Systematic phenomena in Mod p Homotopy Theory

§1. Well, my title is "systematic phenomena in mod p homotopy theory". The words "mod p" remind us of a principle established by Serre; in homotopy-theory, you can consider phenomena at the prime $p=2$ independently of problems at the primes $p=3$ or $p=5$. This insight was carried further by Sullivan with his notion of localisation in homotopy theory, and I will assume all that sort of thing known.

So next I ought to explain what I mean by "systematic phenomena", and the obvious way is to show you one. I shall assume $p>2$ and construct the family of maps which [illegible] calls α_i; following the account in my paper "On the groups $J(X)$ IV".

Let M be a Moore space $M = S^n \cup_p e^{n+1}$ ($p>2$; n suff. large). [Topology 5 (1966) 21–71]

<u>Lemma 1</u>. There is a map $A: S^q M \to M$ ($q = 2(p-1)$) such that in the mapping-cone of A,

$$X = M \cup_A C S^q M,$$

we find the following operation in mod p cohomology:

$$H^n \xrightarrow[\cong]{\beta} H^{n+1} \xrightarrow[\cong]{P^1} H^{n+q+1} \xrightarrow[\cong]{\beta} H^{n+q+2}.$$

The htpy class of this map is unique up to a scalar multiple non-zero mod p.

<u>Proof</u> With such a statement, the obvious proof is by the ASS

$$\mathrm{Ext}_A^{**}(\tilde{H}^*(M;F_p), \tilde{H}^*(M;F_p)) \Rightarrow \{M, M\}_*.$$

We must therefore calculate the relevant Ext. We can construct the beginnings of a resolution as follows.

$$C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} \tilde{H}^*(M) \ \ \text{dim } 0$$

C_0 is the free algebra on c_0 in degree n; ε is ...

C_1 is free on c_1, c_1' in degrees $n+q$, $n+q+1$;
$dc_1 = P^1 c_0$, $\quad dc_1' = P^1 \beta c_0$.

Otherwise no generators in degrees $\leq n + 2q$.

Calculation of Ext in degrees $t-s$ about q.

Cocycle $\quad c_1 \longrightarrow \beta u \quad$ deg $s=1$, $t=q-1$

$\begin{cases} c_1 \longrightarrow u \\ c_1' \longrightarrow \beta u \end{cases} \quad$ deg $s=1$, $t=q$

$\| \quad c_1' \longrightarrow u \quad$ deg $s=1$, $t=q+1$.

This is the answer we want. In the spectral sequence all the relevant differentials go from or to zero groups. $\therefore$ we get

$$\{M, M\}_q = \mathbb{Z}_p, \quad \text{on a generator } f \text{ with}$$

the property sought.

__Remark__. This proof shows that the map exists ... for a mod p ... ; actually it ... (F. R. Cohen and J. A. Neisendorfer, Note on desuspending the Adams map, Math. Proc. Camb. Phil. Soc. 99 (1986) 59–64)

Lemma 2. The map f induces

$$\begin{array}{ccc} K^n(M) & \xrightarrow[\cong]{f^*} & K^n(S^a M) \\ \| & & \| \\ \mathbb{Z}_p & & \mathbb{Z}_p \end{array}$$

Proof. Consider the AHSS

$$\tilde{H}^*(X ; K^*(pt)) \Rightarrow \tilde{K}^*(X).$$

$$K^*(pt) = \begin{cases} \mathbb{Z} & \text{in even dims} \\ 0 & \text{in odd dims} \end{cases}$$

$$\tilde{H}^*(X ; \mathbb{Z}) = \begin{cases} \mathbb{Z}_p & \text{in dims } n+1, n+a+1 \\ 0 & \text{otherwise.} \end{cases}$$

If we work localized at p, the first differential in the AHSS which can possibly be nonzero is d_r for $r = 2a+1 = 2p-1$, and this can be expressed [illegible]

$$\begin{array}{ccc} H^{n+1}(X;\mathbb{Z}) & & H^{n+a+2}(X;\mathbb{Z}) \\ \downarrow \rho & & \uparrow \delta \\ H^{n+1}(X;\mathbb{F}_p) & \xrightarrow{P^1} & H^{n+a+1}(X;\mathbb{F}_p) \end{array}$$

(see JFA, "On the Chern character [illegible]" Proc. Camb. Phil. Soc. 57 (1961) [illegible].)

In our case, P^1 is iso. So the spectral sequence becomes zero from the term $r = 2a+2$ onwards, and $\tilde{K}^*(X) = 0$. Looking at the exact sequence of the cofibring in K^*, we see that $K^*(f)$ is iso.

Coroll 3. For an arbitrarily long composite

$$S^{ia}M \to \cdots \xrightarrow{S^{a}A} S^{2a}M \xrightarrow{S^{a}A} S^{a}M \xrightarrow{A} M$$

the result is essential.

Proof. The composite still induces

$$H^{n+1}(M) \xrightarrow{\cong} H^{n+1}(S^{ia}M)$$

$$\mathbb{Z}_p \qquad\qquad \mathbb{Z}_p$$

We now define the element α_i to be the following composite.

$$S^{ia}M \to \cdots \to S^{2a}M \xrightarrow{S^{a}A} S^{a}M \xrightarrow{A} M$$

$$\uparrow$$

$$S^{n+ia}$$

By construction, it lies in the stable stem of degree $ia-1$, and is annihilated by p.

Prop 4. α_i is non-zero, and of filtration exactly i in the classical ASS.

Proof. Clearly α_i has filt $\geq i$, for I displayed it as a product of i maps, each of which induces the zero map of $H^*(\ ;\mathbb{Z}_p)$ and hence has filt ≥ 1. If it had filtn $\geq i+1$ it would be zero by the result of Adams, which are zero.
[Adams: J(X) IV, Topology 5 (1966) 21–71]
So it is enough to prove it is non-zero.

Xii Bousfield - [illegible]

The wedge-sum operation $\bigvee_\alpha X_\alpha$ (where the sum may be infinite) passes to Bousfield classes, and gives the least upper bound with the ordering relation:

$$\left(\bigvee_\alpha X_\alpha\right) \wedge Z \simeq \bigvee_\alpha (X_\alpha \wedge Z),$$

which is contractible iff $X_\alpha \wedge Z \simeq pt$ $\forall \alpha$.

The smash product also passes to Bousfield classes. Suppose $E_1 \approx E_2$; then $E_1 \wedge F \wedge Z \simeq pt$ iff $E_2 \wedge F \wedge Z \simeq pt$ by the above prop. applied to $F \wedge Z$; this shows $E_1 \wedge F \sim E_2 \wedge F$.

~~However~~ We don't say that [illegible] equivalent [illegible]. However, it may be useful.

Ex. $\langle S^0 \rangle =_{B} \bigvee_p \langle S^0_{(p)} \rangle$

Proof $\pi_*(S^0_{(p)} \wedge X) = \pi_*(X)_{(p)}$,

so $\pi_*(S^0_{(p)} \wedge X) = 0$ $\forall p$ is the same

as $\pi_*(X)_{(p)} = 0$ $\forall p$, which by well-known algebra $\iff$ $\pi_*(X) = 0$.

Coroll. If E is any spectrum

$$\langle E \rangle \simeq_{B} \bigvee_p \langle E_{(p)} \rangle.$$

Proof. Take previous [illegible] & apply $\wedge E$; [illegible] the
[illegible] [illegible] [illegible] [illegible] [illegible] the

Some of Ravenel's problems explicitly concern the theory of Bousfield classes. In any case, Mike Hopkins finds it convenient to process information by turning it into information about Bousfield classes, manipulating it and taking it out again. So we had better prepare a little for such exercises.

Bousfield classes are partially-ordered; we write $(E, I) \geqslant (F, J)$ iff $E_*(Z)$ is I-torsion $\Rightarrow$ $F_*(Z)$ is J-torsion.

Notice that this looks the right way round; if $E = F$ & $I \supset J$, then $E_*(Z)$ is I-torsion $\Rightarrow$ $E_*(Z)$ is J-torsion so $(E, I) \geqslant (E, J)$

Now I'll stick to the case $I = \{1\}$ for a while. The sphere spectrum S^0 is the maximum; $S^0 \wedge Z \simeq pt$ says $Z \simeq pt$, which $\Rightarrow$ $F \wedge Z \simeq pt$ for any F. With $E = S^0$, the E-equivalences are the honest equivalences. Localization changes nothing; every X is S^0-local and $X \xrightarrow{1} X$ will serve as a localization.

The point spectrum pt is the least; $pt \wedge Z \simeq pt$ is always true, and any other statement implies it. With $E = pt$, all maps are E-equivalences; the only E-local spectrum is the point, and localization destroys everything.

an irreducible rep in $\mathcal{S}^{-1}\mathcal{E}$; it is initial among functors which have this property.

