NB 61

The wedge-sum operation $\bigvee_\alpha X_\alpha$ (where the sum may be infinite) passes to Bousfield classes, and gives the least upper bound w.r.t. the ordering relation

$((\bigvee_\alpha X_\alpha) \wedge Z \simeq \bigvee_\alpha (X_\alpha \wedge Z)$, which is contractible iff $X_\alpha \wedge Z \simeq pt$ for $\forall \alpha$.)

The smash product also passes to Bousfield classes. Suppose $E_1 \sim E_2$; then $E_1 \wedge F \wedge Z \simeq pt$ iff $E_2 \wedge F \wedge Z \simeq pt$, by the class. prop. applied to $F \wedge Z$; so $E_1 \wedge F \simeq E_2 \wedge F$.

Finally I give a little lemma. Last Easter I noticed Mike Hopkins using the same handy argument repeatedly. By August he had noticed that Ravenel had already codified the result which the argument proves, and here it is.

Suppose given a map $S^n X \xrightarrow{f} X$. Use it to form a cofibre sequence

$$S^n X \xrightarrow{f} X \longrightarrow C_f$$

and also the direct limit T of

$$X \xrightarrow{S^{-n} f} S^{-n} X \xrightarrow{S^{-2n} f} S^{-2n} X \longrightarrow \ldots$$

Lemma 9 (Ravenel 1.34)

$$\langle X\rangle = \langle C_f\rangle \vee \langle T\rangle .$$

Proof. (i). Suppose $X\wedge Z$ is contractible.

Then $S^nX\wedge Z \xrightarrow{f\wedge 1} X\wedge Z \longrightarrow C_f\wedge Z$

is a cofibering with 2 legs contractible, which shows $C_f\wedge Z$ is contractible. Also $T\wedge Z$ is the limit of

$$X\wedge Z \xrightarrow{S^{-n}f\wedge 1} S^{-n}X\wedge Z \to \cdots$$

which shows $T\wedge Z$ is contractible.

(ii) Suppose C_f is contractible. Then the map

$$S^nX\wedge Z \xrightarrow{f\wedge 1} X\wedge Z$$

is an equivalence. So all the maps of the direct system

$$X\wedge Z \xrightarrow{S^{-n}f\wedge 1} S^{-n}X\wedge Z \xrightarrow{S^{-2n}f\wedge 1} S^{-2n}X\wedge Z \cdots$$

are equivalences. So the maps from the first term to the limit $X\wedge Z \longrightarrow T\wedge Z$

is an equivalence. So if we suppose also that $T\wedge Z$ is contractible, we conclude that $X\wedge Z$ is contractible.

<u>Corollary 10</u>. If f is nilpotent, i.e. $\exists\, m$ s.t. $X \xrightarrow{f} S^{-n}X \xrightarrow{f} S^{-2n}X \xrightarrow{f} \cdots \xrightarrow{f} S^{-mn}X$ is null homotopic, then $\langle X \rangle = \langle C_f \rangle$.

<u>Proof</u>. If f is nilpotent, then

$$T = \varinjlim (X \xrightarrow{f} S^{-n}X \xrightarrow{f} \cdots \text{———})$$

has $\pi_*(T) = 0$ and is contractible.

So $\langle X \rangle = \langle C_f \rangle \vee \langle T \rangle$

$= \langle C_f \rangle$

<u>Example</u>. $\langle CP^2 \rangle = \langle S^0 \rangle$. Proof $\eta^4 = 0$.

Nilpotence (?) seems to run part of this argument the other way. Now a priori the argument seems to fail unless X is a finite spectrum: if T is contractible then the injection $X \to T$ is null homotopic, and if this null htpy takes place in some finite stage of the telescope, say $S^{-mn}X$, then $f^n \simeq 0$; but if X is infinite there is no reason why the null htpy shouldn't use the whole spectrum. And, of course, we like large infinite spectra.

The way forward is that we can say something about maps, which are determined by their values on some finite subspectra.

Example. $\langle HP^2 \rangle = \langle S^0 \rangle$. Proof $\nu^4 = 0$. Ex $\langle CP^2 \wedge HP^2 \rangle = \langle S^0 \rangle$

~~are~~ determined by what they do on a finite subspectrum.

Let R be a ring spectrum. It needs to be homotopy-associative, with unit $S \xrightarrow{\eta} R$ up to homotopy, but no commutativity is needed.
~~Then there is a (1-1) correspondence~~
~~which assigns~~
As we saw, for each element $\alpha \in \pi_a(R)$ we can construct the map

$$S^a \wedge R \xrightarrow{\alpha \wedge 1} R \wedge R \xrightarrow{\mu} R ;$$

this we interpret as an operation "multiplying by α in the coefficient ring". Write $\bar{\alpha} = \mu(\alpha \wedge 1)$. Then $\bar{\alpha}$ is a map of right R-module spectra, & this gives a (1-1) correspondence between elts of $\pi_*(R)$ & maps of right R-module spectra; the correspondence the other way is that $\bar{\alpha}$ arises

~~$S^0 \xrightarrow{\eta} R \xrightarrow{\bar{\alpha}} S^a \wedge R$~~

$$S^a = S^a \wedge S^0 \xrightarrow{1 \wedge \eta} S^a \wedge R \xrightarrow{\bar{\alpha}} R .$$

Composition of maps $\bar{\alpha}$ corresponds to multiplication of elements $\alpha \in \pi_* R$, i.e. $\overline{\alpha\beta} = \bar{\alpha}\,\bar{\beta}$.

Lemma 11. α is nilpotent in $\pi_*(R)$ iff $\mathrm{Tel}(\bar{\alpha}) \simeq pt$.

Proof. If $\alpha^m = 0$, then $\bar{\alpha}^m = 0$, so $\mathrm{Tel}\,\bar{\alpha}$ is contractible.

If $\mathrm{Tel}\,\bar{\alpha}$ is contractible, then

$$S^0 \xrightarrow{1} R \longrightarrow \mathrm{Tel}\,\bar{\alpha}$$

is nullhomotopic, & since S^0 is finite the nullhomotopy must take place in a finite part of the telescope, i.e. $\exists m$ s.t. $\alpha^m = 0$.

Actually this is not yet the form in which we want to use the lemma. We want to suppose given 2 ring-spectra E, R and apply the lemma to $E \wedge R$ — which of course is a ring-spectrum again. But we don't want to do it with a general element of $\pi_*(E \wedge R) = E_*(R)$, but with an element of the following special form.

$$S^a \xrightarrow{\alpha} R \simeq S \wedge R \longrightarrow E \wedge R.$$

Basically, we want to take an elt $\alpha \in \pi_a(R)$ and then take its image in $E_a(R)$ under the E-Hurewicz-homomorphism.

§5. Two spectra $X(n)$. Mahowald's general plan is as follows: given information about the qualitative behaviour of MU_*, to obtain information about the qualitative behaviour of $\pi_* = S^0_*$.

In more detail, his plan is to interpose a sequence of spectra between S^0 and MU:

$$S^0 = X(1) \subset X(2) \subset \cdots \subset X(n) \subset \cdots \subset MU.$$

Then he will argue by induction downwards; at each step he goes from information about the qualitative behaviour of $X(n+1)_*$ to info. about the qualitative behaviour of $X(n)_*$.

This does not contradict the ... MU_* is the strongest theory with which we can really calculate. We don't propose to do real calculations with $X(n)_*$, only to argue about its qualitative behaviour.

To give a rough idea how big $X(n)$ is, the quickest way is to describe its homology. For that I need to recall the description of $H_*(MU)$. Of course it's the same as $H_*(BU)$ via the Thom isomorphism; but first, a bit about Thom spectra.

For a real vector bundle ξ, $S(\xi)$ is the associated bundle with fibre S^{n-1}. Usually this is different from the associated bundle with fibre $U(n)$:

NB

Easy diagram chasing shows that the map

$$\bar{\alpha} : S^a \wedge E \wedge R \longrightarrow E \wedge R$$

which we get is, up to a rearrangement of where you put S^a, the smash product of $1 : E \wedge E$ and $S^a \wedge R \xrightarrow{\alpha} R$.

So $Tel(\bar{\alpha}) = E \wedge Tel(\alpha)$.

__Corollary 12.__ $\alpha \in \pi_a(R)$ becomes nilpotent in $E_a(R)$ iff $E \wedge Tel\,\alpha$ is contractible.

~~In particular, whether α becomes nilpotent in $E_*(R)$ depends only on the Bousfield class of E.~~

Of course you can prove this more directly by cutting out some of the way I built it up.

__Corollary 13.__ Suppose E and F are ring-spectra and $\langle E \rangle = \langle F \rangle$; then $\alpha \in \pi_a(R)$ becomes nilpotent in $E_*(R)$ iff it becomes nilpotent in $F_*(R)$.

