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I. Homotopy

Definition 1.0. A continuous function is called a map.

Definition 1.1. Two maps $f_i\colon X \longrightarrow Y$, $i = 0,1$, are called homotopic if there exists a map $F\colon X \times I \longrightarrow Y$ such that $F(x,i) = f_i(x)$ for $i = 0,1$ and all $x \in X$. Here $I = [0,1]$. Denote this relation by $f_0 \simeq f_1$.

Proposition 1.2. $\simeq$ is an equivalence relation between maps $X \longrightarrow Y$.

Proposition 1.3. $\simeq$ is preserved under composition.

Definition 1.4. $b\colon I \longrightarrow X$ is a path from $b(0)$ to $b(1)$. X is path-connected if ....

Proposition 1.5. Every path-connected space is connected. There is a compact, connected subset of R^2 which is not path-connected.

Proposition 1.6. R^n, S^n, and I^n are path-connected if $n \geq 0$ except for S^0.

Definition 1.7. $f\colon X \longrightarrow Y$ is a homotopy equivalence if $\exists\ g\colon Y \longrightarrow X$ such that $gf \simeq id_X$ and $fg \simeq id_Y$. g is called a homotopy inverse to f.

Proposition 1.8. A homotopy inverse to a homotopy equivalence is a homotopy equivalence. The composition of two homotopy equivalences is a homotopy equivalence.

Definition 1.9. X and Y are of the same homotopy type if $\exists$ a homotopy equivalence $f\colon X \longrightarrow Y$.

Proposition 1.10. Being of the same homotopy type is an equivalence relation.

Proposition 1.11. $\exists$ X and Y of the same homotopy type but X and Y are not homeomorphic.

Definition 1.12. X is <u>contractible</u> if X is of the same homotopy type as a one point space P.

Proposition 1.13. R^n and I^n are contractible if $n \geq 0$.

Proposition 1.14. X is contractible if and only if all maps $X \longrightarrow X$ are homotopic and $X \neq \emptyset$.

Definition 1.15. $f: (X,A) \longrightarrow (Y,B)$ if $f: X \longrightarrow Y$ and $f(A) \subset B$. Define the notion of homotopy between maps of pairs.

Proposition 1.16. $\simeq$ is an equivalence relation.

II. <u>The Fundamental Group</u>

Construction 2.1. If $x \in X$, let $c_x: I \longrightarrow X$ be the path defined by $c_x(s) = x$ for all $s \in I$. If b is a path, let $b^{-1}(s) = b(1-s)$ for all $s \in I$. If b and d are paths in X such that $b(1) = d(0)$, define $b * d$ by $b * d(s) = b(2s)$ for $0 \leq s \leq ½$ and $= d(2s-1)$ for $½ \leq s \leq 1$. Let [b] denote the homotopy class of all paths in X which are homotopic to b, considered as maps $b: (I,\{0\},\{1\}) \longrightarrow (X,\{b(0)\},\{b(1)\})$.

Proposition 2.2. If b is a path in X, then $[b * b^{-1}] = [c_{b(0)}]$ and $[b^{-1} * b] = [c_{b(1)}]$.

Proposition 2.3. If b, d, and e are paths in X such that $b(1) = d(0)$ and $d(1) = e(0)$, then $[(b * d) * e] = [b * (d * e)]$.

Proposition 2.4. If $[b] = [b']$ and $[d] = [d']$ and $b(1) = d(0)$, then $[b * d] = [b' * d']$.

Definition 2.5. A path b in X is a <u>loop</u> at $x_0 \in X$ if

Construction 2.16. Let b be a path in X. Define $b_*: \pi_1(X,b(0)) \longrightarrow \pi_1(X,b(1))$ by $b_*(q) = [b^{-1}] * q * [b]$.

Proposition 2.17. b_* is a homomorphism. If d is a path such that $d(0) = b(1)$, then $d_*b_* = (b * d)_*$. If $[b] = [b']$, then $b_* = b'_*$. $(c_x)_*$ is the identity.

