The Astrophysical Journal, 604:L37–L40, 2004 March 20

ON THE RELATIONSHIP BETWEEN MODIFIED NEWTONIAN DYNAMICS AND DARK MATTER

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Received 2003 November 23; accepted 2004 January 9; published 2004 February 26

ABSTRACT

Numerous astrophysical observations have shown that classical Newtonian dynamics fails on galactic scales and beyond, if only visible matter is taken into account. The two most popular theoretical concepts dealing with this problem are dark matter (DM) and modified Newtonian dynamics (MOND). In the first part of this Letter it is demonstrated that a generalized MOND equation can be derived in the framework of Newtonian DM theory. For systems satisfying a fixed relationship between the gravitational fields caused by DM and visible matter, this generalized MOND equation reduces to the traditional MOND law, first postulated by Milgrom. Therefore, we come to the conclusion that traditional MOND can be also interpreted as a special limit case of DM theory. In the second part of this Letter, a formal derivation of the Tully-Fisher relation is discussed.

Subject headings: dark matter — galaxies: kinematics and dynamics

1. INTRODUCTION

Seventy years ago, Zwicky (1933, 1937) was the first to note that the speed of galaxies in large clusters is much too great to keep them gravitationally bound together, unless they are much heavier than one would estimate on the basis of visible matter. Since those days numerous further astrophysical observations, e.g., Doppler measurements of rotation velocities in disk galaxies, have confirmed the failure of the classical Newtonian theory, if only visible matter is taken into account (Combes et al. 1995; Bertin & Lin 1996; Field 1999; Sanders & McGaugh 2002). Historically, theoretical concepts addressing this problem can be subdivided in two categories. The first category comprises the dark matter (DM) theories (Binney & Tremaine 1994; Sadoulet 1999; van den Bergh 2001; Ostriker & Steinhardt 2003), whereas the second group assumes that Newton's gravitational law requires modification (Milgrom 1983a, 1983b, 1983c).

DM theories are based on the hypothesis that there exist significant amounts of invisible (nonbaryonic) matter in the universe, interacting with ordinary visible matter only via gravity. Since empirically it is very successful, DM has become a widely accepted cornerstone of the contemporary cosmological standard model (Sadoulet 1999; van den Bergh 2001; Ostriker & Steinhardt 2003). Nevertheless, it must also be emphasized that until now DM has been detected only indirectly by means of its gravitational effects on the visible matter or the light.

Aiming to avoid the introduction of invisible matter, an alternative phenomenological concept was proposed by Milgrom (1983a, 1983b, 1983c). Instead of adapting the mass distribution, his approach requires a modified Newtonian dynamics (MOND) in the limit of small accelerations. As extensively reviewed by Sanders & McGaugh (2002), this theory can explain galaxy data, such as the flat rotation curves, in a very compelling way. On the other hand, there also have been some indications in the past that MOND might be an effective or approximate theory, applicable to a limited range of astrophysical problems only (Aguirre 2003). This hypothesis is supported by fundamental difficulties associated with relativistic generalizations of Milgrom's theory (Sanders & McGaugh 2002; Soussa & Woodard 2004; Aguirre 2003). Also, according to Aguirre, Schaye, & Quataert (2001), MOND seems to become less effective on larger scales; e.g., it cannot account for cluster densities and temperature profiles in detail.

The fact that, to some extent, both DM and MOND can successfully explain galactic dynamics favors the possibility that there exists a deeper connection between these two theories (for a general comparison, see Aguirre 2003). Among others, this idea was formulated by McGaugh & de Blok (1998b) and later pursued by Kaplinghat & Turner (2002). Using arguments based on galaxy formation processes in the early universe, the latter authors claim that MOND follows from cold DM theory. In his response, Milgrom (2002) questions these results. Among others, he argues that the predictions made by Kaplinghat & Turner (2002) would conflict not only with astronomical observations of pairs of galaxies (McGaugh & de Blok 1998), but also with numerical results obtained for DM models (Navarro, Frenk, & White 1997). Thus, unclarity still seems to exist about whether or not MOND can in fact be understood in the framework of DM (Aguirre 2003).

