Hyperelliptic curves, L-polynomials and random matrices

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joint work with Kiran Kedlaya

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Distributions of Frobenius traces

Let $E/\mathbb{Q}$ be a non-singular elliptic curve.
Let $t_p = \#E(\mathbb{F}_p) - p + 1$ denote the trace of Frobenius.

Consider the distribution of

$$x_p = -t_p/\sqrt{p} \in [-2, 2]$$

as $p \leqslant N$ varies over primes of good reduction.

What happens as $N \to \infty$?

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Trace distributions in genus 1

1. Typical case (no CM)

All elliptic curves without CM have the Sato-Tate distribution.

[Clozel, Harris, Shepherd-Barron, Taylor, Barnet-Lamb, and Geraghty].

2. Exceptional cases (CM)

All elliptic curves with CM have the same exceptional distribution.

[classical]

Zeta functions and L-polynomials

For a smooth projective curve $C/\mathbb{Q}$ and a good prime p define

$$Z(C/\mathbb{F}_p;T) = \exp\left(\sum_{k=1}^{\infty} N_k T^k/k\right),$$

where $N_k = \#C/\mathbb{F}_{p^k}$. This is a rational function of the form

$$Z(C/\mathbb{F}_p;T) = \frac{L_p(T)}{(1-T)(1-pT)},$$

where $L_p(T)$ is an integer polynomial of degree $2g$. For $g = 2$:

$$L_p(T) = p^2T^4 + c_1pT^3 + c_2pT^2 + c_1T + 1.$$

Unitarized L-polynomials

The polynomial

$$\bar{L}_p(T) = L_p(T/\sqrt{p}) = \sum_{i=0}^{2g} a_i T^i$$

has coefficients that satisfy $a_i = a_{2g-i}$ and $|a_i| \leqslant \binom{2g}{i}$.

Given a curve C, we may consider the distribution of $a_1, a_2, \ldots, a_g$, taken over primes $p \leqslant N$ of good reduction, as $N \to \infty$.

In this talk we will focus on genus $g = 2$.

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The random matrix model

$\bar{L}_p(T)$ is a real symmetric polynomial whose roots lie on the unit circle.

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Every such polynomial arises as the characteristic polynomial $\chi(T)$ of a unitary symplectic matrix in $\mathbb{C}^{2g\times 2g}$.

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Conjecture (Katz-Sarnak)

For a typical curve of genus g, the distribution of $\bar{L}_p$ converges to the distribution of χ in $USp(2g)$.

This conjecture has been proven "on average" for universal families of hyperelliptic curves, including all genus 2 curves, by Katz and Sarnak.

The Haar measure on $USp(2g)$

Let $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_g}$ denote the eigenvalues of a random conjugacy class in $USp(2g)$. The Weyl integration formula yields the measure

$$\mu = \frac{1}{g!}\Big(\prod_{j<k}(2\cos\theta_j - 2\cos\theta_k)\Big)^2 \prod_j \left(\frac{2}{\pi}\sin^2\theta_j d\theta_j\right).$$

In genus 1 we have $USp(2) = SU(2)$ and $\mu = \frac{2}{\pi}\sin^2\theta d\theta$, which is the Sato-Tate distribution.

Note that $-a_1 = \sum 2\cos\theta_j$ is the trace.

$\bar{L}_p$-distributions in genus 2

Our goal was to understand the $\bar{L}_p$-distributions that arise in genus 2, including not only the generic case, but all the exceptional cases.

This presented three challenges:

- Collecting data.
- Identifying and distinguishing distributions.
- Classifying the exceptional cases.

Collecting data

There are four ways to compute $\bar{L}_p$ in genus 2:

1. point counting: $\tilde{O}(p^2)$.
2. group computation: $\tilde{O}(p^{3/4})$.
3. p-adic methods: $\tilde{O}(p^{1/2})$.
4. ℓ-adic methods: $\tilde{O}(1)$.

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4. ℓ-adic methods: $\tilde{O}(1)$.

For the feasible range of $p \leqslant N$, we found (2) to be the best.
We can accelerate the computation with partial use of (1) and (4).