Theorem 2.18. b_* is an isomorphism.

Lemma 2.19. Let $f_0 \simeq f_1: X \longrightarrow Y$. Then $\exists$ b, a path in Y, such that $b_*f_{0\#} = f_{1\#}$.

Theorem 2.20. Let $f: X \longrightarrow Y$ be a homotopy equivalence. Then $f_\#: \pi_1(X,x_0) \longrightarrow \pi_1(Y,f(x_0))$ is an isomorphism. (Compare 2.11.)

Definition 2.21. X is <u>simply connected</u> (or 1-connected) if X is path connected and $\pi_1(X,x_0) = \{1\}$ for some $x_0 \in X$.

Proposition 2.22. If X is simply connected, then $\pi_1(X,x_0) = \{1\}$ for all $x_0 \in X$.

Proposition 2.23. Every contractible space is simply connected.

Theorem 2.24. Let b be a path from x_0 to x_1. Then $b_*: \pi_1(X,x_0) \longrightarrow \pi_1(X,x_1)$ is independent of the choice of b if and only if $\pi_1(X,x_0)$ is abelian.

Definition 2.25. An <u>H-space</u> is a pair (H,e), where H is a space, $e \in H$, and we are given a map $m: H \times H \longrightarrow H$ such that $m(\cdot,e) \simeq m(e,\cdot) \simeq id_{(H,e)}$.

Example 2.26. A topological group is an H-space.

Theorem 2.27. Let (H,e) be an H-space. Then $\pi_1(H,e)$ is abelian.

$b(0) = b(1) = x_0$. $[b]$ denotes the homotopy class of loops homotopic to b.

Theorem 2.6. The homotopy classes of loops at $x_0 \in X$ form a group with the product $[b_1] * [b_2] = [b_1 * b_2]$. This group is called the <u>fundamental group</u> or first homotopy group of X with x_0 as base point and is denoted by $\pi_1(X,x_0)$.

Theorem 2.7. Let $f\colon (X,x_0) \longrightarrow (Y,y_0)$. Define $f_\#\colon \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$ by $f_\#([b]) = [fb]$. Then $f_\#$ is a homomorphism.

Theorem 2.8. $(id_X)_\# = id$. If $f\colon (X,x_0) \longrightarrow (Y,y_0)$ and $g\colon (Y,y_0) \longrightarrow (Z,z_0)$, then $g_\# f_\# = (gf)_\#$.

Theorem 2.9. If $f_0 \simeq f_1\colon (X,x_0) \longrightarrow (Y,y_0)$, then $(f_0)_\# = (f_1)_\#$.

Definition 2.10. $f\colon (X,x_0) \longrightarrow (Y,y_0)$ is a <u>homotopy equivalence relative to the base point</u> if $\exists$ $g\colon (Y,y_0) \longrightarrow (X,x_0)$ such that $gf \simeq id_{(X,x_0)}$ and $fg \simeq id_{(Y,y_0)}$.

Theorem 2.11. If f is a homotopy equivalence relative to the base point, then $f_\#$ is an isomorphism.

Corollary 2.12. $\pi_1(R^n,0) = \{1\}$ and $\pi_1(I^n,0) = \{1\}$ if $n \geq 0$.

Theorem 2.13. $\pi_1(X \times Y,(x_0,y_0)) \cong \pi_1(X,x_0) \times \pi_1(Y,y_0)$.

Proposition 2.14. Let $V \subset X$ be the path component of x_0 in X and let $j\colon (V,x_0) \longrightarrow (X,x_0)$ be the inclusion map. Then $j_\#\colon \pi_1(V,x_0) \longrightarrow \pi_1(X,x_0)$ is an isomorphism.

Theorem 2.15. Let $j\colon I \longrightarrow S^1$ be defined by $j(t) = e^{2\pi i t}$, and let $s_0 = j(0) = 1$. Define $j^\#\colon$ {homotopy classes of maps $(S^1,s_0) \longrightarrow (X,x_0)$} $\longrightarrow \pi_1(X,x_0)$ by $j^\#([b]) = [bj]$. Then $j^\#$ is a bijection.

