CHARACTERISTIC CLASSES IN SG

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0. Introduction.

In this paper, we prove four related theorems concerning the structure of the monoid SG of stable homotopy equivalences of spheres.

In § 2 we study the homomorphisms from $H^*(\mathbb{R}P^\infty)$ to $H^*(SG)$ which preserve the action of the Stenrod algebra. In effect we evaluate the $\mathbb{Z}/2$-vector space $\mathrm{Hom}_A(H^*(\mathbb{R}P^\infty), PH^*(SG))$ where A is the mod 2 Steenrod algebra and $PH^*(SG)$ denotes the primitive elements of the Hopf-algebra $H^*(SG;\mathbb{Z}/2)$.

In § 3 we relate the A-module maps $f: H^*(\mathbb{R}P^\infty) \to PH^*(SG)$ to the Kervaire classes in $PH^*(SG)$, i.e. to the map $SG \to G/TOP$.

In order to do so, we use a certain cohomology operation $Sq^1\varphi$. The operation φ contains exactly half of the terms occuring in the Adem relation corresponding to $Sq^{2^n} Sq^{2^n}$ used by Adams to construct the secondary operations $\varphi_{n,n}$ see [1].

Section 4 deals with the infinite loop map $e: SG \to SJ$ inducing the Adams e-invariant in homotopy. Using homology operations and the main theorem of § 2 we compute the induced map $e: PH^*(SJ) \to PH^*(SG)$.

We finally apply this computation and the results of § 3 to a specific problem motivated by the surgery theory in [4]. In the situation considered there, we are given normal maps of degree 1 between manifolds of dimension $4n+2$

$$(f,\hat{f}): (M,\nu M) \to (N,\nu N)\ ;\ (g,\hat{g}): (N,\nu N) \to (X,\xi)\ .$$

By a theorem of Brown [2], the Kervaire invariant of the composite $(gf,\hat{g}\hat{f}): (M,\nu M) \to (X,\xi)$ is given by the formula

$$K(gf,\hat{g}\hat{f}) = K(g,\hat{g}) + K(f,\hat{f}) + \mu(\Theta^* \sigma w_{2n+3})\ .$$

Here Θ is a homotopy class of maps $N \to SG$, $\sigma w_{2n+3} \in H^{2n+2}(SG)$ is the suspension of the Stiefel - Whitney class, and μ is a certain quadratic form associated with the normal map $(f,\hat{f})$.

In order to use this formula, we have to compute $\Theta^*(\sigma w_{2n+3})$. In § 5 we make this computation for a case considered in [4].

All homology and cohomology groups in this paper have coefficients in $\mathbb{Z}/2$.

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§ 1 The homology of QS^0.

In this section we recall some wellknown facts about the homology of the infinite loop space $QS^0 = \Omega^\infty S^\infty$. For more details and for proofs we refer the reader to [5].

The Dyer - Lashof algebra R is defined as the algebra over $\mathbb{Z}/2$ generated by the symbols $Q^a (a = 0,1,\ldots)$ modulo the Adem relations

1.1
$$Q^a Q^b = \sum_t \binom{t-b-1}{2t-a} Q^{a+b-t} Q^t .$$

Theorem 1.2: Let X be an infinite loopspace. Then R acts on $H_*(X)$, satisfying the following conditions:

(i) $Q^a(x) = 0$ if $a < \deg x$

$Q^a(x) = x \cdot x$ if $a = \deg x$

$Q^a(1) = 0$ if $a > 0$

(ii) $Q^a(xy) = \sum_i Q^i(x) Q^{a-i}(y)$

(iii) $(Sq^a)_* Q^b(x) = \sum \binom{b-a}{a-2t} Q^{b-a+t} (Sq^t)_* x$.

In (iii), the so called Nishida relations, $(Sq^a)_*$ is the vector space dual of the Steenrod square Sq^a.

Let $I = (i_1,\ldots i_k)$ be a sequence of positive integers. If $i_r \le 2i_{r+1}$ then I is called an allowable sequence.

We then define the length $\ell(I) = k$ and the excess $\mathrm{exc}(I) = i_1 - (i_2 + \ldots + i_k)$.

The set of monomials $Q^{i_1} Q^{i_2} \ldots Q^{i_k}$ of length k span a coalgebra $R[k] \subset R$. The dual $R[k]^*$ is a polynomial algebra

$$R[k]^* = \mathbb{Z}/2[\xi_{1k},\dots,\xi_{kk}] .$$

The elements $\xi_{i,k}$ are defined by

$$< \xi_{ik}, \ Q^{I_{ik}} > = 1$$

$$< \xi_{ik}, \ Q^{J} > = 0 \qquad J \neq I_{ik}$$

where

$$I_{ik} = (2^{k-i-1}(2^i - 1), \ 2^{k-i-2}(2^i - 1), \dots, \ 2(2^i - 1), \ 2^i, \dots 2, 1) .$$

In particular, $\deg\xi_{i,k} = 2^{k-i}(2^i - 1)$.

The Nishida relations make $R[k]^*$ into an algebra over the Steenrod algebra A , completely determined by the formulas

1.3 $$Sq^{2^r}\xi_{i,k} = \begin{cases} \xi_{i+1,k} & r = k-i+1 \\ \xi_{i,k}\,\xi_{i,k} & r = k-1 \\ 0 & r \neq k-1, \ k-i-1 \end{cases}$$

Now, consider the space $QS^0 = \Omega^\infty S^\infty$.
It has an obvious infinite loop space structure, which we call the loop sum structure. The associated Dyer - Lashof operations are denoted by Q^a , and the Pontryagin product of x and y by $x_* y$.