Computing L-series of hyperelliptic curves, ANTS VIII, 2008, KS.

Performance comparison

$p \approx 2^k$	points+group	group	p-adic
2^{14}	**0.22**	0.55	4
2^{15}	**0.34**	0.88	6
2^{16}	**0.56**	1.33	8
2^{17}	**0.98**	2.21	11
2^{18}	**1.82**	3.42	17
2^{19}	**3.44**	5.87	27
2^{20}	**7.98**	10.1	40
2^{21}	18.9	**17.9**	66
2^{22}	52	**35**	104
2^{23}		**54**	176
2^{24}		**104**	288
2^{25}		**173**	494
2^{26}		**306**	871
2^{27}		**505**	1532

Time to compute $L_p(T)$ in CPU milliseconds on a 2.5 GHz AMD Athlon

Time to compute $\bar{L}_p$ for all $p \leqslant N$

N	2 cores	16 cores
2^{16}	1	< 1
2^{17}	4	2
2^{18}	12	3
2^{19}	40	7
2^{20}	2:32	24
2^{21}	10:46	1:38
2^{22}	40:20	5:38
2^{23}	2:23:56	19:04
2^{24}	8:00:09	1:16:47
2^{25}	26:51:27	3:24:40
2^{26}		11:07:28
2^{27}		36:48:52

Characterizing distributions

The *moment sequence* of a random variable X is

$$M[X] = (\,\mathrm{E}[X^0], \mathrm{E}[X^1], \mathrm{E}[X^2], \ldots).$$

Provided X is suitably bounded, $M[X]$ exists and uniquely determines the distribution of X.

Given sample values $x_1, \ldots, x_N$ for X, the nth *moment statistic* is the mean of x_i^n. It converges to $\mathrm{E}[X^n]$ as $N \to \infty$.

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If X is a symmetric integer polynomial of the eigenvalues of a random matrix in $USp(2g)$, then $M[X]$ is an *integer* sequence.

This applies to all the coefficients of $\chi(T)$.

The typical trace moment sequence in genus 1

Using the measure μ in genus 1, for $t = -a_1$ we have

$$E[t^n] = \frac{2}{\pi}\int_0^\pi (2\cos\theta)^n \sin^2\theta d\theta.$$

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$$E[t^n] = \frac{2}{\pi}\int_0^{\pi} (2\cos\theta)^n \sin^2\theta d\theta.$$

This is zero when n is odd, and for $n = 2m$ we obtain

$$E[t^{2m}] = \frac{1}{2m+1}\binom{2m}{m}.$$

and therefore

$$M[t] = (1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132, \ldots).$$

This is sequence A126120 in the OEIS.

The typical trace moment sequence in genus $g > 1$

A similar computation in genus 2 yields

$$M[t] = (1, 0, 1, 0, 3, 0, 14, 0, 84, 0, 594, \ldots),$$

which is sequence A138349, and in genus 3 we have

$$M[t] = (1, 0, 1, 0, 3, 0, 15, 0, 104, 0, 909, \ldots),$$

which is sequence A138540.

In genus g, the nth moment of the trace is the number of returning walks of length n on $\mathbb{Z}^g$ with $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_g \geqslant 0$ [Grabiner-Magyar].

The exceptional trace moment sequence in genus 1

For an elliptic curve with CM we find that

$$E[t^{2m}] = \frac{1}{2}\binom{2m}{m}, \qquad \text{for } m > 0$$

yielding the moment sequence

$$M[t] = (1, 0, 1, 0, 3, 0, 10, 0, 35, 0, 126, 0, \ldots),$$

whose even entries are A008828.

An exceptional trace moment sequence in Genus 2

For a hyperelliptic curve whose Jacobian is isogenous to the direct product of two elliptic curves, we compute $M[t] = M[t_1 + t_2]$ via

$$\mathrm{E}[(t_1 + t_2)^n] = \sum \binom{n}{i} \mathrm{E}[t_1^i]\mathrm{E}[t_2^{n-i}].$$

For example, using

$$M[t_1] = (1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132, \ldots),$$
$$M[t_2] = (1, 0, 1, 0, 3, 0, 10, 0, 35, 0, 126, 0, 462, \ldots),$$

we obtain $A138551$,

$$M[t] = (1, 0, 2, 0, 11, 0, 90, 0, 889, 0, 9723, \ldots).$$

The second moment already differs from the standard sequence, and the fourth moment differs greatly (11 versus 3).

