Artin

Measure and Density

1. The planar density of a metal plate.

The mass of a flat metal plate of uniform thickness and uniform composition is proportional to its area, and the *planar density* of the plate is the ratio $\delta = (mass)/(area)$. So

$$(mass) \; = \; \delta \times (area).$$

But if the composition of the plate varies from point to point, and if we define a planar density for the plate, it will also vary.

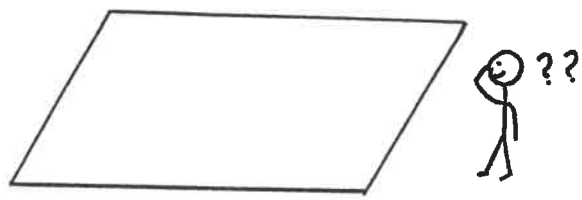


Figure 1

We may pose two problems: To define a density for a plate that is not uniform, and to use this density to compute the mass of the plate.

These problems are closely related to differentiation and integration in one-variable calculus, as is illustrated by the relationship between speed and distance. Speed is the derivative of distance, and the Fundamental Theorem of Calculus tells us that the distance travelled can be computed by integrating speed.

As we know, the speed at a time t is approximated by the ratio $\Delta D/\Delta t$, where Δt is the length of a small time interval containing t and ΔD is the distance travelled. The speed ($\frac{dD}{dt}$) is the limit of this fraction, as the interval shrinks to the point t.

We can make a similar analysis to define a planar density for the plate. We start with a small region ΔR of the plane. Let ΔA be the area of ΔR, and let ΔM the mass of that part of the plate which lies over the region ΔR. The fraction

$$\frac{\Delta M}{\Delta A} = \frac{\Delta mass}{\Delta area}$$

represents an average density of the plate over the region. By definition, the *planar density* δ of the plate at a point P is the limit of this fraction, the limit being taken as the region ΔR shrinks to the point.

When the composition of the plate varies from point to point, the planar density δ becomes a function on the plane (or on the region of the plane occupied by the plate). By analogy with distance and speed in one-variable calculus, you may guess that the mass of the plate can be computed as the double integral of the density:

$$(mass) = \iint \delta \, dA.$$

This is true, and the same principle can be applied in other situations.

2. Measures.

To explain density precisely, we need the concept of a measure, and we'll begin with the *area measure*. This measure associates to a region R of the plane its area $A(R)$.

Figure 1: The area measure

One way to think of the area measure is as a black box: If a description of a region R is inserted into the box, it prints out the area $A(R)$.

Because the area of a region R is the double integral $\iint_R 1\,dxdy$, we can also write the area measure symbolically, as the operator

$$A(\ ) = \iint_{(\ )} dxdy.$$

If a region R is inserted into the space provided, the integral can be evaluated. This is what our black box is supposed to do.

The general concept of a measure is defined similarly: A *measure* μ associates to any region R of the plane a number $\mu(R)$, the μ*-measure* of the region. We can think of such a measure as a different black box. If a description of a region is inserted into this μ-box, it prints out the number $\mu(R)$.

So a measure is a bit more complicated than a function, because we are allowed to evaluate it on any *region* R. (But we restrict ourselves to regions that are bounded by piecewise smooth curves.)

Measures are required to be **additive**. This means that if a region R is decomposed into two regions R_1, R_2, then

$$\mu(R) = \mu(R_1) + \mu(R_2).$$

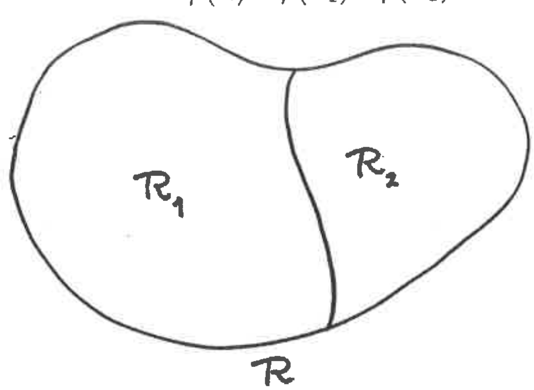


Figure 2: The additive property of a measure

For our second example of a measure, we go back to the plate. The *mass measure* $M(R)$ of a region R is the mass of that portion of the plate that lies above R.