This statement characterizes $\mathcal{S}^{-1}\mathcal{E}$ as the coarsest category of fractions

Then I must explain about Bousfield equivalence. Two pairs (E, I) and (F, J) are Bousfield equivalent if
$$\mathcal{S}(E, I) = \mathcal{S}(F, J).$$

If this is so, then the whole of the rest of the theory of Bousfield localization is the same for both, because the theory only depends on $\mathcal{S}$.

Of course, Bousfield first defined equivalence of spectra because he was considering the case $I = \{1\}$.

If you have a map $f : X \to Y$, and want to see if it lies in $\mathcal{S}(E, I)$ or $\mathcal{S}(F, J)$, you must consider cofibering $X \xrightarrow{f} Y \to Z$. In this way you translate "Bousfield equivalence" into the following canonical condition:

$E_*(Z)$ is I-torsion iff $F_*(Z)$ is J-torsion.

With $I = \{1\}$, $J = \{1\}$ this says,

$E \wedge Z$ is contractible iff $F \wedge Z$ is contractible.

Now I can explain about the category of fractions. Let $\mathscr{C}$ be the category of all spectra. The objects form a class rather than a set, but this is no harm in practice. We will define a new category $S^{-1}\mathscr{C}$. The objects are the same as those of $\mathscr{C}$. To define the maps from X to Y in $S^{-1}\mathscr{C}$, we choose localisation

$$X \xrightarrow{e} X', \qquad Y \xrightarrow{\eta} Y'$$

and take the maps to be $[X', Y']$. Of course we could choose different localisation

$$X \longrightarrow X'', \qquad Y \longrightarrow Y''$$

but then there is a canonical equivalence $X' \simeq X''$, and a canonical equivalence $Y' \simeq Y''$, so there is a canonical way to identify $[X', Y']$ with $[X'', Y'']$.

Composition & identity maps are obvious.

There is a functor λ from $\mathscr{C}$ to $S^{-1}\mathscr{C}$ which is the identity on objects. Viz, given $X \xrightarrow{f} Y$ there is a unique map λf making the following diagram [commute].

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, dashed, "\lambda f"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

<u>Exercise</u>. λ carries every map $s \in S$ to

lie in $\mathcal{T}$ as we have said,

so $[W, Y]_* \cong [W, \mathrm{pt}\rangle_* = 0$.

Thirdly, I still claim that Y is $\mathcal{S}$-local. For suppose $U \xrightarrow{s} V$ is a map s.t. $\mathrm{Ker}\, s_*$ & $\mathrm{Coker}\, s_*$ are $\mathcal{I}$-torsion. Form the cofibering

$$U \xrightarrow{i} V \longrightarrow W;$$

we see that $E_*(W)$ is $\mathcal{I}$-torsion. By the second stage of the proof, $[W, Y]_* = 0$. Applying $[-, Y]_*$ to the cofibering we get

$$[U, Y]_* \xleftarrow[\cong]{s^*} [V, Y]_*,$$

which shows Y is $\mathcal{S}$-local.

This proves Th. 1.

Proof of Th. 1. Let E_*, I be as assumed & constructed* that $\{F_\alpha\}$ be as in Coroll. 7, is ~~as in Coroll.~~ 8, ~~let T be as constructed from the Coroll. 7~~ and let T be defined from the $\{F_\alpha\}$ as above.

By Prop. 4 we have a cofibering

$$X \longrightarrow Y \longrightarrow Z,$$

plotted by the map $X \xrightarrow{\eta} Y$.

First, $Z \in \mathrm{Class}\{F_\alpha\}$ by Prop. 4, so $E_*(Z)$ is I-torsion by Coroll. 8. It follows that $\mathrm{Ker}\,\eta_*$ & $\mathrm{Coker}\,\eta_*$ are I-torsion, so $\eta \in S$.

Secondly, I claim that for any spectrum W s.t. $E_*(W)$ is I-torsion, we have $[W, Y]_* = 0$.

In fact W is T-colocal by Coroll. 8, & the maps $pt \to Y \to pt$

Corollary 7. Any spectrum X s.t. $E_*(X)$ is I-torsion is a direct limit of closed subspectra $X_\beta \subset X$ each equivalent to some F_α.

Proof. X is the direct limit of closed subspectra X_β with at most finitely many cells. By Lemma 6 these are (since $E_*(X_\beta)$ is I-torsion) are colocal, so X is the direct limit of them.

Let τ be defined in terms of the set $\{F_\alpha\}$ as in Coroll 3.

Corollary 8. With this set $\{F_\alpha\}$,

X is τ-colocal $\Leftrightarrow$ $X \in$ class $\{F_\alpha\}$

$\Leftrightarrow$ $E_*(X)$ is I-torsion

Proof. Third $\Leftrightarrow$ by Coroll 5.

Second $\Rightarrow$. The property "$E_*(X)$ is I-torsion" is true for each F_α and is preserved under each of the operations (i) (ii) (iii) (iv) (v) used in constructing Class $\{F_\alpha\}$.

Second $\Leftarrow$. Immediate from Coroll. 7.

and in fact this condition already holds in some finite subcomplex L, $K \subset L \subset X$.

By throwing in at most σ such finite subcomplexes L we can construct

$$A_n \subset A_{n+1} \subset X$$

so that A_{n+1} has $\leq \sigma$ stable cells and for each pair (x, i) the condition $i^m x = 0$ holds in A_{n+1}. This completes the induction. Take $B = \bigcup_n A_n$.

We are now ready to choose $\{F_\alpha\}$. Let $\{F_\alpha\}$ provide at least one representative for each homotopy-equivalence class of spectra F with $\leq \sigma$ stable cells and $E_*(F)$ I-torsion. Clearly we can do this with a set; you can place a bound on its cardinality if you like.

Now let's backtrack and suppose given E_*, I. Let σ be an infinite cardinal s.t. $|\pi_*(E)| \le \sigma$, $|I| \le \sigma$.

The whole idea is to avoid set-theoretic difficulties by using σ to put an a priori bound on the length of our induction and the number of cells we have to consider.
It is easy to show that for any finite spectrum K one has $|E_*(K)| \le \sigma$.
The same conclusion holds if K has $\le \sigma$ cells: for each element $x \in E_*(K)$ must come from some finite subspectrum $K_\alpha \subset K$; there are at most σ of those, and each yields at most σ elements.

<u>Lemma 6</u>. Let X be any spectrum such that $E_*(X)$ is I-torsion. Then any closed subspectrum $A \subset X$ with $\le \sigma$ stable cells is contained in a closed subspectrum $B \subset X$ with $\le \sigma$ stable cells and $E_*(B)$ I-torsion.

<u>Proof</u> We make a construction by induction. Suppose A_n constructed with $\le \sigma$ stable cells, starting from $A_0 = A$. Then there are $\le \sigma$ pairs ($x \in E_*(A_n)$, $i \in I$) and each x comes from a finite subspectrum $K \subset A_n$. Since $E_*(X)$ is I-torsion, there exists m s.t. $i^m x = 0$ in $E_*(X)$

$\Rightarrow$

At this point I will avoid to give any categorical formulation of which [illegible]. I will just harvest such mathematics.

Corollary 5. X is $\perp T$-local $\Leftrightarrow$ $X \in \mathrm{Class}\{f_\alpha\}$.

Proof. $\Leftarrow$ by Coroll. 3. So suppose X is $\perp T$-colocal & let the fibring $X \to Y \to Z$ in Prop. 4 [illegible] $Z \in \mathrm{Class}\{f_\alpha\}$, so Z is $\perp T$-colocal by Coroll. 2. X and Z are $\perp T$-colocal, so Y is $\perp T$-colocal by Lemma 2(iii). $[f_\alpha, Y]_* = 0 \ \forall \alpha$, i.e. the maps

$pt \longrightarrow Y \longrightarrow pt$ are in $\perp T$.

This says $[Y, pt]_* \longrightarrow [Y, Y]_* \longrightarrow [Y, pt]_*$ are iso, so $[Y, Y]_* = 0$, whence $Y \simeq pt$. Thence the map $Z \to \Sigma X$ in the cofibring is an equivalence. But $Z \in \mathrm{Class}\{f_\alpha\}$, so $X \in \mathrm{Class}\{f_\alpha\}$ by using again (i), (ii).

~~Now we will go on and suppose given a homology theory E_* [illegible]. It is best to think of a theory like [illegible] in which the homotopy groups $\pi_*(E)$ are not bounded below; for a connective theory like MU the theory does apply but becomes [illegible] trivial.~~

Then easily we have a cofibration

$$\bigvee_\beta S^{n_\beta} F^i_{\alpha\beta} \longrightarrow Y_i/X \longrightarrow Y_{i+1}/X,$$

so $Y_{i+1}/X \in \mathrm{Class}\{F_\alpha\}$;

we just used the special F_α and more instances of (i) (ii) (iv) (iii).