Composition of maps define a different H - space structure on QS^0 . The Pontryagin product in this structure of x and y is denoted $x\,y$.

On the one - component $(QS^0)_1$ we can extend this to an infinite loop space structure.

The corresponding Dyer - Lashof operations are denoted by $\hat{Q}^a$. They can be extended to operations on all of H_*QS^0 , satisfying the conditions of theorem 1.2.

The relations between the operations belonging to the two structures are complicated.

The space $(QS^0)_1$ with the composition product can be identified with the space SG of orientation preserving stable homotopy equivalences of spheres.

We can now describe the Hopf algebra $H_*(SG) = H_*((QS^0)_1)$.

<u>Theorem 1.4</u>: $H_*(SG) \cong H_*(SO) \otimes \bigotimes_{i \geq 2}^{\infty} C_i$

where

$$C_2 = P(Q^I[1] * [-3] \mid I \text{ allowable, } \ell(I) = 2, \text{exc } I \geq 0)$$

$$C_k = P(Q^I[1] * [1-2^k] \mid I \text{ allowable, } \ell(I) = k, \text{exc } I \geq 1)$$

for $k \geq 3$.

Here $H_*(SO)$ is included in $H_*(SG)$ by the natural map $i : SO \to SG$. Note that the generators are defined using the loop sum structure.

We will eventually need the following

<u>Lemma 1.5</u>:

$$Q^{2^i} Q^{2^{i-1}} \cdots Q^2 Q^1 [1] * [1-2^{i+1}] + \hat{Q}^{2^i} \hat{Q}^{2^{i-1}} \cdots \hat{Q}^4 (Q^2 Q^1 [1] * [-3])$$

is decomposable in the composition product.

Actually, one of the main technical results of [6] is that

$$\hat{Q}^a (Q^I[1] * [1-2^{\ell(I)}]) = \hat{Q}^a (Q^I[1]) * [1-2^{2\ell(I)}] + Q^a Q^I[1] * [1-2^{\ell(I)+1}]$$

modulo decomposables. But if $a = \deg(I) + 1$ then $\hat{Q}^a (Q^I[1]) = 0$ e.g. as in the proof of 8.8 in [3].

The space BSG classifies spherical fibrations. Since these have well-defined Stiefel - Whitney classes, we have universal classes $w_i \in H^i(BSG)$. Let $\sigma w_i \in H^{i-1}(SG)$ be the cohomology suspension of w_i.

Let $k_{4n-2} \in H^{4n-2}(G/TOP)$ be the primitive classes related to the square of the universal Wu - class by the surgery formula as in [8].

Let $p : G \to G/TOP$ be the natural projection.

<u>Theorem 1.6</u>:

$$< \sigma w_i \, , \, Q^I[1]*[1-2^{\ell(I)}] > \; = 0 \qquad \ell(I) > 1$$

$$< \sigma w_i \, , \, Q^{i-1}[1]*[-1] > \; = 1$$

$$< p^*k_{2^n-2} \, , Q^{2^{n-1}-1}Q^{2^{n-1}-1}[1]*[-3] > \; = 1$$

$$< p^*k_{4n-2} \, , \, Q^I[1]*[1-2^{\ell(I)}] > \; = 0 \qquad \text{else} .$$

This follows from [3] 3.3 and 3.5.

By abuse of notation, we will denote $p^*k_{2^n-2}$ by k_{2^n-2}.

Let $Q(C_i)$ be the module of indecomposable elements of C_i. There is an evaluation map

$$ev : R[k] \to C_k \to Q(C_k)$$

defined by

$$ev\,(Q^I) = \text{the class of } \; Q^I[1]*[1-2^{\ell(I)}] \,.$$

The dual of ev is a map

$$ev^* : P(C_k) \cong Q(C_k)^* \to R[k]^* \,.$$

<u>Lemma 1.8</u>: ev^* is injective. The image of ev^* is

$$R[1]^* \oplus R[2]^* \oplus \bigoplus_{k \geq 3} (\xi_{k,k} R[k]^*) .$$

<u>Proof</u>: By 1.4 is

$$ev : R[k] \to \bigoplus_{k=1}^{\infty} Q(C_k)$$

surjective, so ev^* is injective. The image consists of the classes that evaluate trivially on all classes

$Q^I[1]*[1-2^{\ell(I)}]$ with $\ell(I) \geq 3$, $exc(I) = 0$.

The lemma follows from the definition of $\xi_{i,k}$.

§ 2 Maps into $R[k]^*$ and $PH^*(SG)$.