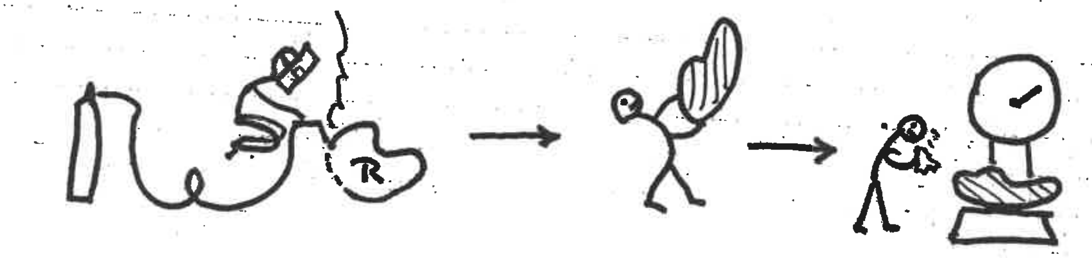


Figure 3 The mass measure operator

One more example: The *double integral* of a continuous function $f(x,y)$ is a measure. Let's write this measure as I. Thus by definition, the I-measure of a region R is the double integral

$$I(R) = \iint_R f(x,y)dxdy.$$

Notice that the additive property $\mu(R) = \mu(R_1) + \mu(R_2)$ is true for the examples we have given.

3. The density of a measure.

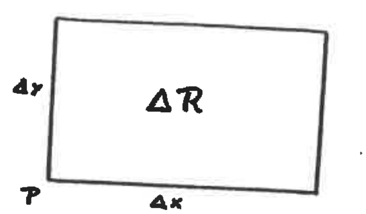
When we are given a measure μ, wehope to express it as the integral of a "density" function δ:

$$\mu(\ \ ) = \iint_{(\ \ )} \delta\, dA.$$

To do so we have to define the density. In analogy with the planar density of a metal plate, we define the *(planar) density* $\frac{d\mu}{dA}$ of a measure at a point P to be the limiting value of

$$\frac{\Delta\mu}{\Delta A} = \frac{\Delta measure}{\Delta area}.$$

The fraction is computed in this way: We choose a small rectangular region ΔR which has P as its "south-west" corner.



Then $\Delta\mu$ is its μ-measure and ΔA is its area (which is equal to $\Delta x \Delta y$). The limit is taken as ΔR shrinks to the point P.

It would also be correct to refer to the planar density as the *derivative of* μ *with respect to area.*

Since we will be discussing measures in the plane, we will to drop the adjective planar from now on.

The limit (and hence the density) may or may not exist. If it exists, the measure is said to be *differentiable.* Then because it depends on the chosen point, the density $\frac{d\mu}{dA}$ is a function on the plane.

Remark: What shape one takes for the small regions ΔR is not important. For all reasonable measures, the limit obtained using rectangles will be the same as the limit obtained with circles centered at P. We use rectangles in the definition because they are easy to piece together, and because rectangular grids are used in the definition of the double integral.

Also, it doesn't matter exactly where the rectangle ΔR is placed, so long as the point P is somewhere in the rectangle. Rectangles centered at P would work just as well.

We'll now describe the densities of a few interesting measures.

The double integral:
As in the previous section, f is a continuous function, and the double integral measure is $I(R) = \iint_R f(x,y)dA$. To determine its density at a point P_0, we let ΔR be a small rectangle with P_0 at its southwest corner. Being continuous, the function f will be approximately constant on ΔR, and therefore $\Delta I = \iint_{\Delta R} f dA$ is approximately equal to the product $f(x_0, y_0)\Delta A$ the value of the function at the point P_0 with the area of ΔR. Then $\frac{\Delta I}{\Delta A} \approx f(x_0, y_0)$, and this approximation becomes an equality in the limit. This shows that the density $\frac{dI}{dA}$ of the integral measure is the given function f.