For a limit ordinal, say j, suppose that for $i < j$ Y_i has been constructed and $Y_i/X \in \mathrm{Class}\{F_\alpha\}$.

We just take $Y_j = \bigcup_{i<j} Y_i$.

Then $Y_j/X = \bigcup_{i<j} Y_i/X$,

and it lies in $\mathrm{Class}\{F_\alpha\}$ by one more instance of (v).

This constructs $Y = Y_\Omega = \bigcup_{i<\Omega} Y_i$.

Consider any map

$$f : S^n F_\alpha \longrightarrow Y.$$

$S^n F_\alpha$ has $\leq \sigma$ stable cells, and each corresponds with some Y_i; therefore f maps to some j of cardinality $\leq \sigma$. Then f becomes null homotopic in Y_{j+1}. This proves $[F_\alpha, Y]_* = 0$, which proves Prop. 4.

The ordinals of cardinal $\leq \sigma$ form an initial segment of the ordinals, and our construction will be by transfinite induction up to, i.e. running to the first ordinal Ω of cardinal $> \sigma$. I don't like that much, even when σ is $\aleph_0$, but if one fears the set-theoretic difficulty then you have to adopt a set-theoretic solution. The whole point is that our induction stops before some cardinal known a priori, say 2^σ.

We take $Y_0 = X$ and construct Y_i by transfinite induction for ordinals $i < \Omega$.

For a successor ordinal, say $i+1$, we suppose Y_i is constructed and Y_i/X lies in $\mathrm{Class}\{F_\alpha\}$. The homotopy classes of maps

$$S^n F_\alpha \longrightarrow Y_i$$

form a set; for each we take a representative map

$$S^{n_\beta} F'_{\alpha\beta} \xrightarrow{q_\beta} Y_i$$

where $F'_{\alpha\beta}$ is a cofinal subspectrum in F_α. We form the cofibering

$$\bigvee_\beta S^{n_\beta} F'_{\alpha\beta} \xrightarrow{S q_\beta} Y_i \longrightarrow Y_{i+1}.$$

[illegible]

Actually this is true with "if" [illegible] but we can't prove it. [illegible]

This is one reason that the class of $\mathcal{T}$-colocal spectra is usable, although of course it does contain some very large spectra.

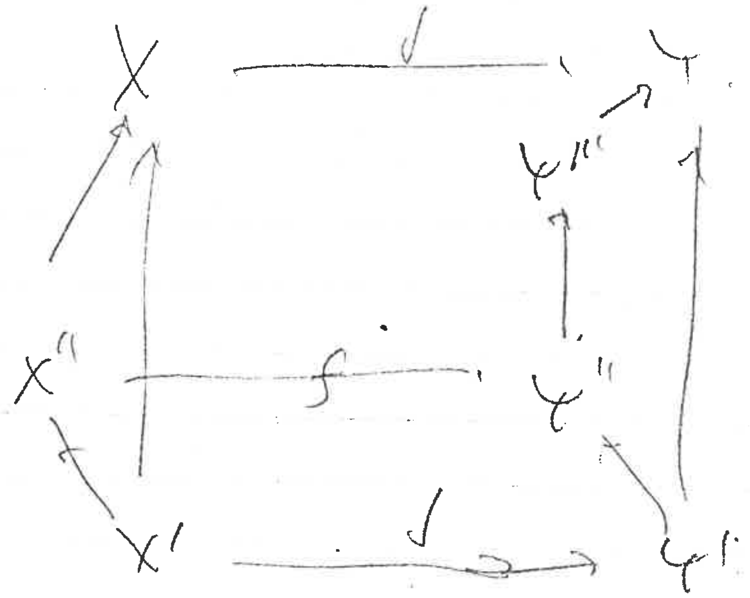
Proof of Corollary 3. We need [illegible] each F_α is $\mathcal{T}$-colocal (which is trivial from the definition) and Lemma 2 does the job.

Proposition 4. For any X there is a cofibering $X \longrightarrow Y \longrightarrow Z$ such that $[F_\alpha, Y]_* = 0$ $\forall \alpha$ and $Z \in \mathrm{Cl}_\Sigma\{F_\alpha\}$.

Proof. $\{F_\alpha\}$ is a definite set of definite spectra, so there is a cardinal σ s.t. each spectrum F_α has $\leq \sigma$ stable cells. In the application σ ~~is the cardinal~~ $\cancel{|\pi_*(E)|}$, which is usually $\aleph_0$. [illegible]

[illegible] σ [illegible] in [illegible]

[illegible]



Then the map g of $Y' \cup_f CX'$ factors through $Y''' \cup_f CX''$, which lies in $\mathcal{H}\mathrm{Class}\{T_\alpha\}$.

(v) Suppose $L = \varinjlim L_\alpha$ and each L_α lies in $\mathrm{Class}\{T_\alpha\}$ and has the property. Any map from a finite spectrum K to L factors through some L_α, so it factors through $Y \in \mathcal{H}\mathrm{Class}\{T_\alpha\}$.

Now consider a finite spectrum K in $\mathrm{Class}\{T_\alpha\}$. By (i) above the identity map $K \xrightarrow{1} K$ factors through some $Y \in \mathcal{H}\mathrm{Class}\{T_\alpha\}$, i.e. K is a summand of some $Y \in \mathcal{H}\mathrm{Class}\{T_\alpha\}$ & K is ... by (i)(iii)(iv)(v)

It factors through $Y' \cup_f CX'$,
where X' is a finite subspectrum
of X and Y' is a finite subspectrum
of Y, so it is enough to consider
$Y' \cup_f CX'$. By hypothesis
$X' \longrightarrow X$ factors through $X'' \in \mathcal{H}(\mathcal{S}(\alpha))$, and Klass $\mathcal{E}(\alpha)$
is closed under cofibrations between finite spectra.
So we can assume $X' \subset X''$
and the diagram

$$\begin{tikzcd}
 & X \arrow[r, "f"] & Y \\
X'' \arrow[ur] & & \\
 & X' \arrow[ul] \arrow[uu] \arrow[r, "f"] & Y' \arrow[uu]
\end{tikzcd}$$

is strictly commutative.

Consider the image of $X'' \cup_{X', f} Y'$ in Y:
this is contained in a finite subspectrum
$Y'' \subset Y$, and by hypothesis the
map $Y'' \hookrightarrow Y$ factors through $Y''' \in \mathcal{H}(\ldots)$

Proof Let $Klass\{f_\alpha\}$ be the class generated by the $\{f_\alpha\}$ by equivalence between finite spectra, suspension and cofibres. I claim that any map from a finite spectrum K to $X \in Class\{f_\alpha\}$ can be factored through $Y \in Klass\{f_\alpha\}$.
The proof is that this is true for the $\{f_\alpha\}$ (clearly) and preserved under operations (i) (ii) (iii) (v)!

(i) Suppose $C' \simeq C$ and C has the prop. Then any map $K \to C'$ factors through C & $\therefore$ through $Y \in Klass$.

(ii) Suppose C has the prop. & an $K \xrightarrow{f} S^n C$. Then $S^{-n} K \xrightarrow{f} C$ factors through $S^{-n} K \to Y \to C$, so f factors through $K \to S^n Y \to S^n C$, & $S^n Y \in Klass$ by defn.

(iii). Suppose given a cofibre sequence

$X \xrightarrow{g} Y \longrightarrow Y \cup_g CX \longrightarrow SX \cdots$

$Y \cup_g CX = Z$ lies in $Class\{f_\alpha\}$ &

Suppose X & Y have the prop., & we take any map of a finite spectrum K to Z.

Omit

(vii). Suppose $X \vee Y \in$ Class $\{T_\alpha\}$. Consider the sequence

$$X\vee Y \xrightarrow{1\vee 0} X\vee Y \xrightarrow{1\vee 0} X\vee Y \to \cdots$$

Then $X \simeq \underrightarrow{\mathrm{Holim}}\,(X\vee Y \xrightarrow{1\vee 0} \cdots)$

$\in$ Class $\{T_\alpha\}$ by (v) (i).

Remark. Suppose the spectra $\{T_\alpha\}$ are all finite. ~~[illegible]~~

~~of contractible [illegible]~~

~~in finite [illegible]. Let Class $\{T_\alpha\}$~~

~~[illegible] Class $\{T_\alpha\}$ [illegible]~~

~~are (H) finite [illegible] that of contractible [illegible],~~

~~at the case may be. Then~~

~~Class $\{T_\alpha\}$ can be [illegible] Class $\{T_\alpha\}$~~

~~by operations~~

Let Fin Class $\{T_\alpha\}$ consist of the finite spectra in Class $\{T_\alpha\}$. Then Fin Class $\{T_\alpha\}$ can be generated from the $\{T_\alpha\}$ by operation (i) restricted to finite spectra, (ii) (iii), (vii).

