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Dear John and Israel,

I think the question of associative comultiplications on a small co-h-space (meaning a two-cell complex localized at an odd prime p) is tied up with the cohomology of the homotopy analyser of an odd dimensional sphere, which is defined below, and I come to the opposite conclusion to that expressed in your draft paper.

I cannot ask you to look critically at my arguments while you have reasonable doubts about the correctness of their conclusions. I therefore have constructed two ad hoc obstruction theory arguments to show that your paper cannot be correct. These form the first two parts of this five part letter.

The first part considers [X,X v X], where X is a small co-h-space: let x,y mean the classes of the inclusions, P(x,y) the obstruction to primitivity of twice the identity map, and D(x,y) a variation in the comultiplication which can be uniquely identified with elements of a homotopy group of the bouquet of two spheres. Then changing the comultiplication by D will change P by

D(2x,2y) - 2D(x,y).

Thus the part D[p] of D involving Whitehead products of weight p will change the corresponding part of P by

(2^p - 2)D[p],

where the coefficient is always divisible by p. This means that if certain kinds of elements involving products of weight p occur in the obstruction P, they cannot be eliminated by changing the comultiplication.

Of course, it is conceivable that no such awkward obstructions arise. The second part deduces from Berstein's Theorem that certain spaces $S^3 \cup_{\alpha_1} e^{2p}$ have no associative comultiplication a similar result for many spaces $S^3 \cup_{\alpha_1 \circ \phi} e^{r+1}$ where the attaching map factors through the previous one. The argument can be used to discuss the lack of primitivity in the same way. The process first considers comultiplications on the second space for which there is a co-h map onto the first. For technical reasons we first use a co-h-map between co-h-spaces in a category of triples, using the mapping cylinder rather than the mapping cone. In this way we can obtain the obstructions for the second space as compositions with the obstructions for the first space. A calculation for Berstein's space shows that certain types of obstruction must be present, and a further calculation for the second space then shows that the obstruction cannot be removed by changing the comultiplication, when p=3.

These primitive methods of attack conceal, in my opinion, what is really involved. In the third part I show that the obstruction to associativity of a small co-h-space is a 3-cocycle in the homotopy analyser of a sphere. The (Lazard) analyser of a functor T can be defined as the result of applying T to the dual Hom(K,Z) of a simplicial free abelian group K(Z,1). For a covariant functor this is a cosimplicial abelian group, and the (Lazard) cohomology of T is the cohomology of the resulting co-complex (it is actually the dual of the homotopy of a simplicial abelian group). To obtain the homotopy analyser of a sphere we construct T by first constructing a functor from free abelian groups (and homomorphisms) to the homotopy category so that a group of rank n goes to a bouquet of n spheres each of dimension m; this is then followed by the graded homotopy group functor. The Lie analyzer is simpler: one just uses the obvious functor from free abelian groups to free Lie algebras. It is a theorem that the cohomology of the homotopy analyser of a sphere of odd dimension is obtained in an easy way from that of the cohomology of the Lie analyser with coefficients in the homotopy groups of spheres and the p-th Hopf Invariant (for the p-primary part). I showed 20 years ago that the third cohomology of the Lie analyser consisted only of torsion, and the only p-torsion is a cyclic group of order p involving products of weight p.

Furthermore, changing comultiplications changes the associator cocycle by a coboundary. Indeed, for every analyser there is a deformation retract which I call the reduced analyser, and there is a 1-1 correspondence between the reduced 3-cocycles in the cohomology class of an obstruction to associativity and the obstructions that arise by changing comultiplications.

The role of Chan's theorem is this. If the spectral sequence of an inclusion is constructed for the inclusion of a sphere in a ball, the E^1 term is the homotopy analyser of the suspension of the sphere,the groups of the E^2 term are those of the cohomology of this homotopy analyser, and the E^∞ terms are the graded groups of a certain filtration of the homotopy of the original sphere. Thus every nonsuspension element on the higher dimensional sphere is mapped nontrivially by some differential. For all known elements in the p-primary part of the homotopy of a sphere of odd dimension either the first or second differential is nonzero. The first differential is the obstruction to primitivity, the second to the existence of an associative comultiplication on the mapping cone.

Thus far the analysis shows the existence of non-associative small co-h-spaces. But (and this is the subject of the fourth part of the letter) we have also a very tight hold on the possible natures of these obstruction cocycles, enabling us to discuss the existence of inverses. The theorem is that if there is an associative comultiplication, it can be chosen to give an abelian cogroup structure (localized at p), and if not the comultiplication can be chosen to have none or one, but not two, of the properties: right inverses exist; left inverses exist; the p-localization is commutative. This should be seen against the background of the theorem I proved in an earlier letter that a simply-connected co-h-complex of finite dimension always has the unique divisibility property, and that for each odd prime p there is a comultiplication whose p-localization is commutative.

Unfortunately these sharp results require knowledge of the p-torsion in the fourth cohomology of the Lie analyser, which the universal coefficient theorem tells us is involved in the third cohomology of the homotopy analyser. Bill Richter and I will be publishing this computation: there is one nontrivial group. However, it turns out to be irrelevant to our problem: the existence of commutative comultiplications implies that the obstruction to associativity cannot lie wholly in this extra summand, and the detailed knowledge of its generator enables us to show that it cannot affect the existence of inverses.

Finally, at least for this introduction, I mention the Lie Analyser Conjecture, which is that the only non-zero groups are those already known. This is, as far as I have calculated, compatible with the p-primary part of the homotopy groups of spheres. It implies that the desuspension spectral sequence referred to converges at E^3, and it also would prove exactly the opposite of the theorem in your paper, that the only small p-local cogroups are suspensions,

Ever yours,

FIRST PART (Rev.1)

Letter:- Barratt to Berstein/Harper

The following theorem will for completeness be proved in an appendix to this part, in order that a generalization can be proved in the SECOND PART.

<u>THEOREM 1</u> (a) If X is a simply-connected co-h-complex of finite dimension, then for any space A and elements a,b of [X,A], the equations

$$ax = b = ya$$

have unique solutions

$$x = a\backslash b, \; y = b/a.$$

(b) For each odd prime p the p-localization of X has a homotopy commutative comultiplication.

Let p denote a fixed odd prime and A the mapping cone of a primitive p-primary map e of S^q to $S = S^m$, where m is odd, $m > 1$ and $q > m$. Such a space A will be called a <u>small co-h-space.</u>

<u>LEMMA 2</u> The pinching map z from A to $S^{(q+1)}$ is always a co-h map; the homomorphism z* from $\pi_{q+1}(B)$ to [A,B] has kernel the image $(\Sigma e)^* \pi_{m+1}(B)$, and the elements of the image of z* commute with, and associate with, the elements of [A,B].

This also is proved in the appendix.

Let B = A v A, and let G denote the subgroup of $\pi_{q+1}(B)$ which is the intersection of the kernels of both retractions of B to A; let G' denote the similarly defined subgroup of $\pi_{q+1}(S \vee S)$. By the Hilton-Milnor theorem G and G' are generated by Whitehead products, and for dimensional reasons the inclusion maps G' isomorphically on G. G and G' will be identified. Since is 2-primary for both A and B, and e is p-primary, z* is a monomorphism for both [A,A] and [A,B].

Let f denote the homotopy class of the comultiplication: the class f' of another comultiplication is a product fz*g, where g is in G. Since m is odd the p-localization of Whitehead products on S are zero: hence products in the p-localization of [A,A] are the same for all comultiplications. For similar reasons, the p-localization of [A,A] is associative (the obstruction to associativity involves Whitehead products on S). By Theorem 1(a) the p-localization of [A,A] is a group, so that for each integer k the k-th power (not iterate) x^k of any p-localized element x of [A,A] is uniquely defined and independent of the comultiplication. It follows that the k-th power y^k of any class y in [A,Y], localized at p, is uniquely defined, as the composition of the k-th power of the class of the identity map of A with y, and the result is the same for all comultiplications.

