HIGHER ORDER SUSPENSION MAPS FOR NON-ADDITIVE FUNCTORS

by

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ABSTRACT

Let R be a commutative ring with identity, let T: R-modules $\longrightarrow$ R-modules be a covariant functor with $T(0) = 0$, and let X be a semi-simplicial R-module. Higher order versions of the Dold-Puppe suspension map $\sigma\colon \pi_* TX \longrightarrow \pi_{*+1} TSX$ are defined and studied in this thesis. These higher order suspension maps are obtained by viewing $\sum_{m,n\geq 0} \pi_m TS^n X$ as a bigraded module over a bigraded algebra, $\sum_{m,n\geq 0} H_m(R,n;R)$, formed from the homology of Eilenberg-MacLane spaces. The general theory of these operations is discussed. Then the properties of those arising from $H_*(Z,2)$ and $H_*(Z_p,1)$ are presented in detail.

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§1. Introduction.

Let R be a commutative ring with identity, and let

$$T\colon \text{R-modules} \longrightarrow \text{R-modules}$$

be a covariant functor with $T(0) = 0$. Dold and Puppe [3] have defined the suspension map

$$\sigma\colon \pi_* TX \longrightarrow \pi_{*+1} TSX ,$$

where X is a semi-simplicial (abbreviated "s.s.") R-module. Our purpose is to define and study certain higher order suspension maps, which we shall call pensions.

More precisely we shall give $\sum_{m,n\geq 0} \pi_m TS^n X$ a left module structure over an algebra $P(R) = \sum_{m,n\geq 0} H_m(R,n;R)$ formed from homology of Eilenberg-MacLane spaces. An element $\alpha \in H_m(R,n;R)$ then gives rise by left multiplication to a pension

$$\alpha\colon \pi_* TX \longrightarrow \pi_{*+m} TS^n X .$$

We remark that there is also a stable theory of pensions, and that the stable version of $P(\mathbb{Z}_p)$ is isomorphic to the dual of the mod-p Steenrod algebra.

In §2 we present the basic (unstable) theory of pensions. Then in §3 and §4 we discuss at length the properties of those pensions which arise from $H_*(Z,2)$ and $H_*(Z_p,1)$.

We do not propose in this thesis to present any applications of pensions to the study of π_*TX for specific functors T. However, such applications are plentiful.

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§2. The Pensions.

2.1. Preliminaries. Let R and T be as in §1. For a set K with basepoint $*$, let AK denote the free R-module generated by K with the relation $* = 0$. If M and N are R-modules, there is a natural R-linear map

$$E: AM \otimes_R TN \longrightarrow T(M \otimes_R N) .$$

We define E as the composition

$$AM \otimes_R TN \xrightarrow{G} T(AM \otimes_R N) \xrightarrow{T(j \otimes 1)} T(M \otimes_R N) .$$

For $m \in M$ the restriction of the "pull-through" map G to $1.m \otimes TN \approx TN$ is induced by the map $N \to AM \otimes_R N$ sending $n \to (1.m) \otimes n$. The map $j: AM \to M$ sends $1.m \to m$.

Now let S_n be the standard s.s. n-sphere; and observe that $\pi_* AAS_n = H_*(R,n)$, where H_* denotes reduced homology with coefficients in R. If X is an s.s. R-module then the map

$$E: AAS_n \otimes_R TX \longrightarrow T(AS_n \otimes_R X)$$

induces

$$E_*: H_*(R,n) \otimes_R \pi_* TX \longrightarrow \pi_* TS^n X ,$$

where $S^n X$ is the n-fold suspension (see [3]) of X. It is desirable to modify E_* so that it will give rise to a stable pairing. Thus, we let

$$E^1_*: H_*(R,n) \otimes_R \pi_* TX \longrightarrow \pi_* TS^n X$$

be the map such that

$$E^1_*(\alpha \otimes \beta) = (-1)^{(m-n)i} E_*(\alpha \otimes \beta)$$

for $\alpha \in H_m(R,n)$ and $\beta \in \pi_i TX$.

2.2. Basic notions. The pension algebra $P(R)$ is the bigraded R-algebra with

$$P(R) = \sum_{m,n \geq 0} H_m(R,n) = \sum_{m,n \geq 0} \pi_m AAS_n$$

and with multiplication given by

$$E^1_*: \pi_* AAS_n \otimes_R \pi_* AAS_q \longrightarrow \pi_* AAS_{n+q} \quad .$$

This pairing E^1_* is as above with $T = A$ and $X = AS_q$. We note that a more "topological" definition of $P(R)$ is suggested by 2.7. The pension algebra $P(R)$ is associative with an identity $1 \cdot [1] \in H_0(R,0) = A(R)$. Furthermore, $\sum_{m,n \geq 0} \pi_m TS^n X$ is a module over $P(R)$ with multiplication given by the pairings

$$E^1_*: H_*(R,n) \otimes_R \pi_* TS^i X \longrightarrow \pi_* TS^{n+i} X \quad .$$

To each element $\alpha \in H_m(R,n)$ there corresponds a pension

$$\alpha: \pi_* TX \longrightarrow \pi_{*+m} TS^n X$$

which is simply left multiplication by α . This pension α is said to be of order n .

Certain pensions are well known.

Example 2.3. For $1.[r] \varepsilon H_0(R,0) = AR$ the pension $1.[r]: \pi_* TX \to \pi_* TX$ is induced by the multiplication map $r: X \to X$.

Example 2.4. Let $\iota_n \varepsilon H_n(R,n) = \pi_n AAS_n$ be the canonical element corresponding to the non-degenerate n-simplex of S_n . Then $\iota_n: \pi_* TX \to \pi_{*+n} TS^n X$ is the n-fold Dold-Puppe suspension σ^n .

Proof. Since $\sigma: \pi_* TX \to \pi_{*+1} TSX$ is induced by the "pull-through" $G: AS_1 \otimes_R TX \to T(AS_1 \otimes_R X)$, 2.4 easily follows for $n = 1$. Hence for $\alpha \varepsilon H_r(R,s)$ the product $\iota_1 \cdot \alpha$ in $P(R)$ equals $\sigma(\alpha) \varepsilon H_{r+1}(R,s+1)$. Therefore $\iota_n = \sigma^{n-1}(\iota_1) = (\iota_1)^n$ in $P(R)$, so the general case is proved from the case $n = 1$.

We shall discuss other less familiar pensions in §3 and §4; but we now continue with generalities.

2.5. The pension algebra $P(R)$ is anti-commutative in the following sense. Let $\alpha \varepsilon H_{m+n}(R,n)$ and $\beta \varepsilon H_{p+q}(R,q)$. Then

$$\alpha . \beta = (-1)^{mp}(-\mu)^{nq} . \beta . \alpha$$

where $\mu = 1 \cdot [-1] \varepsilon H_0(R,0)$. Thus if nq is even then

$\alpha \cdot \beta = (-1)^{mp} \beta \cdot \alpha$.

For $\beta \in H_{p+q}(R,q)$ the product $\mu \cdot \beta$ need not equal $-\beta$ (see 4.13) . However $\mu \cdot \iota_1 = -\iota_1$ for the element $\iota_1 \in H_1(R,1)$. Hence in the above statement of anti-commutativity, if either α or β is the suspension of another element then $\alpha \cdot \beta = (-1)^{mp} \beta \cdot \alpha$.