A volume measure:
We are given the graph $w = f(x,y)$ of a continuous function. Then for a region R of the plane, we let $\text{v}(R)$ denote the volume of the three-dimensional region which is bounded above by the graph, below by the plane $z = 0$ and which lies over R.

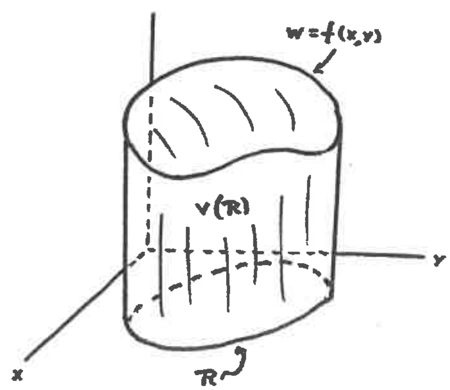


Figure 4: A volume measure

We have already learned that this volume is the double integral

$$\text{v}(R) = \iint_R f\, dA.$$

So the density of the volume measure is the given function f.

The area measure A:
The density of the area measure A is $\frac{dA}{dA} = 1$, because it is approximated by the fraction $\Delta A/\Delta A$, which is always 1.

Rainfall:
The rainfall measure $\text{r}(R)$ measures the total amount of rain that falls onto a region R of the globe in a given time period.

Say that we measure total rainfall in *inches*^{3}, and area in *inches*^{2}. To find the density of this measure, our definition tells us to choose a small region ΔR, and to compute $\Delta \mathbf{r}/\Delta A$. I like to imagine setting a rectangular bucket over the point P to collect the rain. So ΔR is the region covered by the bottom of the bucket, ΔA is its area, and and $\Delta \mathbf{r}$ is the volume of rain that accumulates. Then $\Delta \mathbf{r}/\Delta A$ is equal to the depth, in inches, of the water in the bucket. The density of total rainfall can be measured in inches of depth. This is the number "inches of rain" that one hears on a weather report.

Population:
The population measure $\mathbf{p}(R)$ measures total population in a region R of the globe.

Because people are not distributed continuously on the globe, the population measure is only approximately differentiable. Its approximate density is what is called density of population, and is often measured in "people per square mile".

Surface area:
We are given the graph $w = f(x, y)$ of a function. For a region R of the plane $\mathbf{s}(R)$ is the surface area of that part of the graph that lies over R.

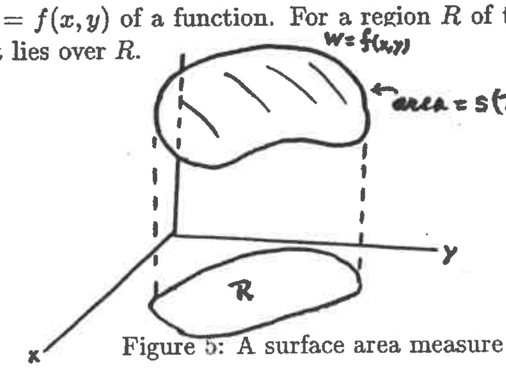


Figure 5: A surface area measure

???its density here???

We mention in passing that there are important measures which are not differentiable, such as the *delta function* δ_P of a point P. (*??? I'm unable to define the delta function.*)

4. The Fundamental Theorem of Calculus for Measures.

The fundamental theorem theorem tells us that we can recover a measure from its density.

Theorem. *(Fundamental Theorem of Calculus for Measures)*
A measure μ is the integral of its density:

$$\mu(R) = \iint_R \frac{d\mu}{dA}\, dA,$$

provided that it has the properties (a) and (b) which are stated below.

The properties referred to in the theorem are

(xxx)(a) μ is uniformly differentiable, and its density is a continuous function on the plane, and

(b) μ has the approximation property.
We know the meaning of the first condition, except for the word *uniformly* that appears in it. This is a technical requirement that we needn't worry about. It means that, in a bounded region, the approximation $\frac{\Delta\mu}{\Delta A}$ approaches the limit at a uniform rate.

