^2	$c^{1/3}$
S	s	$-i$	ζ_3	$c^{1/3}$
T	t	i	ζ_3	$\zeta_3 c^{1/3}$

If we now let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ denote the first three generators of $\mathrm{ST}(C_2)$ listed in the corollary, ψ_{C_2} identifies their images in $\mathrm{ST}(C_2)/\mathrm{ST}^0(C_2)$ with elements of $\mathrm{Gal}(K_A/\mathbb{Q})$ as indicated below, where $\delta = \sqrt{c\sqrt{-3}}$:

g	$\sigma = \psi_{C_2}(g)$	$\sigma(i)$	$\sigma(\zeta_3)$	$\sigma(c^{1/3})$	$\sigma(\delta)$
$\mathcal{R}$	$\mathscr{r}$	$-i$	ζ_3^2	$c^{1/3}$	$i\delta$
$\mathcal{S}$	$\mathscr{s}$	$-i$	ζ_3	$c^{1/3}$	δ
$\mathcal{T}$	$\mathscr{t}$	i	ζ_3	$\zeta_3\, c^{1/3}$	δ

We note that, unlike their restrictions r and s, the automorphisms $\mathscr{r}$ and $\mathscr{s}$ do not commute, they generate a dihedral group of order 8 inside $\mathrm{Gal}(K_{C_2}/\mathbb{Q})$. The three automorphisms $\mathscr{r}, \mathscr{s}, \mathscr{t}$ together generate $\mathrm{Gal}(K_{C_2}/\mathbb{Q})$. Their restrictions to K_A are the generators r, s, t for K_A, and R, S, T are the projections of $\mathcal{R}, \mathcal{S}, \mathcal{T}$ to $\mathrm{ST}(A)$. The automorphisms $\mathscr{r}$ and $\mathscr{s}$ both restrict to the non-trivial element of $\mathrm{Gal}(K_E/\mathbb{Q})$, and both $\mathcal{R}$ and $\mathcal{S}$ project down to J in $\mathrm{ST}(E)$. The restriction of $\mathscr{t}$ to K_E is trivial, as is the projection of $\mathcal{T}$ to $\mathrm{ST}(E)$. To complete the proof it suffices to verify that the map

$$\mathrm{ST}(C_2)/\mathrm{ST}^0(C_2) \simeq \langle \mathcal{R}, \mathcal{S}, \mathcal{T}\rangle/\langle -1\rangle \xrightarrow{\psi_{C_2}} \langle \mathscr{r}, \mathscr{s}, \mathscr{t}\rangle \simeq \mathrm{Gal}(K_{C_2}/\mathbb{Q})$$

we have explicitly defined is indeed an isomorphism. One can check that both sides are isomorphic to the finitely presented group

$$\langle \mathcal{R}, \mathcal{S}, \mathcal{T} \,|\, \mathcal{R}^2, \mathcal{S}^2, \mathcal{T}^3, \mathcal{RSRSRSRS}, \mathcal{RTRT}, \mathcal{STST}^2 \rangle,$$

via maps that send generators to corresponding generators (in the order shown). □

REMARK 5.9. To treat non-generic values of c, simply remove the third generator containing Z_3 from the list of generators for $\mathrm{ST}(C_2)$ in Corollary 5.8 when c is a cube in $\mathbb{Q}^*$.

Using the explicit representation of $\mathrm{ST}(C_2)$ given in Corollary 5.8, one may compute moments sequences for the characteristic polynomial coefficients a_1, a_2, a_3 using the techniques described in §3.2 of [**FKS16**]; the first eight moments are listed below:

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
a_1:	0	2	0	30	0	720	0	20650
a_2:	2	10	75	784	9607	126378	1721715	23928108
a_3:	0	11	0	2181	0	660790	0	224864661

The a_1 moments closely match the corresponding moment statistics listed in Table 2 in the cases where c is generic, as expected. We also computed moment statistics for a_1, a_2 and a_3 by applying the algorithm of [**HS16**] to the curve $y^2 = x^7 - 2x$ over primes $p \le 2^{30}$. The a_1-moment statistics listed below have less resolution than Table 2, which covers $p \le 2^{40}$ (with this higher bound we get $M_8 \approx 20649$, very close to the value 20650 predicted by $\mathrm{ST}(C_2)$).