~~Proof We will [illegible] if the T_α~~

~~are all finite, the [illegible] preserve~~

~~[illegible]~~

~~(iii) and [illegible] $\{S^n T_\alpha\}$ [illegible]~~

~~do (iii) if [illegible] compositions may [illegible]~~

~~of these. Then you do (v) and pass~~

Omit

Remark. On the other hand, if you want to use more operations, then $\mathrm{Class}\{F_\alpha\}$ is also closed under the following operations:

(xvi) If $X \in \mathrm{Class}\{F_\alpha\}$ & Y arb., then $X \wedge Y \in \mathrm{Class}\{F_\alpha\}$. (ideal)

(xvii) If $X \vee Y \in \mathrm{Class}\{F_\alpha\}$, then $X \in \mathrm{Class}\{F_\alpha\}$ (closed under retracts)

Proof (xvi) If Y is finite we can proceed by induction on the number of cells in Y. Suppose Y is constructed by a cofibering

$$S^n \longrightarrow Y' \longrightarrow Y \longrightarrow S^{n+1}$$

where Y' has less cells than Y; then we get a cofiber

$$X \wedge S^n \Longrightarrow X \wedge Y' \longrightarrow X \wedge Y \longrightarrow X \wedge S^{n+1};$$

If $X \in \mathrm{Class}\{F_\alpha\}$ then $X \wedge S^n \in \mathrm{Class}\{F_\alpha\}$ by (ii) & $X \wedge Y' \in \mathrm{Class}\{F_\alpha\}$ by ind. hyp. so $X \wedge Y \in \mathrm{Class}\{F_\alpha\}$ by (iii).

For the general case we can write $Y = \varinjlim_\alpha Y_\alpha$ where Y_α runs over the finite subspectra of Y. Then

$$X \wedge Y = \varinjlim_\alpha X \wedge Y_\alpha,$$

which lies in $\mathrm{Class}\{F_\alpha\}$ by (v).

Remark. If you want to describe the same class by using less operations it is sufficient to use the operations (i)(ii)(iii)(iv). To see this, we have to reduce operation (v) to the others.

Consider first the case of a limit indexed over the integers $\{1, 2, 3 \dots 9\}$. Working stably we can add & subtract maps, & so replace the telescope T by the following construction:

$$\bigvee_{1}^{\infty} L_n \xrightarrow{\;g\;} \bigvee_{1}^{\infty} L_n \longrightarrow T$$

where the nth component of g is $i_n - 1_n$.

So this uses (iv) twice, (iii) once & (i) once.

~~Now in the construction of a [illegible]~~
~~the p~~

Now we look at the construction of a [illegible] limit, as in the proof of [illegible].

$$T^{a-1} \longrightarrow T^{a} \longrightarrow \bigvee_{i_0 < i_1 < \dots < i_a} S^{a} L_{i_0}$$

so we can construct T^a from the L_i by induction over a using (ii) (iv) (iii). & Now we can construct $T = \varinjlim_a T_a$ by the first case of limits.

~~Corollary 3. If $X \in$ Cla. St.~~

~~then X [illegible] T [illegible]~~

$$
\begin{tikzcd}
0 \arrow[r] & \varprojlim\limits_a{}^1 [T^a, X]_* \arrow[r] \arrow[d, "\cong", "t_*"'] & {[T, X]_*} \arrow[r] \arrow[d, "t_*"] & \varprojlim\limits_a{}^0 [T^a, X]_* \arrow[r] \arrow[d, "\cong", "t_*"'] & 0 \\
0 \arrow[r] & \varprojlim\limits_a{}^1 [T^a, Y]_* \arrow[r] & {[T, Y]_*} \arrow[r] & \varprojlim\limits_a{}^0 [T^a, Y]_* \arrow[r] & 0.
\end{tikzcd}
$$

The outside vertical arrows are iso, so by the 5-lemma, the middle one is iso. This holds for all such T, so T is E-colocal. This proves Lemma 2.

All right, now we'll backtrack and suppose given a set of spectra $\{F_\alpha\}$. This is a definite set and under our control. ~~in particular, there is a cardinal κ such that all the spectra F_α have at most κ stable cells.~~ Let $f : X \to Y$ be a map; we write "$f \in -T$" to mean that

$$[F_\alpha, X]_* \xrightarrow{f_*} [F_\alpha, Y]_*$$

is iso for each F_α, or [illegible].

~~... meaningful statement if the ...~~
~~F_α ... considered ...~~ We introduce Class $\{F_\alpha\}$, the class of spectra obtained from the class $\{F_\alpha\}$ by iterating the constructions (i) (ii) (iii) (iv) (v) of Lemma 2.

~~Corollary 2. If $X \in$ Class $\{F_\alpha\}$, then X is E-colocal~~

$L_{i_0} \times \sigma(i_0, i_1, \dots, i_a)$

The rest of the construction carries over. We still have

$$\pi_*(T) = \varinjlim_i \pi_*(L_i)$$

so if we are in the case of inclusions, the map $T \to L$ is an equivalence.

Now we filter T by taking T^a to be the part [illegible] each simplex of dim $\leq a$. Then we get a cofibre [illegible]

$$T^{a-1} \to T^a \to \bigvee_{i_0 < i_1 < \dots < i_a} [illegible]$$

We [illegible] assume each L_i is T-[illegible]

so $\bigvee_{i_0 < i_1 < \dots < i_a} S^a L_{i_0}$ is T-colocal [illegible]

by (ii) & (iv). By induction on a, starting with $T^0 = pt$ & using (iii), we see that T^a is T-colocal.

Now take $X \xrightarrow{f} Y$ in T.

We use Milnor's $\lim^0$ - $\lim^1$ [illegible] on both sides are $\varinjlim_n \pi_*(L_n)$. Thus $T \to L$ is an equivalence. If you aren't in the case of inclusions, then T is the best choice you've got to take limit anyway.

All this goes over to CW spectra: you do the construction above on each term of the spectrum.

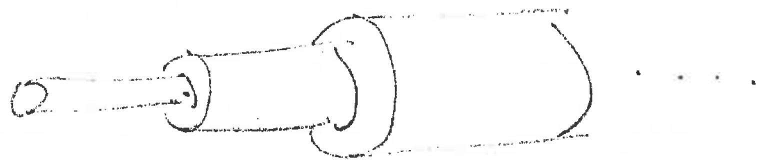
Now we want to look at the case of a more general set of indices I. In this case we modify the telescope construction, and replace $[1, \infty)$ by the following infinite simplicial complex K. It has vertices corresponding to the indices i, and it has a q-simplex corr. to each totally-ordered subset

$$i_0 < i_1 < i_2 < \dots < i_q .$$

We replace $L_n \times [n, n+1]$ by

inside $L \times [1,\infty)$ you have

$$T = \bigcup_n L_n \times [n, n+1)$$



You can also construct it by taking

$$\bigsqcup_n L_n \times [n, n+1),$$

and identifying $(x_n, n+1) \in L_n \times [n, n+1)$

with $(i_n x_n, n+1) \in L_{n+1} \times [n+1, n+2)$,

where i_n is the inclusion map. If a directed system has just one set of maps $f_n : L_n \longrightarrow L_{n+1}$ instead of inclusions, you just put $f_n x_n$ instead of $i_n x_n$.

For CW-complexes with basepoint you replace $L_n \times [n, n+1)$ by $\dfrac{L_n \times [n, n+1)}{pt \times [n, n+1)}$.

If you are in the case of inclusions

... induces an isomorphism of homotopy groups, because the homotopy groups ...

Using the second sequence of the cofibrations we get
$$[C, F]_* = 0.$$
Using the second sequence of the fibring sequence, we get
$$[C, X)_* \xrightarrow{t_*} [C, Y) \quad \text{iso}.$$
This holds for all $t \in \mathcal{T}$, so C is $\mathcal{T}$-colocal. Similarly if condition 2 are the objects claim to be $\mathcal{T}$-colocal — [illegible] from (v). This is the converse.

You can go a long way around, Bousfield [illegible] avoiding the [illegible] it, and he got it right when he first wrote it out, but he got it wrong when he wrote it out for publication. So he had to write a correction. With the current technique we can march straight ahead.

But first we must recall about limits. Usually we begin with the case in which the set of indices is $\{1, 2, 3, \ldots\}$, and the directed system is an increasing sequence
$$L_1 \subset L_2 \subset L_3 \subset \cdots \subset L = \bigcup_n L_n.$$
In the category of CW-complexes, the usual construction of the homotopy direct limit is the telescope;

are T-colocal by [illegible]

using (iii); $\bigvee_\alpha C_\alpha$ is the direct limit

of the finite wedges, so it is T-colocal

by (v). However, it is stacked [illegible]

Remark [illegible]

Proof. (i), (ii) & (iv) are trivial [illegible] I

leave them to you.