III. Covering Spaces

Definition 3.1. A space X is called locally pathwise connected if for every $x \in X$ and every open set U with $x \in U$, there exists a pathwise connected open set V such that $x \in V \subset U$.

Definition 3.2. Let $f: Y \longrightarrow X$ and let U be an open set of X. Then U is evenly covered by f if there is a non-empty indexing set J and for every $j \in J$, an open set $U_j \subset Y$ such that

(i) $\bigcup_{j \in J} U_j = f^{-1}(U)$,

(ii) if $j \neq k$, $U_j \cap U_k = \emptyset$,

(iii) $f|U_j: U_j \longrightarrow U$ is a homeomorphism for $j \in J$.

Definition 3.3. A map $\pi: X^\wedge \longrightarrow X$ is a covering of X if

(i) π is onto,

(ii) $X^\wedge$ is pathwise connected and locally pathwise connected,

(iii) for every $x \in X$, $\exists$ an open U which is evenly covered by π and $x \in U$.

Proposition 3.4. The following are covering spaces.

(i) $id_X: X \longrightarrow X$, if X is pathwise connected and locally pathwise connected,

(ii) $\pi: R^1 \longrightarrow S^1$ given by $\pi(r) = e^{2\pi i r}$,

(iii) $\pi_n: S^1 \longrightarrow S^1$ given by $\pi_n(e^{ir}) = e^{inr}$, for $n \neq 0$ an integer,

(iv) $\pi: S^n \longrightarrow RP^n$ = {pairs of antipodal points of S^n} with the quotient topology (RP^n is called real n-dimensional projective space).

Notation. l.c. stands for pathwise connected and locally pathwise connected.

Proposition 3.5. Let $U \subset X$ be evenly covered by $f: Y \longrightarrow X$ and let Y be l.c. Then U is locally pathwise connected.

Proposition 3.6. Let $\pi: X^\wedge \longrightarrow X$ be a covering. Then X is l.c.

Proposition 3.7. Let $U \subset X$ be evenly covered by $f: Y \longrightarrow X$. Let Z be connected and let $g: Z \longrightarrow U$. Let $y \in Y$, $z \in Z$ be such that $f(y) = g(z)$. Then $\exists! \; g': Z \longrightarrow Y$ such that $g'(z) = y$ and $fg' = g$.

Proposition 3.8. Let $\pi: X^\wedge \longrightarrow X$ be a covering. Let b be a path in X and let $x^\wedge \in X^\wedge$ be such that $\pi(x^\wedge) = b(0)$. Then $\exists!$ path $b^\wedge$ in $X^\wedge$ such that $x^\wedge = b^\wedge(0)$ and $\pi b^\wedge = b$.

Proposition 3.9. Let $\pi: X^\wedge \longrightarrow X$ be a covering. Let Z be pathwise connected and let $g_i: Z \longrightarrow X^\wedge$ be maps such that $\pi g_0 = \pi g_1$ and $g_0(z) = g_1(z)$ for some $z \in Z$. Then $g_0 = g_1$.

Proposition 3.10. Let $\pi: X^\wedge \longrightarrow X$ be a covering, let $g: I^2 \longrightarrow X$ be a map, and let $x^\wedge \in X^\wedge$ be such that $\pi(x^\wedge) = g(0,0)$. Then $\exists! \; g^\wedge: I^2 \longrightarrow X^\wedge$ such that $g^\wedge(0,0) = x^\wedge$ and $\pi g^\wedge = g$.

Proposition 3.11. Let $\pi: X^\wedge \longrightarrow X$ be a covering. Then $\pi_\#: \pi_1(X^\wedge,x^\wedge) \longrightarrow \pi_1(X,\pi(x^\wedge))$ is one to one.

IV. Classification of Coverings with Base Point

Definition 4.1. $\pi: (X^\wedge,x^\wedge) \longrightarrow (X,x)$ is a covering with base point if $\pi: X^\wedge \longrightarrow X$ is a covering.