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
a_1:	0.00	2.00	0.00	30.00	0.04	719.62	2	20636
a_2:	2.00	10.00	74.97	783.59	9600.64	126281.75	1720266	23906297
a_3:	0.00	11.00	0.04	2179.67	19.68	660247.53	8549	224645654

6. Galois endomorphism types

As recalled in §2, up to dimension 3, the Galois endomorphism type of an abelian variety over a number field is determined by its Sato-Tate group. In this section, we derive the Galois endomorphism type of $\mathrm{Jac}(C_2)$ from $\mathrm{ST}(C_2)$ for generic values of c (in the sense of §5.2). The case of $\mathrm{Jac}(C_1)$, although leading to slightly larger diagrams, is completely analogous.

Let $G := \mathrm{ST}(C_2)$ and $V := \mathrm{End}(\mathrm{Jac}(C_2)_{K_{C_2}})$, and set $V_\mathbb{C} := V \otimes_\mathbb{Z} \mathbb{C}$ and $V_\mathbb{R} := V \otimes_\mathbb{Z} \mathbb{R}$. As described in the proof of [**FKRS12**, Prop. 2.19]:

- $V_\mathbb{C}$ is the subspace of $\mathrm{M}_6(\mathbb{C})$ fixed by the action of G^0;
- $V_\mathbb{R}$ is the subspace of $V_\mathbb{C}$, of half the dimension, over which the Rosati form is positive definite;
- If $L/\mathbb{Q}$ is a subextension of $K_{C_2}/\mathbb{Q}$, corresponding to the subgroup $N \subseteq \mathrm{Gal}(K_{C_2}/\mathbb{Q}) \simeq G/G^0$, then $\mathrm{End}(\mathrm{Jac}(C_2)_L) \otimes_\mathbb{Z} \mathbb{R} \simeq V_\mathbb{R}^N$.

The matrices $\Phi \in \mathrm{M}_6(\mathbb{C})$ commuting with $G^0 \simeq \mathrm{U}(1)$, embedded in $\mathrm{USp}(6)$, are matrices of the form $\Phi = (\phi_{i,j})$ with $i, j \in [1,6]$ such that $\phi_{i,j} \in \mathbb{C}$ is 0 unless $i \equiv j$

mod 2. The condition of the Rosati form being positive definite on $V_{\mathbb{R}}$ amounts to requiring that

$$\mathrm{Trace}(\Phi H^t \Phi^t H) \geq 0$$

for every $\Phi \in V_{\mathbb{R}}$, where H is the symplectic matrix given in (6). Imposing the above condition on Φ, we find that

$$\Phi = \begin{pmatrix} \alpha & 0 & \beta & 0 & \gamma & 0 \\ 0 & \overline{\alpha} & 0 & \overline{\beta} & 0 & \overline{\gamma} \\ \delta & 0 & \epsilon & 0 & \phi & 0 \\ 0 & \overline{\delta} & 0 & \overline{\epsilon} & 0 & \overline{\phi} \\ \lambda & 0 & \mu & 0 & \nu & 0 \\ 0 & \overline{\lambda} & 0 & \overline{\mu} & 0 & \overline{\nu} \end{pmatrix} \qquad \text{with } \alpha, \beta, \ldots, \mu, \nu \in \mathbb{C}\,.$$

We thus deduce that $V_{\mathbb{R}} \simeq \mathrm{M}_3(\mathbb{C})$.

We now proceed to determine the sub-$\mathbb{R}$-algebras of $V_{\mathbb{R}}$ fixed by each of the subgroups of $\mathrm{Gal}(K_{C_2}/\mathbb{Q}) \simeq G/G^0$. With notations as in the proof of Corollary 5.8, these subgroups are listed (up to conjugation) in Figure 6, where normal subgroups are marked with a *. We can then reconstruct the Galois type of $\mathrm{Jac}(C_2)$ (see Figure 7) from the information in Table 3.

