18.438 Advanced Combinatorial Optimization October 6, 2009

Lecture 7

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In this lecture, we give a new proof of the relationship between total dual integrality and integrality of polytopes, using Kronecker's Approximation Theorem. We then come back to the matching polytope, and show that Edmonds' description using odd cardinality subset constraints is TDI (a result due to Cunningham & Marsh) and hence is integral and defines the matching polytope.

1 Total Dual Integrality

Consider the linear program defined as

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq b \end{array} \tag{1}$$

where A and b are rational and the associate dual program

$$\begin{array}{ll} \min & y^\top b \\ \text{s.t.} & A^\top y = c \\ & y \geq 0 \end{array} \tag{2}$$

Definition 1 *The system of inequalities defined by $Ax \leq b$ is* Total Dual Integral *(TDI) if for all integral vectors c, the dual program has an integral solution y whenever the optimal value is finite.*

In the previous lecture we showed the following result:

Theorem 1 *If $Ax \leq b$ is TDI and b is integral, then $P = \{x : Ax \leq b\}$ is integral.*

Today we will give another proof of this result, going through Kronecker's Approximation Theorem.

1.1 Kronecker's Approximation Theorem

We begin by proving a theorem of Kronecker.

Definition 2 *Let $a_1, \cdots, a_n$ be rational vectors. A* lattice *$L(a_1, a_2, \ldots, a_n)$ is the set $\{\sum_i^n a_i x_i : x_i \in \mathbb{Z} \;\; \forall i\}$, i.e. is an additive group finitely generated by linearly independent vectors.*

(The assumption that it is finitely generated by *linearly independent* vectors is needed; otherwise, it would not be discrete (consider the 1-dimensional case generated by 1 and $\sqrt{2}$).

Theorem 2 (Kronecker Approximation Theorem (1884)) *$\exists x \in \mathbb{Z}^n$ s.t. $Ax = b$ if and only if $\forall y, y^\top b$ is an integer whenever $y^\top A$ is an integral vector.*

Proof: To prove the forward implication, take an integral solution x^*. For any y, since $y^\top A x^* = y^\top b$, if $y^\top A$ is integral then $y^\top b$ must be an integer too.

We next prove the converse. First, suppose that $Ax = b$ does not have a solution (over the reals). Then, there is a solution to $y^\top A = 0$ (integral) with $y^\top b \neq 0$ and, by scaling y appropriately, we can get $y^\top b \notin \mathbb{Z}$. Thus, we can assume that $Ax = b$ has a solution, and by getting rid of redundant equalities, we can assume that A has full row rank.

Let a_j denote the jth column of A. The statement that $\exists x \in \mathbb{Z}^n$ s.t. $Ax = b$ is equivalent to saying that $b \in L(a_1, \ldots, a_n)$, or $L(A)$, generated by the columns of A. We will perform a series of column operations to A resulting in the matrix $A' = [B \;\; 0]$, where B is lower triangular. Each operation will leave the lattice unchanged, i.e. $L(A') = L(A)$, so that $Ax = b$ will have an integral solution iff $A'x = b$ has an integral solution. The operations will also be shown to preserve integrality of $y^\top b$ and $y^\top A$, so that it will suffice to consider A' instead of A in showing the result.

Observe that B must be nonsingular because we have assumed that A has full row rank. Stacking the $y^\top$ to form a matrix, we have that for all matrices Y, if YA is an integral matrix then Yb is an integral vector. Letting $Y = B^{-1}$, note that $B^{-1}A' = [\; I \quad 0 \;]$ is integral. Hence, $B^{-1}b$ must be integral. Since

$$A' \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} = b, \tag{3}$$

we have found an integral solution to the system $A'x = b$. All that remains is to (a) describe the operations on the matrix A and show that they preserve integrality, and (b) show how to use them to produce A' from A.