<u>2.6.</u> <u>For any</u> $\gamma \in H_m(R,n)$ <u>the pension</u> $\gamma \colon \pi_* TX \to \pi_{*+m} TS^n X$ <u>is stable</u> in the sense that $\gamma \bullet \sigma = \sigma \bullet \gamma$ and both equal the pension $\sigma(\gamma) \in H_{m+1}(R,n+1)$.

The following lemma is useful in computing the multiplication in the pension algebra $P(R)$.

<u>Lemma 2.7.</u> For $m,n \geq 1$ let $f \colon AS_m \wedge AS_n \to AS_{m+n}$ be a map of set complexes such that

$$f^* \colon H^{m+n}(AS_{m+n};R) \longrightarrow H^{m+n}(AS_m \wedge AS_n;R)$$

maps the fundamental class ι^{m+n} to the cohomology product $\iota^m \wedge \iota^n$. Then

$$f_* \colon H_*(AS_m) \otimes_R H_*(AS_n) \longrightarrow H_*(AS_{m+n})$$

equals the map E_* .

<u>Proof.</u> It is clear from definitions that E_* is induced by the map $g \colon AS_m \wedge AS_n \to AS_m \otimes_R AS_n$ with $g(x \wedge y) = x \otimes y$ for each $x \in (AS_m)_i$, $y \in (AS_n)_i$.

The diagram

$$\begin{array}{ccc} AAS_m \otimes_R AAS_n & \xrightarrow{A(g)} & A(AS_m \otimes_R AS_n) \\ & {\scriptstyle j \otimes j} \searrow \quad \swarrow {\scriptstyle j} & \\ & AS_m \otimes_R AS_n & \end{array}$$

commutes, where $j\colon AM \to M$ denotes the R-linear map sending $1 \cdot m \to m$. Applying $\mathrm{Hom}(\cdot, R)$ to the diagram and passing to cohomology, we see that

$$g^*(\iota^{m+n}) = \iota^m \wedge \iota^n \in H^{m+n}(AS_m \wedge AS_n; R)$$

where

$$\iota^{m+n} \in H^{m+n}(AS_{m+n}; R) = H^{m+n}(AS_m \otimes_R AS_n; R) \ .$$

This cohomology property determines g up to homotopy; and since f also has this property, f also induces E_* .

<u>Remark 2.8.</u> In view of 2.7 , the multiplication in $P(Z_p)$, p prime, may be computed using the Steenrod algebra.

<u>2.9.</u> <u>Pensions for functors of finite degree</u>.

Recall that

$$T\colon \text{R-modules} \longrightarrow \text{R-modules}$$

is of <u>degree $\leq r$</u> if its $r+1$-fold cross-effect functor is zero. For functors of finite degree, we shall show in

2.10 that certain pensions are trivial for dimensional reasons. We do this by appropriately factoring $E\colon AM \otimes_R TN \longrightarrow T(M \otimes_R N)$, where M and N are R-modules.

Let $i\colon M \to AM$ be the function with $i(m) = 1\cdot m$. For $m_1,\ldots,m_k \in M$ let $i(m_1 \top \cdots \top m_k)$ denote the k^{th} deviation of i ([4], p. 75). Thus $i(m_1 \top m_2) = 1\cdot(m_1 + m_2) - 1\cdot m_1 - 1\cdot m_2$, etc..

For $r \geq 1$, the <u>universal functor of degree</u> r , A^r , is given by letting $A^r M$ be the quotient of AM by the submodule generated by elements $i(m_1 \top \cdots \top m_{r+1})$ for $m_1,\ldots,m_{r+1} \in M$.

Then

$$A^r\colon \text{R-modules} \longrightarrow \text{R-modules}$$

is indeed a functor of degree $\leq r$. If T is of degree $\leq r$, there exists a unique map

$$E^r\colon A^r M \otimes_R TN \longrightarrow T(M \otimes_R N)$$

such that

$$\begin{array}{ccc} AM \otimes_R TN & \xrightarrow{\quad E \quad} & T(M \otimes_R N) \\ & {\scriptstyle j^r \otimes 1} \searrow \quad \nearrow {\scriptstyle E^r} & \\ & A^r M \otimes_R TN & \end{array}$$

commutes, where j^r is the natural epimorphism.

Proposition 2.10. Let T be of degree $\leq r$, X an s.s. R-module, and $\alpha \varepsilon H_m(R,n)$. If $m > rn$ then $\alpha: \pi_* TX \longrightarrow \pi_{*+m} TS^n X$ is zero.

Proof. In view of the above diagram, it suffices to show that $\pi_m A^r(AS_n) = 0$ for $m > rn$. This follows by a result of [3] since A^r is of degree $\leq r$ and the normalization $N(AS_n)$ of AS_n is trivial above dimension n.

Remark 2.11. For T additive all pension maps are trivial except for multiples of iterated suspension maps.

Remark 2.12. In proving more specific vanishing results than 2.10, the following analysis of the functors A^r is often useful.

Let $R = Z$ and let K be a set complex with base point. If $SP^k AK$ is the k-fold symmetric tensor product of AK, then there is a homomorphism

$$h: \sum_{k=1}^{\infty} SP^k AK \longrightarrow AAK$$

with $h(1 \cdot x_1 \otimes \ldots \otimes 1 \cdot x_k) = i(1 \cdot x_1 \top \cdots \top 1 \cdot x_k)$ for $1 \cdot x_1 \otimes \ldots \otimes 1 \cdot x_k \ \varepsilon \ SP^k AK$. In fact by [3], h is a group homotopy equivalence if K connected. For any K the composition

$$j^r \bullet h: \sum_{k=1}^{\infty} SP^k AK \longrightarrow A^r(AK)$$

is an epimorphism with kernel $\sum_{k=r+1}^{\infty} SP^k AK$.

Let $R = Z_p$, p prime, and let K be as above. Let $SP_p^k AK$ be the quotient of $SP^k AK$ by the relation $x_1 \otimes \cdots \otimes x_k = 0$ if $x_1 = \cdots = x_p$ where $x_1, \ldots, x_k \in AK$ with $k \geq p$. Then the map

$$h: \sum_{k=1}^{\infty} SP_p^k AK \longrightarrow AAK$$

defined as above is an isomorphism. The composition

$$j^r \circ h : \sum_{k=1}^{\infty} SP_p^k AK \longrightarrow A^r(AK)$$

is an epimorphism with kernel $\sum_{k=r+1}^{\infty} SP_p^k AK$.

§3. The Integral Pensions of Second Order.

Throughout this section we shall work over the ring Z of integers. The integral pensions of orders 0 and 1, i.e., those arising from $H_*(Z,0)$ and $H_*(Z,1)$, were identified in 2.3 and 2.4. We now turn to the second order pensions.

Notation 3.1. For $r \geq 1$ let $\varepsilon_r \in H_{2r}(Z,2) = Z$ denote the element dual to the cohomology power $(\iota^2)^r \in H^{2r}(Z,2)$.

Let T: Abelian groups ⟶ Abelian groups be a covariant functor with $T(0) = 0$; and let X be an s.s. abelian group. We shall now present the properties of the pensions

$$\varepsilon_r \colon \pi_* TX \longrightarrow \pi_{*+2r} TS^2 X .$$

Proposition 3.2.

(i) If $\alpha \in P(Z)$ and $r \geq 1$, then $\alpha \cdot \varepsilon_r = \varepsilon_r \cdot \alpha$

(ii) $\varepsilon_1 = \sigma^2$

(iii) $\sigma \cdot \varepsilon_r = 0$ unless $r = p^j$, $j \geq 0$, p prime.