[continued]

Let a,b denote the homotopy classes of the inclusions of A in B, localized at p. Let P(k) be the element of G be defined by

$$z*P(k) = ((ab)^k)\backslash((a^k)(b^k)).$$

P(k) is the obstruction to primitivity of the k-th power of the identity map of A, using the comultiplications in f, everything being p-local. If the comultiplications in f' were used, the new value of ab would be $(ab)z*g$; since $z*g$ commutes with all elements, the new value of $(ab)^k$ would be

$$((ab)^k)z*(kg),$$

where products are defined by the original comultiplication.

Since a^k and b^k map S by maps of degree k on the two retracts S of S v S in B, it follows that the new value of $(a^k)(b^k)$ is

$$((a^k)(b^k))z*h_*g,$$

where h is k times the identity map of S v S.

Therefore the new value of P(k) is

$$P(k) + h_*g - kg.$$

Let P(k)[d] and g[d] denote the parts of P(k) and g, respectively, that involve Whitehead products of weight d. Then the new value of P(k)[d] is

$$P(k)[d] + (k^d - k)g[d].$$

In particular, by Fermat's theorem, the alteration in P(k)[p] is always divisible by p.

Of course, this does not prove that the obstruction to primitivity cannot be killed by changing the comultiplication: it might be that P(k)[p], for example, is divisible by p. This is **not** the case in Berstein's examples ($m=3$, $q=2p$) and, I will show in the next part for $k = 2$ and $p = 3$, other cases where $m=3$ as well. The point of this is that a simply-connected co-h-space that has an associative comultiplication has a comultiplication that gives a commutative co-group structure, for which all powers of the identity map are primitive.

SECOND PART [1] (Rev.3)

Letter:- Barratt to Berstein/Harper

I will use the notations of the FIRST PART. Implicitly all spaces and maps are localised at p. For any space Y let Y(n) mean the bouquet of n copies of Y, and let a,b,c stand for the classes of the three inclusions of A in A(3). Let G(n) denote the subgroup of π_{q+1}(A(n)) which is the intersection of all the retractions on A(n-1); for the reasons given before this is isomorphic to the similarly defined subgroup of π_{q+1}(S(n)), where S = S^m, and these groups are to be identified. Also, as before, z* is a monomorphism of π_{q+1}(A(n)) into [A,A(n)].

The <u>universal associator ☿</u> of the comultiplication on A is the element of G(3) defined by

z*☿ = ((ab)c)\(a(bc)).

This is the obstruction to associativity of the comultiplication.

Let u be a primitive map S^r to S^q, and let e' be the composition e ∘ u. Let A' be the mapping cone of e' and let P'(k), ☿' be the obstructions to primitivity and associativity (respectively) for A'. I will prove in the second chapter of this part

<u>THEOREM 3</u> Suppose u is a suspension. For each comultiplication on A there is a comultiplication on A' and a co-h-map U from A' to A extending the identity map of S.
Moreover, the obstructions to primitivity and associativity for this comultiplication on A' are the compositions of the obstructions for A with the suspension of ±u, modulo the images of (e v e)$_*\pi_{r+1}$(S^q(2)) and (e v e v e)$_*\pi_{r+1}$(S^q(3)) respectively.

For technical reasons I have to prove this theorem first for certain co-h-spaces in a category of triples. This is a device to force the obstructions to remain (more or less) in the homotopy of S(2) and S(3); the stated theorem then follows quickly. The indeterminacy (which will not affect the next theorem) arises from the delicate argument not being quite delicate enough: it will follow from the THIRD PART that the result is true without the indeterminacy. Theorem 3 is proved in the second chapter.

I now specialise to the case where m = 3, q = 2p, and e is a generator of the p-component of π_{2p}(S^3); u is to be any suspension element in the p-component of π_r(S^2p). Berstein proved that A is a co-h-space without an associative comultiplication. Let p = 3.

<u>THEOREM 4</u> If the suspension of u is not divisible by p then the co-h-spaces A,A' cannot have associative comultiplications, and in both cases the square of the identity map is never primitive.

The first interesting case comes by taking u to be the appropriate suspension of e. The restriction to p = 3 is to simplify the computation: the theorem will be proved true for all odd primes as part of a stronger result in the THIRD PART.

[continued]

Theorem 4 will be deduced from Theorem 3 and Berstein's Theorem. The latter implies that the obstruction to associativity for A is never zero modulo 3. Next, the existence of a commutative comultiplication is used to express P(2) in terms of the obstruction to associativity, and hence to prove that it too is never zero modulo 3. Theorem 3 is then invoked to show that these obstructions are non-zero for A' with the compatible comultiplications. Finally it is deduced that no comultiplication on A' has non-zero obstructions for associativity or primitivity of the squaring map.

With p = 3 and q = 6, the group G(3) is generated by triple Whitehead products in a,b,c, and in particular by

x = [a,[b,c]], y = [c,[a,b]].

Here, and subsequently, I use a,b,c in Whitehead products to mean their restrictions to S.

Thus ☿ must be some linear combination

☿ = kx + k'y

for some integers k,k'. Also G(2) is generated by triple products in a,b (the product [a,b] is composed with a coefficient from a group with no p-primary part, and so vanishes on localization at p). A variation in the comultiplication is defined by any element

g(a,b) = d[a,[a,b]] + d'[b,[b,a]],

where d,d' are integers.

Obviously changing the comultiplication by z*g(a,b) will change a(bc) to

(a(bc))z*(g(a,bc) + g(b,c))

and will change (ab)c to

((ab)c)z*(g(a,b) + g(ab,c)).

Here the products are computed in the original comultiplication. In addition, the homotopy groups have been written both multiplicatively and additively, which will now be corrected to additive notation. The effect is that the obstruction ☿ to associativity has been modified by the addition of

g(b,c) - g(a+b,c) + g(a,b+c) - g(a,b).

Using linearity and anticommutativity of Whitehead products, and the Jacobi relation, we obtain for this expression

-d([a,[b,c]]+[b,[a,c]]) + d'([b,[c,a]]+[c,[b,a]])

= -d(2x + y) - d'(2y + x).

The new obstruction to associativity is therefore

(k - 2d - d')x + (k' - d - 2d')y.

It follows that the congruence class (mod.3) of k + k' is the same for all comultiplications. Since we can choose d = k' , d' = 0 we can suppose that k' was initially zero; to preserve this state we must have d = -2d', which will change k to k + 3d'. Hence k (and so k+k') can be changed by any multiple of 3; since k,k' cannot be made simultaneously zero it follows that k + k' must be congruent to 1 or -1 modulo 3. This proves that the obstruction to associativity is non-zero modulo 3.

[continued]

According to Theorem 1(b) in the FIRST PART, there is a comultiplication whose 3-localization is commutative. On writing the universal associator as a function of the three arguments a,b,c we have

(a^2)(b^2) = (((a^2)b)b)z*¤(2a,b,b),

where

((a^2)b)b = (b(a^2))b = (((ba)a)b)z*¤(b,a,a),

in which

((ba)a)b = b(a(ba)) = ((ba)^2)z*¤(b,a,a+b).

This proves that P(2) can be expressed as the sum

¤(2a,b,b) + ¤(b,a,a) + ¤(b,a,a+b),

which reduces to

-(k + k')[b,[b,a]] - 2k'[a,[a,b]].