(a) The operations are (i) exchanging two columns, (ii) multiplying a column by -1 and (iii) subtracting an integral multiple of one column from another column. These are unimodular transformations, i.e. $A' = AT$ where $T \in \mathbb{Z}^{n\times n}$ and $\det(T) = \pm 1$, so $T^{-1} \in \mathbb{Z}^{n\times n}$. Thus, any sequence of these operations leaves the lattice intact. Indeed if $Ax = b$ where $x \in \mathbb{Z}^n$ then $A'x' = b$ for $x' = T^{-1}x \in \mathbb{Z}^n$, and vice versa if $A'x' = b$ for $x' \in \mathbb{Z}^n$ then $Ax = b$ for $x = Tx' \in \mathbb{Z}^n$. Also, for any y, $y^T AT$ is an integral vector iff $y^T A$ is an integral vector, so the operations also preserve the property that $y^\top b \in \mathbb{Z}$ whenever $y^\top A \in \mathbb{Z}$.

(b) Using these elementary operations, we can transform A into the form

$$A' = [\; B \quad 0 \;] \tag{4}$$

with B lower triangular as follows. For the first row, we can take any two non-zero entries, make them positive by possibly multiplying the column by -1, and compute their gcd using Euclid's algorithm,

$$\gcd(x, y) = \begin{cases} \gcd(x - y, y) & \text{if } x \geq y \\ \gcd(y, x) & \text{if } x < y \\ x, y & \text{if } y = 0 \end{cases} \tag{5}$$

Since these operations are elementary, we can perform them on the columns and reduce the first row to one non-zero entry. We can then put this column as column 1 and proceed to the next row, leaving column 1 fixed. Proceeding in this manner results in the desired form for A'. □

1.2 Proof of Theorem 1

Before showing Theorem 1, we give a corollary of Theorem 2 that we will be using.

Definition 3 *$H = \{x : c^\top x = \alpha\}$ is a* supporting hyperplane *of the polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ if $H \cap P$ is a non-empty face of P.*

Corollary 3 *$P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is integral if and only if every supporting hyperplane of P contains an integral vector.*

Proof: The forward implication is immediate because every supporting hyperplane contains a vertex of P. For the converse, suppose x^* is a non-integral vertex of P. We will demonstrate a hyperplane that does not contain any integral vector. Since x^* is a vertex, there exists a subset $\hat{A}$ of the rows of A (that define P) such that x^* is a unique solution of the system $\hat{A}x = \hat{b}$. Applying Kronecker's approximation theorem, (since x^* is the unique solution and is *not* integral) there must exist a vector $y \in R^m$ such that $y^\top \hat{b}$ is non-integral and $y^\top \hat{A}$ (a vector) *is* integral. Assuming that A and b are rational, we can add an integral constant to the components of y to make y nonnegative while mantaining that $y^\top \hat{b}$ is non-integral and $y^\top \hat{A}$ is integral. Let $c = \hat{A}^\top y$ and $\alpha = y^\top \hat{b}$, and consider $H = \{x : c^\top x = \alpha\}$. Since $\hat{A}x \leq \hat{b}$ for all $x \in P$, multiplying both sides by y^T (≥ 0) we obtain that $y^\top \hat{A}x \leq y^\top \hat{b}$, i.e. $c^\top x \leq \alpha$ for all $x \in P$. The inequalities are tight for x^*, which is in H. We conclude that H is a supporting hyperplane of P. However, c is integral and α is not, so it follows that H cannot contain any integral vector. □

We now finish the proof of Theorem 1.

Proof of Theorem 1: If $Ax \leq b$ is TDI and b is integral, pick any integral c. We will show that the supporting hyperplane with normal c contains an integral vector. First, we can assume that the c_j's are relatively prime (otherwise divide by their common gcd). By our TDIness assumption, $\max c^\top x$ subject to $Ax \leq b$ will have value an integer α, and $c^\top x = \alpha$ is a supporting hyperplane. We need to show that it contains an integral vector.