Let M denote $H^*(\mathbb{R}P^\infty)$ considered as a graded module over the Steenrod algebra. Then is $M = \mathbb{Z}/2[t]$, $\deg(t) = 1$, and the action of the Steenrod algebra A is given by

2.1 $$Sq^i\, t^j = \binom{j}{i}\, t^{i+j}$$

Recall from 1. that

$$R[k]^* = P(\xi_{1,k} \dots \xi_{k,k}) \text{ , } \deg(\xi_{i,k}) = 2^{k-i}(2^i - 1) \text{ .}$$

<u>Definition 2.2</u>: Let $\chi_{i,k} \in R[k]^*$ be the component in degree i of the formal power series

$$\xi_{k,k} / (1 + \xi_{1,k} + \dots \xi_{k,k}) \text{ .}$$

In § 1 we introduced the dual of the evaluation map. It gives an embedding

$$ev^*: PH^*(SG) \to \bigoplus_{k=1}^{\infty} R[k]^* \text{ .}$$

It follows from 1.8 that $\chi_{i,k} \in \mathrm{Im}(ev^*)$, say $\chi_{i,j} = ev^*(p_{i,j})$. The main result in this section is

<u>Theorem 2.3</u>: There is an isomorphism

$$\rho : \prod_{k=1}^{\infty} (\mathbb{Z}/2)_k \to \mathrm{Hom}_A(H^*(\mathbb{R}P^\infty) \text{ , } PH^*(SG))$$

$$\rho(a_k)_{k=1}^{\infty}\; t^j = \sum_{i=1}^{\infty} a_k\, (ev^*)^{-1}\, \chi_{i,k} \text{ .}$$

In order to prove 2.3 we first show that $\mathrm{Hom}_A(M, R[k]^*) \approx \mathbb{Z}/2$. We begin by proving that a given $f : M \to R[k]^*$ has to be unique.

Suppose $f : M \to R[k]^*$ is a nonzero map. For dimensional reasons, $f(t^{2^i-1}) = 0$ when $i < k$, and $f(t^{2^k-1}) = \xi_{k,k}$ since $\xi_{k,k}$ is the only nontrivial element of dimension 2^k-1 in $R[k]^*$.

Our first result is

<u>Lemma 2.4</u>: For all $m \geq 0$

(i) $Sq^{2^i} f(t^{2^m-1}) = 0$ if $i < k-1$

(ii) $Sq^{2^k-1} f(t^{2^m-1}) = (Sq^{2^{k-1}} f(t^{2^{m-1}-1}))^2$.

<u>Proof</u>: Let $M_k \subset M$ be the A-submodule generated by $t, t^3 \ldots t^{2^s-1} \ldots t^{2^k-1}$.

We have already remarked that these generators are mapped trivially, so that $f(M_k) = 0$.

One gets from 2.1

$$t^{2^m + 2^{(i-1)} - 1} = Sq^{2^{m-1}} \ldots Sq^{2^{i-1}} t^{2^i-1}$$

so $$t^{2^m+1}, t^{2^m+3}, \ldots t^{2^{(k-2)}-1} \in M_k .$$

But $Sq^j t^{2^m + 2^i - 1} = t^{2^m + 2^i + j - 1}$

when $j \leq 2^i-1$, so $t^{2^m+j} \in M_n$ for $1 \leq j \leq 2^{(k-2)} - 2$.

Hence $f : H^{2^m+j}(\mathbb{R}P^\infty) \to R[k]^*$ is zero for $1 \leq j \leq 2^{(k-2)} - 2$.

In particular $Sq^{2^i} f(t^{2^m-1}) = f(Sq^{2^i} t^{2^m-1}) = 0$ proving assertion (i).

Furthermore, if $m > k$ then

$$Sq^{2^k-1}\, t^{2^m-1} = t^{2^m+2^k-2}$$

$$Sq^{2^{k-1}}\, t^{2^{m-1}} = t^{2^{m-1}+2^{k-1}-1} .$$

Thus

$$Sq^{2^k-1}\, t^{2^m-1} = Sq^{2^{k-1}+2^{m-1}-1}\,(Sq^{2^{k-1}}\, t^{2^{m-1}-1})$$

proving assertion (ii).

We find it convenient to make the following

Definition 2.5: $A_r = \bigcap_{i=1}^{k} \operatorname{Ker} Sq^{2^{i-1}} : R[k]^* \to R[k]^*$.

From the Cartan formula we have that $Sq^{2^{r-1}} : A_{r-1} \to A_{r-1}$ is a derivation so inductively, $A_r = \operatorname{Ker}\ Sq^{2^{r-1}} : A_{r-1} \to A_{r-1}$ is a ring.

Note that 2.4 (i) is equivalent to the statement: $f(t^{2^m-1}) \in A_{k-1}$ for all m .

The point of 2.5 is that we can evaluate Sq^{2^k-1} on A_{k-1} in the appropriate dimensions. In particular we evaluate Sq^{2^k-1} on $f(t^{2^m-1})$.

Lemma 2.6:

$$A_{k-r} = P(\xi_{1,k};\ldots;\xi_{r-1,k};\xi^2_{r,k};\xi^{2^{k-r}}_{k-1,k};\xi_{k,k})$$

Proof: $A_1 = \operatorname{Ker}(Sq^1 : R[k]^* \to R[k]^*) =$

$$= P(\xi_{1,k};\ldots;\xi_{k-2,k};\xi^2_{k-1,k};\xi_{k,k}) .$$

The derivation Sq^2 on A_1 is given by the formulas

$$Sq^2 \xi_{i,k} = 0 \qquad i \neq k-2,\ k-1$$

$$Sq^2 \xi_{k-2,k} = \xi_{k-1,k}$$

$$Sq^2 \xi^2_{k-1,k} = \xi^2_{k,k}$$

We set

$$K_2 = p(\xi_{1,k}; \ldots; \xi_{k-3,k}; \xi^2_{k-2,k}; \xi^4_{k-1,k}; \xi_{k,k}) .$$

Then $K_2 \subset A_2$, and we claim that in fact $K_2 = A_2$.

