Some relations between

HOMOTOPY THEORY and DIFFERENTIAL GEOMETRY

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Abstract

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The purpose of the thesis is to obtain topological information on a space X (and related spaces), by solving the differential geometric problem of calculating the critical points of a real valued functional on the iterated loop space $\Omega^i X$. This is motivated by Morse theory. There are two ways of obtaining information – either from the critical points themselves, or by considering whole collections of critical points (usually critical points will not be isolated).

In part I we give first a Clifford algebra version of some well known results on "distance functions" on X, where X is a classical compact symmetric space embedded in some euclidean space. To the Clifford algebra C_k we associate a symmetric space E_k, and embed this in the tangent space to E_{k+1} (at the basepoint) in a natural way. Second, we examine the critical point theory of characters of representations of Lie groups, this being an obvious generalisation of the preceding problem.

In the main part of this thesis, part II, we discuss functionals on $\Omega^i X$ for $i > 1$, where X is a classical compact homogeneous space. We give two different examples (in chapters IV and V), where the absolute minima can be calculated. We find they form a manifold Y, and that the inclusion $Y \to \Omega^i X$ induces an isomorphism $\pi_j Y \to \pi_j \Omega^i X$ for certain values of j. The first example uses a Dirichlet functional on $\Omega^i E_k$; here $Y = E_{k-i}$. The second example uses the energy functional on $\Omega^2 X$, where X is a complex flag manifold. This time Y consists of minimum energy harmonic maps, which turn out to be precisely the holomorphic maps.

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Introduction

There are several connections between Algebraic Topology and Differential Geometry, which, so far as they have been developed, are well understood. When one restricts attention to specific manifolds—even very familiar ones, such as the classical matrix groups—problems arise which (as suggested by the fact that they remain unsolved) are often interesting and difficult. We shall discuss several aspects of one such problem.

The underlying principle will be the consideration of the critical points of real-valued functionals on the iterated loop space $\Omega^i X$ of a compact homogeneous space X. For $i > 1$ the situation is already complicated and no general theory exists, despite attempts (especially during the 1960's) to extend the original work (in the 1930's) of M. Morse on the space of paths. Our aim is to present a number of concrete results, some new and some merely new versions of old results, which indicate that the problem is interesting and far from hopeless.

A functional on $\Omega^i X$ will have various critical points, whose calculation is basically a problem in Differential Geometry, and the properties of these critical points will then give topological information. The case $i = 0$ will be interpreted as the study of functionals on the homogeneous space X itself. There are two main ways of obtaining topological information; the first is by considering the properties of the individual critical points (for example one may regard critical maps as canonical representatives of their homotopy classes), and the second is by looking at the space of all critical points in relation to the whole space (for example as in Morse theory). The device of admitting the case $i = 0$ may seem to be somewhat artificial, and in view of this the first two chapters are labelled PART I, and the remaining three PART II. However the two parts are not really independent (for example, Chapter IV will rely heavily on results proved in Chapter I).

PART I

Chapter I discusses Morse functions on classical compact symmetric spaces. All the functions considered arise as "distance functions" for certain embeddings in euclidean space. These have been dealt with already by several authors, and our purpose has been to condense the results in a single general formulation based on Clifford algebras. We begin with a specific example, namely the real part of the trace function on $U(n)$, and then go on to point out some relations between Clifford algebras and symmetric spaces for which no coherent account seems to exist. It is then possible to give very elementary proofs of some well known results concerning Morse functions on these spaces. We give many details as these are required later in chapter IV. The classical compact symmetric spaces can be described as homogeneous spaces of groups of unitary elements in Clifford algebras (unitary with respect to some suitable involution); E_k denotes the quotient of the unitary elements in C_k by those in C_{k-1} . There is then a natural embedding of E_k in the tangent space $\mathcal{E}_{k+1}$ of the next space E_{k+1} . This is very convenient for describing certain properties of symmetric spaces, eg complex structures, totally geodesic submanifolds, stable homotopy groups etc., whereas the usual Lie-algebraic or matrix-theoretic descriptions are not. In chapter I we put a number of well known results in this framework.

Chapter II digresses to consider a natural question arising from the discussion of Morse functions in chapter I . For the Lie groups $O(n)$, $U(n)$, $Sp(n)$, the Morse functions of chapter I all arise from the standard representation of these groups (i.e. the trace function may be viewed as the character of the standard representation). One may then ask whether other representations give interesting Morse functions on compact Lie groups. It turns out that representations do not necessarily give nondegenerate functions, and even when they do, the critical point theory is not particularly straightforward. We make some conjectures and give some examples, mainly for Lie groups of low rank.

PART II

In chapter III we make some remarks on the study of functionals on the iterated loop space $\Omega^i X$ (with $i > 0$), before proceeding to the main results of the thesis in chapters IV and V . We consider an element of $\Omega^i X$ to be a map $S^i \longrightarrow X$; the precise restrictions on the map will be quite important. The prototype result, which we wish to generalise, is that of Bott [5]. In the notation of chapter I , this says that the absolute minima for the length functional on the space of paths ΩE_k form a space homeomorphic to E_{k-1} , and that the inclusion $E_{k-1} \to \Omega E_k$ induces an isomorphism $\pi_i E_{k-1} \longrightarrow \pi_i \Omega E_k$ for i up to some dimension. In each of chapters IV and V we prove a generalisation of this to the iterated loop space $\Omega^i X$. In both cases we find that

(a) the absolute minima of some functional on $\Omega^i X$ form a finite dimensional manifold Y , and

(b) the inclusion $Y \to \Omega^i X$ induces an isomorphism in homotopy groups up to some dimension .

In Bott's work [5] , (b) is deduced from (a) by Morse theory . For $i > 1$, no suitable generalisation of Morse theory exists, so we must resort to other methods.

Chapter IV is the more elementary; we use the "ith order energy functional" on $\Omega^i E_k$, and find that the absolute minima are totally geodesic maps forming a space homeomorphic to E_{k-i} . Bott periodicity arises in this way because of the periodicity of the symmetric spaces E_k . The topological information obtained here is of course not new, rather, the interest lies in being able to calculate critical points of a functional on a space of maps. The methods is to impose carefully chosen "boundary conditions" on the maps involved, which permits the reduction of the problem to the case $i = 1$. The explicit description of the critical points is of independent interest; they provide totally geodesic spheres in the spaces E_k , and the Bott periodicity map is seen to be induced by taking a "generalised Samelson product" with such spheres. Finally, there is a connection between these i-spheres in E_k and maximal tori in E_{k-i+1}, which is related to the well known isomorphism $KO(S^{8n}) \cong RO\ Spin(8n)/RO\ Spin(8n+1)$ [2].

In chapter V we take $i = 2$ and X a complex flag manifold $U(K)/U(K_1)\times \ldots \times U(K_n)$, $K_1 + \ldots + K_n = K$. For appropriate components of maps $S^2 \longrightarrow X$, the minimal energy maps turn out to be precisely the holomorphic maps. To prove (a) and (b) above it is therefore necessary to examine in detail the space of holomorphic maps $S^2 \longrightarrow X$, and so the methods are very different to those in chapter IV. The case $X = P_n\mathbb{C}$ was dealt with by G. Segal [21], and we extend this to certain flag manifolds. The results give information on the topology of the space of holomorphic maps $S^2 \longrightarrow X$.

Contents

Chapter 1

In this chapter (see §3, §4) we give a new description of a number of results – many of them well known – concerning the application of Morse theory to compact Lie groups and symmetric spaces. Section §1 contains a motivating example, and §2 and §3 describe a general way of obtaining Morse functions. Here we condense the material of a large number of recent papers which describe various special cases of this procedure. In section §4 we give simple algebraic proofs of some of the results of §2, and make some remarks which will be generalised in chapters III and IV .

§1. A Morse function on the unitary group $U(n)$

To motivate much of what follows, we shall investigate the information provided on the group $U(n)$ of unitary $n \times n$ matrices by the function f,

$$f : U(n) \longrightarrow \mathbb{R} \qquad X \longmapsto \mathrm{Re}\,(\mathrm{trace}\ X).$$

First, f has been shown to be a "Morse-Bott" function (Frankel [12]), ie the critical points of f form a collection of nondegenerate critical submanifolds. (We shall use the terminology of Morse theory from now on without further reference; see [18], [5].) We shall begin by giving this simple calculation.

Let T be the standard maximal torus of $U(n)$, ie the diagonal matrices $\{\mathrm{diag}\,(e^{i\theta_1}, \ldots, e^{i\theta_n})\}$. The following lemma shows that we only need consider the function $f|_T$:

<u>Lemma 1.1</u>: 1) Any critical point of f is conjugate to a critical point lying in T.

(Frankel's Lemma)

2) grad f is tangent to T (grad is defined using the bi-invariant inner product $\langle A,B \rangle = \mathrm{tr}\, A B^*$ on $T_I U(n)$ = skew hermitian matrices).

<u>Proof</u>: 1) is clear as f is a class function.

2) is true when grad f is evaluated at a regular element, since grad f is orthogonal to the orbit (under conjugation) of the element, and it is known that

such orbits cut tori orthogonally. But the regular elements are dense.

The function f is $\mathrm{diag}(e^{i\theta_1},\ldots,e^{i\theta_n}) \longmapsto \sum_{j=1}^{n} \cos\theta_j$, so its critical points are given by $\sin\theta_j = 0$, ie $e^{i\theta_j} = \pm 1$. Two such points are conjugate iff they have the same numbers of $+1$ and -1, so we have, in $U(n)$, $n+1$ connected critical submanifolds, the (i-1)th being homeomorphic to the complex grassmannian $U(n)/U(i) \times U(n-i)$.

Nondegeneracy follows from the next proposition:

Proposition 12: For the critical point ~~up to conjugation~~ $\sigma = \mathrm{diag}(1,\ldots,1.-1,\ldots,-1)$ (with the element 1 repeated i times), the closure of the unstable manifold (ie the set of flow lines of grad f which move out of σ) is $\sigma . U(i) \times I$ ($\subsetneq U(n)$). The closure of the stable manifold is $\sigma . I \times U(n-i)$.= $I \times U(n-i)$

Proof: First we show that the two submanifolds quoted above are invariant under the flow of grad f. We must show that grad f is tangential to each manifold. We know .by §2 of Frankel grad f is tangential to ~~σ~~ $U(i) \times U(n-i)$, so take a curve in this manifold which is orthogonal to σ. ~~σ~~ $U(i) \times I$, and consider the derivative of f in the direction of this curve. We must show that this derivative is zero. Let such a curve be $t \longmapsto \sigma . I \times a(t)$, where $a(t)$ is a curve in $U(n-i)$. We have:

$$f(I \times -a(t)) = \mathrm{Re\ tr}\, I - \mathrm{Re\ tr}\, a(t)\ ;\ df/dt\,\big|_{t=0} = 0 - \mathrm{Re\ tr}\ \dot{a}(0) = 0$$

(Note that the matrix $\dot{a}(0)$ is skew symmetric.) The case of $\sigma . I \times U(n-i)$

is similar. Thus, both the manifolds $\sigma . U(n) \times I$ and $\sigma . I \times U(n-i)$ are invariant under the flow.