It is not necessary to discuss at this point what k,k' can be for a commutative comultiplication: it is sufficient to observe that the first term is non-zero modulo 3, and by the FIRST PART this will be true for all comultiplications, since that part of the obstruction consisting of triple products can be changed only by elements which are divisible by 3.

The indeterminacies described in Theorem 3 produce elements which factor through Whitehead products of maps which are then composed with e, and so by an expansion theorem of Barcus and Barratt a Whitehead product of weight n will factor through (suspensions of) the n-th iterate of e, because the primitivity of e implies the vanishing of all its Hopf invariants. The second iterate of e is zero on the 5-sphere, because, as Hilton and I showed, it is of order dividing 2 on the 6-sphere and hence zero there, while suspension of the p-component on S^5 is a monomorphism. Hence all the indeterminacy is zero in this application.

By Theorem 3, the obstructions P'(2) and ¤' for comultiplications on A' compatible with comultiplications on A will have non-zero Whitehead products of weight 3 which are not divisible by 3, provided the suspension of u is not divisible by 3. By the FIRST PART, such terms in P'(2) can only be changed by an element divisible by 3, and so cannot be made zero. The argument that ¤ cannot be made zero by a change in the comultiplication follows from the observation that only triple Whitehead products in the change g need be considered, and for these we have the same calculation as was made for A, where d,d' are now drawn from some homotopy group. The result is the same, that the sum of the coefficients of x and y can only be changed by elements divisible by 3, and so ¤' is never zero.

Thus Theorem 4 is true if Theorem 3 is true.

SECOND PART [2] (Rev.2) Proof of Theorem 3

Letter:- Barratt to Berstein/Harper

Consider the categories J of pointed topological spaces K, J' of pairs (K,L), and J" of triples (K;L,L') where L,L' are in K and contain the base point *. J will be embedded in J' by sending K to (K,*), and J' will be embedded in J" by sending (K,L) to (K;L,*). Clearly, the <u>pinching functor</u> which sends (K;L,L') to (K/L';L/L',*) retracts J" on J'. Likewise, J' is retracted on J by the <u>forgetful functor</u> which ignores the restrictions on maps of the subspace L.

The n-fold bouquet Y(n) of any triple Y = (K;L,L') is (K(n);L(n),L'(n)). Thus a co-h-triple is a triple Y and a class of comultiplications in [Y,Y(2)]. The following theorem is sharp enough for our needs, although it can be generalized.

<u>THEOREM 5</u> The conclusions of Theorem 1 apply to co-h-triples (K;L,L') where L,L' are simply-connected subcomplexes of a simply-connected CW complex K of finite dimension, and the intersection of L,L' consists only of the base-point.

The proof starts by applying Theorem 1 to L and to L', which proves it for the case when K is the union of L and L'. Then the proof of Theorem 1 is followed mechanically except that the skeleta of K have L and L' added at each stage of the induction.

Suppose P,Q are simply-connected pointed CW complexes of finite dimension. The cone functor C and the mapping cylinder functor M are both to be reduced and so parametrised by the real variable t that t=0 in CP is identified with the base-point and, for a map e of P to Q, Me is the union of Q and P x I with (y,1) identified with e(y). The subsets t = 1 will be called the bottom of the cone or the bottom of the mapping cylinder. It is convenient to identify P with the top of Me. Clearly Q is a deformation retract of Me.

Consider the triple Te = (Me;Q,P), where e maps P to Q, and of course P,Q also mean the top and bottom of Me. Let Te/Q be the triple (Me/Q;*,P), and let z be the identification map of Te to Te/Q: Me/Q is naturally homeomorphic to CP by the map which reverses the direction of the parameter t. Thus [Te/Q,(K;L,L')] is isomorphic to [(CP,P),(K,L')], which is a relative homotopy group of (K,L') when P is a sphere.

Suppose s',s are comultiplications on P,Q with respect to which e is primitive. Thus there is a homotopy H connecting s ° e and (e v e) ° s'. Such a map induces a map R of Te to Te(2), extending s' on P and s on Q, by the formulae

R(y)	= s(y)	if y is in Q,
R(y,t)	= (s'(y),2t)	if y is in P and t <= 1/2,
R(y,t)	= H(y,2t-1)	if y is in P and t >= 1/2.

[continued]

LEMMA 6 The homotopy H can be chosen so that R is a comultiplication.

The cylinder P x I is equipped with two commuting operations on maps: one is the cylinder on the comultiplication s', and the other uses the parameter t. Let these be written +' and + respectively. Let R{1} and R{2} denote the composition of the identification map of P x I into Me , followed by the map R and the projection of Me(2) on the first and second copies of Me respectively. There are homotopies F{1},F{2} from the identity map 1 of P to the sums 1+'0, 0+'1 of 1 and the constant map 0 of P to the base-point *, respectively. Let F'{i} denote F{i} with the parameter t reversed. A variant of H is defined by the formula

((F{1} +' F{2}) + (R{1} +' R{2})) + ((F'{1} +' F'{2}) + H),

where the additions + are based on dividing the interval I into four parts. It is an easy exercise to show that the new map R defined by this new homotopy followed by either projection of Me(2) to Me is homotopic to the identity, whence R is a comultiplication.

REMARK This construction of a comultiplication will be referred to again in the proof of Chan's Theorem in the THIRD PART.

By applying the **pinching functor** we obtain also a comultiplication on the pair (A,Q), where A is the mapping cone of e. By applying the **forgetful functor** we obtain in turn a comultiplication on the mapping cone A. These functors turn the map z for Te to the map z for A. It follows that, when P,Q are spheres, the obstructions to associativity and primitivity of the squaring map for Te will be mapped by these functors to the corresponding obstructions for the mapping cone A.

From here onwards, Q = S = S^m and P = S^q.

The values of the obstructions to associativity and primitivity of the square of the identity map lie in a relative homotopy group of (Me(n),P(n)), where n = 3 or 2, for the case Te ; in the other cases they lie in an absolute homotopy group of A(n). However, looking at the exact homotopy sequence for the pair (Me(n),P(n)) we find that the (q+1)-dimensional homotopy group of P(n) is 2-primary, and the obstructions map under the boundary homomorphism to the corresponding obstructions for the sphere P, and so to zero. Hence an obstructions for Te is the image of a unique element in a homotopy group of Me(n), which is isomorphic to the corresponding homotopy group of Q(n) under the deformation retraction. In fact, we can suppose the obstruction lies in G(n), which is mapped isomorphically by the pinching and forgetful functors. In brief, the obstructions are the same for Te, (A,Q) and A.

[continued]

Suppose now that P' is another sphere S^r and that u mapping P' to P is a suspension. Thus we can suppose u is a strict co-h-map and that the functions F'{i} for P' and F{i} for P commute with u in the obvious sense. Let e' be the composition e ° u, and for this construct a homotopy H' by composing H with the cylinder on the map u. Let U be the obvious map of Te' to Te (it is the identity on Q and u on P'), and let R' mapping Te' to Te'(2) be constructed from H' as R was constructed from H. Obviously

R ° U = (U v U) ° R'.

Now modify the homotopy H' by the process described above for H, and at the same time modify H again. This will create comultiplications R on Te and R' on Te' which still satisfy

R ° U = (U v U) ° R'.

Hence U is a co-h-map. Also, from the definition of U, the composition of z with U, mapping Te' to Te/Q, is the composition C'u ° z, where C' is the cone functor with reversed parametrisation. This proves that the obstructions for Te', whose values lie in relative homotopy groups of (Te'(n),P'(n)) for n = 3 or 2, are obtained from those for Te by composition with C'u, for they lie in the image of the homotopy of Te'(n) which, like Te(n), has Q(n) as a deformation retract. The only complicating factor is that after composing with C'u the elements in the relative homotopy groups may no longer pull back to unique elements of the absolute homotopy groups. The kernel is the image of the (r+1)-st homotopy group of P(n); on following the inclusion of P(n) into Te(n) with the deformation retraction on Q(n) we find that this map is the bouquet of n copies of e. This gives the indeterminacy stated in the theorem.