(iv) $p\sigma \cdot \varepsilon_{p^j} = 0$ for $j \geq 1$, p prime.

Proof. These results are obvious from §2 and from well-known homology suspensions [1].

Let $U(X_1,\ldots,X_r)$ be a covariant functor from abelian groups to abelian groups, and suppose $U(X_1,\ldots,X_r) = 0$ if any $X_i = 0$. Let X be an s.s. abelian group.

Proposition 3.3. The pension

$$\varepsilon_t\colon \pi_* U(X,\ldots,X) \longrightarrow \pi_{*+2t}U(S^2X,\ldots,S^2X)$$

is zero for $t < r$, and ε_r equals the composition

$$\pi_* U(X,\ldots,X) \xrightarrow{\sigma^2} \pi_{*+2}U(X,\ldots,S^2X) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_{*+2r-2}U(X,S^2X,\ldots,S^2X) \xrightarrow{\sigma^2} \pi_{*+2r}U(S^2X,\ldots,S^2X)$$

of double suspensions in each variable. Thus if U is additive, then ε_r is an isomorphism.

Proof. The map $E\colon AAS_2 \otimes U(X,\ldots,X) \to U(AS_2\otimes X,\ldots,AS_2\otimes X)$ equals the composition

$$AAS_2 \otimes U(X,\ldots,X) \xrightarrow{A(\Delta_r)\otimes 1} \otimes^r AAS_2 \otimes U(X,\ldots,X)$$

$$\xrightarrow{1\otimes E} \otimes^{r-1}AAS_2 \otimes U(X,\ldots,AS_2\otimes X) \longrightarrow \cdots$$

$$\cdots \xrightarrow{E} U(AS_2\otimes X,\ldots,AS_2\otimes X)$$

where $\Delta_r\colon AS_2 \longrightarrow AS_2 \wedge \cdots \wedge AS_2$ with $\Delta_r(y) = y \wedge \cdots \wedge y$ for $y \in AS_2$. The first part of 3.3 follows from the fact that

$$\Delta_{r*}\colon H_*(AS_2;Z) \longrightarrow H_*(AS_2;Z)\otimes \cdots \otimes H_*(AS_2;Z)$$

maps $\Delta_{r*}(\varepsilon_t) = 0$ for $t < r$ and $\Delta_{r*}(\varepsilon_r) = \varepsilon_1 \otimes \cdots \otimes \varepsilon_1$. The final statement follows from the fact [3] that σ is an isomorphism for additive functors in one variable.

Now let T and X be as before.

Proposition 3.4. If T is of degree $\leq r$ and $s > r$, then $\varepsilon_s: \pi_* TX \longrightarrow \pi_{*+2s} TS^2X$ is the zero map.

Proof. See 2.10.

The s.s. abelian group X is trivial above n if X is group homotopy equivalent to an s.s. abelian group Y such that $(NY)_i = 0$ for $i > n$, where NY is the normalization of Y .

Theorem 3.5. Let T be of degree $\leq r$ and X be trivial above n . Then $\varepsilon_r: \pi_i TX \longrightarrow \pi_{i+2r} TS^2X$ is an isomorphism for $i > n(r-1) + 1$ and a monomorphism for $i = n(r-1) + 1$. However, ε_r need not be monic for $i = n(r-1)$.

Proof. The final statement is obvious. For example,

$$\varepsilon_r: \pi_{n(r-1)} \otimes^{r-1} AS_n \longrightarrow \pi_{n(r-1)+2r} \otimes^{r-1} AS_{n+2}$$

is the zero map by 3.4, where $\otimes^{r-1}$ is the r-1 fold tensor power functor.

The proof of the main result is by induction on n .

Case n = 0 . By [3] , TX is trivial above 0 and

TS^2X is trivial above $2r$. Hence $_r$ is an isomorphism for $i \geq 1$.

<u>Inductive step.</u> Given that 3.5 is true for $n = k-1$, we now show that it is true for $n = k$. Let X be trivial above k . We may suppose that $(NX)_i = 0$ for $i > k$. There exists a dimensionwise splittable short exact sequence $0 \to Y' \to Y \to X \to 0$ of s.s. abelian groups with Y and Y' trivial above $k-1$. Using the cross-effect exact sequence of Kan and Whitehead one obtains exact sequences of s.s. abelian groups

(I)

$$\cdots \longrightarrow AS_{2r} \otimes T_s(Y',\ldots,Y') + AS_{2r} \otimes T_{s+1}(Y,Y',\ldots,Y') \longrightarrow \cdots$$

$$\cdots \longrightarrow AS_{2r} \otimes T(Y') + AS_{2r} \otimes T_2(Y,Y')$$

$$\longrightarrow AS_{2r} \otimes T(Y) \longrightarrow AS_{2r} \otimes TX \longrightarrow 0$$

(II)

$$\cdots \longrightarrow T_s(AS_2 \otimes Y',\ldots,AS_2 \otimes Y') + T_{s+1}(AS_2 \otimes Y, AS_2 \otimes Y',\ldots, AS_2 \otimes Y) \longrightarrow \cdots$$

$$\cdots \longrightarrow T(AS_2 \otimes Y') + T_2(AS_2 \otimes Y,\ AS_2 \otimes Y')$$

$$\longrightarrow T(AS_2 \otimes Y) \longrightarrow T(AS_2 \otimes X) \longrightarrow 0 \ .$$

We construct a map $F: (I) \longrightarrow (II)$ as follows. Let $i_r: AS_{2r} \to AAS_2$ be an s.s. abelian map such that $i_{r*}(\iota_{2r}) = \varepsilon_r$ in $\pi_{2r}AAS_2$. If $s \geq 1$ and $Y_1,\ldots,Y_s$ are s.s. abelian groups, let f_s denote the composition

$$AS_{2r} \otimes T_s(Y_1,\ldots,Y_s) \xrightarrow{i_r \otimes 1} AAS_2 \otimes T_s(Y_1,\ldots,Y_s) \longrightarrow$$

$$T_s(AS_2 \otimes Y_1,\ldots,AS_2 \otimes Y_s)$$

where the second map is the obvious generalization of E . The map $F: (I) \longrightarrow (II)$ is built from the maps f_s .

To prove the inductive step it suffices to show

$$f_{1*}: \pi_i(AS_{2r} \otimes TX) \longrightarrow \pi_i T(AS_2 \otimes X)$$

is iso for $i > 2r + k(r-1) + 1$ and mono for $i = 2r + k(r-1) + 1$. Using $F: (I) \longrightarrow (II)$, this follows by a spectral sequence argument from the facts:

(i) The functor T_s is zero for $s > r$, since T of degree $\leq r$.

(ii) $f_{r*}: \pi_*(AS_{2r} \otimes T_r(Y',\ldots,Y')) \to \pi_* T_r(AS_2 \otimes Y',\ldots,AS_2 \otimes Y')$

is iso, since T_r is additive and we may apply 3.3.

(iii) For $s \geq 1$,

$$f_{s*}: \pi_i(AS_{2r} \otimes T_s(Y',\ldots,Y')) \to \pi_i T_s(AS_2 \otimes Y',\ldots,AS_2 \otimes Y')$$

and

$$f_{s*}: \pi_i(AS_{2r} \otimes T_s(Y,Y',\ldots,Y')) \to \pi_i T_s(AS_2 \otimes Y,\ AS_2 \otimes Y', \ldots, AS_2 \otimes Y')$$

are iso for $i > 2r + (k-1)(r-1) + 1$ and mono for $i = 2r + (k-1)(r-1) + 1$. This follows from the inductive hypothesis applied to the functor T and the complexes $Y' + \cdots + Y'$ and $Y + Y' + \cdots + Y'$.