Next we show that the restriction of f has a nondegenerate absolute maximum (respectively, a nondegenerate absolute minimum) at σ. This will show our manifold has the same tangent space as the unstable (respectively, stable) manifold, and from invariance under the flow we can conclude that the former contains the latter. To see that no "extra" flow lines have been included, one can look at the intersection with a maximal torus where this is easy to check.

Take $\sigma . U(i) \times I$. We show that the second derivative of f in the direction of any curve $\sigma . \Theta(t) \times I$ is strictly negative at σ. For the purpose of computing this derivative, we may assume that Θ is a 1-parameter subgroup of $U(i)$. So we take $\Theta(t) = \mathrm{diag}\,(e^{a_1 t}, \ldots, e^{a_i t})$, and hence:

$$f(\Theta(t)) = \left(\sum_{j=1}^{i} \cos a_j t\right) - (n-i) \quad \text{and} \quad f''\Big|_{t=0} = -\sum_{j=1}^{i} a_j^2 < 0$$

The case of $\sigma . I \times U(n-i)$ is proved similarly.

Summarising, we have :

<u>Theorem 1.3 (Frankel [12])</u>: The function $f : U(n) \to R$ has $n+1$ nondegenerate critical submanifolds, the $(i-1)$th being of index i^2, and homeomorphic to $U(n)/U(i) \times U(n-i)$.

From this theorem one can derive well known facts concerning the homology of $U(n)$; for example one can show that it has no torsion. We shall not be concerned here with such well documented results; instead we shall investigate further :

a) the negative bundles

b) the flow lines .

a) Negative bundles

The critical submanifold containing σ , ie $M_\sigma = U(n)/U(i) \times U(n-i)$, has a vector bundle of real fibre dimension i^2 associated to it, called the negative bundle, whose fibre over m ($= X\sigma X^{-1}$ for some $X \in U(n)$) is the subspace of $T_m M_\sigma$ spanned by the negative eigenvalues of the Hessian bilinear form. The closure of the unstable manifold at σ is an embedded $U(i)$, on which the subgroup $U(i)$ of $U(n)$ acts by conjugation. Hence the negative bundle is the homogeneous vector bundle associated to the homogeneous space $U(n)/U(i) \times U(n-i)$ by the representation (Ad, Trivial): $U(i) \times U(n-i) \longrightarrow \mathrm{Aut}\ u(i)$ where $u(i) = T_I U(i)$, (so if λ denotes the standard representation of $U(i)$, then the complexification of the negative bundle is the bundle induced by $(\lambda \otimes \bar{\lambda}, \text{Trivial})$).

b) Flow lines of grad f

The proof of proposition 1.2 above does not imply that the flow lines forming the unstable manifold $\sigma . U(i) \times I$ are the 1-parameter subgroups of $U(i)$. We shall calculate the flow lines in this section.

Lemma 1.4: If one identifies $T_X U(n)$ with all elements $X.S$ where $S \in T_I U(n)$ is a skew symmetric matrix, then:

$$\text{grad } f(X) = X. \tfrac{1}{2}(X^* - X)$$

Proof: Let $\text{grad } f(X) = X. \phi(X)$. Then by definition $\langle X. \phi(X), X.V \rangle = df(X.V)$, for all $V \in T_I U(n)$, so:

$$\begin{aligned} \langle \phi(X), V \rangle &= \text{Re tr } X.V \\ &= \tfrac{1}{2}\text{tr}(XV + V^* X^*) = \tfrac{1}{2}\text{tr}(XV + X^* V^*) \\ &= \tfrac{1}{2}\langle X^* - X, V \rangle \text{ for all } V \in T_I U(n) \end{aligned}$$

So we must have $\phi(X) = \frac{1}{2}(X^* - X)$, as required.

(Observe that the critical points are given by the zeros of grad, ie the points X for which $X^* = X$ and $X^* = X^{-1}$, ie the square roots of I in $U(n)$.)

By lemma 1.1, we only need to compute the flow lines in the torus T of the function $f|_T$. Let Θ be a curve in T:

$$\Theta : t \longmapsto \begin{pmatrix} e^{if_1(t)} & & \\ & \ddots & \\ & & e^{if_n(t)} \end{pmatrix}$$

This is a flow line ($\ni$ the point $t=0$) iff

$$\Theta'(t) = \text{grad } \Theta(t)$$

ie $\quad f_i'(t) = \sin f_i(t)$ for all i.

The solutions to this are:

1) $f_i(t) = k\pi$, $k \in \mathbb{Z}$

2) $f_i(t) = 2\tan^{-1}(A_i e^t)$ where $A_i > 0$.

Observe that this gives us a flow line through the point with $t = 0$, which is never a critical point, and the the flow line "starts" and "ends" (ie for $t = -\infty$ and $t = \infty$) on critical points.

It is now clear that a flow line can be "re-parametrised" to give a 1-parameter subgroup precisely when the following condition holds: $|A_i|$ is independent of i. In this case, the corresponding geodesics are:

$$u \longmapsto \begin{pmatrix} e^{\pm iu} & & & & \\ & \ddots & & & \\ & & e^{\pm iu} & & \\ & & & \pm 1 & \\ & & & & \ddots \\ & & & & & \pm 1 \end{pmatrix}$$ and its conjugates.

(The geodesics passing through I are obtained by taking such curves with no constant terms -1 on the diagonal.) We use "re-parametrisation" in its widest sense here, ie we are only interested in the underlying set of points. Therefore, we are only concerned with "prime" geodesics. We summarise this result:

<u>Proposition 1.5:</u> For the function $f: U(n) \longrightarrow R$,

$$\{\text{Geodesics}\} \cap \{\text{Flow lines}\} = \{\text{Paths } u \longmapsto K\cos u + E\sin u\}$$

where $K, E \in U(i)$ and $K^2 = I$, $E^2 = -I$, and where $U(i)$ is included in $U(n)$ by a standard homomorphism.

The prime geodesics occuring here will be of special significance later; those with $i = n$ we call planar geodesics (they are intersections of planes in $M_n(\mathbb{C})$ with $U(n)$), and those with $i < n$ sub-planar geodesics. (Any prime geodesic of the form $U(n) \cap (\text{Plane in } M_n\mathbb{C})$ is clearly of the form $X(I\cos u + E\sin u)$ where $X \in U(n)$ and $E^2 = -I$.) It is of interest to note the following relation between the index and the number of geodesic flow lines in the unstable manifold:

Proposition 16: For the function $f: U(n) \longrightarrow R$, the number of geodesic flow lines in the intersection of a maximal torus with the unstable manifold of the critical point of index i^2 is $3^i - 1$ (there are $-1 + \frac{1}{2}(i+1)(i+2)$ conjugacy classes of flow lines).

Proof: Clear from the calculation above.

§2. Distance functions and embeddings

The function f of §1 can be interpreted as a "distance function". There is a hermitian form $\langle A,B\rangle = \operatorname{tr} AB^*$ on $M_n\mathbb{C}$, with associated norm $\| \ \|$, and we have:

$$f(X) = \frac{\| X+I\|^2 - 2n}{2} \quad \text{for} \quad X \in U(n) \subsetneq M_n\mathbb{C}.$$

So f has the same critical points and underlying flow curves as the function:

$$d_{-I}(X) = \| X+I \|^2 = (\text{distance of } X \text{ from } -I)^2$$

It is not surprising that this example generalises to include a) all classical symmetric spaces and b) all distance functions on these spaces with respect to some natural embedding in euclidean space. To do this, take a compact riemannian globally symmetric space G/K, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, and let $X = \mathrm{Ad}(K)(E)$ be the orbit of $E \in \mathfrak{m}$ under the adjoint action of K on $\mathfrak{m}$. The tangent space $\mathfrak{m}$ has an Ad-invariant metric, so for any $P \in \mathfrak{m}$ one has the distance function $d_P : X \longrightarrow \mathbb{R}$,

$$d_P : x \longmapsto (\text{distance of } P \text{ from } x)^2.$$

Theorem 2.1 (Bott): 1) For all nonsingular points P, d_P is a Morse function (ie it has isolated nondegenerate critical points).

2) The critical points are $X \cap \mathfrak{h}$, where $\mathfrak{h}$ is the unique Cartan subalgebra of $\mathfrak{m}$ which contains P.

Proof: See chapter II of [6] for details. The critical points are given by normals from P to the orbit X, and we know that Cartan subalgebras cut orbits orthogonally. Given a normal, it must cut the orbit of P itself orthogonally, so the normal lies in the Cartan subalgebra through P, and conversely.

Theorem 2.2: 1) For all P, d_P is a Morse–Bott function
(ie it has nondegenerate critical manifolds)

2) The critical points are $X \cap \left(\bigcup_{\substack{\mathfrak{h} \text{ a Cartan subalgebra} \\ \text{and } \mathfrak{h} \ni P}} \mathfrak{h}\right)$.

Proof: The same proof as for theorem 1.1 extends to this case.

Example: For the situation of §1, we have $G = U(2n)$, $K = U(n) \times U(n)$, $\mathfrak{m} = M_n\mathbb{C}$, $E = I$, the Ad action is given by $AD(X,Y)(E) = XEY^{-1}$, and f is essentially the function d_{-I}, as pointed out above. A typical Cartan subalgebra containing $-I$ is $\tau = \{\text{diagonal matrices in } M_n\mathbb{R}\}$, and the others are obtained by conjugating τ by elements of $U(n)$. (NB: $-I$ does not commute with all elements of $\mathfrak{m} = M_n\mathbb{C}$ in the sense of the Lie bracket!)

Theorems 2.1 and 2.2 are particularly useful because of the following fact:

Theorem 2.3: All compact classical irreducible symmetric spaces can be obtained as Ad orbits of other such spaces.

Proof: By inspection.

The embeddings of classical symmetric spaces into euclidean spaces which arise in this way have the following special property:

Theorem 2.4: (Kobayashi and Takeuchi [17]): The embeddings of symmetric spaces as Ad orbits described above are minimal in the sense of having minimal total curvature (see below). In particular, every compact irreducible classical symmetric space admits a minimal embedding into euclidean space.

Recall that the total curvature of an immersion of a manifold $\phi : M \longrightarrow \mathbb{R}^N$ is defined as follows. There is a canonical map ν, from the sphere bundle B of the normal bundle of $\phi(M)$, to the unit sphere S^{N-1} in $\mathbb{R}^N$. Let $d\sigma$ be the volume element of S^{N-1}. Then the total curvature τ is defined by:

$$\tau(M, \phi, \mathbb{R}^N) = (1/\mathrm{vol}\, S^{N-1}) \int_B \nu^*(d\sigma)$$

Define the Morse number of a manifold M to be the least number of critical points which a Morse function on M can have. Then one has the following interesting relation between minimal embeddings and Morse theory:

Proposition 2.5 (Kobayashi and Takeuchi [17]): If ϕ is an embedding of M into $\mathbb{R}^N$, then ϕ is minimal iff the Morse number of M is equal to the number of critical points of the function $X \longmapsto \langle X , V \rangle$ for almost all $V \in \mathbb{R}^N$.