Corollary 10. If f is nilpotent, i.e. $\exists\, m$ s.t. $X \xrightarrow{f} S^{-n}X \xrightarrow{f} S^{-2n}X \xrightarrow{f} \cdots \xrightarrow{f} S^{-mn}X$ is null homotopic, then $\langle X \rangle = \langle C_f \rangle$.

Proof. If f is nilpotent, then

$$T := \varinjlim \left(X \xrightarrow{f} S^{-n}X \xrightarrow{f} \cdots \right)$$

has $\pi_*(T) = 0$ and is contractible.

$$\text{So } \langle X \rangle = \langle C_f \rangle \vee \langle T \rangle = \langle C_f \rangle$$

Example. $\langle CP^2 \rangle = \langle S^0 \rangle$. Proof $\eta^4 = 0$.

Nilpotence works to run part of this argument the other way. Now a priori the argument seems to fail unless X is a finite spectrum: if T is contractible the injection $X \xrightarrow{i} T$ is null htpic, and if this null htpy takes place in some finite stage of the telescope, say $S^{-mn}X$, then $f^m \simeq 0$; but if X is infinite there is no reason why the null htpy shouldn't use the whole spectrum. And, of course, we have to use infinite spectra.

The way forward is that we can say something for maps which are determined by their values on some finite subspectra

Example. $\langle HP^2 \rangle = \langle S^0 \rangle$. Proof $\nu^4 = 0$. Ex $\langle CP^2 \wedge HP^2 \rangle = \langle S^0 \rangle$.

NB → And so on to the end, alas.

Lemma 11. α is nilpotent in $\pi_*(R)$ iff $\mathrm{Tel}(\bar{\alpha}) \simeq pt$.

Proof. If $\alpha^m = 0$, then $\bar{\alpha}^m = 0$, so $\mathrm{Tel}\,\bar{\alpha}$ is contractible.

If $\mathrm{Tel}\,\bar{\alpha}$ is contractible, then

$$S^0 \xrightarrow{?} R \longrightarrow \mathrm{Tel}\,\bar{\alpha}$$

is nullhomotopic, & since S^0 is finite the nullhomotopy must take place in a finite part of the Tel escope, so $\exists\, m$ s.t. $\alpha^m = 0$.

Actually this is not yet the form in which we want to use the lemma. We want to suppose given 2 ring-spectra E, R and apply the lemma to $E \wedge R$ — which of course is a ring-spectrum again. But we don't want to do it with a general elt of $\pi_*(E \wedge R) = E_*(R)$, but with an element of the following special form.

$$S^a \xrightarrow{\alpha} R \simeq S^0 \wedge R \longrightarrow E \wedge R.$$

Basically, we want to take an elt $\alpha \in \pi_a(R)$ and then take its image in $E_a(R)$ under the E-Hurewicz-homomorphism.

H_* by some other homology theories.

__Prop__ (Ravenel) There is a sequence of maps of ring spectra

$$S^0 = X(1) \subset X(2) \subset \cdots \subset X(n) \subset \cdots \subset MU$$

so that $H_*(X(n))$ is identified, as a Hopf ring, with

$$\mathbb{Z}[b_1, b_2, \ldots, b_{n-1}].$$

We use a filtration idea: for any map of a CW-complex $W \xrightarrow{f} BU$ gives you a Thom spectrum over W. In fact, you can filter W as the union of subcomplexes $W(n)$ so that

$$W(n) \xrightarrow{f} BU(n) \subset BU,$$

& we can use f to pull back a complex n-plane bundle ξ_n over $W(n)$; [illegible] there is a sequence of Thom spaces

$W(n)^{\xi_n}$, and with these obvious maps

then form a spectrum. Changing the choice of the filtration $W(n)$, or changing the map f by a homotopy, may change the spectrum, but only up to [illegible] but here is one case in which they are the same, namely, $n=1$ & $S^1 = U(1)$. The universal S^1-bundle over $\mathbb{C}P^\infty$ is the usual bundle $S^\infty \to \mathbb{C}P^\infty$, & the pullback over $\mathbb{C}P^{n}$ is $S^{2n+1} \to \mathbb{C}P^n$.

This gives $S(\xi)$ in the two cases. $D(\xi)$ is homotopy equivalent to the base via the projection, so we get

$$MU(1) = (\mathbb{C}P^\infty)^\xi \simeq \mathbb{C}P^\infty$$ (contracting the contractible space S^∞ down to a point again)

$$(\mathbb{C}P^{n+1})^\xi \simeq \mathbb{C}P^n.$$

We have $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$, $x \in H^2$;

$H_*(\mathbb{C}P^\infty)$ has a basis β_i, $\beta_i \in H_{2i}$; $\langle x^i, \beta_i \rangle = 1$.

$MU(1)$ is term 2 of MU, so

$$H_{2i}(\mathbb{C}P^\infty) \longrightarrow H_{2i-2}(MU)$$

$$\& \quad \beta_i \longmapsto b_{i-1} \in H_{2i-2}(MU)$$

Since MU is a ring-spectrum, $H_*(MU)$ is a ring, and in fact

$$H_*(MU) = \mathbb{Z}[b_1, b_2, b_3, \ldots]$$

Indeed, we can copy this account ($b_0 = 1$), replacing

If I make a specific choice of cancelling htpys, then I need to specify to get a map of Thom spectra

$$\begin{array}{ccc} X^f \wedge Y^g & \xrightarrow{\bar\xi \wedge \bar g} & MU \wedge MU \\ \bar a \downarrow & & \downarrow \bar\mu \\ Y^g & \xrightarrow{\bar g} & MU \end{array}$$

to homotopy commutative. A priori you might worry that $\bar\mu$ depends on the choice of cancelling htpy for the orig. diagram, but don't worry, $\pi_*(BU)$ is 0 in odd dims & in the application $H^*(X \times Y; \mathbb{Z})$ is 0 in odd dims, so by obstruction theory there is essentially only one choice of cancelling homotopy.

Under these conditions there is a htpy unit $S \to X^f$ (namely the injection of any fibre S into X^f).

If we want a htpy assoc. prop

$$\begin{array}{ccc} X^f \wedge X^f \wedge Y^g & \xrightarrow{\bar a \wedge 1} & X^f \wedge Y^g \\ \downarrow 1 \wedge \bar a & & \downarrow \bar a \\ X^f \wedge Y^g & \xrightarrow{\bar a} & Y^g \end{array}$$

Probably at this point I should say a bit about products also.

Suppose we have 3 CW-complexes X, Y, Z with bundles ξ, η, ζ over them, so that we get $\xi \times \eta$ over $X \times Y$, and we have a map

$$X \times Y \longrightarrow Z$$

and a map of bundles

$$\xi \times \eta \longrightarrow \zeta$$

over it.

Then evidently we get a map of Thom complexes

$$X^{\xi} \wedge Y^{\eta} = (X \times Y)^{\xi \times \eta} \longrightarrow Z^{\zeta}$$

Now I said this for actual finite-dim'l bundles, but we expect to be able to replace them by maps into BU, which will be [illegible] for.

Suppose for example that I have a map

$$X \times Y \xrightarrow{\mu} Y$$

and 2 maps

$$X \xrightarrow{f} BU,$$

$$Y \xrightarrow{g} BU$$

s.t. the following diagram is homotopy-commutative:

$$\begin{array}{ccc} X \times Y & \xrightarrow{f \times g} & BU \times BU \\ \mu \downarrow & & \downarrow \\ Y & \xrightarrow{g} & BU \end{array}$$

Now we have a sequence of inclusions

$$BSU(1) \subset BSU(2) \cdots \subset BSU(n) \subset \cdots \subset BSU.$$

$BSU(1) = pt$.

Applying Ω^2, we get

$$\Omega^2 BSU(1) \subset \Omega^2 BSU(2) \subset \cdots \subset \Omega^2 BSU(n) \subset \cdots \subset \Omega^2 BSU$$

$\Omega^2 BSU(1) = pt$, $\Omega^2 BSU \cong BU$ (Bott)

You have to believe that the Bott map preserves all the structure you want, eq it's a [illegible] [illegible] [illegible]

Pass to Thom spectra: we get a sequence of ring spectra

$$X(1) \subset X(2) \subset \cdots \subset X(n) \subset \cdots \subset MU$$

$X(1) = pt$

These are all ring-spectra with units, htpy assoc, & e htpy comm because these were all Ω^2-maps / ~~[illegible]~~ details /.

$\Rightarrow$ Clearly $H_*(X(n)) \cong H_*(\Omega SU(n))$ (Thom iso)

$$\cong \mathbb{Z}[g_1, g_2, \ldots g_{n-1}]$$

$g_i \in H_{2i}$ (Bott)

For the inclusion $\Omega SU(n) \subset BU$ the first non-zero relative htpy group is π_{2n}; we can ~~[illegible]~~ easily check

Then evidently we shall ask 2 things: first, the square

$$\begin{array}{ccc} X \times X \times Y & \xrightarrow{\mu \times 1} & X \times Y \\ \downarrow{\scriptstyle 1 \times \mu} & & \downarrow{\scriptstyle \mu} \\ X \times Y & \xrightarrow{\mu} & X \times Y \end{array}$$

be homotopy-commutative, & secondly, when we push this homotopy into BU using g, we get something compatible with the associativity homotopy in BU which comes from the Milnor construction.