It is therefore the main purpose of the present Letter to explicitly demonstrate that the MOND equations (if considered as modified Newtonian gravity) can be derived from classical Newtonian dynamics, provided one also takes into account the gravitational influence of a DM component. In particular, it is shown that the characteristic threshold acceleration, $a_0 \approx 1.2 \times 10^{-10}$ m s^{-2}, below which MOND effects begin to dominate, also can be interpreted as the asymptotic value of a more general acceleration field, characterizing the difference between the gravitational forces caused by visible matter and DM, respectively.

2. MOND FROM NEWTONIAN DYNAMICS WITH DM

As a starting point, consider the Newtonian equations of motion of a pointlike test particle

$$m\ddot{\boldsymbol{x}} = -m\nabla[\Phi_v(\boldsymbol{x}) + \Phi_d(\boldsymbol{x})], \qquad (1)$$

where $\Phi_v(\boldsymbol{x})$ and $\Phi_d(\boldsymbol{x})$ denote the gravitational potentials due to visible and DM, respectively. Both potentials are solutions of Poisson equations,

$$\nabla^2\Phi_{v/d} = 4\pi G\rho_{v/d}, \qquad (2)$$

where $\rho_{v/d}(\boldsymbol{x})$ is the corresponding mass density and G denotes the gravitational constant. For convenience, we define the accelerations

$$\boldsymbol{g}_{v/d}(\boldsymbol{x}) = -\nabla\Phi_{v/d}(\boldsymbol{x}). \tag{3}$$

Thus, equation (1) simplifies to

$$\ddot{\boldsymbol{x}} = \boldsymbol{g}_v + \boldsymbol{g}_d = \boldsymbol{g}. \tag{4}$$

Now let us additionally assume that the acceleration vectors $\boldsymbol{g}_v$ and $\boldsymbol{g}_d$ point in the same direction, denoted by

$$\boldsymbol{g}_v \uparrow\uparrow \boldsymbol{g}_d. \tag{5}$$

Note that in this case also $\boldsymbol{g}_{v/d} \uparrow\uparrow \boldsymbol{g}$. Roughly speaking, the assumptions (eq. [5]) mean that the visible mass distribution ρ_v and the DM distribution ρ_d behave very similarly. Next, we rewrite equation (4) as

$$\ddot{\boldsymbol{x}} = \left(1 + \frac{g_d}{g_v}\right)\boldsymbol{g}_v, \tag{6}$$

where $g_{v/d} = |\boldsymbol{g}_{v/d}|$ with

$$g_v = g - g_d \geq 0, \tag{7}$$

if condition (5) holds. Inserting this into equation (6) yields

$$\ddot{\boldsymbol{x}} = \left(1 + \frac{1}{g/g_d - 1}\right)\boldsymbol{g}_v. \tag{8}$$

Thus, by virtue of equation (4), we find that

$$\boldsymbol{g}_v = \left(\frac{\epsilon}{\epsilon + 1}\right)\boldsymbol{g} = \tilde{\mu}(\epsilon)\boldsymbol{g}, \tag{9}$$

where we have introduced

$$\epsilon(\boldsymbol{x}) = \frac{g(\boldsymbol{x})}{g_d(\boldsymbol{x})} - 1 \geq 0. \tag{10}$$

Equation (9) can be compared to the fundamental MOND formula (Milgrom 1983a, 1983b, 1983c; Sanders & McGaugh 2002)

$$\boldsymbol{g}_v = \mu\left(\frac{g}{a_0}\right)\boldsymbol{g}, \tag{11}$$

where, because of empirical reasons, the function $\mu(\xi)$ is *postulated* to have the asymptotic behavior

$$\mu(\xi) = \begin{cases} 1, & \xi \gg 1, \\ \xi, & \xi \ll 1. \end{cases} \tag{12}$$