Let $x \in A_2 \setminus K_2$.

We can write $x = \xi_{k-2,k} x' + x''$ where $\xi_{k-2,k}$ occurs with even degrees in all monomials of x' and x'' .

Since $x \in A_2$

$$0 = Sq^2 x = \xi_{k-1,k} x' + \xi_{k-2,k} Sq^2 x' + Sq^2 x'' .$$

It follows that $x' = 0$, because $\xi_{k-1,k}$ occurs with even degree in x' , $Sq^2 x'$, and because $Sq^2 x'' \in A_1$.

So $x = \xi^2_{k-1,k} x^{(3)} + x^{(4)}$ for some $x^{(3)}, x^{(4)} \in K_2$, and $0 = Sq^2 x = \xi^2_{k,k} x^{(3)}$. Thus $x = x^{(4)} \in K_2$, contrary to assumption.

This procedure can be repeated and gives an inductive proof of the lemma.

It is an easy and well-known consequence of formula 2.1 that the elements $t^{2^i} - 1 \in H^*(\mathbb{R}P^\infty)$ are the A-module generators. We have the following inductive result

<u>Theorem 2.7</u>: Let $f : H^*(\mathbb{R}p^\infty) \to R[k]^*$ be a map of A-modules. Then f is uniquely determined by the formula

$$f(t^{2^{k+\ell}-1}) = \sum_{i=1}^{\min(\ell,k)} \xi_{i,k}^{2^\ell} f(t^{2^{k+\ell-i}-1}) .$$

<u>Proof</u>: Let $p_{2^i-1} = f(t^{2^i-1})$. From lemma 2.4(i) we have $p_{2^i-1} \in A_{k-1}$ where by lemma 2.6

$$A_{k-1} = p(\xi_{1,k}^2 ; \xi_{2,k}^4 ; \ldots ; \xi_{k-1,k}^{2^{k-1}} ; \xi_{k,k}) .$$

Since $\deg(\xi_{i,k}) = 2^{k-1}(2^i-1)$

$$\deg(\xi_{i,k}^2) \equiv \deg(\xi_{2,k}^4) \equiv \ldots \equiv \deg(\xi_{k-1}^{2^{k-1}}) \equiv 0 \quad (2^k) .$$

But $\deg(p_{2^i-1}) \equiv -1 \ (2^k)$ so $p_{2^i-1} = \xi_{k,k}\, x$

where $x \in p(\xi_{1,k}^2 ; \xi_{2,k}^4 ; \ldots ; \xi_{k-1,k}^{2^{k-1}} ; \xi_{k,k}^{2^k})$

and then

$$Sq^{2^k-1} p_{2^i-1} = \xi_{k,k}\, p_{2^i-1} .$$

Indeed,

$$Sq^{2^k-1} p_{2^i-1} = Sq^{2^k-1}(\xi_{k,k}\, x) = (Sq^{2^k-1}\xi_{k,k})x =$$

$$= \xi_{k,k}^2\, x = \xi_{k,k}\, p_{2^i-1}$$

We can now prove the theorem by induction. Because, assume it is true for ℓ . Then, by lemma 2.4 (ii)

$$\xi_{k,k}\, P_{2^{k+\ell+1}-1} = (Sq^{2^{k-1}}\, P_{2^{k+\ell}-1})^2$$

$$= (\sum_i Sq^{2^{k-1}}(\xi_{i,k}^{2^\ell}\, P_{2^{k+\ell-i}-1}))^2 =$$

$$= (\xi_{k-\ell,k}^{2^\ell}\, P_{2^k-1})^2 + \sum_i (\xi_{i,k}^{2^\ell})^2\, \xi_{k,k}\, P_{2^{k+\ell-i+1}-1} =$$

$$= \sum_i \xi_{i,k}^{2^{\ell+1}}\, \xi_{k,k}\, P_{2^{k+(\ell+1)-i}-1}\,.$$

__Theorem 2.8__: $\mathrm{Hom}(H^*(\mathbb{R}P^\infty), R[k]^*) \cong \mathbb{Z}/2$, and the generator is given by $f(t^i) = \chi_{i,k}$ where $\chi_{i,k}$ is the i-dimensional component of the formal power series

$$\xi_{k,k}/(1+\xi_{1,k}+\ldots+\xi_{k,k}) \in R[k]^*$$

__Proof__: By theorem 2.7 it is enoguh to show that f is a map of modules over the Steenrod algebra. This is equivalent to the identity

$$Sq^i\, \chi_{j,k} = \binom{j}{i}\, \chi_{i+j,k}$$

where $\binom{j}{i}$ as usual are the binomial coeffecients.