If we have $X = Y$, $f = g$ we expect to get a ring-spectrum; if we want it to be homotopy-commutative, we need similar data.

Evidently, to check all this in detail we need details about the smash-product in the cat. of spectra.

At this stage I find better pictures of even the spectrum $X(2)$ is pretty unavailable.

Of course we have

$$H_*(X(2)) = \mathbb{Z}[b_1]$$

So

$$X(2) = S^0 \cup e^2 \cup e^4 \cup e^6 \cup e^8 \cup \cdots$$

But don't think that $X(2)$ is anything like $\mathbb{CP}^\infty$. We can see that by looking at the operations. Let's look at mod 2 cohomology. We have

$$H^0(X(2);\mathbb{F}_2) \xrightarrow[\cong]{Sq^2} H^2(X(2);\mathbb{F}_2)$$

iso because H_0, H_2 come from $H_2(\mathbb{CP}^\infty)$, $H_4(\mathbb{CP}^2)$. So now consider the map of ring-spectra

$$X(2) \longrightarrow MU \longrightarrow H$$

Clearly we have $b_1 \longrightarrow \xi_1^2$,

So this map identifies $H_*(X(2);\mathbb{F}_2)$ with the sub comodule of $H_*(H)$ given by $\mathbb{F}_2[\xi_1^2]$. This tells you all the structure of $X(2)$. In particular, every operation Sq^{2n} is non-zero on $H^0(X(2);\mathbb{F}_2)$.

The inclusion $CP^{n-1} \to CP^\infty = BU(1) \to BU$

into $\Omega SU(n)$. This gives a map

of the Thom complex CP^n into

term 2 of the spectrum $X(n)$.

So $b_1, b_2, \ldots, b_{n-1}$ lie in the

image of $H_*(X(n)) \to H_*(\Omega U)$.

So this image contains $\mathbb{Z}\{b_1, b_2, \ldots, b_{n-1}\}$.

Well. Then by counting ranks the image

can be no larger & the kernel must be zero.

We don't even need to look at Bott's

algebra as to see if they are the same

on b_i. =

We don't act to $X(n)$ to $X(n+1)$

all in one jump, we do it by filling

$X(n+1)$

Defn 2 (Hopkins) There is a sequence of

maps of module spectra over $X(n)$

$X(n) = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{?} \subset \cdots \subset X(n+1)$

Proof We have a fibring $SU(n) \to SU(n+1) \to S^{2n+1}$ hence

$$\Omega SU(n) \longrightarrow \Omega SU(n+1) \longrightarrow \Omega S^{2n+1}.$$

Here ΩS^{2n+1} is weakly equivalent to the James complex

$$J(S^{2n}) = S^{2n} \cup e^{4n} \cup e^{6n} \cup \cdots.$$

This is filtered by

$$J_n(S^{2n}) = S^{2n} \cup e^{4n} \cup \cdots \cup e^{n.2n}.$$

Form the pull back

$$\begin{array}{ccc} \Omega SU(n) & = & \Omega SU(n) \\ \downarrow & & \downarrow \\ F'_n & \longrightarrow & \Omega SU(n+1) \\ \downarrow & & \downarrow \\ J_n(S^{2n}) & \longrightarrow & \Omega S^{2n+1} \end{array}$$

Let F_n be the Thom space of F'_n.

The $\Omega SU(n)$ acts on F'_n & the maps of both with BU preserve this action,

so $X(n)$ acts on F_n.

The operation Sq^2 is given by

$$H^0 \xrightarrow{\cong} H^2 \xrightarrow{0} H^4 \xrightarrow{\cong} H^6 \xrightarrow{0} \cdots$$

It follows that: we can filter $X(2)$ so that successive subquotients all look like[^1] suspensions of CP^2. In fact, we can filter it more coarsely, so that successive subquotients have 4 cells and all look like[^2] $CP^2 \wedge HP^2$. [illegible] (Hopkins needs to do something like this in greater generality.

In fact, the domain of the map $X(n)$ to $X(n+1)$ all in one jump is done by filtering $X(n+1)$).

<u>Dfn 2</u> (Hopkins) There is a sequence of (maps of module-spectra over $X(n)$)

$X(n) = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \subset X_{n+1}$

s.t. $H_*(F_n)$ is identified, as H_* reps, with the free module over $Z[b_1, \ldots b_{n-1}]$ on the elements: $1, b_n, b_n^2, \ldots b_n^h$.

[^1]: (inserted above "look like") ud 2,
[^2]: (inserted above "look like") ud 2

More precisely the spaces [illegible]

$$\begin{array}{ccc} JS^{2n} & \xrightarrow{\Delta} & J(S^{2n}) \times J(S^{2n}) \\ \uparrow & & \uparrow \\ J_r S^{2n} & \xrightarrow{\Delta} & \bigcup_{i+j=r} J_i(S^{2n}) \times J_j(S^{2n}) \end{array}$$

so the spaces are those we need

$$\begin{array}{ccc} F'_r & \xrightarrow{\Delta} & \bigcup_{i+j=r} F'_i \times F'_j \\ & & \downarrow \\ & & \bigcup_{i+j=r} F'_i \times J_j(S^{2n}) \end{array}$$

and passing to Thom complexes

$$F_r \xrightarrow{\Theta} \bigcup_{i+j=r} F_i \wedge (J_j(S^{2n}) \cup p)$$

$$\cong$$

$$\bigvee_{i+j=r} F_i \wedge S^{2nj}$$

[illegible]

calculating everything [illegible]

modulo over $Z[b_1, b_2 \ldots b_{n-1}]$,

and we have.

Prop 3. At the prime p, we have a cofibration

$$F_{mp^k-1} \longrightarrow F_{(m+1)p^k-1} \longrightarrow S^{mp^k 2n} F_{p^k-1}.$$

This shows that consecutive chunks

$F_{(m+1)p^k-1} / F_{mp^k-1}$ all look the same (up to suspension).

Proof. Consider

$$\Omega SU(n+1) \xrightarrow{\Delta} \Omega SU(n+1) \times \Omega SU(n+1) \xrightarrow{1 \times \text{proj}} \Omega SU(n+1) \times \Omega S^{2n+1} \xrightarrow{1 \times c} \Omega SU(n+1) \longrightarrow BU.$$

Everything in a compact H-space [illegible]. Pass to Thom complexes [illegible] we get a map of Thom spectra

$$X(n+1) \xrightarrow{\theta} X(n+1) \wedge (\Omega S^{2n+1}_{+}) \cong \bigvee_{j \geq 0} \Sigma^{2nj} X(n+1) \wedge S^{2nj}$$

Now, I claim that this map is filtration-preserving.

Prop 4.4. At the prime 2 there is a ring spectrum R, namely $R = S^\infty(\Omega^2 S^{2l+1} \cup pt)$, where $l = 2^h \cdot 2n$, so that F_{2^h-1} is a module-spectrum over R, and there is a map

$$S^\infty(S^{2l-1}) \longrightarrow S^\infty(\Omega S^{2l} \cup pt) \longrightarrow S^\infty(\Omega^2 S^{2l+1} \cup pt)$$

(say Y) so that when we define α by

$$S^{2l-1} \wedge F_{2^h-1} \xrightarrow{Y \wedge 1} R \wedge F_{2^h-1} \longrightarrow F_{2^h-1}$$

we have $C_\alpha \simeq F_{2^{h+1}-1}$.

Proof. I have to go back to the work of James on suspension-theory. At 2 there is a fibering

$$J_{2^h-1} S^{2n} \longrightarrow \Omega S^{2n+1} \longrightarrow \Omega S^{2l+1}$$

when $l = 2^h \cdot 2n$.

This gives the following diagram.

$$b_n^r \longrightarrow \sum \frac{r!}{(r-j)!\,j!}\, b_n^{r-j} \wedge b_n^j$$

$$\bmod\ (b_1, b_2, \dots b_{n-1})$$

so if we take $j = mp^k$, all the binomial coefficients are prime to p and we end up

$$F_{(m+1)p^k-1} \longrightarrow F_{p^k-1} \wedge S^{mp^k \cdot 2n}$$

with fibre F_{mp^k-1}.

At this point I will specialise to the case $p=2$.