Since the entries of c are relatively prime, we can find an integral vector x contained in the supporting hyperplane. (Indeed, it can be shown easily by induction on n that if the gcd of the entries of c is g then there is an integral solution to $c^\top x = g$.) Therefore, we conclude that $Ax \leq b$ is integral. □

2 Back to matchings

Given a graph G and a matching M define a vector $\chi^M \in \mathbb{R}^{|E|}$ as

$$\chi_e^M = \begin{cases} 1 & e \in M \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

The *matching polytope* is the convex hull of all such incidence vectors. Edmonds [2] showed that the following is equivalent to the matching polytope:

$$P = \left\{ (x_e \in \mathbb{R})_{e \in E} \;:\; \begin{array}{ll} \sum_{e \in \delta(v)} x_e \leq 1 & \forall v \in V \\ \sum_{e \in E(U)} x_e \leq \frac{|U|-1}{2} & \forall U \in \mathcal{P}_{odd} \\ x_e \geq 0 & \forall e \in E \end{array} \right\}.$$

where $\mathcal{P}_{odd} = \{U \subseteq V \;:\; |U| \text{ is odd}\}$ denotes the odd cardinality subsets.

Cunningham and Marsh [1] showed that P is TDI, providing yet another proof that P is the matching polytope, since TDIness implies that all vertices of P are integral vectors and any valid integer solution of P is a matching. Consider the dual of $\max_{x \in P} c^\top x$, where the dual variables are y_v for every vertex $v \in V$ and z_U for every set $U \in \mathcal{P}_{odd}$:

$$\begin{aligned} \min & \sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{odd}} \frac{|U|-1}{2} z_u \\ \text{s.t.} & \sum_{v \,:\, e \in \delta(v)} y_v + \sum_{U \in \mathcal{P}_{odd} : e \in E(U)} z_U \geq c_e \;\; \forall e \in E \\ & y \geq 0 \end{aligned}$$

TDIness can be stated as follows.

Theorem 4 (Cunningham-Marsh) *For all $c \in \mathbb{Z}^{|E|}$, there exist integral vectors y and z that are feasible and $\sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{odd}} \frac{|U|-1}{2} z_u \leq \nu(c)$, where $\nu(c)$ is the maximum cost of any matching.*

The proof given here is from Schrijver's book, while Cunningham and Marsh prove it algorithmically.

Proof: We show the result by induction on $|V| + |E| + c(E)$ (recall that $c(E)$ is integral). We can assume that $c(e) \geq 1$ for $e \in E$ (otherwise, delete the edge) and that the graph is connected (otherwise, apply proof to the components). The base case of $|V| = 2, |E| = 1, c(e) \geq 1$ is trivially shown by setting $y_1 = c(e)$ and $y_2 = 0$. Let $D(y, z)$ denote the value of the dual objective.

<u>Case 1</u> $\exists v \in V$ such that every maximum cost matching for c covers v. Define the modified edge costs $c'(e) = c(e)$ for $e \notin \delta(v)$ and $c'(e) = c(e) - 1$ for every $e \in \delta(v)$. By the assumption, the cost of maximum matching $\nu(c')$ is $\nu(c) - 1$. By induction, there exists integral y', z' such that $D(y', z') \leq \nu(c')$. Define $y_v = y'_v + 1$, and $y_u = y'_u$ for $u \neq v$. y and z are feasible since the only constraints changed are for $e \in \delta(v)$ and both $c(e)$ and y_v have increased by 1 above $c'(e)$ and y'_v, respectively. Also, $D(y, z) = D(y', z') + 1 \leq \nu(c') + 1 = \nu(c)$, finishing the induction step.

<u>Case 2</u> Otherwise, $\forall v, \exists$ some maximum cost matching for c that does not cover v. Define the modified edge costs $c'(e) = c(e) - 1 \ \forall e \in E$. We will show that all maximum matchings M for c' miss at least 1 vertex. Let M be a maximum matching for c' with $|M|$ as large as possible. Suppose for contradiction that M covers all vertices. Let N be a maximum cost matching for c that does not cover some vertex. Then,

$$c'(N) = c(N) - |N| > c(N) - |M| \geq c(M) - |M| = c'(M) = \nu(c'),$$

which contradicts the definition of ν (the first inequality is because M covers at least one more vertex than N, and the second inequality is because N is optimal for c).