Remark 3.6. Under the conditions of 3.5, Dold and Puppe [3] have shown that TX is trivial above nr, so that $\pi_i TX = 0$ for $i > nr$. Proposition 3.5 describes a stability phenomenon in a range below these trivial groups. Under the conditions of 3.5, the sequence

$$\pi_i TX \xrightarrow{\varepsilon_r} \pi_{i+2r} TS^2X \xrightarrow{\varepsilon_r} \pi_{i+4r} TS^4X \longrightarrow \cdots$$

stabilizes for each i.

The pensions ε_r satisfy a "Cartan formula" when applied to homotopy products arising from tensor products of functors. In particular, let

$$T,\ T': \text{Abelian groups} \longrightarrow \text{Abelian groups}$$

be covariant functors with $T(0) = T'(0) = 0$, and let X be a semi-simplicial abelian group. We consider the Eilenberg-Zilber product

$$\pi_i TX \otimes \pi_j T'X \longrightarrow \pi_{i+j}(T \otimes T')X$$

Proposition 3.7. If $\alpha \in \pi_i TX$ and $\beta \in \pi_j T'X$ then $\varepsilon_1(\alpha\cdot\beta) = 0$ and for $r \geq 2$

$$\varepsilon_r(\alpha\cdot\beta) = \sum_{k=1}^{r-1} \varepsilon_k(\alpha) \cdot \varepsilon_{r-k}(\beta) .$$

Proof. Clearly the diagram

$$\begin{array}{ccc}
AAS_2 \otimes TX \otimes T'X & \xrightarrow{\quad E \quad} & T(AS_2 \otimes X) \otimes T'(AS_2 \otimes X) \\
\Big\downarrow A(\Delta)\otimes 1 & & \Big\uparrow E \otimes E \\
AAS_2 \otimes AAS_2 \otimes TX \otimes T'X & \xrightarrow{1\otimes \tau \otimes 1} & AAS_2 \otimes TX \otimes AAS_2 \otimes T'X
\end{array}$$

commutes, where $\Delta\colon AS_2 \longrightarrow AS_2 \wedge AS_2$ with $\Delta(y) = y \wedge y$ for $y \in AS_2$, and where τ is the twisting map. The map $\Delta_*\colon H_*(Z,2) \longrightarrow H_*(Z,2) \otimes H_*(Z,2)$ gives

$$\Delta_*(\varepsilon_r) = \sum_{k=1}^{r-1} \varepsilon_k \otimes \varepsilon_{r-k}$$

as can be seen by passing to cohomology. Thus 3.7 follows from the diagram.

Remark 3.8. The following consequence of 3.4 and 3.7 is often applicable. Let $\{T^r\}_{r\geq 1}$ be a set of functors with T^r of degree $\leq r$ and with given pairings $T^r \otimes T^s \longrightarrow T^{r+s}$ for all $r, s \geq 1$. Then

$$\sum_{r=1}^{\infty} \varepsilon_r\colon \sum_{r=1}^{\infty} \pi_* T^r X \longrightarrow \sum_{r=1}^{\infty} \pi_{*+2r} T^r S^2 X$$

preserves the induced multiplication.

We shall now study the action of the ε_r on homotopy products arising from compositions of functors. Let T, T', and X, be as in 3.7. Then for $m, j \geq 0$ the composition product

$$\pi_m T'X \times \pi_j T(AS_m) \longrightarrow \pi_j(T \circ T'X)$$

is defined. We note for this product that

(i) $\beta \circ (\alpha_1 + \alpha_2) = \beta \circ \alpha_1 + \beta \circ \alpha_2$

(ii) $(\beta_1 + \beta_2) \circ \sigma\alpha = \beta_1 \circ \sigma\alpha + \beta_2 \circ \sigma\alpha$

(iii) $\sigma(\beta \circ \alpha) = \sigma\beta \circ \sigma\alpha$.

The action of the ε_r on composition products is somewhat complicated, and we shall devote the rest of §3 to its study.

Theorem 3.9. If $\beta \in \pi_m T'X$, $\alpha \in \pi_{j-1} T(AS_{m-1})$, and $s \geq 1$, then

$$\varepsilon_s(\beta \circ (\sigma\alpha)) = \sum_{r=1}^{s} \sum_{i_1+\cdots+i_r=s} \varepsilon_r(\beta) \circ \left(\varepsilon_{i_1} \cdots \varepsilon_{i_r}(\sigma\alpha)\right)$$

where the second sum ranges over all $(i_1,\ldots,i_r)$ with $i_1,\ldots,i_r \geq 1$ and $i_1 + \cdots + i_r = s$.

Proof. The map

$$E\colon AAS_2 \otimes (T \circ T'X) \longrightarrow T \circ T'(AS_2 \otimes X)$$

equals the composition

$$AAS_2 \otimes (T \circ T'X) \xrightarrow{G} T(AAS_2 \otimes T'X) \xrightarrow{T(E)} T \circ T'(AS_2 \otimes X)$$

where G is the "pull-through" map.

For $r \geq 1$ let $i_r\colon AS_{2r} \longrightarrow AAS_2$ and $j_r\colon AAS_2 \longrightarrow AS_{2r}$ be abelian s.s. maps such that $i_{r*}(\iota_{2r}) = \varepsilon_r$ in $\pi_{2r}AAS_2$ and $j_{r*}(\varepsilon_r) = \iota_{2r}$ in $\pi_{2r}AS_{2r}$. Denote the composition of

$$AAS_2 \otimes (T \circ T'X) \xrightarrow{G} T(AAS_2 \otimes T'X) \xrightarrow{T(i_r \cdot j_r \otimes 1)} T(AAS_2 \otimes T'X)$$

$$\xrightarrow{T(E)} T \circ T'(AS_2 \otimes X)$$

by F_r. There are induced maps

$$E_*, F_{r*}\colon \pi_* AAS_2 \otimes \pi_*(T \circ T'X) \longrightarrow \pi_*(T \circ T'(S^2X)) .$$

It is not hard to show that

$$E_*(\varepsilon_s \otimes (\beta \circ (\sigma\alpha))) = \sum_{r=1}^{s} F_{r*}(\varepsilon_s \otimes (\beta \circ (\sigma\alpha))) .$$

As a step toward analyzing the maps F_r, we let

$$k_r\colon AS_2 \longrightarrow AS_{2r}$$

be the set complex map with $k_r(y) = j_r(1 \cdot y)$ for $y \in AS_2$. Also consider

$$AS_2 \xrightarrow{\Delta_r} \underbrace{AS_2 \wedge \cdots \wedge AS_2}_{r \text{ times}} \xrightarrow{g_r} \underbrace{AS_2 \otimes \cdots \otimes AS_2}_{r \text{ times}} \xrightarrow{h_r} AS_{2r}$$

where $\Delta_r(y) = y \wedge \cdots \wedge y$ for $y \in AS_2$, where $g_r(y_1 \wedge \cdots \wedge y_r) = y_1 \otimes \cdots \otimes y_r$ for $y_1, \ldots, y_r \in AS_2$, and where

h_r is a group homotopy equivalence of the canonical homotopy class. We claim that k_r and $h_r \circ g_r \circ \Delta_r$ are homotopic. Observe that $k_r^*: H^{2r}(AS_{2r};Z) \longrightarrow H^{2r}(AS_2;Z)$ maps $k_r^*(\iota^{2r}) = (\iota^2)^r$. This follows because the composition $AAS_2 \xrightarrow{A(k_r)} AAS_{2r} \xrightarrow{j} AS_{2r}$ equals j_r , where j is the linear map with $j(1\cdot x) = x$ for $x \in AS_{2r}$. Using the proof of 2.7 it is clear that

$$(h_r \circ g_r \circ \Delta_r)^*(\iota^{2r}) = (\iota^2)^r \text{ , so } k_r \text{ and } h_r \circ g_r \circ \Delta_r$$

are homotopic as claimed.