The action of A on $R[k]^*$ is given in principle by 1. We note that

$$Sq^{2^{k-i}-2^{k-j}}\, \xi_{j,k} = \xi_{i,k}$$

$$Sq^{2^{k-1}}\, \xi_{j,k} = \xi_{1,k}\,\xi_{j,k}$$

$$Sq^{2^k+2^{k-j}-2^{k-a}-2^{k-b}}\, \xi_{j,k} = \xi_{a,k}\,\xi_{b,k}\;;\; a+1 \le j \le b$$

$$Sq^i\, \xi_{j,k} = 0 \qquad \text{else}$$

In particular $Sq^i \chi_j = \binom{j}{i} \chi_{j+i}$ for $j \leq 2^k-1$ since $\chi_{2^k-1} = \xi_{k,k}$; $\chi_{2^{k+1}-2^{k-j}-1} = \xi_{j,k}\, \xi_{k,k}$.

Assume inductively that the claim is true for $n < m$. We will show that $Sq^u \chi_m = \binom{m}{u} \chi_{m+u}$.

$$Sq^u \chi_m = \sum_{i=1}^{k} \sum_{v=0}^{u} Sq^v(\xi_{i,k}) \binom{m-2^k+2^{k-i}}{u-v} \chi_{m-2^k+2^{k-i}+u-v}$$

The coefficient of $\xi_{a,k}\, \xi_{b,k}\, \chi_{m-2^{k+1}+2^a+2^b}$ in this sum is

$$\sum_{s=a+1}^{b} \binom{m-2^k+2^{k-s}}{u-2^k-2^{k-s}+2^{k-a}+2^{k-b}} = \binom{m-2^k+2^{k-b}}{u-2^k+2^{k-b}} + \binom{m-2^k+2^{k-a}}{u-2^k+2^{k-a}}$$

since for each pair x,y we have the identity between mod 2 binomial coefficients

$$\sum_{s=q}^{p-1} \binom{x+2^s}{y-2^s} = \binom{x+2^p}{y-2^q} + \binom{x+2^q}{y-2^p}$$

(This follows e.g. by an easy inductive argument).

We collect terms to obtain

$$Sq^u \chi_m = \sum_{i=1}^{k} \left[\binom{m-2^k+2^{k-i}}{u-2^k+2^{k-i}} + \sum_{j=1}^{i} \binom{m-2^k+2^{k-j}}{u-2^{k-j}+2^{k-i}}\right] \xi_{i,k} \chi_{m+u-2^k+2^i}$$

$$= \sum_{i=1}^{k} \binom{m}{u} \xi_{i,k}\, \chi_{m+u-2^k+2^i} = \binom{m}{u} \chi_{m+u}$$

We can now give the proof of theorem 2.3.

Let $f_k : \mathbb{Z}/2 \to Hom_A(H^*(\mathbb{R}P^\infty), R[k]^*)$ be the isomorphism specified in theorem 2.8. Then

$$\rho : \prod_{k=1}^{\infty} (\mathbb{Z}/2)_k \to \mathrm{Hom}_A(H^*(\mathbb{R}P^{\infty}) , PH^*(SG))$$

is the composite of the two isomorphisms

$$\prod_{k=1}^{\infty} f_k : \prod_{k=1}^{\infty} (\mathbb{Z}/2)_k \to \prod_{k=1}^{\infty} \mathrm{Hom}_A(H^*(\mathbb{R}P^{\infty}) , R[k]^*)$$

$$\sigma : \prod_{k=1}^{\infty} \mathrm{Hom}_A(H^*(\mathbb{R}P^{\infty}) , R[k]^*) \to \mathrm{Hom}_A(H^*(\mathbb{R}P^{\infty}) , PH^*(SG))$$

where $\sigma(z_k)_{k=1}^{\infty} = \sum_{k=1}^{\infty} ev^*(z_k)$.

§ 3 On $H^*(\mathbb{R}P^\infty) \to R[2]^*$

Let $f : H^*(\mathbb{R}P^\infty) \to R[2]^*$ be the unique nonzero map constructed in 2.8.

Put $f_i = f(t^{2^i-1})$. Then by 2.7 f is characterized by the inductive formula

3.1 $$f_i = \xi_{1,2}^{2^{i-2}} f_{i-1} + \xi_{2,2}^{2^{i-2}} f_{i-2} .$$

Recall from 1.6 that the classes $\xi_1^{2^{i-1}-1}$ can be identified with the Kervaire classes k_{2^i-2} . It is known from [6] that these geometrically important classes generate $R[2]^*$ as a module over the Steenrod algebra.

In this paragraph we give the following explicit expression for f_i as a sum of Steenrod operations on the Kervaire classes

3.2 $$f_i = Sq^1 k_{2^i-2} + \Big(\sum_{2\le 2j\le i-2} Sq^{2^i-2^{i-2j}+1} Sq^{2^{i-2j}} \Big) k_{2^{i-1}-2}$$

Note that only two Kervaire classes occur in this expression.

The main step in the proof on 3.2 is the calculation of

$$Sq^{2^i-2^{i-2j}} Sq^{2^{i-2j}} k_{2^{i-1}-2}$$

as a polynomial in $\xi_{1,2}$ and $\xi_{2,2}$.