<u>Case 2a</u> Suppose there exists a maximum cost matching M' for c' such that $|M| = \frac{|V|-1}{2}$, i.e. $|V|$ is odd and M misses precisely one vertex. By induction, there exists integral y', z' such that $D(y', z') \leq \nu(c')$. Let $z = z'$ and $y = y'$, with the exception that $z_V = z'_V + 1$ (for the odd set $V \in \mathcal{P}_{odd}$). Since z_V is included in every constraint and both it and $c(e)$ were increased by one, y, z are feasible. Furthermore, $D(y, z) = D(y', z') + \frac{|V|-1}{2} \leq \nu(c') + \frac{|V|-1}{2} \leq \nu(c)$, where the last inequality is because we can construct a matching for c using the matching for c'. This finishes the induction step for $|M| = \frac{|V|-1}{2}$.

<u>Case 2b</u> Suppose all maximum cost matchings for c' miss at least two vertices. Let M be such a matching with $|M|$ maximum and unmatched vertices u and v closest. Note that u and v cannot be adjacent, i.e. $d(u, v) \geq 2$, since otherwise we could have added this edge to make the matching larger. Let t be the second node on the shortest path from u to v in the graph. Note that t must be matched in M, as otherwise we could increase $|M|$ by matching u and t.

Let N be a maximal matching for c, $c(N) = \nu(c)$, such that t is *unmatched* in N. Look at $M \triangle N$. Since t is matched in M and unmatched in N, $M \triangle N$ has a component P with t as an endpoint. Since every vertex in P has degree at most 2, P must be a path. Let M' be the symmetric difference $M \triangle P$ and $N' = N \triangle P$. Since P is a path containing alternatively edges from M and N, M' and N' are both matchings. Also, $|M'| \leq |M|$ because the last edge of the path (connecting to t) is from M. However,

$$\begin{aligned} c(M) + c(N) &= c(M \triangle P) + c(N \triangle P) \implies && (7) \\ c'(M) + |M| + c(N) &= c'(M \triangle P) + |M \triangle P| + c(N \triangle P) \implies && (8) \\ c'(M) + |M| &\leq c'(M \triangle P) + |M \triangle P|, && (9) \end{aligned}$$

where the last step was because $c(N) = \nu(c) \geq c(N \triangle P)$. However, since $c'(M) = \nu(c') \geq c'(M \triangle P)$ and $|M| \geq |M'|$, Eq. (9) must be an equality and we can conclude that $c'(M') = c'(M) = \nu(c')$ and $|M| = |M'|$. Note that t is unmatched in M'. Also, P cannot cover *both* u and v, since neither u nor v are covered by M and only one of them (if in N) can be the other endpoint of the path. Thus, $M' = M \triangle P$ does not cover u or v (or both). Suppose it does not cover u. Then, since t is between u and v on the shortest path, we have that $d(u, t) < d(u, v)$. This contradicts our choice of M, u, v since we had assumed that u and v were uncovered vertices with the shortest distance. □

Although we have fully characterized the matching polytope, it has exponentially many constraints. Padberg and Rao [3] give an efficient separation algorithm for the odd cardinality subset constraints that, given x, in polynomial time decides if $x \in P$. If $x \notin P$, their algorithm produces a hyperplane that separates x from P. The separation algorithm works by finding the minimum odd cut in a suitable graph.

An open question is whether it is possible to give a compact polyhedral description of the matching polytope, e.g. by a suitable lifting of the variables. The interested reader is referred to Yannakakis [4] for partial results and characterizations of the size of the smallest such extended formulation.

References

[1] W.H. Cunningham and A.B. Marsh, III, "A primal algorithm for optimum matching", in Mathematical Programming Study, 8, 50–72, 1978.

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