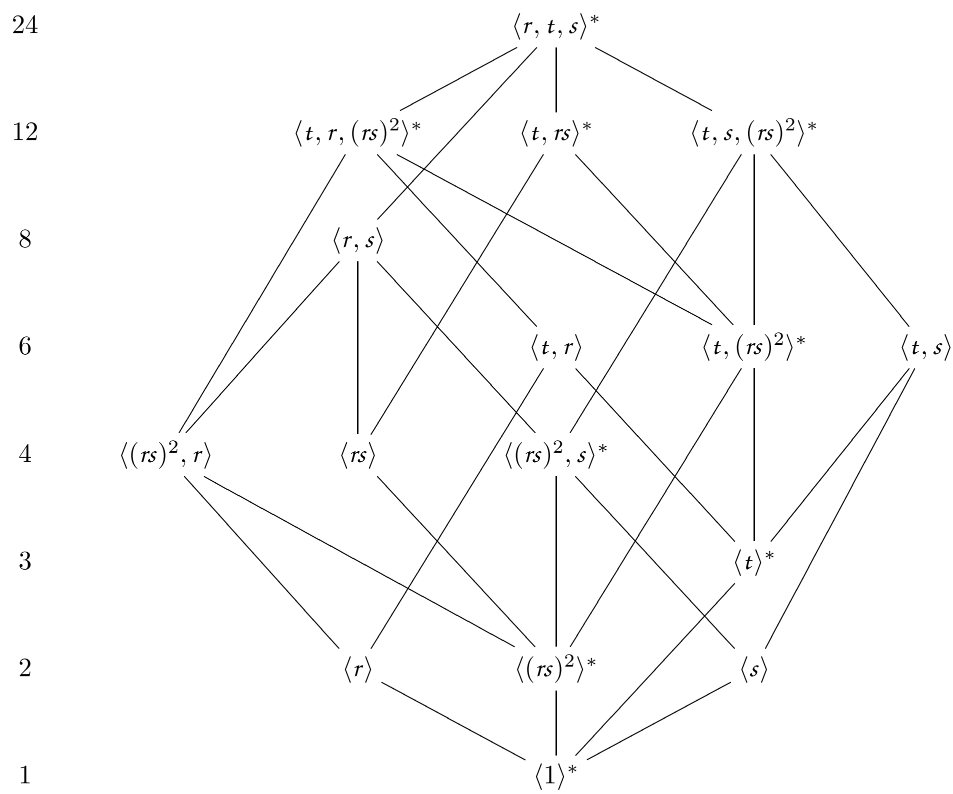


FIGURE 6. Lattice of subgroups of $\mathrm{Gal}(K_{C_2}/\mathbb{Q})$.

N	Condition on Φ	$V_{\mathbb{R}}^N$
$\langle \mathcal{R} \rangle$	$\alpha, \beta, \dots, \mu, \nu \in \mathbb{R}$	$\mathrm{M}_3(\mathbb{R})$
$\langle \mathcal{S} \rangle$	$\alpha = \overline{\epsilon}, \beta = \overline{\delta}, \phi = i\overline{\gamma}, \mu = -i\overline{\lambda}, \nu \in \mathbb{R}$	$\mathrm{M}_3(\mathbb{R})$
$\langle \mathcal{T} \rangle$	$\beta = \gamma = \delta = \phi = \lambda = \mu = 0$	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$
$\langle (\mathcal{R}\mathcal{S})^2 \rangle$	$\gamma = \phi = \lambda = \mu = 0$	$\mathrm{M}_2(\mathbb{C}) \times \mathbb{C}$
$\langle (\mathcal{R}\mathcal{S})^2, \mathcal{S} \rangle$	$\gamma = \phi = \lambda = \mu = 0, \alpha = \overline{\epsilon}, \beta = \overline{\delta}, \nu \in \mathbb{R}$	$\mathrm{M}_2(\mathbb{R}) \times \mathbb{R}$
$\langle \mathcal{R}\mathcal{S} \rangle$	$\gamma = \phi = \lambda = \mu = 0, \alpha = \epsilon, \beta = \delta$	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$

TABLE 3. Some subgroups of $\mathrm{Gal}(K_{C_2}/\mathbb{Q})$ with the respective fixed sub-$\mathbb{R}$-algebras of $V_{\mathbb{R}}$.