Definition 4.2. Two coverings $\pi_i: (X^\wedge_i,x^\wedge_i) \longrightarrow (X,x)$ are homeomorphic if $\exists$ a homeomorphism $h: (X^\wedge_1,x^\wedge_1) \longrightarrow (X^\wedge_2,x^\wedge_2)$ such that $\pi_2 h = \pi_1$.

Proposition 4.3. Being homeomorphic is an equivalence relation on the set of coverings with base point.

Proposition 4.4. Let π_1 and π_2 be homeomorphic coverings. Then $(\pi_1)_\#(\pi_1(X\^_1,x\^_1)) = (\pi_2)_\#(\pi_1(X\^_2,x\^_2)) \subset \pi_1(X,x)$.

Proposition 4.5. Let π be a covering with base point and let $g\colon (Z,z) \longrightarrow (X,x)$ be a map such that $\mathrm{Im}\ g_\# \subset \mathrm{Im}\ \pi_\#$. Let b and d be paths in Z such that $b(0) = d(0) = z$ and $b(1) = d(1) = z_1$. Let b^ and d^ be the unique paths in X^ such that $\pi b\^ = gb$, $\pi d\^ = gd$, and $b\^(0) = d\^(0) = x\^$. Then $b\^(1) = d\^(1)$.

Proposition 4.6. Let π be a covering with base point, let Z be l.c., and let $g\colon (Z,z) \longrightarrow (X,x)$ be a map such that $\mathrm{Im}\ g_\# \subset \mathrm{Im}\ \pi_\#$. Then $\exists!\ g\^\colon (Z,z) \longrightarrow (X\^,x\^)$ such that $\pi g\^ = g$.

Proposition 4.7. Two coverings π_1 and π_2 are homeomorphic if and only if $\mathrm{Im}\ (\pi_1)_\# = \mathrm{Im}\ (\pi_2)_\#$.

Definition 4.8. X is <u>semi-locally 1-connected</u> if for every $x \in X$ there is an open set U, $x \in U$, such that $\pi_1(U,x) \longrightarrow \pi_1(X,x)$ is the trivial homomorphism.

Definition 4.9. Let $G \subset \pi_1(X,x)$ be a subgroup. Two paths b and d are <u>G-homotopic</u> if $b(0) = d(0) = x$, $b(1) = d(1)$, and $[b * d^{-1}] \in G$.

Proposition 4.10. G-homotopy is an equivalence relation on paths in X which start at x.

Construction 4.11. Let X_G denote the set of G-homotopy classes of paths in X which start at x. Let $\pi\colon X_G \longrightarrow X$ be defined by $\pi(\{b\}) = b(1)$. Let $V \subset X$ be open and path connected. Let b be a path in X such that $b(0) = x$ and $b(1) \in V$. Define $U(b,V) \subset X_G$ to consist of the G-homotopy classes of paths of the form $b * d$ where $d(0) = b(1)$ and $d(t) \in V$. Take the $U(b,V)$ as a subbase for the open sets of X_G.

Theorem 4.12. Let X be l.c. and semi-locally 1-connected. Then the homeomorphism classes of coverings of (X,x) are in one to one correspondence (i. e. $\exists$ a bijection) with the subgroups of $\pi_1(X,x)$.

V. Classification of Coverings and Their Translations.

Definition 5.1. Two coverings $\pi_i\colon X\hat{}_i \longrightarrow X$, $i = 1,2$, are homeomorphic if there is a homeomorphism $h\colon X\hat{}_1 \longrightarrow X\hat{}_2$ such that $\pi_2 h = \pi_1$.

Proposition 5.2. Being homeomorphic is an equivalence relation on the set of coverings of X.

Definition 5.3. Two subgroups H, K, of a group G are conjugate if there is a $g \in G$ such that $K = g^{-1}Hg$.

Proposition 5.4. Being conjugate is an equivalence relation on the set of subgroups of G.