The distance functions described above for symmetric spaces, which correspond 1:1 to regular points of the tangent space, do in fact give Morse functions which have the minimum number of critical points. We shall give one more geometrical property of these embeddings. Observe that because the Ad action is by isometries, each of our symmetric spaces is embedded in a sphere in the appropriate euclidean space. In general, for an embedding $\phi : M \longrightarrow S^{N-1}(r)$ of a manifold in a sphere of radius r in euclidean space, ϕ is said to be minimal (in the sense of mean curvature) if the following "mean curvature normals" :

$$\xi_m = \sum_{i=1}^{n} \alpha(e_i, e_i) \,, \quad \text{(where } \alpha \text{ is the second fundamental form)} \quad \text{(and } \{e_1, \ldots, e_n\} \text{ is a basis for } T_m M \text{)}$$

vanish for all points m in M. (The second fundamental form α of the embedding ϕ gives a bilinear map $\alpha_m : T_m M \times T_m M \longrightarrow N_m$, where N is the normal space at m.)

Theorem 2.6 (Kobayashi and Takeuchi [17]): The embeddings of symmetric spaces described above are all minimal in the sense of mean curvature.

Just as we gave an equivalent definition of minimal total curvature in proposition 2.5, we have the following description of minimality in the sense of mean curvature.

Proposition 2.7 (Kobayshi and Takeuchi [17]): Let $\phi : M \longrightarrow S^{N-1}$ be an isometric immersion, with coordinate functions obtained by taking the composition $M \longrightarrow S^{N-1} \longrightarrow R^N$ denoted y_i, $i = 1, \ldots, N$. Then ϕ is minimal in the sense of mean curvature iff $\Delta y_i = -(n/r^2)\, y_i$ where Δ is the Laplacian for M. $(r = \| E \|)$

In §4 of this chapter we shall put the above theorems in a more general framework, which will help to explain why they are true. We remark here that a number of authors have examined various special cases of these theorems, and for convenience we give a bibliography as an appendix to this section.

The last section of this bibliography ([A15] to [A 21]) deals with Morse functions (primarily on grassmannians, and in particular on projective space) whose critical points have been calculated by direct calculation in local coordinates. For example, one has the function of Milnor [18]:

$$P_n\mathbb{C} \longrightarrow \mathbb{R} \qquad (x_0, \ldots, x_n) \longrightarrow \sum_{i=0}^{n} \lambda_i |x_i|^2 \qquad \lambda_0, \ldots, \lambda_n \in \mathbb{R} .$$

This is a case of theorem 2.2 ; we take $P_n\mathbb{C}$ embedded in $T_I(U(n+1))$ as the Ad orbit of X, and then consider the function d_R , where:

$$X = \begin{pmatrix} -i & & \\ & i & \\ & & \ddots & \\ & & & i \end{pmatrix} \qquad R = \begin{pmatrix} i\lambda_0 & & \\ & \ddots & \\ & & i\lambda_n \end{pmatrix}$$

Alternatively, one could embed $P_n\mathbb{C}$ in $U(n)$ by the quadratic embedding induced by

$$X \longmapsto \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} X \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} X^{-1}$$

and then consider the restriction of the function $X \longrightarrow \mathrm{Re}\ \mathrm{tr}\,(RX)$.

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§3. Symmetric spaces, Geometric transformation groups, and Clifford algebras

This section is quite elementary but we shall establish some notation. There is a sense, to be explained, in which Clifford algebras classify symmetric spaces; to the Clifford algebra C_k we shall associate a symmetric space E_k. Our viewpoint in the remainder of this chapter, and especially in chapter IV, will be that as far as classical spaces are concerned, there is no loss of generality in working with the spaces E_k. The descriptions of these spaces are sufficiently specific, however, to reveal certain features not easily obtained by the usual matrix-theoretic methods. In this chapter we shall look at Morse functions and embeddings, for example, and in chapter IV we shall consider the iterated loop spaces $\Omega^j E_k$. Before introducing the spaces E_k, we make a few remarks on symmetric spaces and their transformation groups.

Symmetric spaces:

We begin with some general remarks – the reference is [16]. A riemannian symmetric space M has a "symmetry" s (reflection in the origin o), which is an isometry and satisfies $s^2 = I$. If G is the identity component of the group of isometries, and M is compact, then G is compact and acts transitively. Let the isotropy at o be H, so that $M \cong G/H$. Note that s is not necessarily in G; the action of s on G/H is given by $s(gH) = (sgs)H$.

To classify such spaces up to connected components and coverings one converts to a Lie algebra problem, and to do this one must first replace the pair (M, s) by the pair (G, σ) where σ is the automorphism of G defined by $g \longmapsto sgs$. (One must avoid mentioning s, as it is not necessary that $s \in G$.) We have $\{\text{fixed points of } \sigma\} \supseteq H \supseteq \{\text{identity component of fixed points of } \sigma\}$, so that (G, σ) determines only the simply connected cover of M. In table I (pages 33/4) we give a list of the compact irreducible symmetric spaces of classical type, together with the involution σ; for notational convenience we do not always take a simply connected or even a connected representative.

Observe that in most cases σ is conjugation by an element of G, whose square is I or $-I$. In the remaining cases one can embed G in another compact group G' so that σ is conjugation by an element of G'. (There are fives cases: $O(n) \times O(n) \leq O(2n)$ and σ is conjugation by $\left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right)$, and similarly for $U(n) \times U(n)$, $Sp(n) \times Sp(n)$; next $U(n) \subseteq Sp(n)$ and $\sigma : X \longmapsto \overline{X}$ is conjugation by j; and finally $U(2n) \subseteq O(4n)$ and σ is conjugation by an element of $O(4n)$.) H acts on $M \cong G/H$; call this the AD action. By differentiation, H acts on the tangent space $\mathfrak{m}$ at o; this is the Ad action. Observe that any orbit of either action is again a symmetric space; we pointed out above (theorem 23) that any of the classical spaces arises in this way as an orbit of Ad, and one can see this by inspecting Table I. The converse procedure, ie how to find a symmetric space for which a given space is an orbit of Ad, can also be described directly, in terms of geometric tranformation groups.

Geometric transformation groups:

It is possible that a symmetric space M admits the action of a transitive Lie group which is larger than the group of isometries. If L is such a group (connected), and K is the isotropy at o, then L/K is a new presentation of M which may be attributed to a "geometric structure" on M. (L is the "transformation group" of this structure.) For example, the standard way of demonstrating that the complex grassmannian $U(n) / U(p) \times U(q)$ has a complex structure is to observe that it admits a transitive action of $Gl(n,\mathbb{C})$.

Nagano [19] has classified pairs (L,M) of this form, and in particular one has the following theorem:

Theorem 3.1 (Nagano [19]): Suppose a compact irreducible symmetric space M admits the action of a transformation group L, which is irreducible, has a centre of dimension ≤ 1, and is larger than the isometry group of M. Then the involutive isomorphism σ of G extends to L, and L/G is an irreducible symmetric space of noncompact type; G is a maximal compact subgroup of L. Moreover, the centre of the isotropy of L on M (at o) is nondiscrete, and contains an element Z such that M is embedded in L/G as the AD orbit of Z. (At the Lie algebra level, there is an element z in the Lie algebra $\mathfrak{l}$ of L such that the Ad orbit of z gives an embedding of M in $\mathfrak{p}$, the tangent space to M at o.) The linear transformation $Ad(z)$ has eigenvalues $+1,-1$ (on $\mathfrak{g}$) and 0 (on $\mathfrak{p} = \mathfrak{l}/\mathfrak{g}$).) Conversely, M can be recovered from the triple (L,G,Z).

To see that this is the converse to the procedure exhibited in table I, we must consider the compact form L' of the non compact group L. This L' is obtained as follows:

$\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{p}$ is the decomposition associated with L/G

$\mathfrak{l}_c = \mathfrak{l} \oplus i\mathfrak{l}$ is the complexification of $\mathfrak{l}$.

Then L' is the group with Lie algebra $\mathfrak{g} \oplus i\mathfrak{l}$. L'/G is a compact symmetric space with the associated decomposition $\mathfrak{l}' = \mathfrak{g} \oplus i\mathfrak{p}$. The Ad orbit of iz for the space L'/G is the space M, and the involutive isomorphism σ for M is AD(Z') where Z' = exp(iz) is the symmetry s; this is the involution σ appearing in table I.

We see that it is only possible to have the symmetry in G if there exists an element Z of G such the AD(Z) = AD(Z'), ie when there exists an element c in the centre of L' such that $c.Z' \in G$. By inspection of table I, this occurs for five of the ten spaces; these are precisely the spaces G/H for which H is of maximal rank (alternatively, those of strictly positive Euler characteristic). We shall see later (proposition 3.5) that the "geometric structures" possessed by these spaces have a simple description. In fact we shall show that a classical space G/M can always be identified with an Ad orbit of some other classical space L'/G, regardless of its geometric structure.

Clifford algebras:

C_k denotes the Clifford algebra with generators $e_1, \ldots, e_k$ satisfying the usual relations $e_i^2 = -1$, $e_i e_j + e_j e_i = 0$ $(i \neq j)$; these elements generate a real vector space $R^k \subseteq C_k$ and are orthonormal vectors in the unit sphere $S^{k-1} \subseteq R^k$ with respect to the usual metric. C_k^c is the complexification over R of C_k.

Table II gives a list of the C_k and the C_k^c in terms of classical matrix algebras, for example we have: $C_{2k}^c = M(2^k, \mathbb{C})$, $C_{2k+1}^c = M(2^k, \mathbb{C}) \oplus M(2^k, \mathbb{C})$. Let us see how to make an explicit identification in this case. The algebra C_{2k}^c acts on itself by multiplication on the right; let R_x denote the action of x.

Define:

$$\Delta_{2k} = \bigcap_{j=1}^{k} \{(-1) - \text{eigenspace for } R_{i e_{2j-1} e_{2j}}\}$$

Then Δ_{2k} is a complex 2^k-dimensional left C_{2k}^c-module. Our explicit identification is given by $x \longmapsto R_x$. The map $\phi_{2k-1}: C_{2k-1}^c \longrightarrow C_{2k}^c$, $e_i \longmapsto e_i e_{2k}$, identifies C_{2k-1}^c with $(C_{2k}^c)^{\text{even}}$. Thus C_{2k-1}^c acts on Δ_{2k} via ϕ_{2k-1}, and under this action $\Delta_{2k} = \Delta_{2k-1}^+ \oplus \Delta_{2k-1}^-$ (this is the decomposition into the even and odd parts, respectively). This provides an explicit identification $C_{2k-1}^c = M(2^{k-1}, \mathbb{C}) \oplus M(2^{k-1}, \mathbb{C})$.

Next, $\phi_{2k-1} \circ \phi_{2k-2}: C_{2k-2}^c \longrightarrow C_{2k}^c$, $e_i \longmapsto e_i e_{2k-1}$, allows C_{2k-2}^c to act on Δ_{2k}; it acts on each of Δ_{2k-1}^+ and Δ_{2k-1}^- but the two actions are equivalent (to see this, conjugate by e_{2k}). This identifies C_{2k-2}^c with $M(2^{k-1}, \mathbb{C})$, and we have the map:

$$\phi_{2k-1} \circ \phi_{2k-2}: M(2^{k-1}, \mathbb{C}) \longrightarrow M(2^k, \mathbb{C}) \qquad X \longmapsto \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$$

We shall denote this map by ϕ^2 (ie we drop the suffices). This map restricts to the corresponding unitary groups, and in general we have $\phi^j : U(2^{k-j}) \longrightarrow U(2^k)$. These explicit descriptions will be used later, but for the moment we shall just need the definition of C_k in terms of its generators.