One readily observes that this is exactly the *natural* asymptotic behavior of $\tilde{\mu}(\epsilon)$ for $\epsilon \to 0$ and $\epsilon \to \infty$, respectively. Hence, if we identify μ with $\tilde{\mu}$ and introduce an acceleration field $a(\boldsymbol{x})$ by

$$\frac{g(\boldsymbol{x})}{a(\boldsymbol{x})} = \epsilon(\boldsymbol{x}) = \frac{g(\boldsymbol{x})}{g_d(\boldsymbol{x})} - 1, \tag{13}$$

then it becomes obvious that equation (9) is the natural generalization of the MOND postulate (eq. [11]). The only difference is that we have a local acceleration field $a(\boldsymbol{x})$ in equation (9), whereas $a_0 = \text{const}$ was postulated in the MOND formula (11). Note that equation (13) also can be written in the equivalent form

$$\begin{aligned} \frac{1}{a(\boldsymbol{x})} &= \frac{1}{g_d(\boldsymbol{x})} - \frac{1}{g(\boldsymbol{x})} \\ &= \frac{1}{g_d(\boldsymbol{x})} - \frac{1}{g_v(\boldsymbol{x}) + g_d(\boldsymbol{x})}. \end{aligned} \tag{14}$$

Thus, the special MOND case

$$a(\boldsymbol{x}) \equiv a_0 \tag{15}$$

implies a fixed relation between the acceleration fields due to visible and DM. In particular, since the characteristic MOND acceleration a_0 is relatively small, one can further infer from equation (14) that galaxies satisfying the MOND limit are DM dominated.

3. AXISYMMETRIC DISK GALAXIES AND TULLY-FISHER LAW

In the following, let us concentrate on the quasi–two-dimensional problem of axisymmetric disk galaxies. It is an experimental observation that for many such systems the Tully-Fisher relation holds (Sanders & McGaugh 2002; McGaugh & de Blok 1998a, 1998b)

$$v_\infty^4 = \lim_{r\to\infty} v^4(r) \propto L \propto M, \tag{16}$$

where L denotes the luminosity and M is the *visible* (baryonic) mass of the galaxy. The quantity $v(r)$ is the absolute velocity of stars or gaseous components, rotating in the disk plane around the galactic center (r is the distance from the galactic center, defining the origin of the coordinate system). Equating centripetal acceleration v^2/r and $g(r)$, we find

$$v_\infty^2 = \lim_{r\to\infty} rg(r) = \lim_{r\to\infty} r\sqrt{a(r)\ g_v(r)}. \tag{17}$$

Note that the second equality holds, only if one additionally assumes that $\epsilon(r) \ll 1$ for $r \to \infty$. The reason is that, according to equation (9), only in this very case is the approximation $g^2 \approx ag_v$ valid. Physically, the condition $\epsilon(r) \ll 1$ reflects a dominating DM influence, as implied by equations (10) and (13), respectively.

The Tully-Fisher law (eq. [16]) follows directly from the right-hand side of equation (17). Assuming that $a(r) \to a_\infty$ for $r \to \infty$ and, in agreement with the standard procedure, a Keplerian behavior $g_v(r) \simeq GM/r^2$ for $r \to \infty$, we find the desired result

$$v_\infty^4 = a_\infty GM. \tag{18}$$

For the special case $a_\infty = a_0$, this is the well-known MOND formula. Note that according to our approach, equation (18) represents, at least formally, a derived result, whereas it plays the role of a postulate in the original MOND papers (Milgrom 1983a; Sanders & McGaugh 2002). It might be worthwhile to emphasize here once again the crucial aspect, which is that the function $\tilde{\mu}$ from equation (9) naturally satisfies the MOND postulates equation (12).

Nevertheless, one must be aware of the fact that the above derivation of equation (18) was essentially guided by the knowledge of the empirical Tully-Fisher law (eq. [16]). More precisely, the DM paradigm in its current form does *not* provide any explanation for the fact that in many disk galaxies, visible and DM have arranged in such a way that $a(r)$ rapidly converges to a constant nonvanishing value.