(iii). Let $X \xrightarrow{f} Y$ be any [illegible] T

and use f to build a fibre sequence

$$F \longrightarrow X \xrightarrow{f} Y.$$

(Fibre sequences are actually [illegible] sequences in the category of spectra, but [illegible] may be clearer [illegible] Hill it's relevant). If A is T-colocal

then $[A, X]_* \xrightarrow{f_*} [A, Y]_*$ is iso.

By the long exact sequence of the fibring, we have

$$[A, F]_* = 0.$$

Similarly, if B is T-colocal [illegible]

$$[B, F]_* = 0.$$

[illegible]

a [illegible]

[illegible]

All these are provable on the basis that they are statements $P(t) \Rightarrow Q(t)$ and it doesn't matter what t is.

<u>Lemma 2</u>. (i) If C is T-colocal and $C' \simeq C$ then C' is T-colocal.

(ii) If C is T-colocal then $S^n C$ is T-colocal for $n \in \mathbb{Z}$.

(iii) If $A \longrightarrow B \longrightarrow C$ is a cofibring and two of these are T-colocal then so is the third.

(iv) If each C_α is T-colocal then $\bigvee_\alpha C_\alpha$ is T-colocal.

(v) Suppose L is the direct limit of some direct system of spectra L_α indexed over a directed set. If ~~the~~ each L_α is T-colocal, then so is L.

<u>Remark</u>. Clearly (iv) follows from the others: If each C_α is T-colocal then finite wedges $C_{\alpha_1} \vee C_{\alpha_2} \vee \cdots \vee C_{\alpha_n}$

This construction shows that Z is well constructed.
This difficulty becomes more obvious when the author goes on to assume that the map $X \longrightarrow Z$ is in $\mathcal{S}$. If so, this argument is circular; what he is purporting to construct was already among the things he assumed known for the purpose of constructing it. This will not do.
However, the use of the notion of T [illegible] of [illegible] seek [illegible]

Let C be a spectrum. We will say "C is T-colocal"

if $\forall (t : X \longrightarrow Y) \in T$

$$\Rightarrow [C, X]_* \xrightarrow{t_*} [C, Y]_* \text{ iso}$$

Example. $C = pt$ is T-colocal for all T.

Otherwise this may seem not to be an effective definition, because: C, there's no easy way to check through all the t in T; but that won't stop us...

that's a meaningful statement.

(ii) Nevertheless, we cannot place any upper bound on the cardinal number of maps t which may satisfy $t \in T$. That depends on the size of the spectra X and Y you are willing to consider, just as for $\mathcal{S}$.

This has the following implications:

(iii) There is no objection to using the class T in definitions. A definition is just a way of writing a long, complicated statement in a shorter and more comprehensible way; if the long statement is meaningful and provable, so is the short one.

(iv) Some uses of T which may appear to be contradictions are not in fact contradictions. Suppose for example that some author takes X and forms

$$Z = \operatorname{Holim} Y \mid X \xrightarrow{t} Y \text{ in } \mathcal{S}.$$

We may assume the author does the usual construction of Holim, for he sees nothing to be careful about. Then the cardinal number of (stable) cells in Z will be at least equal to that in any Y: $|Z| \geq |Y|$. Certainly, if Z is well constructed, the number of its cells must be some cardinal, however large it is. But if you show me that cardinal, I can at once construct $X \xrightarrow{t} Y$ in $\mathcal{S}$ so that $|Y|$ is larger, which is a contradiction.

Theorem (after Bousfield)   Every spectrum

has an S-localization.

Now we know that S-localizations exist, we can concentrate on their properties. Since L is S-local, η is terminal among maps in S with source Y. Since $\eta \in S$, η is initial among maps from X to S-local spectra. Either property characterizes η up to a canonical isomorphism of its target L.

These properties are formally exactly the same as the formal properties of localization in the sense of Sullivan.

In principle, the pattern of our construction is as follows. We assume given a definite E_* and I as above. In terms of these we define the class S. We also define a set of spectra $\{F_\alpha\}$. Then what we define another class T of maps of spectra.

When I say "class", I mean that it shares the following properties with S

(i) The ... $t \in T$ is ...

In fact, it is defined ... that

$$[F_\alpha, X]_* \xrightarrow{t_*} [F_\alpha, Y]_*$$ is iso

for each F_α. If the F_α are ...

~~see what I'm doing is I want to make sure that my definitions and constructions are written wholly in terms of BP_*, but nevertheless have the same effect as Bousfield would get by putting $E_* = K(n)_*$.~~

~~Of course, the theory I give will include Bousfield's, because I can always take any E_* and put $R = ?$, $I = \{1\}$. The benefit I get is a bit of extra flexibility in the ~~applications~~.~~

I don't think I shall need it in what follows, but one can check that the set $\mathcal{S}$ is closed under composition and contains identity maps. So we think of it as the analogue of a multiplicative set in algebra; this is the collection of things you propose to use as denominators when you form fractions.

Now I can give some definitions. We say that a spectrum L is $\mathcal{S}$-local if

$$[X, L]_* \xrightarrow{s^*} [Y, L]_*$$

is iso for each $s \in \mathcal{S}$. An <u>$\mathcal{S}$-localisation</u> of X is a map $X \xrightarrow{\eta} L$ such that

(a) $\eta \in \mathcal{S}$

(b) L is $\mathcal{S}$-local.

Therefore, the theory I am going to set up will include this. However, I gain a bit of flexibility.

Example (ii) $E_* = BP_*$.

For each r in the coefficient ring $\pi_*(BP) = Z_{(p)}[v_1, v_2, \ldots]$

I get an operation $x \mapsto rx$ on E_*;

I write r for this operation too. I take

$I = \{p, v_1, v_2, \ldots, v_n\}$.

Of course, if a BP-module M over $\pi_*(BP)$ is I-torsion in the sense above, then the same [illegible] to be [illegible] ideal $(p, v_1, v_2, \ldots, v_n)$ [illegible] here [illegible], for any $r \in (p, v_1, v_2, \ldots, v_n)$ and $x \in M$ $\exists$ e s.t. $r^e x = 0$. Thus I is suggested by the usual notion of I-torsion for ideal I.

Now you see what I'm doing; I want to make sure that my definitions and constructions can be written wholly in terms of BP_* but nevertheless they have the same effect as Bousfield would get by putting $E_* = E(n)_*$.

Following Bousfield, I suppose given a generalised homology theory E_* satisfying the usual axioms. ~~What I add~~ In addition, I suppose that I am given some set I of stable operations $i : E_* \to E_*$; they may be of any degrees they like. I say that $E_*(Y)$ is I-torsion if

$$\forall\, i \in I \ \&\ \forall\, x \in E_*(Y)\ \exists\, m$$

s.t. $i^m x = 0$. I now define a class S of maps of spectra: I put a map $s : X \to Y$ in the class S if the kernel & cokernel of

$$E_*(X) \xrightarrow{s_*} E_*(Y)$$

are I-torsion.

Example (i). For any theory E_* I can take $I = \{1\}$. Then "I-torsion" means "zero", so S is the class of maps s s.t.

$$E_*(X) \xrightarrow{s_*} E_*(Y)$$

is iso. These maps are usually called E_*-equivalences. This is the context which Bousfield originally considered.

difficulties, which I shall explain [illegible] in setting up the proposed category [illegible] bordisms. Hence Bousfield [illegible] [illegible] didn't. I [illegible] [illegible] to [illegible] [illegible] I have been persuaded anyway by aesthetic considerations rather than proof, but there we are.

I shall now [illegible] on [illegible] of Bousfield's work. The two references

A.K. Bousfield, "The localization of spectra with respect to homology", Topology 18 (1979) 257-281.

A.K. Bousfield, "The Boolean algebra of spectra", Comm. Math. Helv. 54 (1979) 368-377.

~~Bousfield supposes [illegible] a [illegible] homology theory E_* satisfying the usual axioms. He defines $\mathcal{S}$ to be the class of maps $s : X \to Y$ between spectra s.t.~~

~~$$E_*(X) \xrightarrow{s_*} E_*(Y)$$~~

~~is iso. We wish to think of $\mathcal{S}$ as being like a multiplicative set in algebra, in particular closed under composition. [illegible] [illegible] to [illegible] [illegible] [illegible] [illegible] of $\mathcal{S}$.~~

6.4. Bousfield localization

We have seen that the homotopy category ... In some lectures in Chicago in 1973 I ... the following question: can one make any sense out of the version of homotopy theory which the E-theory can see? Let me give details. Suppose given a spectrum E and the resulting homology ~~cohomology~~ theories E_*. And suppose that we wish to construct the Adams spectral sequence using E-homology

$$\mathrm{Ext}^{**}_{?}(E_*(X), E_*(Y)) \Rightarrow ?\,.$$

~~Then~~ suppose we have the maps

$X \xrightarrow{f} X'$ and $Y \xrightarrow{g} Y'$ which induce isomorphisms of E-homology. Then f and g will certainly commute with all the structure you can possibly use in defining $\mathrm{Ext}^{**}_{?}$, so such maps f and g ought to induce an isomorphism of the whole SS. Therefore it ought to induce an isomorphism of the RHS. Therefore the best thing that the SS could possibly converge to would be ~~category of the~~ not $[X, Y]$, but maps in some category of fractions which you obtain by inverting those maps f and g which induce an isomorphism of E-homol.