In this case we have a cofibering

$$F_{2^k-1} \longrightarrow F_{2^{k+1}-1} \longrightarrow S^{2^k \cdot 2n} \wedge F_{2^k-1} \xrightarrow{\alpha} S^1 \wedge F$$

It is essential to our purposes to get a good geometrical hold on α so that we can prove $Tel(\alpha) \simeq pt$ and infer $\langle F_{2^{k+1}-1} \rangle = \langle C_\alpha \rangle = \langle F_{2^k-1} \rangle$.

The statement $Tel(\alpha) \simeq pt$ covers geometrical ingredients like the nil potence of η and ν.

Explain: geometric ideas in SS don't know.

The pullback of ξ over E must be fibrewise in the F direction and trivial in the LE direction [illegible] because LB is contractible, so after choosing an iso, we get

$$\begin{array}{ccc} \dot{O} \times \xi & \longrightarrow & \xi \\ & & \downarrow \\ LB \times F & \longrightarrow & E \end{array}$$

Restricting over the path over $b_0 \in B$, we get

$$\begin{array}{ccc} \Omega B \times F & \xrightarrow{\;0 \times \xi\;} & F^{\xi} \\ \downarrow & & \downarrow \\ LB \times F & \xrightarrow{\;0 \times \xi\;} & E^{\xi} \end{array}$$

Thomifying we get

$$\begin{array}{ccc} S^{\infty}(\Omega B \sqcup pt) \wedge F^{\xi} & \longrightarrow & F^{\xi} \\ \downarrow & & \downarrow \\ S^{\infty}(S^{0}) \wedge F^{\xi} & \longrightarrow & E^{\xi} \end{array}$$

This works with $B = \Omega S^{2k+1}$,

$\Omega B = \Omega^{2} S^{2k+1}$,

$R = S^{\infty}(\Omega^{2} S^{2k+1} \sqcup pt)$, so it arises as the action of R on $F^{\xi} = F_{[illegible]}$

$$\begin{array}{ccccc} \Omega SU(n) & \xrightarrow{1} & \Omega SU(n) & & \\ \downarrow & & \downarrow & & \\ F'_{2^n-1} & \longrightarrow & \Omega SU(n+1) & \searrow & \\ \downarrow & & \downarrow & & \\ J_{2^n-1}(S^{2n}) & \longrightarrow & \Omega S^{2n+1} & \longrightarrow & \Omega S^{2n+1} \end{array}$$

Write $F = F'_{2^n-1}$, $E = \Omega SU(n+1)$, $B = \Omega S^{2n+1}$.

Then we have a map $E \xrightarrow{\xi} BU$

such that $F^{\xi} \simeq F_{2^n-1}$.

We can assume without much risk that $F \to E \to B$ is a Hurewicz fibration. So by the homotopy lifting property we can arrange the following diagram.

$$\begin{array}{ccc} F & \longrightarrow & F \\ \downarrow & & \downarrow \\ LB \times F & \longrightarrow & E \\ \downarrow & & \downarrow \\ LB & \longrightarrow & B \end{array}$$

$$\begin{array}{ccc} S^{\infty}(\Omega S^{2n} \sqcup pt) \wedge F^{q} & \longrightarrow & F^{q} = F_{2^{n}-1} \\ \downarrow & & \downarrow \\ S^{\infty}(S^{0}) \wedge F^{q} & \longrightarrow & E^{q} = F_{2^{n+1}-1}. \end{array}$$

Well, here is a natural map

$$\begin{array}{ccc} S^{\infty} S^{2n-1} & \longrightarrow & \Omega^{\infty}(S^{2n} \sqcup pt) \\ & \searrow^{0} & \downarrow \\ & & S^{\infty}(S^{0}) \end{array}$$

which [illegible] accurately [illegible]

With that choice of Y, you [illegible]

and get a map $C_{\alpha} \longrightarrow E^{q} = F_{2^{n+1}-1}$

and check it is a homology isomorphism.

Warning: The obvious map

$S^{2n-1} \longrightarrow \Omega S^{2n}$ doesn't preserve the hopf.

You need I suspect to get a map [illegible] the hopfs [illegible] it [illegible] but [illegible] is not [illegible] ~~[illegible]~~

~~[illegible]~~ [illegible]

To go further I invoke that if I just want to assemble $F_{2^{n+1}-1}$, I don't need all of the fibrings I used above. I can restrict the James fibring to

$$J_{2^n-1} \longrightarrow J_{2^{n+1}-1} \longrightarrow S^{2n} \subset \Omega S^{2n+1}.$$

This gives a diagram just like the previous

$$\begin{array}{ccccc}
\Omega SU(n) & \xrightarrow{1} & \Omega SU(n) & & \\
\downarrow & & \downarrow & & \\
F'_{2^n-1} & \longrightarrow & F'_{2^{n+1}-1} & \searrow & \\
\downarrow & & \downarrow & & \\
J_{2^n-1}(S^{2n}) & \longrightarrow & J_{2^{n+1}-1}(S^{2n}) & \longrightarrow & S^{2n}
\end{array}$$

Write $F = F'_{2^n-1}$, $E = F'_{2^{n+1}-1}$, $B = S^{2n}$.

Thus to lift this to map my own smaller $(B \times F)$ into my own smaller E is the thing I get

~~choice can be forced~~ by a stable map $S^a \to S^0$.
~~That Y and α is congruent~~
plausible, ~~but needs~~ details of the rest we
~~suppress~~ → Proof.

It follows that the composite α^m is given by

$$S^\infty(S^{m(2\ell-1)+1})\wedge F_{2^{\ell}-1} \xrightarrow{Y^m \wedge 1} R \wedge F_{2^{\ell}-1} \to F_{2^{\ell}-1}$$

Now we have to look at $R =$
$S^\infty(\Omega^2 S^{2\ell+1}_{(2)} \sqcup pt)$.

According to V.P. Snaith, A stable decomposition of $\Omega^n S^n X$, JLMS 7 (1974) 577–583, there is a stable equivalence, which ~~[illegible]~~ for our case we can write

$$S^\infty(\Omega^2 S^{2\ell+1} \sqcup pt) \simeq S^\infty \bigvee_{m \geq 0} \mathrm{Snaith}_m(2\ell+1)$$

Snaith doesn't have the bpt separate but I have compensated for that by writing $\bigvee_{m\geq 0}$ instead of $\bigvee_{m\geq 1}$. Also Snaith doesn't explicitly discuss the ring-spectrum structure, but his work has been taken up by later authors who I have trusted not to miss a thing like that.

We need to know that Y^m is being either of the bottom sphere in $\mathrm{Snaith}_m(2\ell+1)$.

Prop 5. At 2 Tel(α)\ı̄ pt.

Suspend $m(2l-1)$ times.

$$\begin{array}{ccccccc} S^{(m+1)(2l-1)+r} & \xrightarrow{\theta} & S^{(m+1)(2l-1)} F_{2^h-1} & \xrightarrow{\alpha} & S^{m(2l-1)} F_{2^h-1} & \xrightarrow{\alpha^m} & F_{2^l-1} \\ & \searrow^{0} & & & \downarrow & \nearrow & \\ & & & & S^{m(2l-1)} H \wedge F_{2^l-1} & & \end{array}$$

$Smith_m(2l+1)$

Well, we can easily take m so that $(2l-1)+r < \frac{1}{2}m$

and so replace $S^{m(2l-1)}H$ by $Smith_m(2l+1)$.

But now α^m extends. Then θ is annihilated

by α^{m+1}. This holds for all $\theta \in \pi_*(F_{2^l-1})$

so $Tel(\alpha)$ is contractible.

~~Moks~~ This is the argument which Hopkins sketched at Durham; it replaces quite a lot of what he told me last Easter. At Durham, Hopkins said almost nothing about the case $p>2$ and his choice of the argument. It seems unclear that some changes would be required for $p>2$. To begin with, we have to replace S^{2l}, as the base of our bundle E, by $J^{p-1}(S^{2l})$. Now it is quite familiar that bundles over S^{2l} are described by clutching functions, but it is not all familiar that bundles over $J^{p-1}(S^{2l})$ can be so described. However we don't want to describe the whole bundle, Next, we need to quote from M. E. Mahowald: A new infinite family in ${}_2\pi^S_*$, Topology 16 (1977) 249–256:

$$\tilde{H}^*(\mathrm{Smith}_m(2l+1);\, F_2)$$

is a free A-module, on its one generator of degree $m(2l-1)$, in degrees

$$\leq m(2l-1) + \tfrac{1}{2}m$$

Thus we can start the proof proper.

Take any element $\theta \in \pi_{qp}(F_{2^n-1})$.

We know that the map

$$S^{2l-1}F_{2^n-1} \xrightarrow{\alpha} F_{2^n-1}$$

induces the zero map of mod 2 homology, because the map $F_{2^n-1} \longrightarrow F_{2^{n+1}-1}$ induces a monomorphism in mod 2 homology.

So the composite

$$S^{2l-1}_{\wedge} S^r \xrightarrow{\theta} S^{2l-1}F_{2^n-1} \xrightarrow{\alpha} F_{2^n-1}$$

$$\downarrow$$

$$H \wedge F_{2^n-1}$$

does so; i.e.