Since g_v and g_d reflect the distributions of visible and dark matter, and because of

$$\frac{1}{a_\infty} = \lim_{r\to\infty}\left[\frac{1}{g_d(r)} - \frac{1}{g_v(r) + g_d(r)}\right], \tag{19}$$

the quantity a_∞ gives us information about the asymptotic mass distributions. According to Milgrom (1983a, 1983b, 1983c) and Sanders & McGaugh (2002), for several disk galaxies the experimental value is given by the MOND value, $a_\infty = a_0$. From the point of view adopted in this Letter, this indicates that the composition of these galaxies is generally similar.

In contrast, at least for some clusters of galaxies the actual value of $a(\boldsymbol{x})$ seems to essentially deviate from the MOND value a_0. As mentioned earlier, Aguirre et al. (2001) have shown that the experimentally observed radial temperature profiles of Coma, A2199, and Virgo *cannot* be fitted if one assumes a globally constant value $a(\boldsymbol{x}) \equiv a_0$. Furthermore, these authors report satisfactory agreement when they apply standard DM models instead. With regard to our above considerations, the latter procedure simply corresponds to using a locally varying field $a(\boldsymbol{x}) \not\equiv a_0$. On the one hand, this supports the hypothesis that MOND should be viewed as a special limit case of DM theory; on the other hand, one is led to ask why $a(\boldsymbol{x})$ is approximately constant in disk galaxies but seems to vary in clusters. According to the author's opinion, the answer to this question can be given only by an improved DM theory, yet to be developed. In particular, such a theory must predict the dynamics of dark and visible mass components in detail.

Finally, we still note that if $g_v(\boldsymbol{x}) \ll g_d(\boldsymbol{x})$ holds, then one can expand equation (14) yielding

$$a(\boldsymbol{x}) \approx \frac{g_d(\boldsymbol{x})^2}{g_v(\boldsymbol{x})}. \tag{20}$$

For spherical matter distributions, this means that

$$a(r) \approx \frac{[GM_d(r)/r^2]^2}{GM_v(r)/r^2}, \tag{21}$$

where $M_{v/d}(r)$ denotes the visible/dark mass contained within radius r. For the special case $a(r) \approx a_0$, this is equivalent to

$$\frac{1}{M_v(r)}\left[\frac{M_d(r)}{r}\right]^2 \approx \frac{a_0}{G} \approx 2\,\frac{\mathrm{kg}}{m^2} \approx 10^3\,\frac{M_\odot}{\mathrm{pc}^2}, \tag{22}$$

which implies a strong correlation between the distributions of visible and DM in the MOND limit. It should be mentioned here that the possibility of such a connection was already suggested by McGaugh & de Blok (1998b) and, later, also more extensively discussed by McGaugh (2000).

4. SUMMARY AND CONCLUSIONS

It has been shown that the generalized MOND (eq. [9]) can be derived from Newtonian dynamics, if one adds a DM contribution Φ_d to the (baryonic) Newtonian potential Φ_v, such that $\Phi_{v/d}$ leads to equally directed accelerations $\boldsymbol{g}_{v/d} = -\nabla\Phi_{v/d}$. Compared to the traditional MOND law (eq. [11]), the only formal difference consists in the fact that the constant threshold value a_0 is replaced by the more general acceleration field $a(\boldsymbol{x})$ from equation (14). In the DM picture, $a(\boldsymbol{x})$ reflects the local difference between the gravitational forces caused by dark and visible matter, respectively. In order to exactly regain the traditional MOND law (eq. [11]), one additionally has to demand that $a(\boldsymbol{x}) \equiv a_0$. Thus, MOND can in principle also be interpreted as a DM theory, satisfying the two additional conditions (5) and (15).

Therefore, it seems reasonable to assume that the traditional MOND theory represents a special limit case of Newtonian DM theory. Adopting this point of view, one can further conclude that MOND successfully explains the rotation curves of disk galaxies because for such objects the above conditions (5) and (15) are fulfilled. If this is true, then, as also discussed above, the MOND constant a_0 can be interpreted as the asymptotic value of the field $a(r)$ as $r \to \infty$.