We still have to explain what we mean by the approximation property:

In computing a double integral over a region R, the first step is to choose a fine rectangular grid in the plane, and to approximate the region by a region R' made up of rectangles from the grid. We'll call R' a *grid region.* Then

$$\iint_{R'} f\, dA \approx \iint_R f\, dA\,,.$$

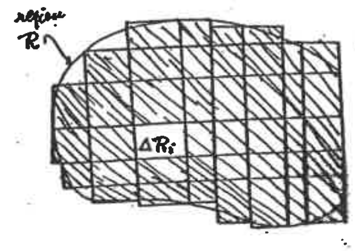


Figure 6

We see then that if a measure is a double integral, it must be true that

(xxx) $$\mu(R') \approx \mu(R),$$

whenever R' is a grid region approximating R. This is what we call the *approximation property* of a measure.

As is the case for the Fundamental Theorem of Calculus in one variable, the theorem can be proved by going carefully over the definition of the double integral.

Here is one important consequence of the theorem: The values that a differentiable measure μ takes on regions of some particular shape, for example on discs, determines the measure completely, provided (a) and (b) are true. In other words, if μ_1 and μ_2 are two such measures, and if $\mu_1(R) = \mu_2(R)$ whenever R is a disc, then $\mu_1(R) = \mu_2(R)$ for every region R. (We say then that the measures themselves are equal.) The reason is that these measures can be expressed as the double integrals of their density functions, say $\mu_i(R) = \iint_R \delta_i dA$. Two continuous functions which have the same integral over *every disc* are equal. So $\delta_1 = \delta_2$ and $\mu_1 = \mu_2$.

Another way to see the same thing is this: If we know the measure on discs, we can compute its density by passing to the limit over discs. Then we can recover the μ-measure of R as the integral over R of the density.

Example: An oracle provides us with the formula $\mu(R) = xya^2$ when R is a disc of radius a about the point $P = (x, y)$. In return for this information, we must compute the value of $\mu(R)$ when R is a rectangle, or we are doomed.

To compute $\frac{d\mu}{dA}$ at the point P, we take for ΔR a disc of small radius r about P. Then $\frac{\Delta\mu}{\Delta A} = xyr^2/\pi r^2 = xy/\pi$. This does not depend on r, so it is equal to its limiting value: $\frac{d\mu}{dA} = xy/\pi$. The Fundamental Theorem asserts that $\mu(R) = \frac{1}{\pi}\iint_R xy\, dxdy$ for all regions R, including rectangles.

However, our oracle is mischievous. Sometimes it supplies us with a bad formula – one which can't be recovered as a double integral. Not all formulas give measures, even if they seem to allow a density to be computed. (The usual problem is that the additive property fails).

To be on the safe side, and also to check our work, we had better evaluate the integral $\frac{1}{\pi}\iint xy\, dxdy$ on a disc, to verify that it returns the formula that we were given. To do this, we shift coordinates to a point $P_0 = (x_0, y_0)$ and change to polar coordinates: $x + x_0 = r\, cos\theta$, $y + y_0 = r\, sin\theta$. The integral is

$$\iint = \frac{1}{\pi}\int_0^a \int_0^{2\pi} (r\, cos\theta - x_0)(r\, sin\theta - y_0)\, r\, d\theta\, dr.$$

Only one term remains after integrating $d\theta$, and

$$\iint = 2\int_0^a x_0 y_0 r dr = x_0 y_0 a^2, \qquad \text{which checks.}$$

Example: We are given the formula $\phi(R) = ab(x + \frac{1}{2}a)y$ for a rectangle R with southwest corner (x, y) and side lengths a, b, and we are asked to decide if this formula defines a measure. Let ΔR be the rectangle with side lengths Δx and Δy and corner (x, y). Then $\frac{\Delta\phi}{\Delta A} = (x + \Delta x)y$, which approaches xy in the limit. So the density of the purported measure is $\delta = xy$. We now compute the double integral $\iint_R xy\, dA$, replacing the variable point (x, y) by (x_0, y_0):

$$\int_{x_0}^{x_0+a} \int_{y_0}^{y_0+b} xy\, dy dx = (ax_0 + \tfrac{1}{2}a^2)(by_0 + \tfrac{1}{2}b^2).$$

So $\phi(R)$ is not recovered by integration, and this shows that ϕ does not define a measure. The measure with density xy is given by the formula $\phi(R) = ab(x + \frac{1}{2}a)(y + \frac{1}{2}b)$ on a rectangle.