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~~At the price we can't quite achieve this, because you have to switch together p things at which we have 2~~ So we replace $S^0 \cup_q e^2$ as our quickening example by

$$X = S^0 \cup_\alpha e^a \cup_\alpha e^{2a} \cup_\alpha \cdots e^{(p-1)a}$$

How are we to prove $\langle S^0 \rangle = \langle X \rangle$?
Well, Hopkins-Smith, you have

$$X = S^0 \cup_\alpha e^a \cup_\alpha e^{2a} \cup_\alpha \cdots e^{(p-1)a}$$

$$\downarrow f$$

$$S^a X = \quad S^a \cup_\alpha e^{2a} \cup_\alpha \cdots e^{(p-1)a} \cup e^{pa}$$

andform $C_f = S^1 \cup_\beta e^{pq}$.

The map f is clearly nilpotent, in fact $f^p \simeq 0$, so $\mathrm{Tel}\, f \simeq pt$ and $\langle X \rangle = \langle C_f \rangle$ ::

but $C_f = C_\beta$ and β is nilpotent, so $\langle C_\beta \rangle = \langle S^1 \rangle$.

We can copy this. The argument above applies a cofibering at p

$$F_{pk,1} \longrightarrow F_{pk+1,-1} \longrightarrow S^{pk \cdot 2n} F_{(p-1)pk,1}$$

because the binomial coefficients [illegible] Therefore we act a map

$$F_{pk+1,-1} \xrightarrow{f} S^{pk \cdot 2n} F_{pk+1,-1}$$

We have an immediate corollary.

Theorem 6 (Hopkins) At 2

$$\langle X(n) \rangle = \langle F_{2^n - 1} \rangle \quad \forall n.$$

Proof. Since $Tel(\alpha) \simeq pt$ we have

$$\langle F_{2^n - 1} \rangle = \langle C_\alpha \rangle = \langle F_{2^{n+1} - 1} \rangle$$

and now induction on n each your way from $X(\infty) = F_0$.

Of course this argument requires some change at a prime $p > 2$. To begin with, we have to switch to cyclic p things at a the vale than 2.

It is also plausible that we can find a suitable map Y. Of course suspension theory at an odd prime ~~stopped~~ is different from at $p=2$, so we remove the reference to James & substitute a reference to Toda. We see that to construct α & Y for $p=2$ we really only needed a bit of fibering

$$J_{2^n-1}(S^{2n}) \longrightarrow J_{2^{n+1}-1}(S^{2n}) \to S^{2l} \subset \Omega S^{2l+1}$$

The smallest ring-spectrum we could use would be $S^\infty(\Omega S^{2l} \cup p t) = \bigvee_{r \geq 0} S^{r(2l-1)}$
At an odd prime the analogue would be

$$J_{p^n-1}(S^{2n}) \to J_{p^{n+1}-1}(S^{2n}) \longrightarrow J_{p-1}(S^{2l}) \subset \Omega S^{2l+1}.$$

By the work of Toda we have a fibering

$$S^{2l-1} \to \Omega J_{p-1}(S^{2l}) \to \Omega S^{p.2l-1}$$

and the obstruction to lifting $S^{p.2l-2}$ vanishes after 2 suspensions, so stably we have at least one candidate

$$S^\infty(S^{p.2l-2}) \to S^\infty \Omega J_{p-1}(S^{2l}) \to S^\infty \Omega^{2l+1}$$

Of course, we still need to check that the bottom cell of $S^\infty J_{p}(2l+1)$ at p survives $p^{n}(2l+1)$ to β.

by ... special case don't know...

so that $C_f \simeq C_\beta$ for a suitable cofibring

$S^1_{p^m}F_{p^k-1} \longrightarrow C_\beta \xrightarrow{S^1} S^{p^{k+1}2n} = F_{p^m-1}$.

The map f is clearly nilpotent – in fact $f^p \simeq 0$, so we get $\langle F_{p^{k+1}-1}\rangle = \langle C_f\rangle = \langle C_\beta\rangle$.

If we prove $Tel(\beta) \simeq pt$, then $\langle C_\beta\rangle = \langle F_{p^k-1}\rangle$, and then we can proceed by induction over m as before.

(Unwanted ... $Tel(\alpha) \simeq pt$) ←

~~Now we must look at the proof that $Tel(\alpha) \simeq pt$. So I am back to $p = 2$. First I will give a bit of usual background knowledge.~~

~~Suppose we have a fibring $F \to E \to S^{q+1}$ over a sphere and a v-bundle ξ over E. Then I get a map Thom complexes $F^\xi \to E^\xi$, where we think ξ really means $\xi|F$ the restriction of ξ to E. What is the cofibre? Well,~~

I can divide E into 2 parts,

as

§6. In this section we want to show how $H\phi h$ins acts to info. about the qualitative behaviour of $X(n \not\in l)_*$-homol. to info. about the qualitative behaviour of $(F_{p^n-1})_*$-homology. By the last section, that will carry qualitative info. about the qualitative ich. of $X(n)_{\hat{\epsilon}}$ local homot. type. h ir.

<u>Theorem 1</u>. If Y is a spectrum bdd below, then there exists an ASS to computing $\pi_*(Y)$ based on $E_*(-)$ where $E = X(n \not\in l)$

$$Ext^{**}_{E_*E}(E_*(S^0), E_*(Y)) \Rightarrow [S^0, Y]_*.$$

<u>Proof</u>. Basically this works because $E = X(n \not\in l)$ behaves so much like MU that the construction of the Novikov SS in any Chicago blue book goes over. On the LHS we have a spectrum S^0 s.t. $E_*(S^0)$ is already projective ove $\pi_*(E)$ - in fact, it's free of one

This is worst [illegible] be [illegible] bit harder than for $p = 2$, because we are dealing with secondary quadratic rather than primary ones.
After that, we have to check that $\mathrm{Sp}_{\mathrm{im}_\phi}(2\ell+1)$ (at p) approximates to H_0.
I suggest that after scanning the extensive of the machine, we accept

Thm 7 (Hopkins) At p

$$\langle X(n) \rangle = \langle F_{p^n-1} \rangle \quad \forall n.$$

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E-homology under control we can keep that under control; all we have to do is to: write the correct algebraic account of $E \times t^{xv}$ ~~so that~~ $E \times E$

copy our algebraic theory off the topology so that what we have done qualifies as a resolution, and the E_2-term qualifies as an Ext.

One must admit that the algebra does have to be ~~generalised~~ upgraded how what way be regarded as classical, but this happens already for $E = MU$ and higher or worse for $E = X(n+1)$.

calculate – so we don't need to resolve that, we only need to resolve Y. Then, if we can

$$Y = Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_s$$

we can certainly take

$$Y_{s+1} \longrightarrow Y_s = S^0 \wedge Y_s \longrightarrow E \wedge Y_s$$

& take Y_{s+1} to be the cofibre of that:

Since $\pi_0(E) = \mathbb{Z}$, the map

$$\pi_*(Y_s) \longrightarrow \pi_*(E \wedge Y_s)$$

is iso on the bottom dim where it's non-zero, so the Hurewicz dimension of Y_{s+1} is one higher than of Y_s and we have no trouble with our convergence.

The E_1-term of our spectral sequence is $\pi_*(E \wedge Y_s) = E_*(Y_s)$ and if we assume we can keep

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All right, let's show that $X(n)$ behaves like MU here:

We have a map

$$S^{-2}\mathbb{C}P^h \longrightarrow X(n) \text{ if } h \leq n,$$

and each gets to be exact if I decrease h or increase n. That says I get an orientation

$$x \in X(n)^2(\mathbb{C}P^h) \text{ if } h \leq n,$$

and each goes to the next if I decrease h or increase n.

Then the AHSS says:

$$X(n)^*(\mathbb{C}P^h) = X(n)^*(pt)[x]/x^{h+1}$$

& $X(n)_*(\mathbb{C}P^h)$ is a free module on $\pi_*X(n)$

on generators $\beta_0, \beta_1, \ldots \beta_h$

with $\langle x^i, \beta_j \rangle = \delta_{ij}$.

These generators map to the corr. if I increase h or increase n.

This gives the generators

Well, we choose to work with the coalgebra $C = E_* E = \pi_*(E \wedge E)$ rather than the algebra of operations $E^* E$, but we need the same point afterwards; $\Lambda = \pi_* E$ acts on the left of C, and also on the right of C, but the 2 actions aren't the same; one is the effect on π_* of the map

$$E \wedge (E \wedge E) \xrightarrow{\mu \wedge 1} E \wedge E$$

& the other

$$(E \wedge E) \wedge E \xrightarrow{1 \wedge \mu} E \wedge E$$

The net effect of this is that for our algebra to work we need to prove that $C = E_* E$ is free as a left module over $\Lambda = \pi_* E$ and also that it is free as a right module over $\Lambda = \pi_* E$; these are different statements although either follows easily from the other.