Having identified the obstructions for Te' we complete the proof by applying first the pinching functor, which converts the map C'u to the suspension of u (perhaps with sign reversed), and then the forgetful functor.

At this point attention can be drawn to the need to use triples to limit the indeterminacy, for composition of the obstructions for A with the suspension of u often produces zero in the homotopy of A(n).

THIRD PART [Ø] (Rev.1) **Introduction**

Letter:- Barratt to Berstein/Harper

The concept of **analyser** was invented by Michel Lazard for algebraic functors and (in ignorance of his work) by me for the homotopy functor. The spectral sequence of a desuspension is a special case of my spectral sequence of an inclusion, recently re-discovered and generalized by Mike Hopkins. The E^1 term of the former is the homotopy analyser, and the E^2 term is the cohomology of the homotopy analyser. There is a spectral sequence whose E^Ø term is the homotopy analyser which converges to its cohomology. The E^1 term of this intermediate spectral sequence is the cohomology of the Lie analyser with coefficients in homotopy groups of spheres, in the case of the homotopy analyser of a sphere of odd dimension. In an old paper I calculated the first three cohomology groups of the Lie analyser. This is enough to show that the intermediate spectral sequence has only one more differential, for the p-primary part, determined by the p-th Hopf invariant. A recent calculation, provoked by the current problem and to be published with Bill Richter, of the fourth cohomology group of the Lie Analyser is relevant to the analysis in the FOURTH PART: of more general interest is the conjecture that the other cohomology of the Lie analyser is zero, which has striking implications for the p-primary part of the homotopy groups of spheres, described in the fifth chapter below.

The connection with small co-h-spaces (and this can be generalised) is the fact that the universal associator for a comultiplication is a (reduced) 3-cocycle in the homotopy analyser of the sphere S, and changes in the comultiplication change this cocycle by any (reduced) coboundary. Thus the cohomology class of the associator is an invariant of the underlying space. It turns out that the cohomology group has two summands, one being a homotopy group of a certain sphere, reduced mod p; the other can be proved from the existence of commutative comultiplications to be irrelevant.

Chan's theorem, proved in his Manchester thesis and reproved here, identifies the cohomology class of the associator with the second differential in the desuspension spectral sequence. The Lie analyser conjecture mentioned above implies that the desuspension spectral sequence converges at E^3, and this would imply that a small co-group is a suspension. In any case, all known elements in the p-primary homotopy of a sphere of odd dimension (through approximately the 2p^3 stem) that are trivial under the first two differentials are suspensions.

This THIRD PART is divided into chapters. The first revises the definition of analyser and explains the term reduced analyser (dual to reduced chains in homology theory). The second chapter describes from published work what was known about the cohomology of the Lie analyser. The third defines the homotopy analyser and, using the previous chapter, analyses its cohomology in the case of a sphere of odd dimension. The fourth chapter defines the associator, proving it to be a reduced 3-cocycle whose cohomology class is independant of the comultiplication. The fifth chapter defines the desuspension spectral sequence. The sixth chapter proves Chan's theorem, and the last chapter applies the analysis to prove various small co-h-spaces to be incapable of carrying a co-group structure. A more detailed analysis of what properties they can have is the subject of the FOURTH PART.

THIRD PART [1] (Rev.1) Analysers

Letter:- Barratt to Berstein/Harper

NOTATION Superscripts will be indicated by a caret ^, and subscripts by a pair of braces {}. The symbol @ will mean tensor product.

Let K denote a simplicial free abelian group K(Z,1), and let K* denote the cosimplicial free abelian group Hom(K,Z). The (Lazard) analyser of a functor T from the category of free abelian groups to a category of abelian groups is TK*. If T (as will here be assumed) is covariant, TK* is a cosimplicial abelian group. The (Lazard) cohomology of the analyzer of T is the cohomology of the cochain complex whose n-th group is TK*{n} with coboundary ∂* the alternating sum of the coface operators d*{i}. This is in fact the dual of the homotopy of a simplicial abelian group, and could have been called the cohomotopy of the analyzer.

A convenient version of K* makes K*{n} a free abelian group on n generators x{i} for i = 1 to n. The operators on K*{n} are

d*{0}(x{i}) = x{i+1} for all i,
d*{n+1}(x{i}) = x{i} for all i,
d*{k}(x{i}) = x{i} if i < k, = x{k} + x{k+1} if i = k, = x{i+1} if i > k, for k = 1 to n.
s*{k}(x{i}) = x{i} if i < k+1, = 0 if i = k+1, = x{i-1} if i > k+2, for k = 0 to n-1.

For the connection with the desuspension spectral sequence it is convenient to use an alternate basis: let y{i} be the sum of the generators x{j} for j > i-1. Obviously, d*{i} for i < n+1 is the monotone map which omits y{i+1} from the image values, while d*{n+1} maps y{j} to y{j} - y{n+1} for all j.

One example of a suitable functor T is the tensor algebra functor, which is also known as the free associative algebra functor when restricted to free abelian groups. If the image of T is made a graded group, graded by the degree (or weight) of products, the cohomology of the analyser is also graded. Lazard proved that the cohomology is a polynomial algebra on one generator, so that all the bigraded groups are zero, except for H{n,n} = Z for all n >= 1.

The reduced analyser is defined to be the sub-cosimplicial complex obtained by taking the intersection of all the kernels of the codegeneracy operators in each dimension.

THEOREM 3.1.1 The reduced analyser is a sub-cosimplicial abelian group of the analyser; the inclusion of the reduced analyser in the analyser of a covariant functor T is a cochain equivalence.

This, first observed by Lazard (private communication), is the dual of the theorem that projection on reduced chains is a chain equivalence, and is proved in a similar way, by applying to TK*{n} the cochain deformation operator

[1-∂*s*{0}-s*{0}∂*][1+∂*s*{1}+s*{1}∂*]...[1+((-1)^n)(∂*s*{n-1}+s*{n-1}∂*)],

where ∂* denotes the coboundary operator. As the contents of each brace, beginning on the right, are applied in turn, the complex is mapped into the intersection of the kernels of an increasing number of operators s*{i}, ending with s*{0}. The computational details are omitted.

In the analysers of interest the reduced analyser has a complementary summand consisting of the submodule spanned by the images of all d*{i}TK*{n} for all i < n+1 and all n. This is the kernel of the above cochain deformation. It is possible, therefore, to formulate the theorem in terms of the quotient by this subcomplex rather than the reduced analyser; the coboundary then reduces to the last coface operator. This is of relevance for the spectral sequence of a desuspension.

THIRD PART [2] (Rev.1) Lie Analyser Cohomology

Letter:- Barratt to Berstein/Harper

The Lie algebra functor turns a free module of rank n into the free Lie algebra on n generators. The Lie algebra is embedded as a module in the tensor algebra of the original module (its enveloping algebra). Thus the Lie algebra can be made into a graded algebra using the grading by weight (or degree or number of tensor products). Then the Lie analyser is a cosimplicial free graded abelian group, and its cohomology is a bigraded group. I will use L^n for the cohomology group defined using products in n variables, and L^(n,k) for the part using products of weight k.