Sieving for exceptional curves

We surveyed the $\bar{L}_p$-distributions of genus 2 curves

$$y^2 = x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

$$y^2 = b_6x^6 + b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0,$$

with integer coefficients $|c_i| \leqslant 64$ and $|b_i| \leqslant 16$, over 10^{10} curves.

We initially set $N \approx 2^{12}$, discarded about 99% of the curves (those whose moment statistics were "unexceptional"), then repeated this process with $N \approx 2^{16}$ and $N \approx 2^{20}$.

We eventually found 30,000 non-isomorphic curves with apparently exceptional distributions, many of which coincided.

Representative examples were computed to high precision $N \approx 2^{26}$.

Survey highlights

- The moment statistics always appear to converge to integers.
- 20 distinct trace distributions (eventually found 23 of 24 predicted). This exceeds the possibilities for $\mathrm{End}(\mathrm{Jac}(C))$, $\mathrm{Aut}(C)$, or $\mathrm{MT}(C)$.
- The same $\bar{L}_p$-distribution can arise for split and simple Jacobians.
- The density of zero traces can be any of

$$\{0, 1/6, 1/4, 1/2, 7/12, 5/8, 3/4, 13/16, 7/8\}.$$

Density 0 occurs in several exceptional cases.

Survey highlights (new results)

- The moment statistics always appear to converge to integers.
- 26 distinct $\bar{L}_p$-distributions (out of 26 predicted). This exceeds the possibilities for $\mathrm{End}(\mathrm{Jac}(C))$, $\mathrm{Aut}(C)$, or $\mathrm{MT}(C)$.
- The same $\bar{L}_p$-distribution can arise for split and simple Jacobians.
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  Density 0 occurs in several exceptional cases.
- Distinct $\bar{L}_p$-distributions may have identical trace distributions.
  As of 2/15/2011, we have identified 30 distinct $\bar{L}_p$-distributions.

Random matrix subgroup model

Conjecture

For a genus g curve C, the distribution of $\bar{L}_p$ converges to the distribution of χ in some infinite compact subgroup $H \subseteq USp(2g)$.

*Equality holds if and only if C has large Galois image.**

*image of $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(T_\ell(C))$ Zariski dense in $GSp(2g, \mathbb{Z}_\ell)$.

Representations of genus 1 distributions

The Sato-Tate distribution has $H = USp(2g)$, the typical case.

For CM curves, consider the subgroup of $USp(2) = SU(2)$:

$$H = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} i\cos\theta & i\sin\theta \\ i\sin\theta & -i\cos\theta \end{pmatrix} : \theta \in [0, 2\pi] \right\}.$$

This is a compact group (the normalizer of $SO(2)$ in $SU(2)$).

Its Haar measure yields the desired moment sequence.

Candidate subgroups in genus 2

Let $G_1 = SU(2)$ and $G_2 = N(SO(2)) \subset SU(2)$.

- $USp(4)$ — generic genus 2 curve.
- Index 2 subgroup K of $N(SO(2) \times SO(2))$ — genus 2 CM curve.
- $G_1 \times G_1$, $G_1 \times G_2$, $G_2 \times G_2$ — products of 2 elliptic curves.
- $J(G_1 \times G_1)$ (but not $J(G_2 \times G_2)$ [Serre]).
- $G_i \otimes G_0$ for some finite subgroup G_0 of $SU(2)$ —
  "twisted" product of an elliptic curve with itself (22 cases!).

We require elements of G_0 to have traces whose squares lie in $\mathbb{Z}$.
We may assume $-I \in G_0$.

A very recent example

The genus 2 curve

$$y^2 = 297x^6 - 324x^5 - 2970/37x^4 + 720/37x^3 + 1980/1369x^2 - 144/1369x - 88/50653$$

found by Fité and Lario in December 2010 has $\bar{L}_p$-distribution matching $G_2 \otimes G_0$, where G_0 is a binary dihedral group of order 24.