First of all, our coefficient ring has to be $\Lambda = \pi_*(E)$ and this isn't a field. Well, too bad, we shall just have to assume that things are projective, or flat, over Λ where we need it, and then check that they are so in the applications.

Secondly, Steenrod operations Sq^k are linear over the ground ring $\Lambda = \mathbb{F}_2$, but Novikov operations r^I aren't linear over the ground ring $\Lambda = \pi_*(MU)$; they satisfy

$$r^I(c.x) = \sum_{J+K=I} (r^J c)\cdot(r^K x)$$

and the action of r^J on $\pi_*(MU)$ is highly non-trivial.

$\pi_* E \to \pi_* MU$. But this level is zero in dimension $\le 2n$, and we only want the [illegible] surface in dim $\le 2n$, so [illegible] the same as in $MU_* MU$.

<u>Corollary 2</u>. The coproduct is given by

$$\psi b_k = \sum_{i+j=k} \left(\sum_{h \geq 0} b_h \right)_i^{j+1} \otimes b_j .$$

Eg

$$\psi b_1 = b_1 \otimes 1 + 1 \otimes b_1$$

$$\psi b_2 = b_2 \otimes 1 + 2 b_1 \otimes b_1 + 1 \otimes b_2 \text{ etc.}$$

The augmentation ε (ie the map of π_* induced by $E \wedge E \to E$) is given by $\varepsilon(b_i) = 0$ $(i > 0)$.

The canonical antiautomorphism c (ie the map of π_* induced by $E \wedge E \xrightarrow{T} E \wedge E$) maps b_n to $-b_n$ mod decomposables in $\mathbb{Z}[b_1, b_2 \ldots b_n]$

Eg

$$c\, b_1 = -b_1$$
$$c\, b_2 = -b_2 + 2b_1^2 \text{ etc.}$$

<u>Corollary 3</u>. The monomials $b_1^{i_1} b_2^{i_2} \ldots b_n^{i_n}$ provide a base for $E_* E$ not only as a left Λ-module, but as a right Λ-[illegible]

$b_i \in X(n)_{2i}\ X(n)$ if $i < k \leq n$

and again, each map to the corresponding one of I increase to ∞ in increase n. The AHSS shows

$$X(n)_* X(k) = \pi_* X(n)\,[b_1, \ldots, b_{k-1}];$$

in particular, with $n = \infty$,

$$E_* E = X(\infty)_* X(\infty)$$

$$= \Lambda\,[b_1, \ldots, b_n]$$

(as a left Λ-module), $\Lambda = \pi_* E$).

Of course we want to know formulae for the structure of $E_* E$. However, we can deduce them from the corresponding formulae for $MU_* MU$, by using the map

$$E_* E \longrightarrow MU_* MU.$$

- This determines the formulae in $E_* E$, up to coefficients in the kernel of

A comodule M over C is a quotient M with a map $M \xrightarrow{\psi} C\otimes_\Lambda M$ which has the formal properties of $E_* X$.

So ψ is a map of left Λ-modules & the following diagrams commute:

$$\begin{array}{ccc} M & & \\ \psi\downarrow & \searrow & \\ C\otimes_\Lambda M & \xrightarrow{\varepsilon\otimes 1} & \Lambda\otimes_\Lambda M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\psi} & C\otimes_\Lambda M \\ \psi\downarrow & & \downarrow\Delta\otimes 1 \\ C\otimes_\Lambda M & \xrightarrow{1\otimes\psi} & C\otimes_\Lambda(C\otimes_\Lambda M) \end{array}$$

$\Rightarrow$

An extended A-module is an A-module of the form $A\otimes_\Lambda V$ where V is a Λ-module, if V is free over Λ, then $A\otimes_\Lambda V$ is a free A-module.

Dually, an extended comodule is a comodule of the form $C\otimes_\Lambda V$ where V is a Λ-module, with the structure map $C\otimes_\Lambda V \xrightarrow{\Delta\otimes 1} C\otimes_\Lambda(C\otimes_\Lambda V)$.

although $m.\lambda \neq \lambda.m$).

Proof. On the left we have it; but ε sends the left action into the right action.

Now I can probably complete my remarks about algebras. An algebra with unit is a gadget A with maps $\Lambda \xrightarrow{\eta} A$, $A \otimes_\Lambda A \xrightarrow{\varphi} A$ satisfying the usual properties. A coalgebra is a gadget C with maps $C \xrightarrow{\varepsilon} \Lambda$, $C \xrightarrow{\psi} C \otimes_\Lambda C$ which have the formal properties of $C = E_*E$, $\Lambda = \pi_*E$. So Λ can act differently on the left & right of C; but ε & ψ commute with both actions: &

$$\begin{array}{ccccc} & & C & & \\ & \swarrow & \downarrow \psi & \searrow & \\ C\otimes_\Lambda \Lambda & \xleftarrow{1\otimes\varepsilon} & C\otimes_\Lambda C & \xrightarrow{\varepsilon\otimes 1} & \Lambda\otimes_\Lambda C \end{array}$$

$$\begin{array}{ccc} C & \xrightarrow{\psi} & C\otimes_\Lambda C \\ \downarrow \psi & & \downarrow \psi\otimes 1 \\ C\otimes_\Lambda C & \xrightarrow{1\otimes\psi} & C\otimes_\Lambda C\otimes_\Lambda C \end{array}$$

comm.

because $E_*(E)$ is flat as a right module over Λ, so $E_*(E \wedge Y_s)$ is an extended comodule. All right, we get a resolution of $E_*(Y)$ by extended comodules. When we apply $\mathrm{Hom}_C(\Lambda, \text{———})$

we get

$$\mathrm{Hom}_C(\Lambda, C \otimes_\Lambda E_*(Y_s))$$

$$= \mathrm{Hom}_\Lambda(\Lambda, E_* Y_s)$$

$$= \pi_*(E \wedge Y_s)$$

so we have succeeded in describing the E_1 term correctly.

Here I should explain that an extended comodule $C \otimes_\Lambda V$ is ~~like~~ not ~~an abs~~ an injective object in the category of comodules, you require V injective as a Λ-module, which we ~~really~~ we don't want to do. So we

We have an adjunction

$$\mathrm{Hom}_A(A \otimes_\Lambda V, M) = \mathrm{Hom}_\Lambda(V, M);$$

dually, we have

$$\mathrm{Hom}_C(L, C \otimes_\Lambda V) = \mathrm{Hom}_\Lambda(L, V).$$

We can use these ideas to describe algebraically what happens when we [illegible] our resolutions

$$Y_s = S^0 \wedge Y_s \longrightarrow E \wedge Y_s \longrightarrow S^1 Y_{s+1}.$$

When we apply E-homology we get

$$E \wedge Y_s = E \wedge S^0 \wedge Y_s \to E \wedge E \wedge Y_s \to E \wedge S^1 \wedge Y_{s+1}$$

with $\mu \wedge 1 : E \wedge E \wedge Y_s \to E \wedge Y_s$, and an arrow labelled η between the first two terms.

so we get split short exact sequences

$$0 \to E_*(Y_s) \to E_*(E \wedge Y_s) \to E_*(S^1 \wedge Y_{s+1}).$$

~~The~~ We have an isomorphism

$$E_*(E) \otimes_\Lambda E_*(Y_s) \xrightarrow{\cong} E_*(E \wedge Y_s)$$

Put crudely, if you want to go to an Adams ad. comput. $[S^0, Y]_*$ by using the homology theory π_*, it doesn't matter whether you write the E_2-term

$$\mathrm{Hom}_{\pi_*(S^0)}(\pi_*(S^0), \pi_*(Y))$$

or $$\mathrm{Hom}_{Z}(Z, \pi_*(Y)).$$

Proof. It is implicit in the statement that $B = Z[b_1, \ldots, b_n]$ is a sub-coalgebra of $E_* E$ ($E = X(n+1)$) but we use Z instead of $\Lambda = \pi_*(E)$; this is clear from Coroll. 2. N.B.: this is the thing you can do for MU & $X(n+1)$ that you can't do for BP.

It's also implicit in the statement that $E_*(Y)$ can be considered as a comodule over $B = Z[b_1, \ldots, b_n]$;

use a resolution using relative injective values. [illegible] absolute ones [illegible] However, relative injectives of [illegible] algebra comodules are good enough to compute $Ext(L, -)$ correctly provided the only L's you use are projective over Λ, and we only want $L = \Lambda$, so that's OK.

That more or less justifies Th 1.

As we decrease n, the theory

$E_* = X(n+1)_*$ keeps more and more of its information in its ring of coefficients $\Lambda = \pi_* E$, and less and less in any sort of operation other than multiplication by scalars $\lambda \in \pi_* E$.