We claim that

$$F_r: AAS_2 \otimes (T \circ T'X) \longrightarrow T \circ T'(AS_2 \otimes X)$$

is group homotopic to the composition

$$AAS_2 \otimes (T \circ T'X) \xrightarrow{A(\Delta_r)} \otimes^r(AAS_2) \otimes (T \circ T'X) \xrightarrow{E \circ \cdots \circ E} T(\otimes^r(AS_2) \otimes T'X)$$

$$\xrightarrow{T(h_r \otimes 1)} T(AS_{2r} \otimes T'X) \xrightarrow{T(i_r \otimes 1)} T(AAS_2 \otimes T'X) \xrightarrow{T(E)} T \circ T'(AS_2 \otimes X)$$

where $E \circ \cdots \circ E$ refers to the composition

$$\otimes^r(AAS_2) \otimes T \circ T'X \xrightarrow{1 \otimes E} \otimes^{r-1}(AAS_2) \otimes T(AS_2 \otimes T'X) \longrightarrow \cdots$$

$$\cdots \longrightarrow AAS_2 \otimes T(\otimes^{r-1}(AS_2) \otimes T'X) \xrightarrow{E} T(\otimes^r(AS_2) \otimes T'X) .$$

This is straightforward to prove in view of the fact that

$$T(j_r \otimes 1) \circ G = E \circ (A(k_r) \otimes 1) : AAS_2 \otimes (T \circ T'X) \to T(AS_{2r} \otimes T'X)$$

and the fact that $A(k_r): AAS_2 \to AAS_{2r}$ is group homotopic to $A(h_r \circ g_r \circ \Delta_r)$.

The map

$$\Delta_{r*}: H_*(AS_2;Z) \longrightarrow \otimes^r H_*(AS_2;Z)$$

gives

$$\Delta_{r*}(\varepsilon_s) = \sum_{i_1+\cdots+i_r=s} \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_r} \quad .$$

It therefore follows from the above paragraph that

$$F_{r*}(\varepsilon_s \otimes (\beta \circ (\sigma\alpha))) = \sum_{i_1+\cdots+i_r=s} \varepsilon_r(\beta) \circ (\varepsilon_{i_1} \cdots \varepsilon_{i_r}(\sigma\alpha))$$

Now 3.9 follows since

$$E_*(\varepsilon_s \otimes (\beta \circ (\sigma\alpha))) = \sum_{r=1}^{s} F_{r*}(\varepsilon_s \otimes (\beta \circ (\sigma\alpha))) \ .$$

<u>Corollary 3.10.</u> Let $\beta \in \pi_m T'X$ and $\alpha \in \pi_j TAS_m$. Then for p prime and $s \geq 0$

$$\varepsilon_{p^s}(\sigma\beta \circ \sigma\alpha) = \sum_{k=0}^{s} \varepsilon_{p^k}(\sigma\beta) \circ ((\varepsilon_{p^{s-k}})^{p^k}(\sigma\alpha)) \ .$$

<u>Proof.</u> This follows from 3.9 using 3.2.

The following theorem is a corollary of 3.9 if α is a suspension.

<u>Theorem 3.11.</u> Suppose that T' is of degree $\leq k$ and T is of degree $\leq h$. If $\beta \in \pi_m T'X$ and $\alpha \in \pi_j T(AS_m)$

then

$$\varepsilon_{hk}(\beta \bullet \alpha) = \varepsilon_k(\beta) \bullet ((\varepsilon_h)^k(\alpha)).$$

Proof. We claim that the compositions

$$E \circ (i_{hk} \otimes 1),\ F_k \circ (i_{hk} \otimes 1) : AS_{2hk} \otimes (T \circ T'X) \longrightarrow AAS_2 \otimes (T \bullet T'X)$$
$$\longrightarrow T \circ T'(AS_2 \otimes X)$$

are group homotopic, where F_k and i_{hk} are as in proof 3.9.

We now prove this claim. Let f denote the composition

$$AS_{2hk} \xrightarrow{i_{hk}} AAS_2 \xrightarrow{A(i)} AAAS_2 \xrightarrow{j^h} A^h AAS_2 ,$$

where $i: AS_2 \longrightarrow AAS_2$ maps $i(x) = 1 \cdot x$ for $x \in AS_2$ and where j^h is the epimorphism of 2.9. It is easy to show that $E \circ (i_{hk} \otimes 1)$ equals the composition

$$AS_{2hk} \otimes (T \circ T'X) \xrightarrow{f \otimes 1} A^h AAS_2 \otimes (T \circ T'X) \xrightarrow{A^h(j^k) \otimes 1} A^h A^k AS_2 \otimes (T \circ T'X)$$

$$\xrightarrow{E^h} T(A^k AS_2 \otimes T'X) \xrightarrow{T(E^k)} T \circ T'(AS_2 \otimes X)$$

where E^r is as in 2.9. Since A^h and A^k are of degree $\leq h$ and $\leq k$ respectively, it is not hard to show that

$$A^h(j^k) \circ f : AS_{2hk} \longrightarrow A^h A^k AS_2$$

is group homotopic to the compsoition ψ of

$$AS_{2hk} \xrightarrow{f} A^h AAS_2 \xrightarrow{A^h(i_k \circ j_k)} A^h AAS_2 \xrightarrow{A^h(j^k)} A^h A^k AS_2 .$$

The proof of this uses the following elementary fact. Suppose V is a functor of degree $\leq r$ and $X_1,\ldots,X_m$ are s.s. abelian groups such that X_i is trivial above n_i with $n_1 \leq \cdots \leq n_m$. Then the cross-effect $V_m(X_1,\ldots,X_m)$ is trivial above $n + (r-m)n_m$ where $n = n_1 + \cdots + n_m$. Our initial claim now follows since $F_k \bullet (i_{nk} \otimes 1)$ equals the composition

$$AS_{2hk} \otimes (T \bullet T'X) \xrightarrow{\psi} A^h A^k AS_2 \otimes (T \bullet T'X) \xrightarrow{E^h} T(A^k AS_2 \otimes T'X) \xrightarrow{T(E^k)} T \bullet T'(AS_2 \otimes X)$$

Using the analysis of

$$F_k\colon AAS_2 \otimes (T \bullet T'X) \longrightarrow T \bullet T'(AS_2 \otimes X)$$

given in the proof 3.9, it follows from our initial claim that

$$\varepsilon_{hk}(\beta \bullet \alpha) = \sum_{i_1+\cdots+i_k=hk} \varepsilon_k(\beta) \bullet \left(\varepsilon_{i_1} \cdots \varepsilon_{i_k}(\alpha)\right).$$

In view of 3.4 this reduces to

$$\varepsilon_{hk}(\beta \bullet \alpha) = \varepsilon_k(\beta) \bullet \left((\varepsilon_h)^k(\alpha)\right).$$

We prove one other unstable composition result. As in 3.9 let T and T' be of arbitrary degree.