Obtaining the data in Table 3 is a straight-forward exercise, let us make just a few specific comments:

- $N = \langle \mathcal{S} \rangle$: One easily checks that the matrices Φ satisfying the required condition form a simple nondivision $\mathbb{R}$-algebra; by Wedderburn's structure theorem, it is of the form $\mathrm{M}_d(D)$, for some division algebra D and $d > 1$; since its real dimension is 9, we must have $d = 3$ and $D = \mathbb{R}$.
- $N = \langle (\mathcal{R}\mathcal{S})^2, \mathcal{S} \rangle$: Note that the $\mathbb{R}$-algebra
$$\mathcal{H} := \left\{ A_{\alpha,\beta} := \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$
is isomorphic to $\mathrm{M}_2(\mathbb{R})$. Indeed, if $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, then
$$\psi \colon \mathcal{H} \to \mathrm{M}_2(\mathbb{R}), \qquad \psi(A_{\alpha,\beta}) = \begin{pmatrix} \alpha_1 + \beta_1 & -\alpha_2 + \beta_2 \\ \alpha_2 + \beta_2 & \alpha_1 - \beta_1 \end{pmatrix}$$
provides the required isomorphism. Alternative, one can reach the same conclusion by noting that $\mathcal{H}$ is the only non-commutative $\mathbb{R}$-algebra of dimension 4 with zero divisors.
- $N = \langle \mathcal{R}\mathcal{S} \rangle$: Note that the $\mathbb{R}$-algebra
$$\mathcal{H} := \left\{ A_{\alpha,\beta} := \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$
is isomorphic to $\mathbb{C} \times \mathbb{C}$ by means of
$$\psi \colon \mathcal{H} \to \mathbb{C} \times \mathbb{C}, \qquad \psi(A_{\alpha,\beta}) = (\alpha + \beta, \alpha - \beta).$$

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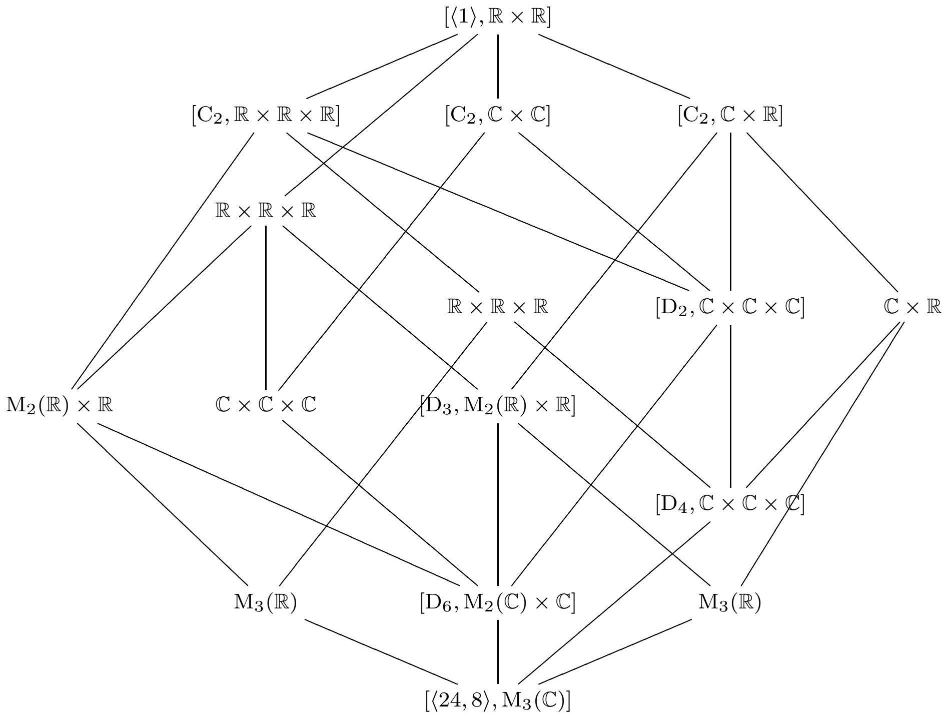
FIGURE 7. Galois endomorphism types of $\mathrm{Jac}(C_2)$. Lattice of fixed sub-$\mathbb{R}$-algebras corresponding to the subgroups of Figure 6.



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