I claim, however, that the first sort of information can be chased out of ~~[illegible]~~ everywhere except $E_*(Y)$, where it's essential. In Thm 1, [illegible] The E_2-term of the SS may be written

$$\left(Ext^{*e}_{Z[b_1,\dots,b_n]}(Z, E_*(Y))\right) \Rightarrow [S^0, Y]_*$$

Next I claim that any extended [illegible] C-comodule $C\otimes_\Lambda V$ can be considered as an extended B-comodule. For this, of course, I have to assume V is a left Λ-module. The proof is obvious: I have a map $B\otimes_Z V \longrightarrow C\otimes_\Lambda V$; and it is iso a before: all I have to do is check that this iso. Moreover the structure map for the one and the structure map for the other. Well,

$$\begin{array}{ccccc}
B\otimes_Z V & \xrightarrow{\ \cong\ } & C\otimes_\Lambda V & & \\
\downarrow{\scriptstyle \psi_B\otimes 1} & & \vdots & \searrow{\scriptstyle \psi_C\otimes 1} & \\
B\otimes_Z B\otimes_Z V & \xrightarrow[\cong]{} & B\otimes_Z(C\otimes_\Lambda V) & \xrightarrow[\cong]{} & C\otimes_\Lambda C\otimes_\Lambda V
\end{array}$$

All right, [illegible]

resolutions we use are C are legitimate resolutions over B.

in fact, any module over $C = E_* E$ can be so considered. ~~In fact, we~~ More generally, to any such comodule M we have a map

$$M \xrightarrow{\psi} C \otimes_\Lambda M$$

$$\psi' : M \to \underbrace{\mathbb{Z}[b_1, \dots, b_n]}_{B} \otimes_{\mathbb{Z}} M \xrightarrow{\cong} C \otimes_\Lambda M$$

and this is iso by Coroll. 3. So I get ψ'. I just have to check that this satisfies the axioms.

Well,

$$M \xrightarrow{\psi} C \otimes_\Lambda M \xrightarrow{\varepsilon \otimes 1} \Lambda \otimes_\Lambda M, \quad \text{composite} = \text{identity}$$

$$\underbrace{\mathbb{Z}[b_1, \dots, b_n]}_{B} \otimes M \xrightarrow{\cong} C \otimes_\Lambda M$$

$$\underbrace{\mathbb{Z}[b_1, \dots, b_n]}_{B} \otimes M \xrightarrow{\varepsilon \otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} M \xrightarrow{\cong} \Lambda \otimes_\Lambda M \quad \checkmark \text{ Coroll. 2}$$

Also

$$M \xrightarrow{\psi'} B \otimes_{\mathbb{Z}} M, \qquad M \xrightarrow{\psi} C \otimes_\Lambda M, \qquad B \otimes_{\mathbb{Z}} M \xrightarrow{\cong} C \otimes_\Lambda M$$

$$M \xrightarrow{\psi} C \otimes_\Lambda M, \qquad C \otimes_\Lambda M \xrightarrow{1 \otimes \psi} C \otimes_\Lambda C \otimes_\Lambda M, \qquad B \otimes_{\mathbb{Z}} M \xrightarrow{1 \otimes \psi} B \otimes_{\mathbb{Z}} (C \otimes_\Lambda M), \qquad B \otimes_{\mathbb{Z}} M \xrightarrow{1 \otimes \psi'} B \otimes_{\mathbb{Z}} (B \otimes_{\mathbb{Z}} M)$$

$$M \xrightarrow{\psi'} B \otimes_{\mathbb{Z}} M, \qquad B \otimes_{\mathbb{Z}} M \xrightarrow{\cong} C \otimes_\Lambda M$$

$$C \otimes_\Lambda M \xrightarrow{\psi \otimes 1} C \otimes_\Lambda C \otimes_\Lambda M, \qquad B \otimes_{\mathbb{Z}} (C \otimes_\Lambda M) \xrightarrow{\cong} C \otimes_\Lambda C \otimes_\Lambda M, \qquad B \otimes_{\mathbb{Z}} (B \otimes_{\mathbb{Z}} M) \xrightarrow{\cong} B \otimes_{\mathbb{Z}} (C \otimes_\Lambda M)$$

$$B \otimes_{\mathbb{Z}} M \xrightarrow{\psi_B \otimes 1} B \otimes_{\mathbb{Z}} (B \otimes_{\mathbb{Z}} M)$$

Hopkins' idea is to compute the Ext by a change-of-rings theorem. Actually this means the theorem for Ext over coalgebras corresponding to the change-of-rings theorem for Tor over algebras, so I had better explain about the classical case first.

Suppose I have an injection of rings $S \to R$ with R free as a right module over S. If L is a S-module, M an R-module I have

$$\mathrm{Ext}^*_R(R\otimes_S L, M) \cong \mathrm{Ext}^*_S(L, i^*M)$$

for the following reason. Take a free resolution of L over S, say

$$\cdots C_q \longrightarrow C_{q-1} \cdots \longrightarrow C_0 \longrightarrow L \to 0$$

Apply $R\otimes_S -$; since R is free as a right module over S, this preserves exactness, & carries free S-modules to free R-modules. I get

$$\cdots \to R\otimes_S C_q \longrightarrow R\otimes_S C_{q-1} \cdots \to R\otimes_S C_0 \to R\otimes_S L \to 0$$

and this is a free resolution of $R\otimes_S L$ over R. I can calculate $\mathrm{Ext}^*_R(R\otimes_S L, M)$ by applying $\mathrm{Hom}_R(-, M)$ and then taking cohomology.

$$\cdots \leftarrow \mathrm{Hom}_R(R\otimes_S C_q, M) \longleftarrow \mathrm{Hom}_R(R\otimes_S C_{q-1}, M) \leftarrow \cdots$$

$$\cdots \leftarrow \mathrm{Hom}_S(C_q, i^*M) \longleftarrow \mathrm{Hom}_S(C_{q-1}, i^*M) \leftarrow \cdots$$

(the two rows are equal term by term)

It just remains to remark that

$$\mathrm{Hom}_C(\Lambda, M) = \mathrm{Prim}(M)$$
$$(= \{m \in M \mid \psi m = 1 \otimes m\})$$
$$= \mathrm{Hom}_B(Z, M).$$

Thus we conclude

$$\mathrm{Ext}_C^{**}(\Lambda, M) = \mathrm{Ext}_B^{**}(Z, M).$$

This enables us to get rid of the nuisance that Λ is difficult on the Z side of C; Z acts the same on both sides of

$$B = Z[b_1, b_2, \dots, b_n],$$

which makes us feel a bit more secure.

<u>Theorem 5.</u> Suppose we work at p and in Thm 1 take

$$Y = F_{p^h - 1} \wedge W$$
$$(c \times (n+1))$$

where $\pi_q(W) = 0$ for $q < c$. Then the ASS has $\mathrm{Ext}^{s,t}(-, -) = 0$ for $t < p^h \cdot 2n \cdot s + c$.

In particular, B is a right comodule over C, so I can form

$$0 \to B \square_C M \longrightarrow B \otimes M \underset{1 \otimes \psi}{\overset{\psi \otimes 1}{\rightrightarrows}} B \otimes C \otimes M.$$

Working with modules, I know that $R \otimes_S M$ is a left R-module; so working with comodules, I want to say that $B \square_C M$ is a left comodule over B. Well, I need some assumptions to get through the diagram-chasing; I need to assume B is flat over the ground ring $\Lambda (= \mathbb{Z}/(p))$. Well, that's no application at all.

Lemma 6 (Change of rings) Suppose B is extended as a right C-comodule, i.e. is a $V \otimes C$, where V projective over $\Lambda (= \mathbb{Z})$ and L projective are $\Lambda (= \mathbb{Z})$; then

$$\mathrm{Ext}_B^{**}(L, B \square_C M) \cong \mathrm{Ext}_C^{**}(j_* L, M).$$

Proof Well, first we have to check the ~~Hom~~, that's the adjunction

$$\mathrm{Hom}_B(L, B \square_C M) \cong \mathrm{Hom}_C(j_* L, M).$$

This goes just like the adjunction / Frobenius, but with all the arrows turned round.

Then the proof is going to go just like that for the change-of-rings thm. We take a resolution $C_0 \to C_{n-1} \to \cdots \to C_0 \to M$

So the cohomology I want is $\mathrm{Ext}_S^*(L, i^*M)$

Now we just have to dualize this. Instead of an injection of rings $S \xrightarrow{i} R$,

I take a surjection of coalgebras $B \xrightarrow{j} C$, which in the applications becomes

$$B = \mathbb{Z}[b_1, \dots, b_n] \longrightarrow \frac{\mathbb{Z}[b_1, \dots b_n]}{(b_1, \dots b_{n-1})} = \mathbb{Z}[b_n] = C.$$

In C we have $\psi b_n = b_n \otimes 1 + 1 \otimes b_n$; all the intermediate terms have gone.