<u>Proposition 3.12.</u> Let $\beta \in \pi_m T'X$ and $\alpha \in \pi_j TAS_m$. Suppose for some integer $s \geq 2$ that $\varepsilon_t(\beta) = 0$ whenever $1 < t < s$. Then

$$\varepsilon_s(\beta \circ \alpha) = \varepsilon_s(\beta) \circ \big((\varepsilon_1)^s(\alpha)\big) + \varepsilon_1(\beta) \circ \varepsilon_s(\alpha) .$$

<u>Proof.</u> Let $\beta\colon AS_m \longrightarrow T'X$ represent $\beta \in \pi_m T'X$. The composition

$$AS_{2s} \otimes TAS_m \xrightarrow{1_s \otimes T(\beta)} AAS_2 \otimes (T \circ T'X) \xrightarrow{E} T \circ T'(AS_2 \otimes X)$$

equals

$$AS_{2s} \otimes TAS_m \xrightarrow{1_s \otimes 1} AAS_2 \otimes TAS_m \xrightarrow{G} T(AAS_2 \otimes AS_m)$$

$$\xrightarrow{T(1 \otimes \beta)} T(AAS_2 \otimes T'X) \xrightarrow{T(E)} T \circ T'(AS_2 \otimes X) .$$

In view of our hypotheses this is group homotopic to

$$AS_{2s} \otimes TAS_m \xrightarrow{1_s \otimes T(\beta)} AAS_2 \otimes (T \circ T'X) \xrightarrow{F_1 + F_s} T \circ T'(AS_2 \otimes X) .$$

Now 3.12 follows from the analysis of F_r given in proof 3.9.

§4. The Mod-p Pensions of First Order.

Let p be a prime. A homomorphism of pension algebras $P(Z) \longrightarrow P(Z_p)$ is induced by the natural map $Z \longrightarrow Z_p$. The results of §3 for the pensions $\varepsilon_r \in P(Z)$ translate into results for the image pensions $\varepsilon_r \in P(Z_p)$. No new proofs are required since a functor

$$T\colon Z_p\text{-modules} \longrightarrow Z_p\text{-modules}$$

may be extended to a functor of the same degree

$$\overline{T}\colon \text{Abelian groups} \longrightarrow \text{Abelian groups}$$

by setting $\overline{T}(M) = T(M \otimes Z_p)$.

Many of the simplest mod-p pensions, however, do not arise from integral pensions. We shall discuss the first order pensions, that is, those from the groups $H_r(Z_p,1;\, Z_p) = Z_p$, $r \geq 1$. We deal with the case $p = 2$ in 4.1 - 4.11, then with the case p odd in 4.12 - 4.19.

Notation 4.1. For $r \geq 1$ let σ_r denote the non-trivial element of $H_r(Z_2,1) = Z_2$.

The pension σ_r is essentially a mod-2 refinement of ε_r.

Proposition 4.2. If $r \geq 1$ then $\sigma_r \bullet \sigma_r = \varepsilon_r$ in $H_{2r}(Z_2,2)$.

Proof. Observe that $\sigma_r \cdot \sigma_r$ is the image of σ_{2r} under the composition

$$H_*(AS_1) \xrightarrow{\Delta_*} H_*(AS_1 \wedge AS_1) \xrightarrow{f_*} H_*(AS_2)$$

where f is as in 2.7 and $\Delta(x) = x \wedge x$ for $x \in AS_1$. This follows since

$$f_* \bullet \Delta_*(\sigma_{2r}) = f_*(\Sigma_{i+j=2r} \sigma_i \wedge \sigma_j) = \Sigma_{i+j=2r} \sigma_i \bullet \sigma_j = \sigma_r \bullet \sigma_r .$$

Clearly $\Delta^* \bullet f^*(\iota^2) = (\iota^1)^2$ in $H^2(AS_1)$. Now $H^*(AS_2)$ is a polynomial algebra generated by ι^2, $Sq^1\iota^2$, $Sq^2Sq^1\iota^2$, $Sq^4Sq^2Sq^1\iota^2$, ... All of these generators except ι^2 go to zero under $\Delta^* \bullet f^*$. Since ε_r is the dual to $(\iota^2)^r \in H^{2r}(AS_2)$ with respect to the obvious basis, it follows by dualizing that $f_* \bullet \Delta_*(\sigma_{2r}) = \varepsilon_r$.

Let

$$T: \quad Z_2\text{-modules} \longrightarrow Z_2\text{-modules}$$

be a covariant functor with $T(0) = 0$, and let X be an s.s. Z_2-module.

Proposition 4.3.

(i) The algebra $P(Z_2)$ is commutative

(ii) $\sigma_1 = \sigma$

(iii) $\sigma \cdot \sigma_r = 0$ unless $r = 2^j$, $j \geq 0$.

Proof. See proof 3.2.

Let $U(X_1,\ldots,X_r)$ be a covariant functor from Z_2-modules to Z_2-modules with $U(X_1,\ldots,X_r) = 0$ if any $X_i = 0$. Let X be an s.s. Z_2-module.

<u>Proposition 4.4.</u> The pension

$$\sigma_t\colon \pi_* U(X,\ldots,X) \longrightarrow \pi_{*+t} U(SX,\ldots,SX)$$

is zero for $t < r$, and σ_r equals the composition

$$\pi_* U(X,\ldots,X) \xrightarrow{\sigma} \pi_{*+1} U(X,\ldots,SX) \to \cdots \xrightarrow{\sigma} \pi_{*+r-1} U(X,SX,\ldots,SX)$$

$$\xrightarrow{\sigma} \pi_{*+r} U(SX,\ldots,SX)$$

of suspensions in each variable. Thus if U is additive, then σ_r is an isomorphism.

<u>Proof.</u> See proof 3.3.

<u>Proposition 4.5.</u> If T is of degree $\leq r$ and $s > r$, then $\sigma_s\colon \pi_* TX \longrightarrow \pi_{*+s} TSX$ is the zero map.

<u>Proof.</u> See proof 3.4.

<u>Theorem 4.6.</u> If T is of degree $\leq r$ and X is trivial above n, then $\sigma_r\colon \pi_i TX \longrightarrow \pi_{i+r} TSX$ is an isomorphism for $i > n(r-1) + 1$ and monomorphism for $i = n(r-1) + 1$.

<u>Proof.</u> See proof 3.5.

Now let

$$T,\ T'\colon\ Z_2\text{-modules} \longrightarrow Z_2\text{-modules}$$

be covariant functors with $T(0) = T'(0) = 0$; and let X be an s.s. Z_2-module.

For the Eilenberg-Zilber product we have

Proposition 4.7. If $\alpha \in \pi_i TX$ and $\beta \in \pi_j T'X$ then $\sigma(\alpha \cdot \beta) = 0$ and for $r \geq 2$

$$\sigma_r(\alpha \cdot \beta) = \sum_{k=1}^{r-1} \sigma_k(\alpha) \cdot \sigma_{r-k}(\beta)$$

Proof. See proof 3.7.

For the composition product we have the following results.

Theorem 4.8. If $\beta \in \pi_m T'X$, $\alpha \in \pi_{j-1}T(AS_{m-1})$, and $s \geq 1$ then

$$\sigma_s(\beta \circ (\sigma\alpha)) = \sum_{r=1}^{s} \sum_{i_1+\cdots+i_r=s} \sigma_r(\beta) \circ \left(\sigma_{i_1} \cdots \sigma_{i_r}(\sigma\alpha)\right)$$

Proof. See proof 3.9.