Unfortunately there were some set-theoretic

§4. Bousfield localisation.

Let me recap. K-theory is good because some problems look easy to the eyes of K-theory although they look hard to ordinary cohomology. As Bott said, the Steenrod algebra doesn't know that the problem of Hopf invariant one has been solved; nobody told it. K-theory does know; the map α_i is of filtration i to the eyes of ord. cohomology and of filtration 1 to K-theory; i.e. it looks subtle to ordinary cohomology theory and crudely obvious to K-theory. But here we see things to which K-theory is more blind; for example, it sees $V(1)$ as a point.

but in general I don't assume any such thing.

Let X be a finite p-torsion complex, localised at $\mathcal{E}(\ )$. Then with rather less effort than at the end of $\S 1$, we have the following dichotomy.

Either (i) X is stably contractible (in which case nothing can be done)

or (ii) There is an n s.t. $\widehat{BP}_*(X)$ is $\{p, v_1, v_2 \dots v_{n-1}\}$-torsion but is not v_n-torsion.

In that case we can make Hopkins' theorem say that there is a map

$$S^{m2(p^n-1)}X \xrightarrow{f} X$$

such that

$$\begin{array}{c} f_*: \widehat{BP}_*(S^{m2(p^n-1)}X) \longrightarrow \widehat{BP}_*(X) \\ \| \\ \widehat{BP}_*(X) \end{array}$$

is multiplication by v_n^m mod $(p, v_1, \dots, v_{n-1})\widehat{BP}_*(X)$. Then we shall have no trouble proving that arbitrary composites of suspensions of f are essential, or drawing any other conclusions we want to draw.

Why should we degrade our information by this process of killing quotients?

Field coefficients are absolutely splendid for doing calculations. However, we don't want to do calculations, we want to give qualitative theorems about what sort of thing goes on. For this purpose, there is no loss, and there may be some gain, in taking qualitative statements about how $K(n)_*(X)$ behaves, and replacing them by the equivalent qualitative statements about how $BP_*(X)$ behaves. Let me illustrate how this applies to the end of §1.

Let R be a ring; I have in mind $R = \pi_*(BP) = Z_{(p)}[v_1, v_2, v_3, \ldots]$.

Let M be a module over R; I have in mind $M = BP_*(X)$. Let I be a subset of R; I have in mind $I = \{p, v_1, v_2, \ldots, v_{n-1}\}$. We will say that M is an I-torsion-module, or I acts nilpotently on M, if $\forall i \in I$ & $\forall m \in M$ $\exists e = e(i,m)$ s.t. $i^e m = 0$. Of course, the same conclusion $j^e m = 0$ follows for every element j of the ideal generated by the elements $i \in I$. If this ideal is finitely-generated then e can be chosen to work for all j; if M is finitely generated then e can be chosen to work for all m

Here are homomorphisms

$$\Theta : \pi_*(B(n)) = \frac{\pi_*(BP)}{(p, v_1, \ldots, v_{n-1})} \longrightarrow \mathbb{F}_p[v_n, v_n^{-1}]$$

which send v_n to v_n. The obvious one sends v_i to zero for $i > n$. However, you don't have to do that; you could send

$$v_{2n} \longmapsto v_n^{p^n+1}.$$

For each such homomorphism you have a homology theory

$$K(n)_*(X) = \mathbb{F}_p[v_n, v_n^{-1}] \otimes_{\pi_*(B(n))} B(n)_*(X)$$

on finite complexes, and you can obtain one on infinite complexes by passing to direct limits.

Any of these theories has coefficients $\mathbb{F}_p[v_n, v_n^{-1}]$, and this is a graded field; any graded module over it is free. This fact helps to give these theories very favourable properties.

Any such theory deserves to be called a Morava K-theory. In practice people always speak of "Morava K-theory" as if there was only one. Probably there ought to be a uniqueness theorem for Morava K-theory, but at the moment I don't know what it ought to say.

For present purposes, however, it seems to me that we don't have to bother with Morava K-theory. Given that BP_* contains all the information we want, you allow, and then doing hackery with them, you can build up representable homology theories whose coefficient groups are certain quotients of $\pi_*(MU)$.

<u>Prop</u> (Johnson-Wilson §2) There is a homology theory $P(n)_*$, represented by a spectrum $P(n)$ which is a module-spectrum over the ring-spectrum BP, and so that

$$\pi_*(P(n)) \cong \frac{\pi_*(BP)}{(p, v_1, \ldots, v_{n-1})}$$

as a module over $\pi_*(BP)$. Moreover $P(n)_*(X)$, as a module over $\pi_*(BP)$, is a module over $\pi_*(BP)/(p, v_1, \ldots, v_{n-1})$.

Since localisation preserves exactness, we get another homology theory by

$$B(n)_*(X) = S^{-1} \cdot P(n)_*(X)$$

where S is the mult. set $\{1, v_n, v_n^2, v_n^3, \ldots\}$

<u>Prop</u> (Johnson-Wilson 3.1) If X is a finite complex then $B(n)_*(X)$ is free over $\pi_*(B(n))$.

It follows that we can get another homology theory by tensoring with any module we like over $\pi_*(B(n))$. In particular, ~~you can choose any homo you like~~

$\phi : \pi_*(BP)$

For that I need a diagram like this.

$$\begin{array}{ccc} S^8 \wedge \Pi U & \xrightarrow{f_1} & S^6 \wedge \Pi U \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_3} \\ S^2 \wedge \Pi U & \xrightarrow{f_1} & \Pi U \end{array}$$

I can easily make such a diagram commutative up to homotopy, and I need to choose a homotopy in order to make a specific construction of C_{f_1, f_3} without homotopy is added

$$\frac{Z[x_1, x_2 x_3 \ldots Z^7[\Sigma \ldots, Z[x_2 \; x_4 \; x_5 \ldots]}{(x_1, x_3)}$$

If I want to kill 3 generators, I shall need a cube of maps f_1, f_3, f_5 and a homotopy between homotopies; to more generators I need higher homotopies.

In general the way to keep higher homotopies in order is to do direct geometrical construction rather than calculate with algebraic invariants. In this case the appropriate geometric tool is the Baas–Sullivan theory of "manifolds with singularities". By describing very explicitly the sort of manifolds with singularity

§3 Morava's Extraordinary K-theories.

The standard source is D. C. Johnson & W. Stephen Wilson, BP operations and Morava's Extraordinary K-theories, Math. Zeit. 144 (1975) 55–75. These authors cite further earlier unpublished work of Morava.

Let me begin with MU. We have said that

$$\pi_*(MU) = \mathbb{Z}[x_1, x_2, x_3, x_4 \ldots].$$

Suppose I don't like the generator x_1; maybe I am interested in the prime p and I really want to keep x_2 of dim $2(3-1)$, x_8 of dim $2(9-1)$ etc. Well, I form

$$S^2 \wedge MU \xrightarrow{x_1 \wedge 1} MU \wedge MU \longrightarrow MU;$$

and form the cofibre

$$S^2 \wedge MU \xrightarrow{x_1} MU \longrightarrow C_{x_1}.$$

In homotopy I get

$$Z[x_1, x_2 \ldots] \xrightarrow{x_1} Z[x_1, x_2 \ldots] \rightarrow \frac{Z[x_1, x_2, x_3 \ldots]}{(x_1)}$$

(exactness)

Now, suppose I don't like x_1 or x_3. Then presumably I would like to form

$$C_{x_1} \xrightarrow{x_3} C_{x_1} \longrightarrow \ldots.$$

a classification scheme for such spaces as may exist, and start to fill it in. The second is to see what can be said in general terms about the relationship between the periodicity phenomena in the algebra, which are clearly very good, and those in the world of spaces and maps, which are presumably more complicated. A start on the second problem was made by

D.C. Ravenel, Localization with respect to certain periodic homology theories, American Jour. Math. 106 [illegible], 351–414.

Very roughly speaking, Ravenel did everything easy and ended by stating about ten significant problems which barred his progress; and Hopkins's achievement is that he has already solved more than half of them.

Our next step is to see what Ravenel meant by "certain periodic homology theories" and what he meant by "localization" with respect to them.

The answer to the first is that to recall the Morava K-theories.

More generally, if there is a space $V(n)$, regardless of the way it may be constructed, then

$$BP_*(V(n)) \approx \frac{\pi_*(BP)}{(p, v_1, v_2, \ldots, v_n)}.$$

Notice that this is indpt of the choice of generators we factor out — p and all the elements of in degrees > 0 & $< 2(p^{n+1}-1)$.