Definition: The standard involution σ on C_k is obtained by extending the map $\ell : R^k \longrightarrow R^k$ given by $e_i \longmapsto -e_i$ to an involution; the standard anti-involution σ^* on C_k is obtained by extending ℓ to an anti-involution.

Note that σ and σ^* depend on the choice of the subspace R^k. We shall of course choose σ and σ^* so that there is consistency with respect to the inclusion ϕ, but it should be pointed out that if one is dealing with a particular C_k then the matrix calculations involved are sometimes simplified by making a different (ie non-standard) choice of σ, σ^*.

Definition: D_k = Unitary elements of C_k with respect to σ^*

$$E_k = D_k / \phi(D_{k-1}) .$$

(Similarly for D_k^c, E_k^c.)

Proposition 3.2: E_k and E_k^c are compact symmetric spaces of classical type, and they are irreducible except for E_k, $k \equiv 1,5(8)$ and for E_k^c, $k \equiv 1(2)$ (when isotropy representation contains a trivial summand).

Proof: This is clear from a comparison of tables I and II. Note that E_k^c is connected, and E_k is connected except when $k \equiv 6,7(8)$ (when it has two components).

We can identify $\mathfrak{e}_k$ (the tangent space of E_k at o) and $\mathfrak{d}_k$ (the Lie algebra of D_k) with subsets of C_k as follows:

$$\mathfrak{d}_k = \{x \in C_k \mid \sigma^* x = -x\} \qquad \mathfrak{e}_k = \{x \in \mathfrak{d}_k \mid \sigma x = -x\}$$

This is consistent with the usual decomposition $\mathfrak{d}_k = \mathfrak{e}_k \oplus \mathfrak{d}_{k-1}$, ie the decomposition into eigenspaces of $d\sigma$ $(=\sigma)$. Observe that we have identified $\mathfrak{d}_{k-1}$ with the space of even elements of $\mathfrak{d}_k$, and $\mathfrak{e}_k$ with the odd ones. With this identification we have $e_1, \ldots, e_k \in \mathfrak{e}_k$.

A great advantage of the Clifford algebra formulation is that one can deal both with symmetric spaces / Lie groups and with their tangent spaces at the same time, by considering suitable embeddings in some C_k. Having done the tangent spaces in the last paragraph, we shall now look at the symmetric spaces themselves. First observe that $\phi(D_{k-1})$ consists of the even elements of D_k (because $\phi(C_{k-1})$ consists of the even elements of C_k, and $\phi(\sigma^*) = \sigma^*$). We now have the embeddings:

$$\mathfrak{d}_{k-1} \subseteq \mathfrak{d}_k \subseteq C_k \qquad \phi(D_{k-1}) \subseteq D_k \subseteq C_k$$

The continuation of this procedure has a simple description. The space $\mathfrak{d}_{k-2}$ is embedded as the even elements of $\mathfrak{d}_{k-1}$ (in C_{k-1}), ie as the fixed points of the involution (of C_{k-1}) defined by $e_i \mapsto -e_i$, $i = 1,\dots,k-1$. On composing with $\phi : C_{k-1} \longrightarrow C_k$, the defining relations for the involution become $e_i e_k \mapsto -e_i e_k$, $i = 1,\dots,k-1$, which is clearly the same as conjugation by e_k. By induction we obtain:

$$\mathfrak{d}_{k-j} = \{x \in \mathfrak{d}_{k-1} \mid x \text{ commutes with } e_k, \dots, e_{k-j+2}\} \subseteq C_k \qquad (j \geq 2)$$

Similarly, $\phi(D_{k-2})$ is the set of even elements of D_{k-1}, and on composing with $\phi : C_{k-1} \longrightarrow C_k$, $\phi^2(D_{k-2})$ is identified with those elements of $\phi(D_{k-1})$ (in C_k) which commute with e_k. In general one has:

$$\phi^j(D_{k-j}) = \{x \in \phi(D_{k-1}) \mid x \text{ commutes with } e_k, \dots, e_{k-j+2}\} \subsetneq C_k \qquad (j \geq 2)$$

<u>Definition:</u> Let σ_k denote the automorphism of D_k defining the symmetric space E_k.

It is now clear that the automorphism σ_{k-1} of $\phi(D_{k-1})$, and its derivative $d\sigma_{k-1}$ on $\mathfrak{d}_{k-1}$, <u>are both given by conjugation by e_k</u>.

<u>Proposition 3.3:</u> For the Ad action of the symmetric space E_k, the orbit of $e_k \in \mathcal{E}_k$ is E_{k-1}.

<u>Proof:</u> We only need to show that the isotropy subgroup of e_k under the action of $\phi(D_{k-1})$ (by conjugation) is D_{k-2}. The isotropy subgroup consists of those elements of $\phi(D_{k-1})$ which commute with e_k, and this is $\phi^2(D_{k-2})$

by the remark above.

Proposition 3.4: E_k is embedded in C_{k+1} as components of $e_{k+1}\phi(D_k)\cap d_{k+1}$. (Note that $D_{k+1}\cap d_{k+1}$ is the set of square roots of $-I$ in D_{k+1}.) .

Proof: Certainly $E_k \subseteq e\phi(D_k)\cap d_{k+1}$. For the tangent spaces at e_{k+1}:

$$e_{k+1} d_k \cap d_{k+1} = \{ x \in e_{k+1} d_k \mid x e_{k+1} = - e_{k+1} x \}$$
$$= e_{k+1} e_k$$

Hence the result, as E_k is a homogeneous space.

Example: The tangent space at the base point to $U(2n)/U(n) \times U(n)$ can be identified with $M(n,\mathbb{C})$. The Ad action of $U(n) \times U(n)$ on $M(n,\mathbb{C})$ is given by $(X,Y) . V = X V Y^{-1}$, and the standard embedding of $U(n)$ in $M(n,\mathbb{C})$ arises as the orbit under this action of the element I.

With E_k embedded in C_{k+1} in this way, it is clear that the action of $\phi^2(D_{k-1})$ by conjugation gives the Ad action of E_k. (One has $h(ge_{k+1} g^{-1})h^{-1} = (hg)(e_{k+1})(hg)^{-1}$ where $h \in \phi^2(D_{k-1})$, $g \in \phi(D_k)$.) There is another well known embedding, this time of E_k in D_k, with a similar property:

Definition: The quadratic embedding $Q: E_k \to D_k$ is the map induced from the map $D_k \to D_k$ given by $x \mapsto \sigma(x)\, x^{-1}$.

One has $Q(E_k) = \exp(e_k)$, where $\exp: d_k \to D_k$ (and $d_k - d_{k-1} \oplus e_k$).

Proposition 3.5: $Q(E_k)$ is embedded in G_k as components of $D_k \cap \{\text{Fixed points of } \sigma\sigma^*\}$.

Proof: If x is in D_k, so are $\sigma(x)$ and $\sigma(x)x^{-1}$. Apply σ^* to $\sigma(x)x^{-1}$; $\sigma^*(\sigma(x)x^{-1}) = (\sigma(x)x^{-1})^{-1} = x\sigma(x)^{-1}$; now apply σ to obtain $\sigma(x)x^{-1}$ again. So $Q(E_k)$ is a subset of $D_k \cap \{\text{Fixed points of } \sigma\sigma^*\}$, in fact an orbit of the action of D_k by conjugation. Since $\mathfrak{e}_k$ is the fixed point set of $\sigma\sigma^*$ on $\mathfrak{d}_k$, $Q(E_k)$ must consist of components of the fixed points of $\sigma\sigma^*$ on D_k.

Remark:. We obtain a number of classical embeddings in this way. For example, $U(n)/O(n)$ can be considered as the symmetric matrices in $U(n)$.

In order to compare the quadratic embedding, and the embedding as an Ad orbit, we shall consider the composite map $\phi \circ Q: E_k \longrightarrow C_{k+1}$. From proposition 3.3, we see that the involution σ_k of $\phi(D_k)$ defining the symmetric space E_k is the restriction of an involution (also to be called σ_k) of D_{k+1} (ie conjugation by e_{k+1}), as predicted by theorem 3.1. (Caution: The involutions σ_k and σ should not be confused!) The relation between E_k (embedded in $\mathfrak{e}_{k+1}$ as the Ad orbit of e_{k+1}) and $\phi \circ Q(E_k) \subseteq C_{k+1}$ is given by the next proposition.

Proposition 3.6: $\phi(Q(E_k)) = e_{k+1}^{-1} E_k$.

Proof: The embedding $\phi \circ Q$ is induced by the map $\phi(D_k) \longrightarrow C_{k+1}$ given by:

$$d \longmapsto \alpha_k(d)\, d^{-1} = e_{k+1}\, d\, e_{k+1}^{-1}\, d$$

This is the result of premultiplying the map inducing the Ad embedding of E_k, by e_{k+1}^{-1} .

<u>Proposition 3.7</u>: 1) The action of $\phi(D_k)$ on E_k – considering $\phi(D_k)$ as a group of isometries of the symmetric space $E_k = \phi(D_k)/\phi^2(D_{k-1})$ – is given by conjugation by $e_{k+1}\, d\, e_{k+1}^{-1}$. $(d \in \phi(D_k))$

2) The AD action of $\phi^2(D_{k-1})$ on E_k is given by conjugation in C_{k+1} , and the Ad action of $\phi^2(D_{k-1})$ on $\mathcal{E}_k$ is given by conjugation in C_{k+1} .

<u>Proof</u>: Trivial.

The point of this proposition is that the operations on the space E_k are described naturally in terms of its Ad embedding in C_{k+1} . We consider $\mathcal{E}_k$ to be the "tangent space at the base point to the quadratic embedding" ; if $e \in E_k$ then $e\mathcal{E}_k$ is to be thought of as the tangent space to the Ad embedding of E_k at the point e .

<u>Example</u>: $E^c_{2k+1} \subseteq \mathcal{E}^c_{2k+2}$ is the usual embedding of the $n \times n$ unitary matrices in $M(n,\mathbb{C})$. The skew-hermitian matrices in $M(n,\mathbb{C})$ represent the tangent space to $U(n)$ at the basepoint I, and the tangent space at any other point is represented by the translate (multiplicatively) of the skew-hermitian matrices by that point. We have $I = e_{2k+2}$, and σ^* is the map $X \longmapsto \overline{X}^t$. (NB: The usual multiplication in $U(n)$ is <u>not</u> induced from the Clifford algebra C^c_{2k+2} !) $(n = 2^k)$

We now have both E_K, and its tangent space at the point e_{K+1} (i.e. $e_{K+1}\mathcal{E}_K$), embedded in $\mathcal{E}_{K+1}$. In fact we have a whole sequence of embeddings:

Proposition 3.8 : 1) $e_{i+1}\mathcal{E}_i$ is the subspace of $\mathcal{E}_{K+1}$ consisting of elements which anticommute with $e_{K+1}, \ldots, e_{i+1}$ $(i = 1,\ldots,k)$.