The reduced Lie analyser consists of the intersection of the kernels of the codegeneracy operators; using the generators x{i} described in the previous chapter, s*{i} sets x{i} to zero. It follows that the reduced analyser is spanned by those Lie monomials which involve all the available generators; I will use LC^(n,k) for the reduced cochain group which has products of weight k in the maximum number n of variables. Since LC^(n,k) must be zero if k < n, it follows that L^(n,k) is also zero for k < n.

The cohomology for weights 1 and 2 are easily computed, for the only nonzero reduced cochain groups are LC^(1,1), generated by x{1}, and LC^(2,2), generated by [x{1},x{2}].

<u>LEMMA 3.2.1</u> All the groups L^(n,k) are torsion groups except

L^(1,1) = Z, L^(2,2) = Z.

That the stated groups are cyclic follows from the previous remarks. The easiest proof that the others are zero when tensored with the rationals comes from the connection with the homotopy groups of spheres (whose rational values are known) and will be given in a later chapter. It can be deduced from results of Lazard [Annales de E.N.S.(3)72(1955)pp299-400], with a rather disagreeable argument for the case n = k.

There is a useful Frobenius or p-th power operator in this cohomology. It is well known that if p is prime and the sum of a number of elements a{i} in a Lie algebra is zero, then in the enveloping algebra the sum of their p-th powers a{i}^p is a sum u + pv where u is again in the Lie algebra. Hence any cochain c in LC^(n,k) defines a unique (mod.p)-cochain fc of weight pk in n+1 variables by the formula

fc = Ð*(c^p) - (Ð*c)^p (modulo p),

where Ð* is the coboundary in either the Lie or associative algebra analyser. Since Ð*fc = -Ð*((Ð*c)^p) (mod.p), this operator f carries (mod.p)-cocycles to (mod.p)-cocycles; since

f(Ð*c) = Ð*((Ð*c)^p) = -Ð*fc (modulo p),

f carries (mod.p)-coboundaries to (mod.p)-coboundaries. Suppose a,b are (mod.p)-cocycles: there is a Lie element w congruent (mod.p) to (a+b)^p - a^p - b^p, so that

f(a+b) = fa + fb + Ð*w (modulo p).

This proves that f induces a homomorphism F between (mod.p)-cohomology groups, which increases dimension by 1 and multiplies the weight by p. I will use the same symbol F for the map of integral groups which precedes F by reduction (mod.p) and β for the Bochstein homomorphism from (mod.p)-cohomology to integral cohomology in one higher dimension.

[continued]

THEOREM 3.2.2 The groups L^(2,k) are all zero for k > 2.
The groups L^(3,k) have no p-torsion except for
L^(3,p) = βFL^(1,1) = Z/pZ.

I proved this in 1960 [Quarterly Journal of Mathematics,(Oxford)(2)11,pp275-86], except for the reference to βFL^(1,1), which can be verified most most easily by embedding the Lie algebras in their envelopes. Lazard proved the envelope has no cohomology in the relevant dimensions, so all Lie cocyles must be coboundaries in the envelope. In particular the (mod.p).generator of L^(2,p)(Z/pZ) can only be the coboundary of x{1}^p, so this group is FL^(1,1).

This theorem is crucial for our problem, as for the understanding of Hopf Invariants on a sphere of odd dimension. I will therefore provide a cleaned-up proof in the FIFTH PART, together with proofs of the following more recent results.

THEOREM 3.2.3 The compositions FF and FβF are trivial in the (mod.p)-cohomology of the Lie analyser.

These come about by contemplating the coboundary of c^(p^2): the details will be given with the proof of the previous theorem.

THEOREM 3.2.4 The groups L^(4,k) have no p-torsion except for
L^(4,2p) = βFL^(2,2) = Z/pZ.

This will be published with Bill Richter. The proof (for at least k < p^2) will be given in the FIFTH PART.

These results and the evidence of the p-primary homotopy groups of spheres suggests the

LIE ANALYSER CONJECTURE

All the groups L^n are zero for n > 4. Thus the only non-zero groups are those described above.

This has remarkable implications for the homotopy groups of spheres, as well as for the present problem.

THIRD PART [3.1] The Homotopy Analyser

Letter:- Barratt to Berstein/Harper

Let A denote an abelian cogroup with a base point, and A(n) the bouquet of n copies of A. Then A(n) is also an abelian cogroup with the obvious comultiplication. For convenience, the groups [A(n),Y] will be written additively in this chapter. Because [A,A(2)] is abelian and associative, inverses exist and all multiples of the class of the identity map of A are primitive and therefore co-h-maps.

Therefore there is a homomorphism of the Z-module of integral n-by-m matrices to [A(m),A(n)] such that the generating matrix e{i,j} is sent to the map which first projects A(n) on the j-th factor A{j}, and then maps this identically on the i-th factor A{i} of A(n). Obviously this sends products of matrices to compositions of homotopy classes: hence

LEMMA 3.3.1 There is a functor from the category of free abelian groups on sets, and all homomorphisms, which sends a group of rank n to the bouquet A(n) and identity maps to identity maps.

If we apply this functor to K* = Hom(K(Z,1),Z) with an arbitrarily chosen basis in each dimension we obtain a cosimplicial space whose homotopy type is independant of the choices of bases. This homotopy type thus merits the name the topological analyser of A.

REMARK I have not had time to work out all the details of a proposed construction of a functor from the category of free abelian groups in the case that A is S^2, and hence for any double suspension. I am indebted to Jeff Smith for raising the question of the existence of such a functor, and hope he will plug the ideological gap first.

The graded homotopy group functor assigns to any pointed space the collection of its homotopy groups: the q-th homotopy group is to have grade q. The homotopy analyser of A is for the moment to be the result £ of applying the graded homotopy group functor to the image of the previously defined bouquet functor using the basis elements x{i} described in chapter 1. Using any other basis will produce isomorphic groups, the isomorphisms being induced by the matrices which express the change of basis. In fact, we will be interested later on in using the alternate basis elements y{i} described in chapter 1.

In future, A will be a sphere S^m of odd dimension, localised at an odd prime p.

By invoking Hilton's theorem, which implies that the q-th homotopy group of A(n) is a direct sum of homotopy groups of spheres embedded by composition with Whitehead products, we obtain a filtration of this group by weight of Whitehead product. If v,w are Whitehead products of the same weight, the difference of a composition (v+w)°g and v°g + w°g is a sum of Whitehead products of greater weight composed with Hilton-Hopf invariants of g.

THIRD PART [3.2] Homotopy Analyser of S^m

Letter:- Barratt to Berstein/Harper

The homotopy analyser £ of S^m has been filtered by the weight of Whitehead products, and, by the last observation of [3.1], in the associated graded cosimplicial group composition with homotopy elements is distributive over addition.

LEMMA 3.3.2 E^0£^(n,q,k) is isomorphic to the group of (n,k)-cochains of the Lie analyser, tensored with the q-th homotopy group of a sphere of odd dimension M{k} = 1+k(m-1).

This easily follows, of course, from the same observation that powered the theorem of Hilton's cited in the previous section of this chapter.

The filtration of the analyser gives a spectral sequence starting at E^0£ and converging to the associated graded group of the cohomology of the analyser. Obviously the first differential is induced by the coboundary in the Lie analyser, whence

LEMMA 3.3.3 E^1£^(n,q,k) = L^(n,k)(π_q(S^M{k})).

Now £^(1,q) is just the q-th homotopy group of S^m and the coboundary maps x{1}°g to

x{2}°g - (x{1}+x{2})°g + x{2}°g,

where x{i} has become the identity map on the i-th sphere in a bouquet. This means that the higher differentials in the spectral sequence are measuring the obstruction to distributivity of composition, and so are determined by the Hilton-Hopf invariants of g. Since m is odd the first Hopf invariant is zero on the p-primary component. By 3.3.3 and theorem 3.2.2 (recording a published result) all the groups E^1£^(2,q,k) for k > 2 are zero except for

E^1£^(2,q,p) = L^(2,p)(π_q(S^M{k}) = Tor(π_q(S^M{k}),Z/p).