Corollary 4.9. If $\beta \in \pi_m T'X$, $\alpha \in \pi_j TAS_m$, and $s \geq 0$ then

$$\sigma_{2^s}(\sigma\beta \circ \sigma\alpha) = \sum_{k=0}^{s} \sigma_{2^k}(\sigma\beta) \circ \left((\sigma_{2^{s-k}})^{2^k}(\sigma\alpha)\right).$$

Proof. This follows from 4.8 using 4.3.

Theorem 4.10. Suppose degree $T' \leq k$ and degree $T \leq h$. If $\beta \in \pi_m T'X$ and $\alpha \in \pi_j TAS_m$ then

$$\sigma_{hk}(\beta \circ \alpha) = \sigma_k(\beta) \circ \left((\sigma_h)^k(\alpha)\right).$$

Proof. See proof 3.11.

Now let T and T' again be of arbitrary degree.

Proposition 4.11. Let $\beta \in \pi_m T'X$ and $\alpha \in \pi_j TAS_m$. Suppose for some integer $s \geq 2$ that $\sigma_t(\beta) = 0$ whenever $1 < t < s$. Then

$$\sigma_s(\beta \circ \alpha) = \sigma_s(\beta) \circ \sigma^s(\alpha) + \sigma(\beta) \circ \sigma_s(\alpha)$$

Proof. See proof 3.12.

We turn now to the case p odd.

The cohomology ring $H^*(Z_p,1;\ Z_p)$ is the tensor product of the exterior algebra generated by ι^1 with the polynomial algebra generated by Bockstein ι^1 . This presentation determines a basis.

Notation 4.12. For $r \geq 1$ let $\eta_r \in H_r(Z_p,1;\ Z_p) = Z_p$ denote the dual to the cohomology generator indicated above.

Let $\mu = 1 \cdot [-1] \in H_0(Z_p,0)$ as in 2.4. We say that an element $\alpha \in P(Z_p)$ is μ-even if $\mu\alpha = \alpha$ and μ-odd if $\mu\alpha = -\alpha$.

Proposition 4.13. For $r \geq 1$ the elements η_{2r} and η_{2r-1} are μ-even if r is even and are μ-odd if r is odd.

Proof. Pass to cohomology and use the description μ in 2.3.

The commutativity relations $\eta_r \cdot \eta_s = \pm \eta_s \cdot \eta_r$ are determined by 2.5 and 4.13. It is clear that many products $\eta_r \cdot \eta_s$ are zero, since a product in $P(Z_p)$ of a μ-even element with a μ-odd element is zero.

Proposition 4.14. If $1 < r,s < 2p$ then $\eta_r \cdot \eta_s = 0$. If $s \geq 2$ then $\eta_1 \cdot \eta_s = \sigma \cdot \eta_s = 0$ unless $s = 2p^j$, $j \geq 0$.

Proof. The second statement follows from familiar homology suspensions [1]. The first statement is easily proved by the method 2.8 using [5].

We suppose henceforth that

$$T, T': Z_p\text{-modules} \longrightarrow Z_p\text{-modules}$$

are covariant functors with $T(0) = T'(0) = 0$, and that X is an s.s. Z_p-module.

Remark 4.15. If $r \geq 1$ and degree $T \leq pr - 1$

$$\eta_{2r} = 0: \pi_* TX \longrightarrow \pi_{*+2r} TSX .$$

If degree $T \leq pr$ then

$$\eta_{2r+1} = 0: \pi_* TX \longrightarrow \pi_{*+2r+1} TSX .$$

These facts are proved from 2.12 using the following, which we state without proof. The groups $\pi_1 SP^s_p AS_1$ are all zero except for

$$\pi_{2r+1}SP_p^{pr+1}AS_1 = Z_p\ , \quad r \geq 0\ ,$$

and for

$$\pi_{2r}SP_p^{pr}AS_1 = Z_p\ , \quad r \geq 1\ .$$

We now consider the action of the η_s on an <u>Eilenberg-Zilber product</u>.

<u>Proposition 4.16.</u> Let $\alpha \in \pi_i TX$ and $\beta \in \pi_j T'X$ so that $\alpha \cdot \beta \in \pi_{i+j}(T \otimes T')X$. Then $\eta_1(\alpha \cdot \beta) = \eta_2(\alpha \cdot \beta) = 0$. If $r \geq 2$ then

$$\eta_{2r}(\alpha \cdot \beta) = \sum_{h+k=r} \eta_{2h}(\alpha) \cdot \eta_{2k}(\beta)\ .$$

If $r \geq 1$ then

$$\eta_{2r+1}(\alpha \cdot \beta) = \sum_{h+k=2r+1} (-1)^{hj} \eta_h(\alpha) \cdot \eta_k(\beta)\ .$$

<u>Proof.</u> The proof 3.7 is easily modified to give 4.16. One uses that

$$\Delta_*: H_*(Z_p,1) \longrightarrow H_*(Z_p,1) \otimes H_*(Z_p,1)$$

maps

$$\Delta_*(\eta_{2r}) = \sum_{h+k=r} \eta_{2h} \otimes \eta_{2k}$$

and

$$\Delta_*(\eta_{2r+1}) = \sum_{h+k=2r+1} \eta_h \otimes \eta_k\ ,$$

where $\Delta: AS_1 \longrightarrow AS_1 \wedge AS_1$ maps $\Delta(x) = x \wedge x$ for $x \in AS_1$. One also must take account of the signs introduced by the twisting map of proof 3.7 and the pairing E_*^1 of 2.1.

Finally we consider the action of the η_s on <u>composition products</u>.

<u>Theorem 4.17.</u> If $\beta \in \pi_m T'X$ and $\alpha \in \pi_{j-1} TAS_{m-1}$ then for $s \geq 1$

$$\eta_s(\beta \circ (\sigma\alpha)) = \sum_{2 \leq 2r \leq s} \sum_{2i_1 + \cdots + 2i_r = s} (-1)^{j-m+sj} \eta_{2r}(\beta) \circ (\varepsilon_{i_1} \cdots \varepsilon_{i_r}(\sigma\alpha))$$

$$+ \sum_{1 \leq 2r+1 \leq s} \sum_{1+2i_1 + \cdots + 2i_r = s} (-1)^{ij+sj} \eta_{2r+1}(\beta) \circ (\eta_1 \varepsilon_{i_1} \cdots \varepsilon_{i_r}(\sigma\alpha))$$

<u>Proof.</u> The proof is a modification of proof 3.9. For $r \geq 1$ let $i_r: AS_r \longrightarrow AAS_1$ and $j_r: AAS_1 \longrightarrow AS_r$ be s.s. Z_p-module maps such that $i_{r*}(\iota_r) = \eta_r$ in $\pi_r AAS_1$ and $j_{r*}(\eta_r) = \iota_r$ in $\pi_* AS_r$. Denote the composition

$$AAS_1 \otimes (T \circ T'X) \xrightarrow{G} T(AAS_1 \otimes T'X) \xrightarrow{T(i_r \circ j_r \otimes 1)} T(AAS_1 \otimes T'X)$$

$$\xrightarrow{T(E)} T \circ T'(AS_1 \otimes X)$$

by F_r. There are induced maps

$$E_*, F_{r*}: \pi_* AAS_1 \otimes \pi_*(T \circ T'X) \longrightarrow \pi_*(T \circ T'(SX))$$