But whether $V(n)$ exists or not, $(p, v_1, v_2, \ldots, v_n)$ in $\pi_*(BP)$ is a sub comodule over $BP_*(BP)$; $\frac{\pi_*(BP)}{(p, v_1, \ldots, v_n)}$ is a valid comodule over $BP_*(BP)$ and

$$\frac{\pi_*(BP)}{(p, v_1, \ldots, v_n)} \xrightarrow{\text{mult. by } v_{n+1}} \frac{\pi_*(BP)}{(p, v_1, \ldots, v_n)}$$

is a valid map of comodules over $BP_*(BP)$ and these statements are valid even for $p=2$.

So the quickest summary of the situation is that all the algebra works as if the spaces $V(n)$ and the cofibre sequence

$$S^{p^n q} V(n) \longrightarrow V(n) \longrightarrow V(n+1)$$

exist and explain what is going on, although they probably do not exist ~~for~~ ~~in fact~~ and certainly do not (at ...) ?

This manifestly raises two problems. The first is to erect so that the cofibre C_0 is of type $V(1)$; by iterating C you can play just the same game with A and B. The spectra I have constructed are unique up to stable homotopy equivalence. Essentially nothing more is known about the existence or non-existence of spaces of type $V(n)$. However, the algebraic structures ready for them if they do exist.

More precisely, we have

$$BP_*(V(0)) = \pi_*(BP)/(p) \quad \text{[illegible]}$$

The map $A: S^q V(0) \longrightarrow S^q V(0)$ induces a map of modules over $\pi_*(BP)$; and it turns out to be multiplication by v_1 up to a scalar non-zero mod p. So

$$BP_*(V(1)) = \pi_*(BP)/(p, v_1) \quad p > 2.$$

It is convenient to interpret p as v_0.

torsion space ^(a spectrum) such that $H_*(X) \cong \Lambda[\ldots]$

then X is said to be of type $V(n)$.

[H. Toda, On spaces realizing certain

parts of the Steenrod algebra, Topology

10 (1971) 53–65]. For example:

the Moore spectrum $S^0 \cup_p e^1$ is of type

$V(0)$, for all p. ~~Then~~ The spectrum

$C_A = V(0) \cup_A C S^a V(0)$ is of type $V(1)$

for $p > 2$; for $p = 2$ our definition

is not appropriate, but there is no

spectrum $V(1)$ anyway. For $p > 3$

there is a map $B : S^{pa} V(1) \to V(1)$

so that the cofibre C_B is of type $V(2)$;

by iterating B you can play games just

like I did with A in the first lecture;

for $p = 3$ [illegible]

For $p > 5$ there is a map $C : S^{p^2 a} V(2) \to V(1)$

W.S. Wilson Brown-Peterson Homology: An introduction and sampler
American Math Soc. 1980 (Regional Conference Series in Mathematics no 48)

The purely algebraic investigation of the E_2-term

$$\mathrm{Ext}_{BP_*(BP)}(BP_*(S^0), BP_*(S^0))$$

are sufficient to bring one in touch with a maze of inter-related periodicity phenomena with different periods. I will try to give you a taste as quickly as I can.

Let H be the Eilenberg-Maclane spectrum ~~[illegible]~~ which corresponds to ordinary homol. & cohomol. with mod p coeffs. Then H_*H is the dual of the mod p Steenrod algebra ~~[illegible]~~ and Milnor told us its structure:

$$H_*H = \Lambda[\tau_0, \tau_1, \tau_2 \cdots] \otimes \mathbb{F}_p[\xi_1, \xi_2 \cdots]$$

$$|\tau_i| = 2p^i - 1, \quad |\xi_i| = 2(p^i - 1)$$

In particular, $\Lambda[\tau_0, \tau_1, \ldots, \tau_n]$ is a left comodule over $H_*(H)$.

If there is a finite p-primary

Applications to complex cobordism and Brown-Peterson cohomology, J. Pure Appl. Alg. 10 (1977) 1-18. Alternatively you use those given by Araki, Typical formal groups in complex cobordism and K-theory, Kinokuniya Book Store 1973, Warning: they are not the same.

If Y is p-local, the Novikov spectral sequence may be written

$$\mathrm{Ext}_{BP_*(BP)}(BP_*(X), BP_*(Y)) \Rightarrow \{X, Y\}_*$$

It is very natural to substitute $X = S^0$, $Y = S^0$ and try to prove results about the E_2-term. Strenuous efforts in this direction were made by a number of authors, who exploited the invariant [illegible] Jack Morava.

H. Miller, D.C. Ravenel & W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, Annals of Math. 106 (1977) 469-516.

D.C. Ravenel, A Novice's guide to the Adams-Novikov spectral sequence, in Lecture Notes in Math. Vol 658, Springer 1978, pp 404-475.

D.C. Ravenel, Complex Cobordism and its application to homotopy theory, Proc. Int. Congress Math. Helsinki [illegible]

491-496.

D C Ravenel, forthcoming book on Adams Novikov S.S.

forming Ext, $MU_*(X)$, $MU_*(Y)$ are regarded as comodules over the coalgebra $MU_*(MU)$. For the homological approach, see JFA, Lectures on Generalised Cohomology, Lecture Notes in Maths vol 99, Springer 1969, pp 1–138.

If we are interested in p-primary problems, then we do not have to use MU as it stands. We can localise MU at p, so as to get $MU_{(p)}$, and then we find that it splits as the wedge of a large number of copies of suspensions of a single spectrum, which is called BP because it had already been invented by Brown and Peterson:

$$MU_{(p)} \simeq \bigvee_{\alpha} S^{q(\alpha)} BP.$$

The splitting is due to D. Quillen, On the formal group laws of unoriented and complex cobordism theory, Bull. Amer. Math. Soc. 75 (1969) 1293–1298. It is expounded in my book, Stable homotopy and generalised homology, Chicago U.P. 1974. The coefficient groups for the theory are much smaller:

$$\pi_*(BP) = Z_{(p)}[v_1, v_2, \ldots, v_i, \ldots]$$

where $|v_i| = 2(p^i - 1)$. If you want an explicit choice of the generators v_i, you use the ones given by M. Hazewinkel, Constructing formal groups III, which apparently dates from about 1977 although...

If X is a CW-complex with bpt [illegible]

then

$$\widetilde{MU}_r(X) = \lim_{n\to\infty} \pi_{r+2n}(MU(n) \wedge X)$$

$$\widetilde{MU}^0(X) = [\Sigma^\infty X, MU]$$

suspension spectrum of X

$$= [S^{2n} \wedge X, MU(n)]$$

if $n >> \dim X < \infty$

while $\widetilde{MU}^r(X) = [\Sigma^\infty X, S^r \wedge MU]$ if $r > 0$

~~& put the sphere on the other side~~

& if $r < 0$ you put the sphere on the other side.

As a pragmatic judgement, we can say that MU-theory is a theory with which one can compute in practice, and among such, it is the most powerful in the sense that it most easily resolves a typical range of problems.

The idea of Novikov, then, was to take the classical Adams spectral sequence and replace ordinary cohomology $H^*(\ ;\mathbb{F}_p)$ by MU-theory. In the form now adopted, it is a spectral sequence

$$Ext_{MU_*(MU)}(MU_*(X), MU_*(Y)) \Rightarrow \{X, Y\}_*$$

Here X and Y may be CW-complexes or spectra of some kind, but ideally are spaces [illegible] MU_* [illegible] reduced MU-homology; and [illegible] purposes of

Milnor remarked that one should regard the sequence of spaces $\{MO(n)\}$ or $\{MSO(n)\}$ as a spectrum. In fact, if ξ_n is the universal bundle over $BO(n)$, then $\xi_n \oplus 1$ is a bundle, which is classified by a map $BO(n) \to BO(n+1)$ & this gives a map

$$MO(n) \wedge S^1 \longrightarrow MO(n+1);$$

~~similarly~~ $\cancel{MSO(n) \wedge S^1 \longrightarrow MSO(n+1)}$.

Thus we can write

$$N_m \xrightarrow{\cong} \varinjlim_n \pi_{n+m}(MO(n)).$$

Similarly the

$$MSO(n) \wedge S^1 \longrightarrow MSO(n+1);$$

In the unitary case it is

$$MU(n) \wedge S^2 \longrightarrow MU(n+1).$$

Milnor calculated the homotopy groups $\pi_*(MU) = \varinjlim_n \pi_{r+2n}(MU(n))$

([J. Milnor, On the cobordism ring Ω^* and a complex analogue, Jour. Amer. Math. Soc. 82 (1960) 505-521.) The result is that the ring $\pi_*(MU)$ is a polynomial algebra $Z[x_2, x_4, x_6, x_8 \dots]$ on generators of degree $2i$, $i = 1, 2, 3, 4, \dots$

Spectra for a Catalogue [illegible]

G.W. Whitehead showed how one may associate to any spectrum a generalised homology theory and a generalised cohomology theory. We may take it as a definition that

JFA. Stable homotopy and generalised homology, Chicago U.P. [illegible] the universal bundle over $BO(n)$, $BSO(n)$ or $BU(n)$ we get the Thom complexes ~~$MSO(n)$,~~ $MO(n)$, $MSO(n)$ and $MU(n)$.