This proves several things. Firstly, the first non-vanishing Hopf invariants are those associated with basic products of weight p, and are fixed multiples of a single element, the coefficient of fx{1} = ϑ*(x{1}^p), which we will call the p-th Hopf invariant H{p}. Secondly, this invariant has order dividing p. Thirdly, that all higher Hopf invariants are determined by the p-th (it is clear that they are zero if H{p} is, and formulae can be given for them). Fourthly, since all the higher derivatives in our spectral sequence arise from the distributivity formula, the only remaining differential in our spectral sequence is the p-th. Lastly, we can compute this differential in terms of the p-th Hopf invariant H{p} and the modular operation F in the cohomology of the Lie analyser. This proves

[continued]

<u>THEOREM 3.3.4</u> The cohomology of the homotopy analyser of S^m, filtered by weight of Whitehead products, has associated graded group the cohomology of {L^(n,k)((S^M{k}))} with respect to the derivative F @ H{p}.

Here the tensor product is indicated as usual by @.

On putting Theorems 3.2.2 and 3.2.3 with this last theorem we find some of the cohomology explicitly. The first cohomology group of the homotopy analyser of S^m is the kernel of H{p}, that is, the primitive elements in the homotopy of S^m.

The second cohomology group has two summands. The first is the kernel of H{p} applied to the homotopy groups of S^(2m-1); it is associated with products of weight 2. The second is the cokernel of H{p} applied to the homotopy groups of S^m; it is associated with products of weight p. This second summand happens to be zero when m=3.

Of more interest for the present question is the third cohomology group. There is a summand obtained from the group L^(3,p), which for grade q is the q-th homotopy group of S^(1+p(m-1)) tensored with Z/p. This is associated with products of weight p, of course.

However, there is more: it also contains elements derived from torsion in L^4. If we now assume Theorem 3.2.4 of the previous chapter we see that this is a summand, associated with products of weight 2p, isomorphic to the cokernel of H{p} applied to the homotopy groups of S^(2m-1).

The truth of Lie Analyser Conjecture would imply that there is no other cohomology in the homotopy analyser of a sphere of odd dimension.

Incidentally, the cohomology of the homotopy analyser of a sphere of even dimension has precisely two non-zero graded groups, being the kernel of the usual Hopf invariant applied to homotopy groups of S^m, associated with the product of weight 1, and its cokernel, associated with products of weight 2. The former is the first, and the latter is the second, cohomology group; all other groups are zero.

[Here the underlying algebras (in the ring-stack sense) are the Twisted Lie algebras with Z coefficients where the symmetric groups act on Z through the sign of a permutation. The Lie algebras arise in a similar way except that the permutation groups act trivially on Z in this case. The current knowledge of the cohomology of the Twisted Lie Algebra analyser is an analogue of Theorem 3.2.2 with the prime p being replaced, for weight k, by the ideal spanned by all twisted binomial coefficients (r,k-r) for 0<r<k. The Dynkyn-Specht-Wever element of "weight" k annihilates this ideal. This is relevant to the analyser of a general double suspension.]

THIRD PART [3.3] The Distributivity Law for S^m

Letter:-Barratt to Berstein/Harper

Even localised at an odd prime p, the composition of maps into S^m with maps of S^m is not distributive in the second factor except when this is a self-map of S^m. Luckily the distributivity law (which is quite complicated for general spaces) has an elegant form for these spheres of odd dimension. Since this is relevant to choosing representative cocycles, I will give the analysis here.

Let LA and AA denote the Lie Algebra and Associative Algebra functors respectively. AA is also called the Tensor Algebra functor; both functors will here be applied only to the free abelian groups K*^n, and their images will be written LA^n and AA^n respectively. There is a natural transformation from LA to AA which assigns each Lie algebra to its envelope: this embeds LA^n as a module on a summand of AA^n. Let AA^(n,k) denote the sub-module of AA^n spanned by homogeneous monomials of weight k, and let LA^(n,k) denote its intersection with LA^n. Let m be a fixed odd integer greater than 1, and let M{k} = 1+k(m-1), an odd integer, for each positive integer k. Let q be a fixed positive integer and let G{k} denote the q-th homotopy group of S^M{k}: G{k} is non-zero for only a finite number of values of k.

Let AAH^n denote the direct sum of all the tensor products AA^(n,k)@G{k}, and let LAH^n denote the summand which is the direct sum of all LA^(n,k)@G{k}, for each positive n.

The Hilton-Hopf invariant H{p} (which actually is, up to sign, the p-th James invariant) is a homomorphism of G{k} to G{pk}. Let H{p,r} denote the r-th iterate of H{p}. For any (homogeneous) element a of AA^(n,k) and any g of G{k} define

    a|g = a@g + (a^p)@H{p}(g) + (a^(p^2))@H{p,2} + ...
         = a@g + (a^p)|H{p}(g).

Since H{p}(g) has order dividing p and t^p is congruent to t modulo p, (ta)|g = a|(tg). Obviously, a|g is linear in g: it is not, however, linear in a, and I now construct a formula describing the deviation from linearity in a in terms of the Lie bracket [a,b] = ab - ba.

Suppose a{1},...,a{s} are in AA^(n,k) and have zero sum. There is a Lie polynomial f{1} in the a{i} such that

    a{1}^p +...+a{s}^p = f^{1}   modulo p,

and Lie polynomials f{r} in the a{i} defined recursively by

    f{r} = a{1}^(p^r) +...+ a{s}^(p^r) - f{1}^(p^(r-1)) -...- f{r-1}^p mod. p.

Thus f{r} is unique modulo p, and hence f{r}|H{p,r}(g) is uniquely defined for each positive r.

<u>LEMMA 3.3.5</u> The sum of a{i}|g for i=1 to s is

(a) the sum of a{i}^p|H{p}(g) for i=1 to s,

(b) the sum of f{r}|H{p,r}(g) for all r > 0.

[continued]

PROOF: The equality (a) follows at once from the second form of the definition of a|g since the sum of the a{i}'s is zero. The equality (b) then follows iterating (a), using the definition of f{r} and the fact that H{p,t} is zero when t is sufficiently large.

I now construct a map of the analyser of LAH to the homotopy analyser £. For each (n,k) select an additive basis of monomials for LA^(n,k) and for each basic monomial a and every g in G{k} map a|g to the composition a°g, where the Lie monomial is interpreted as a Whitehead product.

THEOREM 3.3.6 This maps a|g to a°g for every homogeneous element a, and hence if a collection of Whitehead products a{i} of weight k has zero sum, the sum of all a{i}°g is equal to the sum of all f{r}°H{p,r}(g).

PROOF: It is sufficient to prove this for k=1, for the general result will follow by naturality. It is also sufficient to prove that if a|g = a°g for a=x{1},x{2} then it is also true for a = (x{1}+x{2}); the general result will then follow, by induction and naturality, for any sum of basic monomials. It is not difficult to see that the coboundary Ð*(x{1}|g) is mapped to a cocycle. Now Ð*(x{1}°g) is also a cocycle and, by definition, is f{1}°H{p}(g) modulo products of weight greater than p. Therefore the difference of Ð*(x{1}|g) and Ð*(x{1}°g) is a cocycle in the homotopy analyser, with filtration greater than p. Since the cohomology group is zero for filtration greater than p, this cocycle is a coboundary; since the homotopy analyser is zero in dimension 1 and filtration greater than 1, this difference cocycle must be zero. This completes the proof.