2) E_i can be embedded in $\mathcal{E}_{K+1}$ as the Ad orbit of the point e_{i+1} for the space E_{i+1} (i.e. the orbit of e_{i+1} under the action of $\phi^{K-i+1}(D_i)$ by conjugation). E_i is then embedded in E_K as components of the set of elements which anticommute with $e_{K+1}, \ldots, e_{i+2}$ $(i = 1,\ldots k-1)$.

Proof: Recall that E_K is embedded in $\mathcal{E}_{K+1} \subseteq C_{K+1}$ as the Ad orbit of the point e_{K+1} for the space E_{K+1}. The tangent space to E_K at e_{K+1}, ie $e_{K+1}\mathcal{E}_K$, is embedded in $\mathcal{E}_{K+1}$ as the set of elements anticommuting with e_{K+1}. (To see this, first recall the decomposition $\mathcal{A}_{K+1} = \mathcal{E}_{K+1} \oplus \mathcal{A}_K$. By the remark on page 24 the decomposition $\mathcal{A}_K = \mathcal{E}_K \oplus \mathcal{A}_{K-1}$ consists of the ± 1 eigenspaces of the involution on $\mathcal{A}_K \subseteq \mathcal{A}_{K+1}$ given by conjugation by e_{K+1}.)

We have so far dealt with the first step of the procedure, i.e. the embeddings of E_K and $e_{K+1}\mathcal{E}_K$ in $\mathcal{E}_{K+1}$. Now we must consider E_i and $e_{i+1}\mathcal{E}_i$.

Recall that $\mathcal{A}_i$ is embedded in $\mathcal{A}_K$ as the elements commuting with $e_{K+1}, \ldots, e_{i+2}$ $(i \leq k-1)$, and by the same argument as that used in the first step above, $\mathcal{E}_i$ is the subspace of $\mathcal{A}_i$ consisting of those elements which anticommute with e_{i+1}.

On multiplying by e_{i+1} , we obtain the first assertion of the proposition.

For the second assertion, it is clear that E_i is isomorphic to the Ad orbit of e_{i+1} and that the symmetry on E_i is also given by conjugation by e_{i+1}. We know E_i consists of components of $e_{i+1}\phi(D_i) \cap \mathfrak{p}_{i+1}$ (using proposition 3.4), so the assertion now follows.

This proposition will be very useful in chapter IV. To summarise, we have the following sequence of symmetric spaces, together with a corresponding sequence of the tangent spaces at their respective basepoints:

$$E_1 \subseteq \dots \subseteq E_i \subseteq \dots \subseteq E_k \subseteq \mathcal{E}_{k+1} \qquad e_2\mathcal{E}_1 \subseteq \dots \subseteq e_{i+1}\mathcal{E}_i \subseteq \dots \subseteq e_{k+1}\mathcal{E}_k \subseteq \mathcal{E}_{k+1}$$

Observe that $e_1,\dots,e_i$ are contained in both E_i and $e_{i+1}\mathcal{E}_i$, and that e_{i+1} is in E_i .

We pause here to mention how the "geometric structures" of the classical symmetric spaces can be described in terms of the spaces E_k . From table II we observe that precisely five of the ten series of spaces are of the form G/H with $\mathrm{rank}(G) = \mathrm{rank}(H)$, i.e. the spaces $O(2n)/U(n)$, $Sp(n)/U(n)$, and the three grassmannians. It is well known that these spaces are characterised as being those of positive Euler characteristic, i.e. those for which the symmetry is an inner automorphism (and therefore is actually conjugation by an element of H). If Z_G , Z_H are the centres of G , H respectively, then Z_H / Z_G is either $\mathbb{Z}_2$

or S^1 , and H is the identity component of the centraliser in G of Z_n (see [4]). The latter case gives those spaces which admit homogeneous complex structures (in fact they also possess compatible hermitian metrics and become Kähler manifolds); these are the spaces of the form $G/C(T)$ ($C(T)$ denotes the centraliser of the torus T), i.e. the spaces $O(2n)/U(n)$, $Sp(n)/U(n)$ and $U(2n)/U(n) \times U(n)$.

In terms of the spaces E_k , those of the form G/H with rank (G) = rank (H) are given by the condition $k \equiv 0(2)$, as one can see by inspection. In this case, $\phi(D_{k-1}) \leq D_k$ contains the elements $\pm e_1 \ldots e_k$ ($= \pm \phi(e_1 \ldots e_{k-1})$) in its centre. If we now embed E_k , D_k in C_{k+1} as usual, the symmetry of E_k is given by conjugation by e_{k+1} , and this is the <u>same</u> as conjugation by $(e_1 \ldots e_{k+1})e_{k+1}$, as $e_1 \ldots e_{k+1}$ is in the centre of D_{k+1} . The element $\pm(e_1 \ldots e_{k+1})e_{k+1}$ is $\pm(e_1 \ldots e_k)$, so it actually lies in $\phi(D_k)$ (and in fact in $\phi^2(D_{k-1})$) . This demonstrates directly that the symmetry of E_k is an inner automorphism when $k \equiv 0(2)$ (compare also the remarks following theorem 3.1) . When $k \equiv 1(2)$, this procedure breaks down as $e_1 \ldots e_{k-1}$ is no longer central in D_{k-1} . We can go further and describe the "geometric structures" in the case $k \equiv 0(2)$.

<u>Proposition 3.9</u> : 1) If $k \equiv 2(4)$, multiplication by $e_1 \ldots e_k$ endows E_k with an almost complex structure (i.e. multiplication by $e_1 \ldots e_k$ preserves $e_{k+1} \hat{\ } E_k$ and $(e_1 \ldots e_k)^2 = -1$) .

2) If $k \equiv 0(4)$, multiplication by $e_1 \dots e_k$ endows E_k with an almost product structure (ie multiplication by $e_1 \dots e_k$ preserves $e_{k+1}\mathcal{E}_k$ and $(e_1 \dots e_k)^2 = 1$).

Proof: Recall that the tangent space to E_k at the basepoint e_{k+1} is identified with the subspace $e_{k+1}\mathcal{E}_k$ of $\mathcal{E}_{k+1}$ consisting of elements which anticommute with e_{k+1} (the action of the symmetry $(d\sigma_k = \sigma_k)$ on the tangent space is seen to be -1). If $k \equiv 0(2)$, multiplication by $e_1 \dots e_k$ preserves $\mathfrak{G}_k$, as $\mathfrak{G}_k$ consists of the even elements of $\mathfrak{G}_{k+1}$. Since $\mathcal{E}_k$ and $\mathfrak{G}_{k-1}$ are the eigenspaces of conjugation by e_{k+1}, these are also preserved. Hence $e_{k+1}\mathcal{E}_k$ is preserved. By homogeneity, E_k receives an endomorphism of its tangent bundle, which, by direct calculation, has square -1 or $+1$ depending on the parity of $k/2$. This completes the proof.

Next we shall look at the Lie algebra structure of $\mathfrak{G}_{k+1}$, and its various subspaces. There is a natural inner product on $\mathfrak{G}_{k+1}$, invariant under Ad and unique up to a scalar multiple. This is given by a multiple of the Killing form B :

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$$

$$\langle X,Y \rangle = -\mathrm{tr}\,(XY)$$

(A classical Lie algebra has a trace function, and tr XY is a multiple of the trace of the linear transformation $\mathrm{ad}X \circ \mathrm{ad}Y$.)

In fact, the inner product on $\mathfrak{G}_{k+1}$ is the restriction of the symmetric bilinear form $(X,Y) = \mathrm{tr}(X\sigma^*(Y))$ on C_{k+1} .

<u>Example:</u> $E_{8k+7} = O(8.16^k) \subseteq \mathcal{E}_{8k+8} = M(8.16^k, R)$. Here, $\langle X,Y\rangle = \operatorname{tr} XY^t$.

The Lie algebra $\mathcal{A}_{k+1}$ has its bracket given by: $[X,Y] = XY - YX$, i.e. ordinary commutation in the Clifford algebra. This induces a Lie bracket on each of the spaces $\mathcal{A}_1, \ldots, \mathcal{A}_k$ (i.e. these become subalgebras) . Restricted to $\mathcal{E}_i$, this bracket maps into $\mathcal{A}_{i-1}$ (it maps elements which anticommute with e_{i+1} to elements which commute with e_{i+1}) , and precisely the same is true on restricting to $e_{i+1}\mathcal{E}_i$ (because elements of $\mathcal{E}_i$ anticommute with e_{i+1}) . The last fact is important, because $e_{i+1}\mathcal{E}_i \subseteq \mathcal{E}_{k+1}$, and it is on $\mathcal{E}_{k+1}$ that we wish to do calculations.

<u>Example:</u> Take E_{8k+7} as above; if $X,Y \in O(8.16^k)$ then $[X,Y]$ (which is <u>not</u> to be considered in $\mathcal{E}_{8k+8} = M(8.16^k, R)$!) is $XY^t - YX^t$, and on restricting to $e_{8k+8}\mathcal{E}_{8k+7}$ = skew-symmetric $8.16^k \times 8.16^k$ matrices we obtain $[X,Y] = YX - XY$.

There is a topological property of the spaces E_k which will be important in chapter IV , namely the existence of stable homotopy groups.

<u>Definition:</u> If $k \in \mathbb{N}$, define $C_n(k) = C_n \otimes_R M(k, R)$ and denote the automorphism and antiautomorphism induced respectively by σ and σ^* , by $\sigma(k)$ and $\sigma^*(k)$. Similarly, $\phi^i_{(k)}$ denotes the inclusion $C_n(k) \longrightarrow C_{n+i}(k)$ induced by $\phi^i : C_n \longrightarrow C_{n+i}$.

<u>Definition:</u> $D_n(k)$ = Unitary elements of $C_n(k)$ with respect to $\sigma^*(k)$.

$E_n(k) = D_n(k) / \phi_{(k)}(D_{n-1}(k))$.

One can make similiar definitions for $C^c_n(k)$, $D^c_n(k)$, $E^c_n(k)$, of course. Observe that

1) $E_n(1) = E_n$

2) $E_n(16^k) = E_{n+8k}$.

<u>Definition:</u> The <u>stability</u> map $s:E_n(i) \longrightarrow E_n(i+1)$ is the map induced by the inclusion $M(i,R) \longrightarrow M(i+1, R)$ given by $X \mapsto \left(\begin{smallmatrix} X & \\ & 1 \end{smallmatrix}\right)$.

We shall use the same convention with s^j as with ϕ^j, i.e. we shall drop suffices when the context is clear. Note that the two maps $s, \phi : E_n \longrightarrow E_{n+8}$ are <u>not</u> the same.

<u>Proposition 3.10</u>: The induced map $s_* : \pi_i E_n(j) \longrightarrow \pi_i E_n(j+1)$ is an isomorphism for $i < j.\rho(n) - 1$, where $\rho(n)$ is given by the following table:

n	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$	$8k+6$	$8k+7$	$8k+8$
$\rho(n)$	16^k	2.16^k	4.16^k	4.16^k	8.16^k	8.16^k	8.16^k	8.16^k

<u>Proof:</u> Direct computation, using the known stability ranges for the classical groups. (NB: In some cases, s_* may well be an isom. for higher values of i.)

<u>Remark:</u> The function $\rho(n)$ is already familiar; the sphere $S^{\rho(n)-1}$ has n linearly independent vector fields.