We begin with the definition of some particular elements in $R[2]^*$. To simplify notation, write $a = \xi_{1,2}$ $b = \xi_{2,2}$. Then we have

$$\begin{array}{ll} Sq^1\, a = b & Sq^2\, b = ab \\ Sq^2\, a = ab & Sq^3\, b = b^2 \\ Sq^i\, a = 0 \quad i \neq 0,1,2 & Sq^i\, b = 0 \quad i \neq 0,1,3 \end{array}$$

Define classes

$$x_i = Sq^{2^i} a^{2^i - 1} \qquad \deg x_i = 3\cdot 2^i - 2$$

3.3 $$y_i = Sq^{2^i} x_i \qquad \deg y_i = 4\cdot 2^i - 2$$

$$z_i = Sq^{2^{i+1}} x_i \qquad \deg z_i = 5\cdot 2^i - 2$$

We need the following relations

(i) $$x_i = b^{2^{i-1}} x_{i-1} + a^{3\cdot 2^{i-1} - 1}$$

3.4 (ii) $$y_i = (a\,b)^{2^{i-1}} x_{i-1} + b^{2^{i-1}} z_{i-1} + a^{2^{i-1}-1} b^{2^i}$$

(iii) $$z_i = b^{2^i} y_{i-1} + (a\,b)^{2^{i-1}} z_{i-1} + a^{5\cdot 2^{i-1} - 1}$$

The proof is a standard calculation, using the facts

$$Sq^r(u^{2^i}) = 0 \quad \text{unless} \quad 2^i | r$$

$$Sq^{2^i r}(u^{2^i}) = (Sq^r u)^{2^i}$$

For example, given (i) we have

$$\begin{aligned} y_i = Sq^{2^i} x_i &= Sq^{2^i}(b^{2^{i-1}} x_{i-1} + a^{3\cdot 2^{i-1}-1}) \\ &= (Sq^2 b)^{2^{i-1}} x_{i-1} + b^{2^{i-1}} Sq^{2^i} x_{i-1} + \\ &\quad + (Sq^1 a)^{2^i} a^{2^{i-1}} + a^{2^i} Sq^{2^i} a^{2^{i-1}-1} \\ &= (a\,b)^{2^{i-1}} x_{i-1} + b^{2^{i-1}} z_{i-1} + a^{2^{i-1}-1} b^{2^i} \end{aligned}$$

But this is the formula 3.4 (ii).

From 3.4 we get the following recursive formula for y_i

3.5 $$y_{i+1} = a^{2^{i-1}} b^{2^i} y_i + b^{2^{i+1}} y_{i-1} \qquad i \geq 2$$

Indeed, we can use 3.4 (ii) to express y_{i+1} as a function of x_i and z_i. Expanding this function using 3.4 (i) and 3.4 (ii) we express y_i as a sum. After rearranging

$$y_{i+1} = b^{2^{i+1}} y_{i-1} + a^{2^{i-1}} b^{2^i}((ab)^{2^{i-1}} x_{i-1} + b^{2^{i-1}} z_{i-1} + $$
$$+ a^{2^{i-1}-1} b^{2^i}) = b^{2^{i+1}} y_{i-1} + a^{2^{i-1}} b^{2^i} y_i$$

Next we define classes

$$t_{i,j} = Sq^{(2^j-1)2^i} Sq^{2^i}(a^{2^{i+j-1}-1})$$

These classes satisfy

3.6 $$t_{i,j} = a^{(2^j-4)2^i} y_{i+1} + a^{(2^j-2)2^i} y_i$$

This is a consequence of the following identity

$$Sq^{(2^j-1)2^i} Sq^{2^i}(a^{2^{i+j-1}}) =$$
$$= Sq^{(2^j-1)2^i} Sq^{2^i}((a^{2^{i+j-1}-2^{i+1}}) \cdot a^{2^i} \cdot a^{2^i-1})$$
$$= a^{2^i(2^j-2)} y_i + a^{(2^j-4)2^i}((ab)^{2^i} x_i + b^{2^i} z_i + a^{2^i-1} b^{2^{i+1}})$$

<u>Theorem 3.7</u>: Let f_i be the image of t^{2^i-1} under the unique non-zero A-module map $H^*(\mathbb{R}P^\infty) \to R[2]^* \to PH^*(SG)$. Then we have

$$f_i = Sq^1(k_{2^i-2} + \sum_{2 \le 2j \le i-2} Sq^{2^i-2^{i-2j}} Sq^{2^{i-2j}} k_{2^{i-1}-2})$$

where k_{2^j-2} denotes the Kervaire class in dimension 2^j-2.

<u>Proof</u>: We claim that $b^{2^i} f_i = Sq^1 y_i$. This is true for $i = 1,2$ since

$$y_1 = Sq^2 Sq^2 a = b^2 \ ; \ y_2 = Sq^4 Sq^4(a^3) = ab^4$$

and $f_1 = 0$; $f_2 = b$.

But the classes $Sq^1 y_i$ and $b^{2^i} f_i$ satisfy the same recursion formula, since by 3.5 and 3.1

$$Sq^1 y_{i+1} = a^{2^{i-1}} b^{2^i} Sq^1 y_i + b^{2^{i+1}} Sq^1 y_{i-1}$$

$$b^{2^{i+1}} f_{i+1} = a^{2^{i-1}} b^{2^{i+1}} f_i + b^{2^{i-1}} b^{2^{i+1}} f_{i-1}$$

$$= a^{2^{i-1}} b^{2^i} (b^{2^i} f_i) + b^{2^{i+1}} (b^{2^{i-1}} f_{i-1})$$

We can now rewrite formula 3.1 as

$$f_{i+1} = a^{2^{i-2}} f_i + Sq^1 y_i$$

Equivalently,

$$f_{i+1} = Sq^1 y_i + a^{2^{i-2}} Sq^1 y_{i-1} + \ldots + a^{2^i - 7} Sq^1 y_2 \ ,$$

and 3.7 follows from this and 3.6.