REMARK Suppose w is a homogeneous Lie cocycle of weight k and g is in G{k}. There will be cocycles in the homotopy analyser congruent to c°g modulo filtration k+1. The elements of higher filtration may be required because composition with g is not distributive. The analysis of the distributivity law given above implies that these extra terms can be chosen to have weights kp^r, for r=1,2,... .

THIRD PART [4] The Associator

Letter:-Barratt to Berstein/Harper

I will use the notations of the FIRST and SECOND PARTS. The universal associator ¤ was defined in the SECOND PART for a small co-h-space. In fact there is a generalisation to the case of a central co-h-extension of one abelian co-group by another; many of the observations made here apply to this more general situation, but for simplicity of expression we will stick to small co-h-spaces.

By definition, ¤ is a (3,q+1)-cochain in the homotopy analyser of S^m. Since it is zero when any of its arguments are trivial, it is a reduced cochain.

<u>LEMMA 3.4.1</u> The associator ¤ is a reduced cocycle.

PROOF: Let a,b,c,d be the inclusions of A in A(4). The equations

```
a(b(cd))  =  (a((bc)d))z*¤(b,c,d),
a((bc)d)  =  ((a(bc))d)z*¤(a,b+c,d),
(a(bc))d  =  (((ab)c)d)z*¤(a,b,c),
```

coupled with the equations

```
a(b(cd))  =  ((ab)(cd))z*¤(a,b,c+d),
(ab)(cd)  =  (((ab)c)d)z*¤(a+b,c,d),
```

show that the difference between a(b(cd)) and ((ab)c)d is the image under the monomorphism z* of either side of the equation

```
¤(b,c,d) + ¤(a,b+c,d) + ¤(a,b,c)  =  ¤(a+b,c,d) + ¤(a,b,c+d).
```

However, the left-hand-side minus the right-hand-side is the coboundary of ¤, proving that ¤ is a cocycle.

A change in the comultiplication is the result of multiplying the original comultiplication by z*g, where g = g(a,b) is a (2,q+1)-cochain in the homotopy analyser of S^m. In order that the new map be a comultiplication, g must vanish when either of its arguments is trivial. Thus g can range over all the reduced (2,q+1)-cochains.

It was shown in the SECOND PART that the effect of changing the comultiplication by g changed ¤ by

```
g(b,c) - g(a+b,c) + g(a,b+c) - g(a,b),
```

which is the coboundary of g. This proves

<u>THEOREM 3.4.2</u> The set of all associators ¤ is all the reduced (3,q+1)-cocycles in a cohomology class in the cohomology of the homotopy analyser of S^m. Thus the cohomology class of ¤ is an invariant of the homotopy type of the underlying space.

We have seen that, localised at an odd prime p, there is always a commutative comultiplication. This limits the possible values for ¤, because

[continued]

THEOREM 3.4.3 If a comultiplication is commutative the associator ¤ satisfies

¤(a,b,c) + ¤(c,b,a) = 0,
¤(a,b,c)+¤(b,c,a)+¤(c,a,b) = 0.

Proof. The two equations

(cb)a = a(bc) = ((ab)c)z*¤(a,b,c),
(ab)c = c(ba) = ((cb)a)z*¤(c,b,a),

imply the first relation. The second is implied by the sequence

(bc)a = a(bc) = ((ab)c)z*¤(a,b,c),
(ab)c = c(ab) = ((ca)b)z*¤(c,a,b),
(ca)b = b(ca) = ((bc)a)z*¤(b,c,a).

I have not written out the detailed proof but I think the generalised version of the next theorem can be used to prove that any simply-connected co-group, localised at an odd prime p, admits an abelian co-group structure. (I think this is the real interest of the theorem, for I believe that the only small co-groups are suspensions, and hence double suspensions.)

THEOREM 3.4.4 A small co-h-space that admits an associative comultiplication admits an abelian co-group structure (in which all powers of the identity are primitive).

PROOF: Let a,b,c denote the three inclusions of A in A(3); no confusion will arise from using a,b also for the two inclusions of A in A(2). The first step is to reprove Theorem 1(b), that localisation at an odd prime p allows A to have a commutative comultiplication.

There is a reduced (2,q+1)-cochain g{0}(a,b) such that

ba = (ab)z*g{0}(a,b)

and clearly g{0}(b,a) = -g{0}(a,b). Hence changing the comultiplication by ½g{0} will produce a commutative comultiplication. Any further change by a symmetric (2,q+1)-cochain will produce other commutative comultiplications.

Let ¤ be the associator of a commutative comultiplication. Since there is an associative comultiplication, ¤ is a coboundary Ð*g. We can suppose that g(a,b) is asymmetric in a,b, for otherwise we will change the comultiplication by the cochain -½(g(a,b)+g(b,a)), which will produce another commutative comultiplication for which the associator is the coboundary of the asymmetric cochain ½(g(a,b) - g(b,a)). Since

¤(a,b,c) = g(b,c) - g(a+b,c) + g(a,b+c) - g(a,b),

the asymmetry of g implies that ¤(c,b,a) = Ð*g = ¤(a,b,c). Therefore the first relation in Theorem 3.4.3 implies that ¤ is zero, so that the commutative comultiplication is associative.

The technique just described can be used to clean up the associator in general. For the rest of this chapter I will assume Theorem 3.2.4, that the only p-torsion in the fourth cohomology group of the Lie analyser is L^(4,2p). The interest of the next theorem is that this cannot be involved in an associator, which eventually has implications for the desuspension spectral sequence of S^m.

[continued]

THEOREM 3.4.5 A small co-h-space has a commutative comultiplication for which the associator involves only Whitehead products whose weights are powers of p.

PROOF: The only non-zero cohomology of the homotopy analyser in dimension 3 derives from L^(3,p) and L^(4,2p). Let ¤ be the associator of a commutative comultiplication. Then ¤ is the sum ¤' + ¤" of cochains where ¤' consists only of products whose weights are powers of p, and ¤" is free of such products. It follows from the analysis at the end of chapter 3 that the coboundaries of ¤' and ¤" have similar properties to ¤' and ¤" respectively. Therefore both ¤' and ¤" are cocycles, and furthermore each satsfies the equations in Theoren 3.4.3. We will allow ourselves to change the comultiplication by symmetric cochains which do not include a product whose weight is a power of p; this will change only ¤" and leave ¤' unaltered. To prove the Theorem it suffices to show that we can make ¤" zero in this manner.

We now express ¤" as the sum of a coboundary Đ*g and a carefully chosen representative u of its cohomology class: this class must lie in Tor(L^(4,2p),G{2p}) and so is determined by an element β of order p in G{2p}, the (q+1)-st homotopy group of S^(1+2p(m-1)). For convenience, let a,b,c be the Lie generators x{1},x{2},x{3}. The generator of L^(4,2p) is the Bochstein of the mod. p class Đ*([a,b]^p). There is a Lie polynomial h(a,b) congruent modulo p to Đ*(a^p), as is h(b,a). By replacing h with ½(h(a,b)+h(b,a)) we can ensure that h is symmetric. An easy calculation shows that

Đ*([a,b]^p) = h([a,c],[b,c]) - h([a,c],[a,b]) modulo p,
= h'(a,b,c), say.

Equally easy to prove is that the right hand side is invariant under the permutation which exchanges a and c (since h is a homogeneous polynomial of odd degree). We now use the isomorphism of the analyser with that of LAH in AAH explained at the end of the previous chapter to define u as the coboundary of [a,b]^p|β. Thus u is congruent to the composition of h' with β, modulo elements of higher filtration.