Table I : Classical symmetric spaces and their defining involutions

G	H	σ
$O(n)\times O(n)$	$O(n)$	$\sigma(X,Y) = (Y,X)$
$U(n)\times U(n)$	$U(n)$	$\sigma(X,Y) = (Y,X)$
$Sp(n)\times Sp(n)$	$Sp(n)$	$\sigma(X,Y) = (Y,X)$
$O(n)$	$O(p)\times O(q)$	$\sigma(X) = D\,X\,D^{-1}$, $D = \mathrm{diag}(1,\dots,1,-1,\dots,-1$
$U(n)$	$U(p)\times U(q)$	$\sigma(X) = D\,X\,D^{-1}$, $D = \mathrm{diag}(1,\dots,1,-1,\dots,$
$Sp(n)$	$Sp(p)\times Sp(q)$	$\sigma(X) = D\,X\,D^{-1}$, $D = \mathrm{diag}(1,\dots,1,-1,\dots,$
$O(2n)$	$U(n)$	$\sigma(X) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$
$Sp(n)$	$U(n)$	$\sigma(X) = i\,X\,i^{-1}$
$SU(n)$	$SO(n)$	$\sigma(X) = \overline{X}$
$SU(2n)$	$Sp(n)$	$\sigma(X) = D\,X\,D^{-1}$, $D = \mathrm{diag}(A,\dots,A,-A,\dots,-A$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

In the right hand column, standard inclusions are being tacitly assumed. For example, in the last line, we have $Sp(n) \leq U(2n) \leq O(4n)$. The matrix D is in $O(4n)$, but it is not in the image of $U(2n)$.

Table II: C_* , D_* and E_*

$*$	$8k$	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$	$8k+6$	$8k+7$
C_*	$M(n,\mathbb{R})$	$M(n,\mathbb{C})$	$M(n,\mathbb{H})$	$M(n,\mathbb{H}) \oplus M(n,\mathbb{H})$	$M(2n,\mathbb{H})$	$M(4n,\mathbb{C})$	$M(8n,\mathbb{R})$	$M(8n,\mathbb{R}) \oplus M(8n,\mathbb{R})$
D_*	$O(n)$	$U(n)$	$Sp(n)$	$Sp(n) \times Sp(n)$	$Sp(2n)$	$U(4n)$	$O(8n)$	$O(8n) \times O(8n)$
E_*	$\frac{O(n)}{O(\frac{n}{2}) \times O(\frac{n}{2})}$	$\frac{U(n)}{O(n)}$	$\frac{Sp(n)}{U(n)}$	$Sp(n)$	$\frac{Sp(2n)}{Sp(n) \times Sp(n)}$	$\frac{U(4n)}{Sp(2n)}$	$\frac{O(8n)}{U(4n)}$	$O(8n)$
rank E_*	$n/2$	n	n	n	n	$2n$	$2n$	$4n$
$Z(D_*)$	$\mathbb{Z}_2$	$U(1)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$U(1)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
dim Δ_*	n	$2n$	$4n$	$4n$	$8n$	$8n$	$8n$	$8n$

Here $n = 16^k$, $Z(D_*)$ denotes the centre of D_* , and dim Δ_* denotes the real dimension of the irreducible C_* -module (s) .

The complex version of this table is very simple as $C^c_{2k} = M(n,\mathbb{C})$, $(n = 2^k)$, $C^c_{2k+1} = M(n,\mathbb{C}) \oplus M(n,\mathbb{C})$, so we shall omit it.

§4. An algebraic version of §2

We discuss, initially in terms of the spaces E_k, E_k^c, the following points:

a) Classification of symmetric spaces via Clifford algebras.

b) Distance functions on symmetric spaces (critical points and flow lines).

c) Embeddings of symmetric spaces (minimality properties).

a) Classification of symmetric spaces: We formalise in the following proposition an observation made in §3:

Proposition 4.1: The 10 sequences of spaces $E_k^c(n)$ $(k=1,2)$, $E_k(n)$ $(k=1,\ldots,8)$, give rise to 10 sequences of compact connected irreducible symmetric spaces (on selecting the identity component of $E_k(n)$ for $k\equiv 6,7(8)$, and discarding a factor S^1 from $E_k^c(n)$ for $k\equiv 1(2)$ and from $E_k(n)$ for $k\equiv 1,5(8)$). These are subsequences of the 10 sequences of compact irreducible symmetric spaces of classical type listed in [16].

Recall that the graded Brauer group $GB(k)$ of a field k (of characteristic zero here) is the space of graded central simple algebras over k under graded tensor product, with the equivalence relation $A \sim A \otimes M(n,k)$. (See [22]. A graded central simple algebra over k is a k-algebra $A = A_0 \oplus A_1$ of finite dimension, with no proper graded ideals, and such that $A_0 \cap \text{centre}(A) = k$.)

Proposition 4.2: (Wall [22]): $GB(\mathbb{C}) = Z_2$, $GB(R) = Z_8$ and the equivalence classes of C^c_{2k+1} , C_{8k+1} (respectively)give generators.

Proof: [22].

An algebra $A = A_0 \oplus A_1$ gives rise to a symmetric space by taking G to be the group of unitary elements of A with respect to the involution of A which is I on A_0 and $-I$ on A_1. Conversely, a symmetric space (G,σ) with G classical gives a graded algebra (i.e. the algebra of matrices containing G, graded by σ).

(b) Distance functions on symmetric spaces: We can take "distance functions" with respect to the embedding $E_k \longrightarrow \mathcal{E}_{k+1}$, using the metric introduced in §3 . For $P \in \mathcal{E}_{k+1}$,consider the function:

$$f_P : E_k \longrightarrow R \qquad M \longmapsto \langle M,P \rangle$$

By theorems 2.1 and 2.2 of §2 we know that the critical points are those M satisfying $[M,P] = 0$ i.e. $MP = PM$ (Clifford algebra multiplication induced from C_{k+1}). We can now give an elementary proof of this fact, entirely avoiding Morse theory and properties of Cartan subalgebras, by calculating the gradient of f_P .

First we consider the function $F_P : D_k \longrightarrow R$ defined by $F_P(X) = \langle X,P \rangle = \mathrm{Re}\ \mathrm{tr}\ X\, \sigma^*(P)$. We shall calculate $\mathrm{grad}(F_P)(X) \in X.\mathfrak{d}_k$. By definition we have:

$$\langle \text{grad}(F_P)(X), XV\rangle = F_P(XV) \qquad \text{for all } V \in \mathfrak{d}_k$$
$$= \text{Re tr } XV\sigma^* P$$
$$= -\text{Re tr } \sigma^* PX\sigma^* V$$
$$= -\langle \sigma^* PX, V\rangle$$

i.e. $\langle X^{-1}\,\text{grad}(F_P)X, V\rangle = -\langle \sigma^* PX, V\rangle .$

Here $\langle\ ,\ \rangle$ defines an inner product on $\mathfrak{d}_k$ and hence on $X.\mathfrak{d}_k$ by left invariance. Observing that $(\sigma^* XP - \sigma^* PX)/2$ is the projection of $-\sigma^* PX$ onto the subspace $\mathfrak{d}_k$ of C_k (recall that $\mathfrak{d}_k$ is the -1 eigenspace of σ^*), we deduce:

$$X^{-1}.\,\text{grad}(F_P)X = \frac{\sigma^* XP - \sigma^* PX}{2}$$

The critical points of F_P on D_k are thesfore those points X satisfying $\sigma^* XP = \sigma^* PX$. We shall use this to recover Bott's result on the critical points of f_P.

<u>Theorem 4.3:</u> The critical points of $f_P : E_k \longrightarrow R$ are those points $M \in E_k$ satisfying $MP = PM$.

<u>Proof:</u> Consider $\phi(Q(E_k)) \subsetneq \phi(D_k) \subsetneq C_{k+1}$.

We claim that $\text{grad}(F_R)$ is tangent to $\phi(Q(E_k))$, where F_R is the function on $\phi(D_k)$ described above and $R = e_{k+1}^{-1} P$, $P \in \mathcal{E}_{k+1}$. To show this, we only need prove that F_R is invariant under $\phi(\sigma\sigma^*)$, and then use the fact (proposition 3.5) that $\phi(Q(E_k))$ is embedded in $\phi(D_k)$ as a component of the fixed points of

$\phi(\sigma\sigma^*)$. In the following calculation, we take $X \in \phi(D_k)$ and omit the map ϕ for clarity.

$$
\begin{aligned}
F_R(\sigma\sigma^* X) &= \text{Re tr}(\sigma\sigma^* X \sigma^* R) \\
&= \text{Re tr}(\sigma XR) && (\text{as Re tr } \sigma^* = \text{Re tr}) \\
&= \text{Re tr}(e_{k+1} X e_{k+1}^{-1} R) && (\text{as } \phi(\sigma)X = e_{k+1} X e_{k+1}^{-1}) \\
&= \text{Re tr}(e_{k+1}^{-1} XP) \\
&= \text{Re tr}(XPe_{k+1}^{-1}) \\
&= \text{Re tr}(X\sigma^*(R)) && (\text{as } \sigma^*(e_{k+1}^{-1} P) = Pe_{k+1}^{-1}) \\
&= F_R(X)
\end{aligned}
$$

The critical points of F_R on $\phi(Q(E_k))$ are therefore given by the condition $\sigma^* XR = \sigma^* RX$. By proposition 3.6, the critical points M of f_P on $E_k \subset \mathcal{E}_{k+1}$ are precisely those points for which $e_{k+1}^{-1} M$ is critical for F_R on $\phi(Q(E_k))$. This means the condition to be satisfied by M is $\sigma^*(e_{k+1}^{-1} M)e_{k+1}^{-1} P = \sigma^*(e_{k+1}^{-1} P)e_{k+1}^{-1} M$, i.e. $\sigma^*(M)P = \sigma^*(P)M$. But $M, P \in \mathcal{E}_{k+1}$, so $\sigma^* M = -M$ and $\sigma^* P = -P$. Hence the condition is $MP = PM$, as required.

<u>Flow lines:</u> We can now show that some of the flow lines of grad f_P have the same underlying curves as geodesics in E_k, as we did in the special case discussed in §1.

Let $S(e_1,\dots,e_i)$ be the sphere in E_k consisting of points $\sum_{j=1}^{i}\lambda_j e_j$ with $\sum_{j=1}^{i}\lambda_j^2 = 1$. Let S denote the unit sphere in $\mathcal{E}_{k+1}$. Then we have $S(e_1,\dots,e_{k+1}) \subseteq E_k \subseteq S$. If we extend the definition of f_P to the whole of S, the critical points and flow lines of the restrictions of f_P to the spheres give critical points and flow lines in E_k (on taking the intersection with E_k).

Take $P = e_{k+1}$. The critical points of the function on the spheres are just $\pm e_{k+1}$, and the flow lines are the great circles passing through these points. These points and flow lines lie in E_k. So we obtain in E_k the critical points $\pm e_{k+1}$, and the flow lines between them, i.e. the curves of the form $\theta \longmapsto e_{k+1}\cos\theta + e\sin\theta$ (reparametrised) where $e \in S(e_1,\dots,e_k)$.