§ 4 The map PH*(S J) → PH*(S G)

In this paragraph all spaces are localized at the prime 2.

We use the results of § 2 and § 3 to evaluate the map $e^* : PH^*(SJ) \to PH^*(SG)$ induced by the e - invariant map $SG \to SJ$.

We begin by recalling some facts about the space SJ . For details and proofs see [5], [7].

Let $BO_{\otimes}$ and $BSpin_{\otimes}$ be organized as infinite loop spaces by the tensor product. The map $\Psi^3/1 : BO_{\otimes} \quad BSpin_{\otimes}$ is an infinite loop map. Let SJ denote its fibre with the induced infinite loop space structure.

There is an infinite loop map $e : SG \to SJ$, whose fibre is usually denoted Cok J . It is a consequence of the affirmed Adams conjecture that the fibration splits

$$SG \simeq SJ \times \text{Cok } J$$

but this equivalence is not an equivalence of H - spaces.

Let $i : Spin \to SJ$ be the natural map. Recall that $H_*(Spin)$ is an exterior algebra

$$H_*(Spin) = E(u_r \mid r \neq 2^q)$$

Define
$$t_r = e_*(Q^{2^s} Q^{2^s}[1]*[-3]) \quad \text{if} \quad r = 2^{s+1}$$
$$t_r = e_*(Q^{2^s(n+1)} Q^{2^s n}[1]*[-3]) \text{if} \quad r = 2^s(2n+1)$$
$$x_{2^r} = e_*(Q^{2^r}[1]*[-1])$$

where Q^i denotes the Dyer - Lashof operations in $H_*(QS^0)$ associated with loop sum.

Recall from [5] that

$$H_*(SJ) = E\{i_*u_r \mid r \neq 2^q\} \otimes E\{x_{2^q}\} \otimes P\{t_r \mid r \geq 2\}$$

$$H_*(BO_\otimes) = E\{p_*x_{2^q}\} \otimes P\{p_*t_r \; r \geq 2\}$$

where $p : SJ \to BO_\otimes$ is the natural map.

The vector space of primitive elements $PH^r(SJ)$ has the basis $(i_*u_r)^*$, $(t_r)^*$ if $r \neq 2^q$ resp.

$(x_r)^*$, $(t_r)^*$ if $r = 2^q \geq 2$ by the standard duality with the indecomposables $QH_r(SJ)$.

Let $\chi_{i,k} \in R[k]^*$ be the classes defined in 2.2. Let $\chi_{i,k} = ev^*(P_{i,k})$; $P_{i,k} \in PH^*(SG)$.

Note that $e^*(x^*_{2^r}) = e^*(x^*_1)^{2^r} = ev^{-1}(\xi^{2^r}_{1,1})$; $e^*(u^*_{2^r}) = ev^{-1}(\xi^{2^{r-1}}_{1,2})$

<u>Theorem 4.1</u>: Let $r \neq 2^q$

$$e^*((i_*u_r)^*) = P_{r,1} + \sum_{k=2}^{\infty} (k-1) \; P_{r,k}$$

$$e^*((t_r^*)) = \sum_{k=2}^{\infty} P_{r,k}$$

The rest of § 4 is devoted to a proof of 4.1.

It is wellknown that there are maps of modules over the Steenrod algebra

$$\varphi_1, \varphi_2 : H^*(\mathbb{R}P^\infty) \to PH^*(SJ)$$

$$\varphi_1(t^r) = (i_*t_r)^* \quad r \neq 2^q \qquad \varphi_1(t^{2^q}) = 0$$

$$\varphi_2(t^r) = (i_*u_r)^* \quad r \neq 2^q \qquad \varphi_2(u^{2^q}) = 0$$

The composite maps $e^*\varphi_1$ and $e^*\varphi_2$ are elements of $Hom_A(H^*(\mathbb{R}P^\infty) , PH^*(SG)) \cong \prod_{K=1}^{\infty} (\mathbb{Z}/2)_k$ which we proceed to determine.

Let $e^*\varphi_s \in \mathrm{Hom}_A(H^*(\mathbb{R}P^\infty), PH^*(SG))$ correspond to $(\lambda_i^s)_{i=1}^\infty \in \prod_{k=1}^\infty (\mathbb{Z}/2)_k$. Then by 1.5

$$\lambda_k^s = < e^* \varphi_s(t^{2^k-1}), Q^{2^{k-1}} Q^{2^{k-2}} \ldots Q^2 Q^1 [1]*[1-2^k] >$$

$$= < e^* \varphi_s(t^{2^k-1}), \hat{Q}^{2^{k-1}} \hat{Q}^{2^{k-2}} \ldots \hat{Q}^4 (Q^2 Q^1 [1]*[-3]) >$$

because $e^* \varphi_s(t^{2^k-1})$ is primitive.

