18.438 Advanced Combinatorial Optimization Feb. 27, 2014

Lecture 7

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1 Perfect Graph

To define *perfect graphs* first we need to review several graph parameters. Given a graph $G = (V, E)$, $\chi(G)$ denotes the minimum number of colors required to properly color all vertices of G and $\omega(G)$ denotes the size of the largest clique in G. Since each vertex of a clique should get a distinct color, $\chi(G) \geq \omega(G)$. In this lecture we consider a family of graphs in which the inequality is tight.

Definition 1 (Perfect Graphs) *A graph $G = (V, E)$ is perfect if for all $S \subseteq V$, $\omega(G[S]) = \chi(G[S])$.*

Note that the equality is required to hold for all induced subgraphs of G. In fact if we alternatively only consider G itself and do not put any condition on its induced subgraphs the family will not be interesting. Union of any graph $G = (V, E)$ with K_n where $n \geq |V|$ satisfies $\chi(G \cup K_n) = \omega(G \cup K_n)$.

Example 1 *Any bipartite graph G is perfect.*

This is trivial as (i) any induced subgraph of a bipartite graph is bipartite, and (ii) the largest clique in a bipartite graph is 2 (or 1 if the graph is empty) while the number of colors needed is 2 (or 1 if the graph is empty).

Example 2 *Let G be a complement of a bipartite graph. Then G is perfect.*

Though it is straightforward to show that any bipartite graph is perfect, proving that complement of a bipartite graph is perfect is more involved. In fact it requires to show that for any bipartite graph G, $\chi(\bar{G}) = \omega(\bar{G})$ or equivalently $\alpha(G) = \bar{\chi}(G)$ where $\alpha(G)$ denotes the size of largest stable set[1] in G and $\bar{\chi}(G)$ denotes the smallest number of cliques needed to cover all vertices of G. This is known as König edge covering theorem. There are some other families that are known to be perfect.

Interval graphs Graph $G = (V, E)$ is an interval graph if there is a mapping between vertices of G and intervals in real line ($\phi : V \rightarrow \mathbb{R} \times \mathbb{R}$) such that $(u, v) \in E$ iff $I_v \cap I_u \neq \emptyset$ where $I_v, I_u \subset \mathbb{R}$ are respectively the intervals associated with v and u.

Example 3 *Any interval graph (the complement of any interval graph) is a perfect graph.*

Comparability graphs Graph $G = (V, E)$ is a comparability graph if it has an acyclic transitive orientation. More precisely, graph G is a comparability graph if we can orient its edges such that the resulting digraph $D = (V, A)$ has the following properties:

1. Transitivity: If (u, v) and (v, w) are both in A, $(v, w) \in A$ too.
2. Anti-symmetry: If $(u, v) \in A$, then $(v, u) \notin A$.

Note that these properties guarantee that the oriented graph is acyclic. Moreover, we can interpret a comparability graph as a partial ordered set (poset) over vertices of G; each edge connects two comparable elements. In this way $\omega(G)$ corresponds to the size of the largest chain in G and $\chi(G)$ captures the minimum number of disjoint antichains that cover G.

[1] also known as independent set

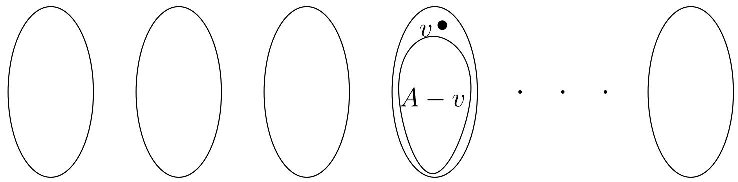

Figure 1: A coloring of G such that v belongs to class A

Lemma 1 *Any comparability graph G is perfect.*

The proof follows from Dilworth's theorem on posets.

It was highly believed that the complement of any perfect graph G is perfect till finally Lovász proved the following remarkable theorem.

Theorem 2 ((Weak) Perfect Graph Theorem [1]) *G is perfect iff $\bar{G}$ is perfect.*

A key part in the proof of (Weak) Perfect Graph Theorem is the following lemma known as the *repetition lemma.*

Lemma 3 (Repetition Lemma) *Let $G = (V, E)$ be a perfect graph and $v \in V$. Then $G_v = (V \cup \{v\}, E \cup \{(v, v')\} \cup \{(v', w) \mid (v, w) \in E\})$ is perfect too.*

Proof: First it is straightforward to see that we only need to show that $\chi(G_v) = \omega(G_v)$ to prove that G_v is perfect. The reason is that if a subgraph H of G_v contains at most one of v and v' it is a subgraph of G as well and the fact that G is perfect implies that $\chi(H) = \omega(H)$. And if H contains both v and v' then it is obtained by repeating v in a subgraph G' of G, and $\chi(H) = \omega(H)$ would then follow by considering G'.
Based on whether v participates in any maximum clique of G, we consider two cases.

- Case 1: there exists a maximum clique of G that contains v.

  Since v' is incident to all neighbors of v, $\omega(G_v) = \omega(G) + 1$. Moreover $\chi(G_v) \leq \chi(G) + 1$ since we can use a coloring of G and give v' a new color. Thus $\chi(G_v) \leq \omega(G_v)$ and we must have equality since the reverse inequality alays hold.

- Case 2: no maximum clique of G contains v.

  Consider a coloring of G as in Figure 1. Since no maximum clique contains v, every maximum clique of G intersect $A \setminus \{v\}$. Thus by removing the set $A - \setminus\{v\}$ the size of maximum clique decreases by one.
  $$\omega(G[V \setminus (A \setminus \{v\})]) = \omega(G) - 1.$$
  Since G is a perfect graph, $\omega(G[V \setminus (A \setminus \{v\})]) = \chi(G[V \setminus (A \setminus \{v\})])$. Now we can add a new color for $\{v'\} \cup (A \setminus \{v\})$to properly color G_v with $\omega(G)$ colors. Thus $\omega(G_v) = \omega(G) = \chi(G) = \chi(G_v)$.

□

References

[1] Lovász, László, Normal hypergraphs and the perfect graph conjecture, *Discrete Mathematics* 2, 253–267, 1972.