The flow lines obtained in this way are precisely the minimal geodesics from e_{k+1} to $-e_{k+1}$. We shall take up this point in chapter III and make a generalisation in chapter IV. The main feature of these flow lines/geodesics is the following. There is a map:

$$\beta : \{\text{Cartan subalgebras of } \mathcal{E}_k\} \to \left\{\begin{matrix}\text{submanifolds of minimal geodesics in } E_k \\ \text{between } e_{k+1} \text{ and } -e_{k+1}\end{matrix}\right\}$$

To define this map, take a Cartan subalgebra of $\mathcal{E}_k$, and write it in the form $d(\mathfrak{h})d^{-1}$, where $d \in \phi(D_{k-1})$ and $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathcal{E}_k$ containing the element $e_{k+1}e_k$. The corresponding set of geodesics is the collection of geodesics of the form $e_{k+1}\cos\theta + de_k d^{-1}\sin\theta$.

Proposition 4.4: 1) Any minimal geodesic between $\pm e_{k+1}$ arises in this way (for some Cartan subalgebra) .

2) The set of all Cartan subalgebras which give rise to collections of geodesics containing the geodesic $e_{k+1}\cos\theta + e_k \sin\theta$ is in 1:1 correspondence with the set of all Cartan subalgebras of $\mathcal{E}_{k-1}$.

Proof: 1) Take a geodesic $e_{k+1}\cos\theta + e\sin\theta$. Then there exists a Cartan subalgebra of $\mathcal{E}_k$ containing $e_{k+1}\, e$, and the image under β of this will clearly contain the geodesic.

2) Those Cartan subalgebras which give rise to $e_{k+1}\cos\theta + e_k\sin\theta$ are precisely the conjugates of a fixed Cartan subalgebra of $\mathcal{E}_{k-1}$ by elements of $\phi^2(D_{k-1})$ which commute with e_k , i.e. by elements of $\phi^2(D_{k-2})$. This corresponds to the collection of Cartan subalgebras of $\mathcal{E}_{k-1}$.

It is this proposition which we shall generalise in chapter IV .

(c) Embeddings of symmetric spaces (minimality properties): By using the Morse functions f_P for regular P and the (known) Betti numbers of the spaces E_k it is easy to check that the embeddings $E_k \rightarrow \mathcal{E}_{k+1}$ are minimal in the sense of total absolute curvature (using proposition 2.5).

Let S denote the unit sphere in $\mathcal{E}_{k+1}$ as usual. To show that the embeddings

$E_k \longrightarrow S$ are minimal in the sense of mean curvature, it is sufficient to do the case $k \equiv 7(8)$ and then use the fact that E_j is embedded in E_k as a totally geodesic submanifold for $j \leq k$. It is sufficient therefore to show that the embedding of $O(n)$ in the unit sphere in $M(n,R)$ is minimal for all n, and this is easy (see [17]).

Chapter II

In chapter I we began with the critical point theory of the function Re(tr) on U(n) and then extended this to the case of a distance function on any classical symmetric space. Although the results are basically well known, our approach via Clifford algebras provided new insight (and will be required later in chapter IV in a different context). In this chapter we shall just consider Lie groups, but we shall use new real-valued functions, namely those of the form $\mathrm{Re}(\chi)$, where χ is any character. For most χ, this problem is essentially different from the case considered in chapter I (χ = character of standard representation), as the functions involved are in general not "linear", and have not previously been investigated.

We do not know whether there are simple necessary and sufficient conditions for these functions to be Morse functions, and we shall give examples and counterexamples to a number of conjecture related to this question. In fact most of this chapter will be concerned with examples, general results being available only for the groups $SO(3)$, $SO(4)$, $Spin(3)$, $Spin(4)$, $U(2)$.

After making some general remarks in §1, we shall examine in detail the groups of rank 1 (in §2) and rank 2 (in §3). In §4 we shall make some conjectures regarding groups of higher rank.

§1. The critical points of Re (χ)

Let G be a compact Lie group, let ρ be an n-dimensional unitary representation of G, and let $\chi = \mathrm{tr}\,\rho$ be the character of ρ. Then we shall examine the critical point theory of the function $f = \mathrm{Re}\,\chi$.

Lemma 1.1 of chapter I applies to this situation as f is a class function, i.e. the critical points of f come in G-orbits (G acts on itself by conjugation), and if T is a fixed maximal torus of G, the critical points of f which lie in T are just the critical points of $f|_T$. We may therefore restrict attention to the torus T. We shall be particularly interested in the following two conditions on f:

I) f has isolated critical points on T

II) f is a nondegenerate function on G (i.e. it has only nondegenerate critical manifolds).

It is not known for which characters the properties I and II hold. We shall give many examples later, but when they do hold one has a critical manifold M_P for each critical point $P \in T$, and this is an embedded symmetric space isomorphic to $G/C(T')$ where T' is the subtorus generated by P. The bundle $TG|_{M_P}$ splits as $T(M_P) \oplus N_+ \oplus N_-$, where N_+ and N_- are the positive and negative bundles respectively. These bundles are clearly homogeneous, i.e. they are induced from the principle bundle $G \longrightarrow G/C(T')$ by representations ρ_+, ρ_- of $C(T')$, and we have $\mathrm{Ad}(G)|_{C(T')} = \mathrm{Ad}(G/C(T')) \oplus \rho_+ \oplus \rho_-$. (The tangent bundle of a homogeneous space G/K is induced from the principal bundle $G \longrightarrow G/K$ by the representation of K obtained by restricting the adjoint action $Ad(G)$ of G to its subgroup K; we denote this by $Ad(G/K)$.) The representations ρ_+, ρ_- can often be identified in practice, as their sum i.e. $Ad(C(T'))$ is usually known. For example, if $C(T')$ is a semisimple group, then one of ρ_+, ρ_- must be zero. At the other extreme, if P is a regular point, then $C(T') = T$, and ρ_+, ρ_- are sums of trivial representations.

To decide condition I for classical groups is in principle a straightforward calculation on the torus, and the same is true of condition II by the following lemma:

<u>Lemma 1.1</u>: If a representation satisfies condition I, and $f|_T$ is nondegenerate, then f is nondegenerate (using the above notation).

<u>Proof</u>: If $f|_T$ has a nondegenerate isolated critical point at $s \in T$, then there exist subspaces t_+, t_- of the tangent space t to T at the identity, such that $t = t_+ \oplus t_-$ and the Hessian of f is positive definite on $s.t_+$ and negative definite on $s.t_-$ (identifying the tangent space to T at s with $s.t$). Let T_+, T_- be the corresponding tori in T. Let $G(T_+)$, $G(T_-)$ denote the largest connected subgroup of G containing T_+, T_- respectively as maximal tori. We have $\dim G(T_+) + \dim G(T_-) = \dim C(s)$.

We claim that the Hessian of f is positive definite in the direction of $s.G(T_+)$ at s, and negative definite in the direction of $s.G(T_-)$. To see this, take a geodesic at s tangent to $s.G(T_+)$ and conjugate it by an element of $C(s)$ to bring it inside $s.T_+$. Since f is invariant under conjugation, the Hessian in the direction of either geodesic is positive definite. The same argument applies to $s.G(T_+)$. But $\dim(s.G(T_+)) + \dim(s.G(T_-)) + \dim(G/C(s)) = \dim G$, so this completes the proof.

The following observations are clear:

a) The set of critical points is invariant under the action of the Weyl group W.

b) The identity element of G is always a point for which f takes its absolute maximum.

As f is the composite of the two functions ρ and Re tr, it is convenient to make the following definition:

<u>Definition</u>: A critical point P of f is of <u>type I</u> if there is a critical point σ of Re tr on $U(n)$ such that $\rho(P) = \sigma$. Otherwise P is of <u>type II</u> (i.e. grad Re tr $(\rho(P)) \neq 0$ but grad Re tr $(\rho(P)) \subsetneq \ker d$).

Let us now establish some notation and some formulae for the critical points of f. The representation ρ has weights $\Lambda_1, \ldots, \Lambda_n$; these are linear functionals on the (Stiefel) diagram of G (i.e. the tangent space to T), and some of them may be zero. They are obtained by simultaneously diagonalising the matrices $\rho(\Theta)$ for $\Theta = (\Theta_1, \ldots, \Theta_\ell)$ in T, so that:

$$\rho(\Theta) = \begin{pmatrix} e^{\Lambda_1(\Theta)} & & \\ & \ddots & \\ & & e^{\Lambda_n(\Theta)} \end{pmatrix} \qquad (\ell = \text{rank } G)$$

By means of a bi-invariant inner product on T_eG we shall identify all weights (in particular the weights $\Lambda_1, \dots, \Lambda_n$ of ρ, and the roots $r_1, \dots, r_k$ of G) with vectors in the diagram.

When $\pi_1 G$ is finite, we shall also identify the tangent space to the appropriate maximal torus $\tilde{T}$ of the universal cover $\tilde{G}$ of G with the diagram. (The root vectors of G, $\tilde{G}$ are identified; the planes of the diagram determine the integer lattice once the centre is known. The integer lattice determines the lattice of weight vectors using the fixed inner product. With these conventions, the projection $\tilde{G} \longrightarrow G$ does not "commute" with exp.) If G is semisimple, the weight lattice of $\tilde{G}$ (which contains that of G) is generated by ℓ basic weights $w_1, \dots, w_\ell$. We define $\beta = w_1 + .. + w_\ell$. In sections §2 and §3 we shall exhibit diagrams of several groups, and so we fix the following notation:

- • denotes an integer lattice point of G
- ⊙ denotes an integer lattice point of $\tilde{G}$
- - - - -> denotes a root vector (as explained above)
- ———> denotes a basic weight vector of $\tilde{G}$ (when G is semisimple)

At least for groups of low rank, the critical points of f are easily calculated directly. The point $(\theta_1, \dots, \theta_\ell)$ is critical if grad $f(\theta) \subseteq \ker d\rho(\theta)$ ie if:

$$\sum_{i=1}^{n} \frac{\partial}{\partial\theta_j}(\Lambda_i) \sin(\Lambda_i) = 0 \quad \text{for } j = 1, \dots, \ell \qquad \text{(i)}$$

If ρ is an irreducible representation, it has a unique positive highest weight $\Lambda = \Lambda_i$

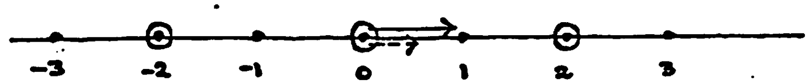
say, and its character is given by the formula of Weyl:

$$\chi(\theta) = \sum_{w \in W}' w \, . \exp(\Lambda + \beta) \Big/ \sum_{w \in W} w \, . \exp(\beta) \qquad \text{(ii)}$$

Here, $w \, . \exp(\mu)$ denotes $\pm \exp(w \, . \mu)$, where W acts on the weights in the usual way, and the sign is chosen according to the parity of w. The critical points of f are then given by differentiating the real part of this expression. In general, neither (i) nor (ii) is easy to deal with, but we shall begin in the next two sections with groups of low rank, where some calculations are possible.