Working with modules over S, the tensor product $L \otimes_S M$ is defined by

$$L \otimes S \otimes M \overset{\varphi \otimes 1}{\underset{1 \otimes \varphi}{\rightrightarrows}} L \otimes M \longrightarrow L \otimes_S M \to 0.$$

Working with comodules over C, the cotensor product $L \square_C M$ is defined by

$$0 \to L \square_C M \longrightarrow L \otimes M \overset{\psi \otimes 1}{\underset{1 \otimes \psi}{\rightrightarrows}} L \otimes C \otimes M$$

Corollary 7. In the ASS of Thms 1, 5 we have

$$Ext^{s,t}_{B}(Z, E_x(F_{p^n-1}, nW)) \quad (B = Z[b_1 \cdots b_n]) \;\cong\; Ext_C(Z, Z\{1, b_n, \ldots b_n^{p^n-1}\} \otimes E_x[W])$$

where $C = Z[b_n]$, $Z\{1, b_n, \ldots b_n^{p^n-1}\} \subset C$ is a left comodule in the obvious way, and the tensor product of left C-comodules becomes a C-comodule in the obvious way using the fact that C is a Hopf algebra.

of M by extended comodules over C.
We want to apply $B\square_C -$ to a resolution

$$B\square_C C_a \longrightarrow B\square_C C_{a-1} \to \cdots \quad B\square_C M \to$$

So first we want to check that $B\square_C -$ carries extended C-comodules to extended B-comodules. That's OK,

$$B\square_C (C\otimes W) \cong B\otimes W$$

preserving the structure. Then we want to check that $B\square_C -$ preserves exactness. For this we assume B is extended; with $B \cong V\otimes C$ we find an additive iso

$$B\square_C M \cong V\otimes M$$

natural in M, which preserves exactness.

With these preliminaries, the classical proof rolls with all its arrows turned round.

~~Wetting in the application ... to~~

~~$E_*(Y) = E_*(F_{p^n} \wedge Z)$.~~

Here B_n = [illegible] ... as the ones used above,

($E_*(F_{p^n-1})$ is ... a) free module over $\Lambda[b_1 \cdots b_{n-1}]$

on generators $1, b_n, b_n^2, \ldots, b_n^{p^n-1}$. In particular, it is free as a left Λ-module; so we have ~~[illegible]~~ [illegible]

$$E_*(Y) = E_*(F_{p^n-1} \wedge W) \leftarrow E_*(F_{p^n-1}) \otimes_\Lambda E_*(W)$$

where Λ acts on the left of the left [illegible].
This says

$$E_*(Y) \cong Z[b_1, \ldots, b_{n-1}]\{1, b_n, \ldots b_n^{p^n-1}\} \otimes_Z E_*(W)$$

and this is an iso. of coalgebras over $B = Z[b_1 \cdots b_n]$ provided the coproduct [illegible] you put in the [illegible] product in the [illegible] [illegible] [illegible] that B is a Hopf algebra).

In order to get a map

$$E_*(Y) \xrightarrow{f} B \,\square_C\, M$$

of B-comodules, it is sufficient by the adjunction [illegible] ... of [illegible]

$$E_*(Y) \xrightarrow{f} M.$$

Proof.

In order to apply lemma (i), I have to check just two things. The first is that

$$B = \mathbb{Z}[b_1, b_2 \ldots b_n]$$

is extended as a right comodule over

$C = \mathbb{Z}[b_n]$, to a V projective over $\mathbb{Z}$.

Well, I have to exhibit a ~~coexhibit~~ map $B \longrightarrow V$, and I exhibit

$$B = \mathbb{Z}[b_1, b_2 \ldots b_n] \longrightarrow V = \mathbb{Z}[b_1, b_2 \ldots b_{n-1}]$$

the map $b_n \longrightarrow 0$. This corresponds to a map of comodules $B \longrightarrow V \otimes C$, namely

$$B \xrightarrow{\psi} B \otimes B \xrightarrow{1 \otimes j} B \otimes C \longrightarrow V \otimes C$$

I have to check this is iso. (Well, it carries $b_1^{i_1} b_2^{i_2} \ldots b_{n-1}^{i_{n-1}} b_n^{i_n}$ to $b_1^{i_1} b_2^{i_2} \ldots b_{n-1}^{i_{n-1}} \otimes b_n^{i_n}$, so it's iso.

The other thing I have to do is to check that the comodule $\mathrm{Ext}_{\Gamma}(\mathbb{F}_p, W)$ with appropriate ... $B \square_C M$ ... am M.

his. ass. $\mathbb{Z}$-module. By scalars again I can reduce to the case $N = \mathbb{Z}$.

Now we want to use the fact that B is Galois, $B \cong V \otimes C$ in the way I have explained. In this case $(V \otimes C) \square_C M$, which is defined as a subobject of $V \otimes C \otimes M$, can be identified with $V \otimes M$ under the maps

$$V \otimes M \xrightarrow{1 \otimes \psi} V \otimes C \otimes M \xrightarrow{1 \otimes \varepsilon \otimes 1} V \otimes M, \qquad \text{composite } 1.$$

So what we have to do is have

$$\mathbb{Z}[b_1 \ldots b_{n-1}]\{1, b_n, \ldots, b_n^{p^h-1}\} \xrightarrow{\bar{f}} B \square_C M$$

$$\downarrow \psi$$

$$B \otimes \mathbb{Z}[b_1 \ldots b_{n-1}]\{1, b_n \ldots b_n^{p^h-1}\} \xrightarrow{1 \otimes f} B \otimes M$$

$$B \square_C M \to B \otimes M, \qquad B \square_C M \to (V \otimes C) \square_C M \to V \otimes C \otimes M \xrightarrow{1 \otimes \varepsilon \otimes 1} V \otimes M$$

$$B \otimes M \to V \otimes C \otimes M, \qquad B \otimes M \xrightarrow{q \otimes 1} V \otimes M$$

$$B \otimes \mathbb{Z}[b_1 \ldots b_{n-1}]\{1, b_n \ldots b_n^{p^h-1}\} \xrightarrow{q \otimes f} \mathbb{Z}[b_1 \ldots b_{n-1}] \otimes \mathbb{Z}\{1, \ldots\} = V \otimes M$$

This is iso.

Now I would like to make an argument which works if you replace $E_*(W)$ by any comodule N over $B = Z[b_1, \ldots b_n]$, provided N is bdd below. I claim we have a natural isomorphism

$$Z[b_1 \ldots b_{n-1}]\{1, b_n, \ldots b_n^{p^k-1}\} \otimes_Z N$$

$$\downarrow \bar{f}$$

$$B \square_C (Z\{1, b_n, \ldots b_n^{p^k-1}\} \otimes_Z N).$$

Well, this is clear; to make a map of B-comodules into $B \square_C$ — I just have to make a map of C-comodules

$$Z[b_1, \ldots b_{n-1}]\{1\, b_n \ldots b_n^{p^k-1}\} \otimes_Z N$$

$$\downarrow f$$

$$Z\{1, b_n, \ldots b_n^{p^k-1}\} \otimes_Z N$$

and I take the obvious map of N tensored with the appropriate map $b_i \mapsto 0$ for $0 < i < n$.

I want to prove the natural map is iso.

Both sides of the natural transformation preserve exactness, so I can filter N by dimension and reduce to the case in which N has the trivial structure as B-comodule. Both sides preserve direct limits, so I can reduce to the case in which N is a

Work over $Z_{(p)}$. Let

<u>Lemma 9</u>. Let $C = Z_{(p)}[x]$ be the Hopf algebra

on one primitive generator x of degree $2d$

$$\left(\psi x^h = \sum_{i+j=h} \frac{h!}{i!\,j!} x^i \otimes x^j\right).$$

Let $L = Z_{(p)}\{1, x, x^2, \ldots x^{p^k - 1}\}$, considered as a comodule over C. Let M be another comodule over C, zero in degrees $< m$. Then

$$\mathrm{Ext}_C^{s,t}(Z_{(p)}, L \otimes M) = 0$$

for $t < m + 2dp^k s$.

i) <u>Proof</u>. We have to consider successive cases.

First: suppose $pM = 0$, so that M is a module over F_p. ~~and M is finite dimensional over F_p in each degree.~~ Then, since any V with $pV = 0$ ... then the extended comodule ... $Z_{(p)}[x] \otimes V$ is the same as $F_p[x] \otimes V$;

so a resolution constructed within over F_p qualifies as one constructed over $Z_{(p)}$, and we get

$$\mathrm{Ext}^{s,t}_{Z_{(p)}[x]}(Z_{(p)}, L \otimes M) \cong \mathrm{Ext}^{s,t}_{F_p[x]}(F_p, L \otimes M).$$

Now we introduce the assumption that M is finite-dimensional over F_p in each degree.