It is not hard to show that

$$E_*(\eta_s \otimes (\beta \circ (\sigma\alpha))) = \sum_{r=1}^{s} F_{r*}(\eta_s \otimes (\beta \circ (\sigma\alpha))) .$$

We now analyse the maps F_{2r} for $r \geq 1$. Let

$$k_{2r}: AS_1 \longrightarrow AS_{2r}$$

be the set complex map with $k_{2r}(y) = j_{2r}(1 \cdot y)$ for $y \in AS_1$. Then k_{2r} is homotopic to the composition

$$AS_1 \xrightarrow{\theta} AS_2 \xrightarrow{\Delta_r} \underbrace{AS_2 \wedge \cdots \wedge AS_2}_{r \text{ times}} \xrightarrow{g_r} \underbrace{AS_2 \wedge \cdots \wedge AS_2}_{r \text{ times}} \xrightarrow{h_r} AS_{2r}$$

where θ is such that $\theta^*: H^*(Z_p,2) \longrightarrow H^*(Z_p,1)$ maps $\theta^*(\iota^2)$ = Bockstein ι^1 ; and where Δ_r, g_r, and h_r are as in proof 3.9. It follows that F_{2r} is group homotopic to the composition

$$AAS_1 \otimes (T \circ T'X) \xrightarrow{A(\Delta_r \circ \theta) \otimes 1} \otimes^r AAS_2 \otimes (T \circ T'X) \xrightarrow{E \circ \cdots \circ E} T(\otimes^r AS_2 \otimes T'X)$$

$$\xrightarrow{T(h_r \otimes 1)} T(AS_{2r} \otimes T'X) \xrightarrow{T(i_{2r} \otimes 1)} T(AAS_1 \otimes T'X) \xrightarrow{T(E)} T \circ T'(AS_1 \otimes X)$$

We next analyse F_{2r+1} for $r \geq 0$. Let

$$k_{2r+1}: AS_1 \longrightarrow AS_{2r+1}$$

be given by $k_{2r+1}(y) = j_{2r+1}(1 \cdot y)$ for $y \in AS_1$. Then k_{2r+1} is homotopic to the composition

$$AS_1 \xrightarrow{\Delta_{r+1}} AS_1 \wedge \cdots \wedge AS_1 \xrightarrow{1 \wedge \theta \wedge \cdots \wedge \theta} AS_1 \wedge AS_2 \wedge \cdots \wedge AS_2$$

$$\xrightarrow{g} AS_1 \otimes AS_2 \otimes \cdots \otimes AS_2 \xrightarrow{h} AS_{2r+1}$$

where $g(y_1 \wedge \cdots \wedge y_{r+1}) = y_1 \otimes \cdots \otimes y_{r+1}$, and h is the group homotopy equivalence of canonical homotopy class. It follows that F_{2r+1} is group homotopic to the composition

$$AAS_1 \otimes (T \circ T'X) \xrightarrow{A(\cdots)\otimes 1} AAS_1 \otimes \otimes^r AAS_2 \otimes (T \circ T'X) \xrightarrow{E \circ \cdots \circ E} T(AS_1 \otimes \otimes^r AS_2 \otimes T'X)$$

$$\xrightarrow{T(h\otimes 1)} T(AS_{2r+1} \otimes T'X) \xrightarrow{T(i_{2r+1}\otimes 1)} T(AAS_1 \otimes T'X) \xrightarrow{T(E)} T \circ T'(AS_1 \otimes X)$$

The map $\theta: AS_1 \longrightarrow AS_2$ has the property that $\theta_*: H_*(Z_p,1) \longrightarrow H_*(Z_p,2)$ maps $\theta_*(\eta_{2s}) = \varepsilon_s$ for $s \geq 0$ and $\theta_*(\eta_{2s+1}) = 0$ for $s \geq 0$. One verifies this using cohomology. It follows that

$$(\Delta_r \circ \theta)_*: H_*(Z_p,1) \longrightarrow \otimes^r H_*(Z_p,2)$$

maps

$$(\Delta_r \circ \theta)_*(\eta_{2s}) = \sum_{i_1+\cdots+i_r=s} \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_r}$$

and

$$(\Delta_r \circ \theta)_*(\eta_{2s+1}) = 0 .$$

Also

$$((1 \wedge \theta \wedge \cdots \wedge \theta) \circ \Delta_{r+1})_*: H_*(Z_p,1) \longrightarrow H_*(Z_p,1) \otimes \otimes^r H_*(Z_p,2)$$

maps

$$((1 \wedge \theta \wedge \cdots \wedge \theta) \circ \Delta_{r+1})_*(\eta_s) = \sum_{i+2i_1+\cdots+2i_r=s} \eta_i \otimes \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_r}$$

It follows from the above that

$$F_{2r*}(\eta_s \otimes (\beta \circ (\sigma\alpha))) = \sum_{2i_1+\cdots+2i_r=s} (-1)^m \eta_{2r}(\beta) \circ (\varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_r}(\sigma\alpha))$$

$$F_{(2r+1)*}(\eta_s \otimes (\beta \circ (\sigma\alpha))) = \sum_{i+2i_1+\cdots+2i_r=s} (-1)^{ij-j} \eta_{2r+1}(\beta) \circ (\eta_i \varepsilon_{i_1} \cdots \varepsilon_{i_r}(\sigma\alpha))$$

These expressions take into account signs introduced by E^1_* of 2.1.

Now 4.17 follows from the fact that

$$E_*(\eta_s \otimes (\beta \circ (\sigma\alpha))) = \sum_{r=1}^{s} F_{r*}(\eta_s \otimes (\beta \circ (\sigma\alpha)))$$

Corollary 4.18. If $\beta \in \pi_m T'X$, $\alpha \in \pi_j TAS_m$, and $r \geq 0$, then

$$\eta_{2p^r}(\sigma\beta \circ \sigma\alpha) = \eta_1(\sigma\beta) \circ \eta_{2p^r}(\sigma\alpha)$$

$$+ (-1)^{j-m} \sum_{k=0}^{r} \eta_{2p^k}(\sigma\beta) \circ ((\varepsilon_{p^{r-k}})^{p^k}(\sigma\alpha)).$$

Proof. This follows from 4.17 using 3.2 and 4.14. We conclude with an unstable composition result.

Proposition 4.19. Let $\beta \in \pi_m T'X$ and $\alpha \in \pi_j TAS_m$. Suppose for some integer $s \geq 2$ that $\eta_t(\beta) = 0$ whenever $1 < t < s$. Then

$$\eta_s(\beta \circ \alpha) = \sigma(\beta) \circ \eta_s(\alpha) + (-1)^{(s-1)(j-m)} \eta_s(\beta) \circ \big((\sigma)^s(\alpha)\big)$$

<u>Proof.</u> Let $\beta\colon AS_m \longrightarrow T'X$ represent $\beta \in \pi_m T'X$. As in proof 3.12 one shows that the composition

$$AS_s \otimes TAS_m \xrightarrow{i_s \otimes T(\beta)} AAS_1 \otimes (T \circ T'X) \xrightarrow{E} T \circ T'(AS_1 \otimes X)$$

is group homotopic to

$$AS_s \otimes TAS_m \xrightarrow{i_s \otimes T(\beta)} AAS_1 \otimes (T \circ T'X) \xrightarrow{F_1 + F_s} T \circ T'(AS_1 \otimes X)$$

where we use the notation of proof 4.17. Now 4.19 follows by the analysis of the F_r in proof 4.17.

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BIOGRAPHICAL NOTE

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