We claim that

$$\lambda_k^s = < \varphi_s(t^3), t_3 > + k < \varphi^s(t^3), u_3 > \qquad (k \geq 2)$$

In fact, let $\hat{Q}$ denote the Dyer-Lashof operations in $H_*(SJ)$. Recall from [5] that

$$\hat{Q}^{2^i}(t_{2^i-1}) = t_{2^{i+1}-1} + u_{2^{i+1}-1} + \text{decomposables}$$

$$\hat{Q}^{2^i}(u_{2^i-1}) = u_{2^{i+1}-1} + \text{decomposables}$$

$$\lambda_k^s = < \varphi_s(t^{2^k-1}), e_* \hat{Q}^{2^{k-1}} \ldots \hat{Q}^4 (Q^2 Q^1 [1]*[-3]) >$$

$$= < \varphi_s(t^{2^k-1}), \hat{Q}^{2^{k-1}} \ldots \hat{Q}^4 \, t_3 >$$

$$= < \varphi_s(t^{2^k-1}), t_{2^k-1} > + k < \varphi_s(t^{2^k-1}), u_{2^k-1} >$$

$$= < \varphi_s(t^3), t_3 > + k < \varphi_s(t^3), u_3 >$$

so that $\lambda_k^1 = 1$, $\lambda_k^2 \equiv (k-1) \bmod(2)$ for $k \geq 2$. But it is easy to see that $\lambda_1^1 = 0$, $\lambda_1^2 = 1$ and the theorem follows.

§ 5 An application.

We apply the results of the previous paragraphs to a situation which occurs in connection with a certain surgery problem, see [4].

Let $\lambda : S^{14} \to SG$ be a map with Kervaire invariant 1, i.e. such that $\lambda^*(k_{14}) \neq 0 \in H^{14}(S^{14})$. Assume that λ extends to a map

$$\lambda^1_* : X = S^{14} \underset{2}{\cup} e^{15} \to SG$$

Let $x \in H_{15}(SG)$ be the image of the top homology class in X. Let $y \in H_{14}(SG)$ be the image of the fundamental class of S^{14}.

The homology Bockstein operator Sq^1 maps x to y. Since y is spherical it is primitive and annihilated by dual Steenrod operations.

We consider the suspension of the universal Stiefel-Whitney classes

$$\sigma w_i \in PH^{i-1}(SG)$$

Then we have the following relation.

Theorem 5.1: $< \sigma w_{16} , x > = < k_{14} , y >$.

Proof: Let $e : SG \to SJ$ be the e-invariant map. We claim that the induced map

$$f^* e^* : H^{15}(SJ) \to H^{15}(X)$$

is the zero map.

Indeed, there is a fibration $SO \to SJ \to BSpin$. Since $\pi_{14}(BSpin) \approx \pi_{14}(SO) \approx 0$, $e\, f$ factors over the collapsing map

$$X = S^{14} \cup_2 e^{15} \to S^{15}$$

Since $\pi_{15}(BSpin) \approx 0$, $e\,f$ factors further over a map $\overline{e\,f} : S^{15} \to SO$.

The only indecomposable class in $H^{15}(SO)$ is the cohomology suspension σw_{16} . Let $g : S^{16} \to SO$ be the desuspension of $\overline{e\,f}$, so

$$< \overline{e\,f}^*(\sigma w_{16}) , [S^{15}] > = < g^* w_{16} , [S^{16}] >$$

The claim follows from the wellknown fact that the Hurewicz map $\pi_{16}(BSO) \to H_{16}(BSO ; \mathbb{Z}/2)$ is the zero map.

Recall from 4.1 that the images of the two primitive classes in $H^*(SJ)$ under e^* are $P_{15,1} + P_{15,3}$ and $P_{15,2} + P_{15,3} + P_{15,4}$. By 1.6 $P_{15,1} = \sigma w_{16}$, 3.2 $P_{15,2} = Sq^1 k_{14}$. It follows that

$$< \sigma w_{14}, x > + < Sq^1 k_{14}, x > = < P_{15,4}, x >$$

In order to finish the proof of 5.1 we use the following

Lemma 5.2: Let $x \in PH_{15}\, SG$.
If $Sq^3 x = 0$ then $< P_{15,4} , x > = 0$.

Proof: By 1.4 the following classes represent a basis for $Q H_{15} ((QS^0)_1)$:

$$Q^{15}[1]*[-1] \; ; Q^8 \, Q^7[1]*[-3] \, , Q^9 \, Q^5[1]*[-3]$$

$$\hat{Q}^7(Q^4 \, Q^3[1]*[-3]) \; , \hat{Q}^8 \, \hat{Q}^4(Q^2 \, Q^1[1]*[-3]) \, .$$

But then the image of $QH_{15}((QS^0)_1)$ in $QH_{14}((QS^0)_1)$ under Sq^1 is represented by $Q^7 \, Q^7[1]*[-3]$.

Let P_7 be the Newton polynomial of degree 7 in the classes

$$Q^1 Q^1[1]*[-3] ; Q^2 Q^2[1]*[-3] ; \ldots Q^7 Q^7[1]*[-3] .$$

Then either $Sq^1 x$ or $Sq^1 x + P_7$ is a decomposable primitive class, and consequently a square z^2 .

Since $(Sq^1 z)^2 = Sq^2 z^2$, z is contained in the ideal I generated by the classes $Q^i[1]*[-1]$; $Q^j Q^j[1]*[-3]$ and $Q^4 Q^3[1]*[-3]$. It follows that $Sq^1 x \in I$, and that the class of x in $QH_{15}(SG)$ written in the basis choosen above, cannot contain $\hat{Q}^8 \hat{Q}^4 (Q^2 Q^1[1]*[-3])$, which proves the lemma.

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