The Pontryagin – Thom construction goes as follows. Suppose given a compact smooth ~~or~~ manifold M of dim m, not necessarily oriented. ~~By~~ Whitney we can ~~use~~ ~~a~~ smooth embedding

$$M^m \longrightarrow S^{n+m};$$

this has a normal bundle ν with a classifying map $M^m \longrightarrow O(n)$.

There is a tubular nbd of M^m in S^{n+m}; by collapsing the complement of the tubular nbd to a point, we get

$$S^{n+m} \longrightarrow (M)^{\nu} \longrightarrow MO(n).$$

Thom showed that this construction is an isomorphism [illegible]

$$\mathfrak{N}_m \xrightarrow{\cong} \pi_{n+m}(MO(n)) \quad (n \gg m).$$

(group of bordism classes of m-mflds, [illegible])

Similarly,

$$\Omega_m \xrightarrow{\cong} \pi_{n+m}(MSO(n)) \quad (n \gg m).$$

(group of bordism classes of oriented closed m-mflds)

§2.

The next step was Novikov's introduction of the spectral sequence which bears his name.

S.P. Novikov: The methods of algebraic topology from the viewpoint of cobordism theories, Izvestija Akademii Nauk SSSR, serija Matematičeskaja 31 (1967) 855–951.

First we should summarise the work of R. Thom. Let ξ be a vector-bundle over the space X. Let $D(\xi)$ be the associated bundle of unit discs and $S(\xi)$ its boundary, the associated bundle of unit spheres; the quotient $D(\xi)/S(\xi)$ is called the "Thom complex" X^{ξ}. Strictly, it is a complex only if X is a CW-complex, when it can be made a CW-complex; also, it appears to depend on a choice of metric in each fibre. However we can equip a construction independent of any choice of metric, as follows: form $E(\xi)^{\wedge}$ by compactifying each fibre with a point at ∞, and form $X^{\xi} = E(\xi)^{\wedge}/(\text{section at } \infty)$. That is homotopy-equivalent. For cohomological purposes you can also consider the pair $E(\xi), E_0(\xi)$ ← (the complement of the zero-section); but then you must ~~either~~ identify $E_0(\xi)$ to a point, or you must take $E(\xi) \cup C\,E_0(\xi)$.

If we proceed in this way with

One of the heroes of [illegible] Hopkins – but not the hired we shall come to – says that this behaviour is general. Let X be a finite p-torsion complex (i.e. $\tilde{H}_*(X; \mathbb{Z})$ [illegible]). Then it is quite possible we may have $\tilde{K}^*(X) = 0$; after all, that happens for C_A. If so, we gain nothing by being told that there is a map $f : S^? X \longrightarrow X$ inducing an isomorphism of K-theory. However, given p, there is a sequence of cohomology theories called Morava K-theories $K(n)_*$ ($n = 1, 2, 3 \cdots$). Then we have the following dichotomy of [illegible]:

(i) X is stably contractible (in which case nothing can be done)

or (ii) There is an n s.t. $\tilde{K}(m)_*(X) = 0$ for $m < n$ but $\tilde{K}(n)_*(X) \neq 0$.

In the second case, Hopkins [illegible] to construct a map

$$S^{m2(p^n-1)} X \xrightarrow{f} X \quad (\text{some } m > 0)$$

s.t.

$$K(n)_*(S^{m2(p^n-1)} X) \xrightarrow{f_*} K(n)_*(X)$$

is iso. In that case, if we can get hold of f, we can exploit it as we did for M.

But I am getting ahead of myself; we need to go back to the historical sequence of development.

Now let's take [illegible] of

$$\begin{array}{ccccc} S^{2a-1}\cup_p e^{2a} & \longrightarrow & S\text{[illegible]} & \longrightarrow & S^{2b} \\ \downarrow & & \downarrow & & \downarrow \\ S^{2a} & \longrightarrow & X & \longrightarrow & S^{2b} \end{array}$$

In K-theory we get

$$\begin{array}{ccccc} \mathbb{Z}/p = \tilde K(S^{2a}\cup_p e^{2a}) & \longleftarrow & \tilde K(\text{S[illegible]}) = \mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & \tilde K(S^{2b}) = \mathbb{Z} \\ \uparrow & & \uparrow & & \uparrow \\ \tilde K(S^{2a}) = \mathbb{Z} & \longleftarrow & \tilde K(X) & \longleftarrow & \tilde K(S^{2b}) = \mathbb{Z} \end{array}$$

Clearly we have $\xi \longmapsto \mu\xi$ with $\mu \not\equiv 0 \bmod p$.

This vertical map commutes with Ψ^k so we get

$$\Psi^k \mu\xi = k^a \mu\xi + \lambda(k^b - k^a)p\xi$$

$$= k^b \mu\xi$$

$$\lambda = \mu/p \not\equiv 0 \text{ and } 1.$$

~~All this goes back~~ to my paper

"On ~~[illegible]~~ $J(X)$ IV"

~~[illegible]~~ [illegible]

If you can find $S^n \cup_p e^{n+1}$ & then a map $S^{ap^{n-1}}\,[illegible] \longrightarrow S^n\,[illegible]$ inducing an iso on [illegible] which you can use in a similar way.

Now we want to relate this to the operations. We have

$$\mathrm{ch}\, \Psi^k \xi = k^a u + \lambda k^b v$$
$$\mathrm{ch}\, k^a \xi = k^a u + \lambda k^a v$$
$$\text{so } \Psi^k \xi = k^a \xi + \lambda (k^b - k^a) \eta .$$

If we choose ξ & η as values of the ... we can read off λ.

Consider now the case we are interested in. The map

$$S^{ia}\Pi \xrightarrow{\ \cdots S^{i}\Lambda\Pi\ } \Pi$$

induces an iso of K-theory, so for the mapping-cylinder we get

$$\tilde{K}(S^{2a-1} \cup e^{2a} \cup e^{2b} \cup e^{2b+1}) = 0$$

$(b-a = i(n-1))$

Decomposing this a different way we see the attaching maps

$$S^{2b} \longrightarrow S^{2a-1} \cup e^{2a} \cup e^{2b}$$

we get

$$\tilde{K}(S^{2b}) \xleftarrow{\ \cong\ } \tilde{K}(S^{2a-1} \cup e^{2a} \cup e^{2b}).$$

So $\tilde{K}(S^{2a-1} \cup e^{2a} \cup e^{2b}) \cong \mathbb{Z}$, gen ξ, with operation on b,

$$\Psi^k \xi = k^b \xi .$$

The composite $S^{2b} \longrightarrow S^{2a-1} \cup e^{2a} \cup e^{2b} \cdots$ is p, so we get $\eta \longmapsto p\xi$.

We will show that α_i is non-zero by studying the K-theory of its mapping cone. We will see that although the K-theory of this mapping cone is additively the same as if $\alpha_i \simeq 0$, nevertheless the K-theory operations are different. This is just like what we saw for the operation P in ord. cohomology a [illegible].

We may as well assume that $n \in |i|$ [?] is even, say $= 2a$, so I will study a cofibring

$$S^{2b-1} \xrightarrow{q} S^{2a} \longrightarrow X = S^{2a} \cup_q e^{2b} \longrightarrow S^{2b} \qquad (b>a)$$

In K-theory I get

$$\begin{array}{ccccc} \tilde K(S^{2a}) & \longleftarrow & \tilde K(X) & \longleftarrow & \tilde K(S^{2b}) \\ \| & & & & \| \\ \mathbb{Z} & & & & \mathbb{Z} \end{array}$$

So $\tilde K(X) = \mathbb{Z} \oplus \mathbb{Z}$, with $\xi \to$ gen of $\tilde K(S^{2a})$, η coming from $\tilde K(S^{2b})$. I can replace ξ by $\xi + N\eta$ if I like. What might the K-theory operations look like? The quickest way is to use

$$ch: K(X) \longrightarrow H^*(X; \mathbb{Q}).$$

Clearly we have

$$ch\,\xi = u + \lambda v$$
$$ch\,\eta = v$$

where u, v are the generators in $H^{2a}(X;\mathbb{Z})$, $H^{2b}(X;\mathbb{Z})$ & $\lambda \in \mathbb{Q}$. Replacing ξ by $\xi + N\eta$ replaces λ by $\lambda + N$, but $[\lambda] \in \mathbb{Q}/\mathbb{Z}$ is an invariant.