As in the proof of the previous theorem we can suppose that the 2-cochain g(a,b) is asymmetric, for otherwise we can change the comultiplication by the symmetric cochain -½(g(a,b)+g(b,a)), which replaces g by ½(g(a,b)-g(b,a)). Since u + Đ*g satisfies the first equation in Theorem 3.4.3, we find (as in the proof of that Theorem) that

u(a,b,c) + u(b,c,a) + 2Đ*g =0.

This proves firstly that Đ*g is zero in filtrations less than 2p, so we can suppose that g itself has filtration at least 2p. Secondly, looking at terms of weight exactly 2p we see that 2h' composed with β is a coboundary modulo elements of higher filtration. This implies that β and so also u is zero. Thirdly, since u is zero so also is Đ*g. Thus ¤" is zero (for this corrected commutative comultiplication), and the theorem is proved.

REMARK This shows that the cohomology class of the associator ¤ lies entirely in the part of the cohomology of the homotopy analyser stemming from L^(3,p). This is of interest in the spectral sequence of the desuspension of S^m.

THIRD PART [5] The Desuspension Spectral Sequence

Letter:-Barratt to Berstein/Harper

Let P be the suspension of a simply-connected space B. The **desuspension spectral sequence of P** to be described here starts with E^1-term the (reduced) graded homotopy of the bouquets P(n) and converges to the graded homotopy of B. Here the reduced homotopy of a bouquet P(n) means the intersection of the kernels of the n obvious retractions on P(n-1). In the case that B is itself a suspension, so that P is an abelian cogroup, the E^1-term is the reduced homotopy analyser of P and the E^2-term is the cohomology of this analyser, as will be proved below. This desuspension spectral sequence is a special case of the **spectral sequence of an inclusion** that I described in the notes published by Aarhus on their 1962 conference. It is the case of the inclusion of B into the cone on B.

Let C mean the reduced cone functor and, for n>1, CB[n] the (n+1)-ad whose total space is the result of identifying with B the bases of n disjoint cones CB{1},...,CB{n}, and whose n subspaces are the results of making similar constructions omitting each of the cones in turn. CB[n] is an excisive (n+1)-ad and hence by a theorem of Toda's it will be (nr+1)-connected if if B is (r-1)-connected.

Let CB'[n] be the result of adding an extra cone CB{n+1} to each of the subspaces of CB[n]. The total space of this (n+1)-ad is that of CB[n+1], and its subspaces are the first n subspaces of that (n+2)-ad. One of the exact sequences of the (n+2)-ad connects the homotopy groups of CB[n+1], CB[n] and CB'[n].

In addition, CB[1] is to be the pair (CB{1},B), CB'[1] is the result of adding a cone CB{2} on B to both the total space and subspace of CB[1], CB[0] is B, and CB'[0] is CB{1}. Thus we may add two more exact sequences to the previous collection, the last being the exact sequence of the pair CB[1].

This collection of exact sequences defines an exact couple in the sense of Blakers and Massey. The associated spectral sequence is the Desuspension Spectral Sequence of P. The reasons for the name are that firstly the spectral sequence converges to the homotopy of B, and secondly the homotopy type of CB'[n] is that of P[n] whose total space is the bouquet P(n) of n copies of P, with subspaces the n copies of P(n-1) in which each of the factors P{i} are omitted in turn. (This is easily seen by pinching the common contractible subspace CB{n+1} to a point.) It is well-known that the q-th homotopy group of the (n+1)-ad P[n] is isomorphic to the quotient of the q-th homotopy group of the bouquet P(n) by the subgroup generated by images of the homotopy group of P(n-1) under the n inclusion maps. This is isomorphic to the summand which is the intersection of the kernels of the n retractions of P(n) on P(n-1).

[continued]

We now turn to the homotopy analyser of P. We take the alternate model of the cosimplicial complex K* using the generators y{i} mentioned in the first chapter. The homotopy analyser defined in this way is equivalent to the one previously defined by the maps induced by the isomorphism of cosimplicial complexes which on K*^n maps x{n} to y{n} and for every i<n maps x{i} to y{i} - y(i+1}. The reduced analyser is isomorphic to the quotient cosimplicial group obtained by dividing out by the subgroup which is generated by the images of all the coface operators except the last. In the case of the homotopy analyser of P this means dividing out by the images of the homotopy of all P(n-1) in P(n), so the groups of the reduced homotopy analyser of P can be identified with the homotopy groups of the (n+1)-ad P[n] in dimension n. This proves the first part of

<u>THEOREM 3.5.1</u> Let P be the suspension of a simply-connected suspension B. Then the E^1 term of the desuspension spectral sequence of P is isomorphic to the reduced homotopy analyser of P.

Moreover, the E^2 term of the desuspension spectral sequence is isomorphic to the cohomology of the homotopy analyser of P.

PROOF: One advantage of dealing with the reduced analyser as a quotient of the analyser is that, having factored out the images of all except the last face operator, the differential reduces to that last face operator, which is induced by the map which on K*^n sends each y{i} to y{i} - y{n+1}. In the desuspension spectral sequence the first differential can be described by means of the map of the total space of CB'[n] to that of P[n+1] which pinches the common subspace B to a point. The i-th factor of P[n] corresponds in CB'[n] to the union of the cone CB{i} with the extra cone CB{n+1}, and the map pinching the common base B to a point maps this to the bouquet of P{i} and P{n+1} through the comultiplication map followed by the identity map onto P{i} and the identity onto P{n+1} with suspension parameter reversed, which therefore is the topological realisation of the map sending y{i} to y{i} - y{n+1}. This establishes the identification of the differential in the homotopy analyser with the first differential in the desuspension spectral sequence, and completes the proof of 3.5.1.

[continued]

The Desuspension Spectral Sequence for a sphere of odd dimension m, localised at an odd prime p, has E^2 term governed by the cohomology of the Lie analyser. The LIE ANALYSER CONJECTURE would imply that there are no groups in E^2 in dimensions above 4, and the spectral sequence has to converge at E^4. In fact, the last result of the previous chapter and Chan's Theorem in the next will enable us to draw a much stronger inference, described at the end of the next chapter.

However, by using the spectral sequence to compute the rational homotopy of even dimensional spheres, which is known, we can prove that the Lie Analyser has no rational cohomology except for L^(1,1) and L^(2,2). The argument uses the fact that the same cohomology appears in the spectral sequence for all odd spheres, and the idea is to select the first unexpected class in an appropriate ordering, and then select a sufficiently large value of m so that this class must create unexpected rational homotopy on S^(m-1), creating a contradiction. I omit the details for the moment.

The desuspension spectral sequence for a sphere of even dimension, or any sphere localised at 2, for similar reasons must reduce to the EHP sequence. This is evidence that the Lie Analyser cohomology localised at 2, and the graded Lie Algebra cohomology where the generators have odd dimension, consists only of the cyclic groups at (1,1) and (2,2). It would be nice to have an algebraic proof of these algebraic facts.

The Desuspension Spectral Sequence has recently been re-discovered and generalised by Mike Hopkins, who has observed that it is the dual of the Milner-Moore-Rothenburg-Steenrod spectral sequence, which gives grounds for hope that it will converge at a finite term if B has finite co-category. He adds that this would predict that for a sphere of even dimension, localised at an odd prime, it should converge at E^2, which is true; for a sphere of odd dimension convergence should be at E^3, which (as we shall see) is indeed implied by the Lie Analyser conjecture. In fact, I believe we can show that the Lie Analyser conjecture must be true if the spectral sequence for S^3 converges at E^3. Convergence at E^3 will produce with Chan's Theorem a corollary that a small co-h-space localised at an odd prime must be a suspension if it admits an associative comultiplication.