Proposition 5.5. Let π be a covering and let $x\hat{}_i \in X\hat{}$ be such that $\pi(x\hat{}_i) = x$, $i = 1,2$. Then $\pi_\#(\pi_1(X\hat{},x\hat{}_1))$ and $\pi_\#(\pi_1(X\hat{},x\hat{}_2))$ are conjugate in $\pi_1(X,x)$.

Proposition 5.6. Let π be a covering, let $x\hat{} \in X\hat{}$, and let $G \subset \pi_1(X,\pi(x\hat{}))$ be conjugate with Im $\pi_\#$. Then $\exists\ x\hat{}_1 \in X\hat{}$ such that $\pi_\#(\pi_1(X\hat{},x\hat{}_1)) = G$.

Theorem 5.7. Let π_i be coverings and let $x\hat{}_i \in X\hat{}_i$ be such that $\pi_1(x\hat{}_1) = \pi_2(x\hat{}_2)$. Then π_1 and π_2 are homeomorphic if and only if Im $(\pi_1)_\#$ and Im $(\pi_2)_\#$ are conjugate in

$\pi_1(X,\pi_1(\hat{x}_1))$.

Theorem 5.8. The homeomorphism classes of coverings of a l.c. and semi-locally 1-connected space X are in one to one correspondence with the conjugacy classes of subgroups of $\pi_1(X,x)$.

Definition 5.9. A <u>translation</u> of a covering π is a homeomorphism $h: \hat{X} \longrightarrow \hat{X}$ such that $\pi h = \pi$. Let $T(\pi)$ denote the set of translations of π.

Proposition 5.10. $T(\pi)$ is a group under composition of translations.

Definition 5.11. Let $H \subset G$ be a subgroup of G. The elements $g \in G$ such that $g^{-1}Hg = H$ form a subgroup N(H), the <u>normalizer</u> of H in G.

Proposition 5.12. H is normal in N(H).

Construction 5.13. Let π be a covering and let $H = \pi_{\#}(\pi_1(\hat{X},\hat{x}))$. Define a function $t: N(H) \longrightarrow T(\pi)$ as follows. Let $q \in N(H)$. Choose a path $\hat{b}$ in $\hat{X}$ such that $\hat{b}(0) = \hat{x}$ and $[\pi\hat{b}] = q$. $t(q)$ is the unique translation such that $t(q)(\hat{x}) = \hat{b}(1)$.

Theorem 5.14. t is well defined. t is a homomorphism onto $T(\pi)$, and Ker t = H.

Corollary 5.15. N(H)/H is isomorphic to $T(\pi)$.

VI. <u>Some Fundamental Groups.</u>

Definition 6.1. A covering π is called <u>universal</u> if $\hat{X}$ is simply connected.

Corollary 6.2. Let π be a universal covering. Then $\pi_1(X,x)$ is isomorphic to $T(\pi)$.

Proposition 6.3. Example 3.4 (ii) is a universal covering.

Theorem 6.4. $\pi_1(S^1,s)$ is isomorphic to the additive group of integers, Z, for any $s \in S^1$.

Example 6.5. Describe the covering space corresponding to each subgroup of $\pi_1(S^1,s)$.

Definition 6.6. $T^n = S^1 \times \ldots \times S^1$, n times, is the <u>n-dimensional torus.</u>

Theorem 6.7. $\pi_1(T^n,t) \cong Z \oplus \ldots\ldots \oplus Z$, n times.

Theorem 6.8. Let X be l.c. and semi-locally 1-connected, let A, $B \subset X$ be open subsets which are l.c. and simply-connected, and let $A \cap B$ be pathwise connected and $A \cup B = X$. Then X is simply-connected.

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Theorem 6.9. S^n is simply-connected for all $n > 1$.

Theorem 6.10. Example 3.4 (iv) is a universal covering and hence $\pi_1(RP^n,p)$ is isomorphic to $Z/2Z$ if $n > 1$.

VII. <u>Higher Homotopy Groups.</u>

Definition 7.1. Y^X denotes the function space of continuous maps $f\colon X \longrightarrow Y$ with the compact-open topology. If $B \subset Y$ and $A \subset X$, then $(Y,B)^{(X,A)} \subset Y^X$ is the subspace of functions such that $f(A) \subset B$.