This distribution was predicted by our model that did not show up in our survey. It also occurs for the curve

$$y^2 = x^6 - 9x^5 - 15x^4 + 30x^3 + 15x^2 - 9x - 1$$

whose coefficients lie just beyond the range of our search.

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The paremetrizations they used (due to Cardona) also yielded two new distributions that were not predicted by our model!

Finite subgroups of $SU(2)$

A finite subgroup of $SU(2)$ is isomorphic to one of the following:

- Cyclic C_n group of order n.
- Binary dihedral group BD_n of order $4n$.
- Binary tetrahedral group BT (order 24).
- Binary octahedral group BO (order 48).
- Binary icosahedral group BI (order 120).

There are 12 groups on this list that are candidates for G_0.

All of these give rise to distributions that match an exceptional $\bar{L}_p$-polynomial distribution in genus 2.

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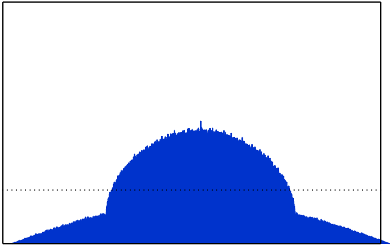
All of these give rise to distributions that match an exceptional $\bar{L}_p$-polynomial distribution in genus 2.

This includes the two new distributions, arising from BT and BO, which only seem applicable to G_2.

H	#	d	$c(H)$	$z(H)$	M_2	M_4	M_6	M_8	M_{10}
$USp(4)$	1	10	1	0	1	3	14	84	594
K	19	2	4	3/4	1	9	100	1225	15876
$G_1 \times G_1$	2	6	1	0	2	10	70	588	5544
$G_1 \times G_2$	3	4	2	0	2	11	90	889	9723
$G_2 \times G_2$	8	2	4	1/4	2	12	110	1260	16002
$J(G_1 \times G_1)$	9	6	2	1/2	1	5	35	294	2772
$G_1 \otimes C_2$	5	3	1	0	4	32	320	3584	43008
$G_1 \otimes C_4$	11b	3	2	1/2	2	16	160	1792	21504
$G_1 \otimes C_6$	4	3	3	0	2	12	110	1204	14364
$G_1 \otimes C_8$	7	3	4	1/4	2	12	100	1008	11424
$G_1 \otimes C_{12}$	6	3	6	1/6	2	12	100	980	10584
$G_1 \otimes BD_1$	11	3	2	1/2	2	16	160	1792	21504
$G_1 \otimes BD_2$	18	3	4	3/4	1	8	80	896	10752
$G_1 \otimes BD_3$	10	3	6	1/2	1	6	55	602	7182
$G_1 \otimes BD_4$	16	3	8	5/8	1	6	50	504	5712
$G_1 \otimes BD_6$	14	3	12	7/12	1	6	50	490	5292
$G_2 \otimes C_2$	13	1	2	1/2	4	48	640	8960	129024
$G_2 \otimes C_4$	21b	1	4	3/4	2	24	320	4480	64512
$G_2 \otimes C_6$	12	1	6	1/2	2	18	220	3010	43092
$G_2 \otimes C_8$	17	1	8	5/8	2	18	200	2520	34272
$G_2 \otimes C_{12}$	15	1	12	7/12	2	18	200	2450	31752
$G_2 \otimes BD_1$	21	1	4	3/4	2	24	320	4480	64512
$G_2 \otimes BD_2$	23	1	8	7/8	1	12	160	2240	32256
$G_2 \otimes BD_3$	20	1	12	3/4	1	9	110	1505	21546
$G_2 \otimes BD_4$	22	1	16	13/16	1	9	100	1260	17136
$G_2 \otimes BD_6$	24	1	24	19/24	1	9	100	1225	15876
$G_2 \otimes BT$	25	1	24	5/8	1	6	60	770	10836
$G_2 \otimes BO$	26	1	48	11/16	1	6	50	525	6426

`smalljac` now available in purple Sage.



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