4. The Circulation Measure, and Green's Theorem.

The most intriguing measures, apart from physical examples, are obtained by path integration. The *circulation measure* **c** associated to a differential $M\, dx + N\, dy$ is defined this way: Let C denote the boundary curve of a region R, oriented in the counterclockwise direction.

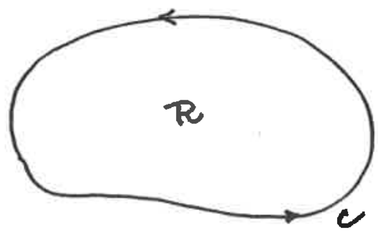


Figure 7

Then the circulation measure of R is the path integral

(xxx) $$\mathbf{c}(R) = \int_C M dx + N dy.$$

This may seem to be a bizarre way to define a measure, because it takes nothing about the region R into account except for its boundary. But it turns out to be differentiable.

We'll first verify that the circulation measure is additive. So we suppose that R is composed of two regions, as in the next figure.

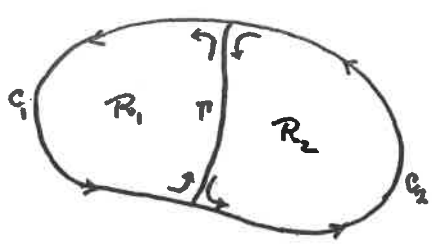


Figure 8: The additive property of the circulation measure

The oriented boundary curves C_1, C_2 of R_1 and R_2 have a common part Γ. But notice that the two boundary curves traverse Γ in opposite directions. Therefore those parts of the two path integrals cancel out, and this shows that

$$\int_C = \int_{C_1} + \int_{C_2}.$$

Theorem xxx. *The circulation measure* **c** *is differentiable, its density is the function*

$$\frac{d\mathbf{c}}{dA} = \frac{dN}{dx} - \frac{dM}{dy},$$

and it has properties xxx(a),(b).

Combining this theorem with the Fundamental Theorem of Calculus for Measures gives us Green's Theorem:

Green's Theorem xxx. *For any region R in the plane, with boundary curve C,*

$$\int_C M dx + N dy = \iint_R (\tfrac{dN}{dx} - \tfrac{dM}{dy}) dx dy.$$

We'll now show the reasoning behind Theorem xxx. First, we'll compute the density of the circulation measure. The two integrands $M\,dx$ and $N\,dy$ can be treated independently. We'll look at $\int_C N\,dy$. Let ΔR be a small rectangle with side lengths Δx and Δy, and with vertices and boundary labelled as in the diagram below:

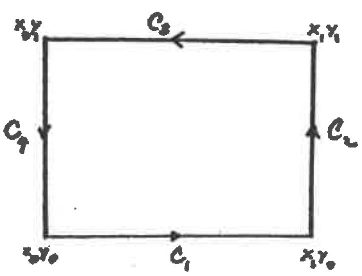


Figure 9

By definition of the path integral, $\int_{C_2} N dy$ is approximately equal to $N(x_1, y_0)\Delta y$. Similarly, $\int_{C_4} N\,dy \approx -N(x_0, y_0)\Delta y$, the minus sign resulting from the fact that C_4 is oriented downwards. On the other hand, $\int_{C_1} N\,dy$ and $\int_{C_3} N\,dy$ are zero because y is constant on the segments C_1 and C_3. So

$$\Delta \mathbf{c} = \int_C N\,dy \approx N(x_1, y_0)\Delta y - N(x_0, y_0)\Delta y.$$

Then because $\Delta A = \Delta x \Delta y$,

$$\frac{\Delta \mathbf{c}}{\Delta A} \approx \frac{N(x_1, y_0) - N(x_0, y_0)}{\Delta x}.$$

As Δx and Δy shrink to zero, this fraction approaches $\frac{dN}{dx}$, as the theorem asserts. Similar reasoning shows that the density of the circulation measure $\int_C M\,dx$ is $-\frac{dM}{dy}$.