More generally speaking, whenever there is a fixed relationship between g_d and g_v (or ρ_d and ρ_v, respectively) such that $a(\boldsymbol{x}) \approx a_0$, then the traditional MOND theory should continue to work successfully. In turn, if a disk galaxy is in the MOND regime, then equation (14) can be used to estimate the DM distribution ρ_d, provided the visible matter distribution ρ_v is known from observations. Furthermore, it was shown that $\mu(\xi) = \xi/(\xi + 1)$ is the natural candidate for the MOND function. Another result of this Letter was the formal derivation of the Tully-Fisher law (eq. [18]) in § 3. This relation should hold whenever the two conditions $g_v \ll g_d$ and $a_\infty > 0$ are satisfied, where $a_\infty = \lim_{r\to\infty} a(r)$. In this context it must be stressed that the current DM model cannot explain in which situations these two conditions are fulfilled and, if so, why this is the case. Therefore, modifications of the conventional DM theory seem inevitably necessary.

We conclude this short Letter with a more general remark. In principle, there seems to be an agreement that Newton's theory applied to visible matter does *not* give a generally correct description of the dynamics of galaxies and, therefore, has to be modified. A first way to do this is to simply consider an additional potential Φ_d and, following the standard strategy, to attach a "generating object" called DM to this potential. As shown above, Milgrom's concept (if considered as modification of gravity) is in fact very similar, even though it seems quite different at first glance. In particular, the MOND equations can also be transformed into a modification of the former potential type, by starting with $a(\boldsymbol{x}) \equiv a_0$ and reversing the above manipulations. The generating object of the related potential can then be named DM as well.

The author is very grateful to Christian Theis for his encouraging support and careful reading of the manuscript. He also wants to thank Stefan Hilbert for numerous very helpful discussions and Stacy McGaugh for valuable comments. This work was, in parts, financially supported by the Studienstiftung des deutschen Volkes.

REFERENCES

Aguirre, A. 2003, in IAU Symp. 220, Dark Matter in Galaxies, ed. R. Ryder, D. J. Pisano, M. Walker, & K. Freeman (San Francisco: ASP), 77
Aguirre, A., Schaye, J., & Quataert, E. 2001, ApJ, 561, 550
Bertin, G., & Lin, C. C. 1996, Spiral Structure in Galaxies—A Density Wave Theory (Cambridge: MIT Press)
Binney, J., & Tremaine, S. 1994, Galactic Dynamics (Princeton: Princeton Univ. Press)
Combes, F., Boissé, P., Mazure, A., & Blanchard, A. 1995, Galaxies and Cosmology (Berlin: Springer)
Field, G. 1999, Rev. Mod. Phys., 71, 33
Kaplinghat, M., & Turner, M. 2002, ApJ, 569, L19
McGaugh, S. S. 2000, in ASP Conf. Ser. 197, Galaxy Dynamics: From the Early Universe to the Present, ed. F. Combes, G. A. Mamon, & V. Charmandaris (San Francisco: ASP), 153
McGaugh, S. S., & de Blok, W. J. G. 1998a, ApJ, 499, 41
McGaugh, S. S., & de Blok, W. J. G. 1998b, ApJ, 499, 66
Milgrom, M. 1983a, ApJ, 270, 365
———. 1983b, ApJ, 270, 371
———. 1983c, ApJ, 270, 384
———. 2002, ApJ, 571, L81
Navarro, J. F., Frenk, C. S., & White, S. D. M. 1997, ApJ, 490, 493
Ostriker, J. P., & Steinhardt, P. 2003, Science, 300, 1909
Sadoulet, B. 1999, Rev. Mod. Phys., 71, 197
Sanders, R. H., & McGaugh, S. S. 2002, ARA&A, 40, 263
Soussa, M. E., & Woodard, R. P. 2004, Phys. Lett. B, 578, 253
van den Bergh, S. 2001, in ASP Conf. Ser. 252, Historical Development of Modern Cosmology, ed. V. J. Martinez, V. Trimble, & M. J. Pons-Borderia (San Francisco: ASP), 75
Zwicky, F. 1933, Helvetica Phys. Acta, 6, 110
———. 1937, ApJ, 86, 217