Theorem 7.2. Define $\alpha\colon Z^X \times Y \longrightarrow (Z^Y)^X$ by $[[\alpha(f)](x)](y) = f(x,y)$. If Y is locally compact and Hausdorff, then α is a bijection.

Definition 7.3. $(X,x_0)^{(I,0)}$ is the <u>space of paths</u> on X starting at x_0. $(X,x_0)^{(I,\{0,1\})}$ is the <u>space of loops</u> on X at x_0 and denoted by $\Omega(X,x_0)$. Define $\Omega^n(X,x_0) =$

$\Omega(\Omega^{n-1}(X,x_0),x_{n-1})$, where x_{n-1} is the constant loop at x_{n-2}.

Definition 7.4. $\pi_n(X,x_0) = \pi_1(\Omega^{n-1}(X,x_0),x_{n-1})$.

Theorem 7.5. $\Omega(X,x_0)$ is an H-space.

Corollary 7.6. $\pi_n(X,x_0)$ is abelian if $n > 1$.

Construction 7.7. Let $f\colon (X,x_0) \longrightarrow (Y,y_0)$. Define $f_1\colon \Omega(X,x_0) \longrightarrow \Omega(Y,y_0)$ by $f_1(b) = fb$. Define $f_n\colon (\Omega^n(X,x_0),x_n) \longrightarrow (\Omega^n(Y,y_0),y_n)$ by induction. Define $f_\#\colon \pi_n(X,x_0) \longrightarrow \pi_n(Y,y_0)$ by $(f_{n-1})_\#\colon \pi_1(\Omega^{n-1}(X,x_0),x_{n-1}) \longrightarrow \pi_1(\Omega^{n-1}(Y,y_0),y_{n-1})$.

Theorem 7.8. $id_\#$ is the identity. If $f\colon (X,x_0) \longrightarrow (Y,y_0)$ and $g\colon (Y,y_0) \longrightarrow (Z,z_0)$, then $g_\# f_\# = (gf)_\#$.

Proposition 7.9. If $f \simeq g\colon (X,x_0) \longrightarrow (Y,y_0)$, then $f_\# = g_\#$.

Corollary 7.10. If (X,x_0) is contractible, then $\pi_n(X,x_0) = 0$ for $n \geq 1$.

Theorem 7.11. $\pi_n(X,x_0) \cong \pi_{n-1}(\Omega(X,x_0),x_1)$.

Definition 7.12. Let $I' \subset I^n$ denote the subspace of all n-tuples for which at least 1 coordinate is 0 or 1. $\pi(I^n,I'; X,x_0)$ denotes the set of homotopy classes of maps from (I^n,I') to (X,x_0).

Theorem 7.13. There is a natural 1-1 correspondence between $\pi_n(X,x_0)$ and $\pi(I^n,I'; X,x_0)$.

VIII. Fibre Spaces.

Definition 8.1. Let $p\colon E \longrightarrow B$. p is a *fibre map* if for every l.c. space X the outer triangle in the diagram below can be filled in by the dotted arrow so that the triangles **commute**

$$\begin{array}{ccccc} & & & & E \\ & & \overset{g}{\nearrow} & \overset{G}{\nearrow} & \downarrow \\ X & \xrightarrow[i_0]{} & X \times I & \xrightarrow[H]{} & B \end{array}$$

where $i_0(x) = (x,0)$. (E,p,B) is called a <u>fibre space</u>, E is the <u>total space</u>, B the <u>base space</u>, and p the <u>projection</u>. If B has a base point b_0, then $F = p^{-1}(b_0)$ is the <u>fibre</u>.

Proposition 8.2. If B is pathwise connected and $E \neq \emptyset$, then p is onto.

Proposition 8.3. Let $E = F \times B$ and let p be the projection onto the second factor. Then (E,p,B) is a fibre space.