The other property that we wish to consider is the approximation property. The circulation measure is the only measure that we have seen for which the approximation property is not intuitively clear. And the fact that it has the approximation property depends on the particular form of the line integral that appears. It is not true for other integrals. For instance, an attempt to define the a *length* measure of a region R to be the length $|C|$ of its boundary C will fail: Neither the approximation property nor the additive property are true for the length of the boundary.

To verify the approximation property for the circulation measure, one must show that, when the boundary curve C is replaced by the boundary of a grid approximation, the line integral over C' is very close to the integral over C. The proof of this follows rather simply from two facts: (i) N remains almost constant over very small regions, and (ii) when N is constant the line integral is path independent.

Exercises

1. A metal plate of uniform thickness and uniform composition lies in partial shade, and its temperature distribution is uneven. For a region R covered by the plate, let $H(R)$ denote the heat required to raise the temperature of the part of the plate over R to an even 100^oC. Give a physical interpretation of the density function $\frac{dH}{dA}$.

2. A pond of uniform depth contains salty water, but the concentration of salt is not uniform. For a region R of the bottom of the pond, let $S(R)$ denote the total amount of salt in the patch of water which lies over over R. Describe the meaning of the density function $\frac{dS}{dA}$.

3. Let R denote the rectangle with southwest corner (x, y) and side lengths a, b. Which of the following formulas define differentiable measures? If they do, find the density.
(a) $\alpha(R) = ab(2x + a)$, (b) $\beta(R) = abxy + \frac{1}{4}a^2b^2$, (c) $\gamma(R) = abe^y$.

4. Let R be a disk of radius c about the point (x, y). Which of the following formulas define differentiable measures?
(a) $\alpha(R) = xy^2c^2$, (b) $\beta(R) = (x + y)c^2$, (c) $\gamma(R) = \pi(x^2 + c^2/4)yc^2$.

5. The measure of a "polar coordinate rectangle" from r to $r + a$ and from θ to $\theta + \alpha$ is given to be $\frac{\alpha a}{r(r+a)}$. Express this measure as a double integral in polar coordinates.

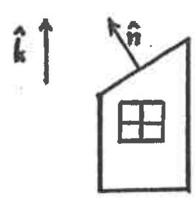
6. Show that the moment of inertia of a region R about the y-axis is given by a measure, and determine its density.

7. *(one-dimensional measures)* The concept of measure has an analogue for the one-dimensional space – the real line. A region in the line is simply an interval $[a, b]$, or a finite union of intervals. The *length measure* L on the line is analogous to the area measure in the plane: If R is the interval $[a, b]$, then $L(R) = b - a$ is the length of R. The derivative $\frac{d\mu}{dL}$ of a one dimensional measure μ is the limit as $\Delta R \longrightarrow 0$, of $\frac{\Delta\mu}{\Delta L}$, where ΔR is a small interval, ΔL is its length, and $\Delta\mu = \mu(\Delta R)$.

Given a function $f(x)$ of one variable, we may define the *difference measure* $D(R) = f(b) - f(a)$ when $R = [a, b]$. Show that the difference measure is additive, and determine its derivative. What does the Fundamental Theorem of Calculus say for this measure?

8.*(surface area) Question: Computation of surface area makes an interesting example, and should be either an exercise or in the handout. Which?*

(a) A shed has a rectangular base and a sloping roof.



Side view of shed

Find a formula for the area of the roof in terms of the area B of the base and the unit normal vector $\hat{n}$ to the roof.

(b) Consider the graph $w = f(x, y)$ of a differentiable function. Obtain a formula for the unit normal vector $\hat{n}$ at a point of the graph by normalizing the gradient of the function of three variables $w - f(x, y)$.

(c) Adapt the computation done in part (a) to a suitably situated rectangle to determine the derivative $\frac{ds}{dA}$ of the surface area measure s.

(d) Express the surface area measure as a double integral.

9. Let T denote temperature in degrees Farenheit. This is a function of position and time: $T = T(P,t)$. Assuming that houses are heated to 68^oF, the rate of heating oil consumption is roughly proportional to $68-T$ if $T \leq 68$, and is zero if $T \geq 68$. Let $f = f(P,t)$ denote the function defined as follows: $f = 68-T$ if $T \leq 68$, and $f = 0$ if $T \geq 68$. The number of "degree days" in a heating season is defined to be the integral

$$D = \int_{oct\,1}^{may\,1} f\,dt.$$

This number still depends on position. Then the annual heating oil use per person is approximately of the form kD, where k depends on things such as efficiency of insulation, average household size, and the proportion of homes that are heated with oil. Treating k as a constant, express the total heating oil consumption in Massachusetts as an integral in terms of k, D, and the population measure.

10. The hemisphere $x^2+y^2+z^2=1$, $z \geq 0$ is covered by a metal shell X. We can compute densities in two ways:

(density relative to the hemisphere) Take a small region ΔS on the shell. Let $A(\Delta S)$ be its surface area, and let ΔM be the mass of the portion of the shell lying over S. The first density $\mathbf{d}_1$ is the limit of $\Delta M/A(\Delta S)$.

(density relative to the plane) Take a small region ΔR in the plane. Let ΔM be the mass of that part of the shell which lies over R, and let ΔA be athe area of ΔR. The second density $\mathbf{d}_2$ is the limit of $\Delta M/\Delta A$.

Explain the relation between these two densities in terms of the surface area measure s of the hemisphere.

11. Let M be the mass measure of a plate, and let μ be another measure. We may define the *derivative of* μ *with respect to the mass measure* $\frac{d\mu}{dM}$ to be the limit of the fractions $\frac{\Delta\mu}{\Delta M}$, taken over rectangles ΔR as in the definition of the density. Relate this derivative to the density of the two measures μ and M, and compare your answer with the chain rule in calculus.

12. Let ρ denote the function of position and time which measures rainfall in $(inches)/(hour)$. So $\rho(P,t)$ is the number of inches of rain falling per hour at the point P.
(a) What is the measure on the three dimensional "point, time" space of which ρ is the density?
(b) If R is a region of the globe, what is the meaning of $\iint_R \rho dA$?

13. By analogy with the antiderivative in one-variable calculus, we call a function $\mathcal{F}(x,y)$ a "mixed antiderivative" of a function $f(x,y)$ if $\frac{\partial^2\mathcal{F}}{\partial x\partial y} = f$.
(a) Find mixed antiderivatives of the following functions: $f(x,y) = xy^2+x^3$, $g(x,y) = \cos x$.
(b) Let $\mathcal{F}$ be a mixed antiderivative of f, and let R be the rectangle with vertices (x_0,y_0), (x_1,y_0), (x_1,y_1), (x_0,y_1). Find a formula for $\iint_R f\,dA$ in terms of $\mathcal{F}$.
(c) Compute the double integral over the rectangle R of the function $f = \cos x$, using a mixed antiderivative.

14. In one-variable calculus, the antiderivative $F(x)$ of a function $f(x)$ is determined up to adding a constant C. What can you say about the mixed antiderivative $\mathcal{F}$?

15. A false oracle claims that the μ-measure of the rectangle R with side lengths a and b and southwest corner (x,y) is $abxe^y$. Determine what the density would be, and use a mixed antiderivative to correct the formula for the μ-measure of the rectangle.

16. *variant of 11.* Let $R_{x,y,a,b}$ denote the rectangle with southwest corner (x,y) and side lengths a,b. Suppose that we are given a formula to compute $\phi(R)$ when $R = R_{x,y;a,b}$. We define a function g of two variables by setting $g(u,v) = \phi(R_{0,0;u,v})$.

(a) Suppose that the formula ϕ defines a measure. Use the additive property and elementary geometry to show that

(xxx) $$\phi(R_{x,y,a,b}) = g(x+a, y+b) - g(x, y+b) - g(x+a, y) + g(x, y).$$

(b) Assuming that ϕ defines a differentiable measure, show that the density of $\frac{d\phi}{dA} = \frac{\partial^2 g}{\partial x \partial y}$.
(c) Prove the converse of (a): If the formula xxx holds for rectangles, and if g is a differentiable function, then ϕ defines a measure.