Proposition 8.4. A covering space $(X^{\wedge},\pi,X)$ is a fibre space.

Theorem 8.5. Let $A,B \subset Y$. Let $E(Y; A,B) \subset Y^I$ be the subspace of those maps such that $f(0) \in A$ and $f(1) \in B$. Define p: $E(Y; A,B) \longrightarrow A \times B$ by $p(f) = (f(0),f(1))$. Then p is a fibre map.

Corollary 8.6. p: $(X,x_0)^{(I,0)} \longrightarrow X$ by $p(f) = f(1)$ is a fibre space with fibre $\Omega(X,x_0)$.

IX. <u>Exact Sequences.</u>

Definition 9.1. A sequence $\ldots \longrightarrow A \xrightarrow[f]{} B \xrightarrow[g]{} C \longrightarrow \ldots$ of abelian groups and homomorphisms is <u>exact</u> if $\operatorname{Im} f = \operatorname{Ker} g$.

Proposition 9.2. $0 \longrightarrow A \longrightarrow 0$ is exact if and only if $A = 0$.

Proposition 9.3. $0 \longrightarrow A \xrightarrow[f]{} B \longrightarrow 0$ is exact if and only if f is an isomorphism.

Proposition 9.4. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if $g\hat{}: B/f(A) \longrightarrow C$ is an isomorphism and f is injective.

"Five Lemma" 9.5. Let

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\ \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \downarrow p_4 & & \downarrow p_5 \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{i'} & E' \end{array}$$

be a commutative diagram with exact rows. If p_1, p_2, p_4, and p_5 are isomorphisms, so is p_3.

X. Relative Homotopy Groups and Exact Sequences.

Definition 10.1. Let $I^{n-1} \subset I^n$ be the subspace where $t_n = 0$. Let $J^{n-1} \subset I^n$ be the closure of $I^n - I^{n-1}$. Let $x_0 \in A \subset X$. For $n \geq 2$, define $\pi_n(X,A)$ to be the set of homotopy classes of maps from $(I^n, I^{n-1}, J^{n-1}) \longrightarrow (X,A,x_0)$.

Proposition 10.2. Let $Z \subset X^I$ be defined by $Z = \{f \mid f(1) = x_0, f(0) \in A\}$. Then $\pi_n(X,A,x_0) \cong \pi_{n-1}(Z,c)$.

Corollary 10.3. $\pi_n(X,A)$ is abelian for $n > 2$.

Definition 10.4. Let $f: (X,A,x_0) \longrightarrow (Y,B,y_0)$. Define $f_\#: \pi_n(X,A,x_0) \longrightarrow \pi_n(Y,B,y_0)$ in the usual way.

Theorem 10.5. $f_\#$ is a homomorphism, $(id)_\# = id$, and $g_\# f_\# = (gf)_\#$.

Definition 10.6. $\partial: \pi_n(X,A) \longrightarrow \pi_{n-1}(A,x_0)$ is defined by $\partial([f]) = [f|I^{n-1}]$.

Theorem 10.7. Let $i: (A,x_0) \longrightarrow (X,x_0)$, $j: (X,x_0,x_0) \longrightarrow (X,A,x_0)$.

Then

$$\cdots \xrightarrow{\partial} \pi_n(A,x_0) \xrightarrow{i_\#} \pi_n(X,x_0) \xrightarrow{j_\#} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \xrightarrow{i_\#} \cdots$$

is exact. (This is called the exact sequence of a pair.)

Theorem 10.8. Let (E,p,B) be a fibre space, $b_0 \in B$, $p\hat{}$: $(E,F) \longrightarrow (B,b_0)$. Then $p\hat{}_\#$: $\pi_n(E,F) \longrightarrow \pi_n(B,b_0)$ is an isomorphism for $n \geq 2$.

Theorem 10.9. Let (E,p,B) be a fibre space, $b_0 \in B$, $e_0 \in F$. Then there is an exact sequence

$$\cdots \xrightarrow{\partial\hat{}} \pi_n(F,e_0) \xrightarrow{i_\#} \pi_n(E,e_0) \xrightarrow{p_\#} \pi_n(B,b_0) \xrightarrow{\partial\hat{}} \pi_{n-1}(F,e_0) \xrightarrow{i_\#} \cdots$$

Here $\partial\hat{} = \partial(p\hat{}_\#)^{-1}$. (This is called the exact sequence of a fibre space.)

Theorem 10.10. Let $(X\hat{},\pi,X)$ be a covering space. Then $\pi_\#$: $\pi_n(X\hat{},x\hat{}) \longrightarrow \pi_n(X,\pi(x\hat{}))$ is an isomorphism for $n \geq 2$.

Corollary 10.11. $\pi_n(S^1,x_0) = 0$ if $n > 1$.

Corollary 10.12. $\pi_n(S^r,x_0) \cong \pi_n(RP^r,(x_0,-x_0))$, if $n > 1$.

11. Polyhedra

Definition 11.1. A simplicial complex K (of dimension at most 2) is a collection of vertices $\{V_i\}$, a collection of edges $\{V_i,V_j\}$, and a collection of triangles $\{V_i,V_j,V_k\}$ satisfying the following condition: given a set in the collection, then a non-empty subset of it is in the collection.

(Intuitive) Definition 11.2. A topological space X is a polyhedron if it can be broken up into vertices, edges, and triangles as in definition 11.1. More precisely, if X is homeomorphic to $|K|$, the geometric realization of K.

Proposition 11.3. I^n, S^n, and R^n are polyhedra (if $n \leq 2$).

Definition 11.4. Given a simplicial complex K. An *edge path* in K is a sequence of vertices of K, $w = V_0 \ldots V_k$, such that each successive pair of vertices is an edge or repeats itself. If $w' = V_k \ldots V_l$, then $w * w' = V_0 \ldots V_l$, the *product of edge-paths*. $w^{-1} = V_k \ldots V_0$ is the *inverse edge-path* to w.

Definition 11.5. We define an equivalence relation on edge-paths with the same beginning and end points as follows. If $w = V_0 \ldots V_i V_i \ldots V_k$, then $w \simeq w' = V_0 \ldots V_i \ldots V_k$. If V_i, V_j, V_l is a triangle in K, then $w = V_0 \ldots V_i V_l \ldots V_k \simeq w' = V_0 \ldots V_i V_j V_l \ldots V_k$. Let [w] denote the equivalence class of w.

Proposition 11.6. Let $\pi(K,V)$ denote the set of equivalence classes of edge-paths in K which start and end at V. Then $\pi(K,V)$ is a group under the operation $[w] * [w'] = [w * w']$. This is called the *edge-path group of K*.

Theorem 11.7. Let X be a polyhedron with corresponding simplicial complex K. Then $\pi(K,V) \cong \pi_1(X,V)$.

Definition 11.8. K is *connected* if |K| is path-connected. K is *simply connected* if |K| is simply connected. L is a *subcomplex* of K if the collections of triangles, edges, and vertices of L are subcollections of those of K and L is a simplicial complex.

Proposition 11.9. Given K, a connected complex. Then there exists a simply connected subcomplex L (having only vertices and edges) of K which contains all the vertices of K. Such a subcomplex is a *maximal tree*.

Definition 11.10. Given K and a subcomplex L. Let G =

$G(K,L)$ be the group generated by g_{ij}, one generator for each edge V_i,V_j in K subject to the relations $g_{ij} = 1$ if V_i,V_j is in L and $g_{ij}g_{jk} = g_{ik}$ if V_i,V_j,V_k is in K.

Theorem 11.11. If L is a maximal tree, then $\pi(K,V) \cong G(K,L)$.

Corollary 11.12. If K is a finite simplicial complex, then $\pi_1(|K|,V)$ is finitely generated and finitely presented.

Corollary 11.13. If K has no triangles, then $\pi_1(|K|,V)$ is free.

Proposition 11.14. π_1 of a figure 8 is free on two generators.

Theorem 11.15. A free group on two generators contains a free group on n generators.