§ 2. Compact Lie groups of rank 1

Let $G = SO(3)$; $\widetilde{G} = S^3$ (ie the unit quaternions) is the only simply connected simple group of rank 1. The diagram of $SO(3)$ (using the notation of §1) is:



A fundamental domain for the maximal torus of $SO(3)$ under consideration is $[-1,1]$, and a fundamental domain for the torus of S^3 is $[-2,2]$. The irreducible representations of S^3 are in 1:1 correspondence with the weights in a fixed Weyl chamber, say $\{n/2 \mid n \in \mathbb{Z}, n \geq 0\}$, and the character of the representation ρ_n with maximal weight $n/2$ is:

$$(z^{(\frac{1}{2}+\frac{n}{2})} - z^{-(\frac{1}{2}+\frac{n}{2})}) / (z^{\frac{1}{2}} - z^{-\frac{1}{2}}) \quad \text{for} \quad z^{\frac{1}{2}} = \exp(\tfrac{\pi}{2} i d), \ d \in [-2,2]$$

(The covering map of maximal tori $\widetilde{T} \to T$ is given by $\pm z^{1/2} \mapsto z$.) The weights of ρ_n are $\pm n/2, \pm(n/2-1), \pm(n/2-2), \ldots$; the dimension of ρ_n is $n+1$.

The representations of $SO(3)$ are given by even values of n; the "spin representation" is given by $n = 1$, and the adjoint representation is given by $n = 2$. All the representations of S^3 and $SO(3)$ are self conjugate, hence their characters are real-valued.

Theorem 2.1 : The nontrivial irreducible representations of S^3 and $SO(3)$ satisfy I and II of §1 , ie their characters are Morse-Bott functions whose nondegenerate critical manifolds intersect the maximal torus in a finite number of points. Moreover:

a) For $\chi(\rho_n)$ on S^3 , the points ± 1 are always critical. There are $2(n-1)$ other critical points on the torus, all of which are regular points, giving $n-1$ critical manifolds $S^3/S^1 \cong S^2$.

b) For $\chi(\rho_n)$ on $SO(3)$ (n even), the points I and $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$ are always critical; I is an isolated maximum and $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$ gives a critical manifold $P_2\mathbb{R}$ There are $n-2$ other critical points on the torus, all of which are regular points giving $\frac{1}{2}(n-2)$ critical manifolds $SO(3)/S^1 \cong S^2$.

For (a) the negative bundles are trivial, and for (b) the negative bundles are trivial except that over $P_2\mathbb{R}$, which is either the Hopf line bundle or the zero dimensional bundle (depending on whether the index is 1 or 0) .

Proof: We shall deal with S^3 as the results for $SO(3)$ follow easily. First, it is clear that 1 and -1 are absolute maxima or minima for all characters. We shall prove they are nondegenerate after treating the regular critical points.

The Weyl character formula gives $\operatorname{Re}\chi(\rho_n)(d) = \sin((1+n)\pi d/2)/\sin(\pi d/2)$. We shall assume the denominator of this expression is nonzero, as we have just excluded the corresponding points $(d = 0,2)$. The critical points are therefore given by:

$$\therefore \quad (1+n)\tan \pi d/2 = \tan\left((1+n)\pi d/2\right)$$

This shows that there are $2(n-1)$ critical points on the torus besides ± 1. We shall calculate $\chi''(\rho_n)(d)$ at a regular critical point. Write $a = \sin\left((1+n)\pi d/2\right)$, $b = \sin \pi d/2$; at a critical point we have $a'b = b'a$:

$$\chi''(\rho_n)(d) = \left(b^2(ab'' + a'b' - b'a' - ba'') - 2bb'(ab' - ba')\right)/b^4$$

$$= (\pi/2)^2 (a/b)(1-(1+n)^2) \quad (\text{as } a'' = -(\pi/2)^2 a, \; b'' = -(1+n)^2(\pi/2)^2 b)$$

$$= (\pi/2)^2(1-(1+n)^2)\chi(\rho_n)(d)$$

Since the right hand side is never zero when $\chi' = 0$ (by a trivial calculation), this shows that all regular critical points are nondegenerate for the restriction $\chi|_{S^1}$. At a regular point, the tangent spaces to the torus and orbit through the point are complementary, so these points are in fact nondegenerate for χ. So far we have not needed to use lemma 1.1. To check nondegeneracy of ± 1 we shall use lemma 1.1 and just deal with $\chi|_{S^1}$. By continuity, the above formula for χ'' holds even at ± 1, hence $\chi|_{S^1}$ is nondegenerate. The negative bundles on S^2 are trivial, either because they are vector bundles of fibre dimension ≤ 1 over S^2, or by the remarks of §1. The same argument gives the result for $P_2 R$, if one notes that the one dimensional negative bundle cannot be orientable. This completes the proof.

Observe that one only has the least number of critical points consistent with the

Morse inequalities for $n = 1$ on S^3, and for $n = 2$ on $SO(3)$. (The $\mathbb{Z}_2$ Morse inequalities have to be used for $SO(3)$.)

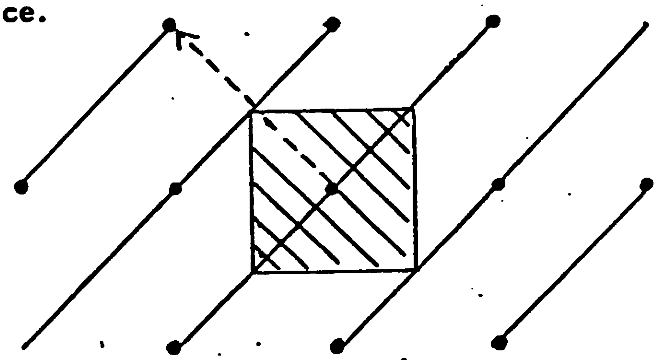
Examples of degenerate reducible representations are easy to produce, for example the k-th power of the spin representation for $k > 1$.

§ 3. Compact Lie groups of rank 2

The simply connected simple groups of rank 2 are $SU(3)$, $Sp(2)$, and G_2; $Spin(5)$ is isomorphic to $Sp(2)$. We shall also deal with the groups $Spin(4)$ (not simple), $SO(4)$ and $SO(5)$ (whose double covers are $Spin(4)$, $Spin(5)$ respectively), and $U(2)$ (not semisimple).

The group U(2)

The diagonal matrices $\mathrm{diag}(z_1, z_2)$ form a maximal torus; if we write $z_j = e^{2\pi i d_j}$ $(j = 1,2)$ and $\exp(d_1, d_2) = \mathrm{diag}(z_1, z_2)$, the roots are $\pm(d_1 - d_2)$. The diagram is the (d_1, d_2)-plane shown below, with a fundamental domain for the torus shaded. The integer points form the integer lattice and using the standard inner product $\langle (d_1, d_2), (d_1', d_2') \rangle = d_1 d_1' + d_2 d_2'$ the integer lattice is identified with the weight lattice.



There are no basic weights (as defined in §1), although the irreducible representations still correspond to the weights in the fundamental chamber (ie (m_1, m_2) with $m_1, m_2 \in \mathbb{Z}$ and $m_1 \geq m_2$), and the Weyl character formula applies with $\beta = (1,0)$. Let $\lambda : U(2) \longrightarrow U(2)$ denote the standard representation, with maximal weight $(1,0)$. Then $\Lambda^2\lambda$ has maximal weight $(1,1)$, and the complex representation ring of $U(2)$ is generated by the representations Λ, $\Lambda^2\lambda$ and $(\Lambda^2\lambda)^*$. The character of the representation with maximal weight (m_1, m_2) is given by $\begin{vmatrix} z_1^{m_1+1} & z_2^{m_1+1} \\ z_1^{m_2} & z_2^{m_2} \end{vmatrix} / (z_1 - z_2) = (z_1 z_2)^{m_2}(z_1^{m_1-m_2} + z_1^{m_1-m_2-2} z_2^2 + \dots)$.

The dual of this representation has maximal weight $(m_2, -m_1)$, so the self-conjugate representations have maximal weights of the form $(m, -m)$. These give precisely the irreducible self-conjugate representations of $SU(2)$, on restriction. The irreducible representation $S^m\lambda$ has maximal weight $(m, 0)$ and this gives the irreducible representation ρ_{2m} of $SU(2)$ on restriction ($(m, -m)$, $(2m, 0)$ have the same restriction.)

Theorem 3.1 : The irreducible self-conjugate representations of $U(2)$ give Morse – Bott functions.

For $m > 0$, the representation $(m, -m)$ has $2m$ nondegenerate critical manifolds, one being the centre S^1 (maximum points) and the others isomorphic to $S^1 \times S^2$. The negative bundles are all trivial, and all the critical manifolds are of index 0 or 1.

Proof: This follows easily from theorem 2.1 because the character of $(m, -m)$ is given by $\sin 2\pi(2m+1)(d_1 - d_2) / \sin 2\pi(d_1 - d_2)$. All critical sets are translates of conjugating (by $U(2)$) the sub-torus $z_1 = z_2$, so (except for the trivial translate, S^1) they are copies of $S^1 \times S^2$. Nondegeneracy in the directions orthogonal to the centre follows from theorem 2.1; and in the direction of the centre there is nothing to check, as the function is constant on the centre (hence this direction is tangent to the critical manifold). The negative bundle over $S^1 \times S^2$ is trivial, because its restriction to S^2 is trivial, having fibre dimension $\leqslant 1$, and the rest of the bundle is given by multiplication by the centre S^1 (note that $S^2 \cap S^1 = \emptyset$ in $U(2)$).

λ, $\Lambda^2\lambda$: Frankel [12] examined λ (see chapter I, §1); the character of $\Lambda^2\lambda$ gives the function $f(d_1, d_2) = \cos 2\pi d_1 \cos 2\pi d_2 - \sin 2\pi d_1 \sin 2\pi d_2$ on the torus. Its critical points are given by $d_1 + d_2 \in \mathbb{Z}$, so condition I of §1 is not satisfied here, ie the critical points are not isolated on the torus. The critical manifolds are two copies of $SU(2)$ (the standard embedding gives absolute maxima and its translate by i gives absolute minima). The function is nondegenerate; it suffices to check this for the manifold of maxima, and to do this one need only calculate the Hessian in the direction of the centre.

The negative bundle over the minima must be trivial, being a line bundle on S^3; Morse theory then shows that $U(2)$ is homeomorphic to $S^1 \times S^3$. The Morse-Bott polynomial is $(1 + t^3) + t(1 + t^3)$ and the Poincaré polynomial of $U(2)$ is $(1 + t)(1 + t^3)$ (over $\mathbb{Z}$), so f is in fact a "minimal" Morse-Bott function.

One can in fact regard the Weyl character formula as the product of characters of irreducible representations of S^3 and S^1:

$$\chi_{m_1, m_2} = \left((z_1/z_2)^{m_1-m_2} + (z_1/z_2)^{m_1-m_2-2} + \dots + (z_1/z_2)^{-(m_1-m_2)} \right) (z_1^{m_2} z_2^{m_1})$$

Ad (U(2)): This is not irreducible; in fact its character is $1 + \left(\frac{z_1}{z_2} + 1 + \frac{z_2}{z_1}\right)$, and the function f is $2\cos 2\pi(d_1 - d_2) + 2$. Its critical point theory is the same as that of the representation with maximal weight $(1, -1)$, which has been dealt with by theorem 2.1 .

It is clear that the nondiscrete centre of $U(2)$ is responsible for the failure of property I ; we shall therefore not consider any more representations of $U(2)$, and proceed to the semisimple groups.

The group SU(3)

The diagonal matrices $\mathrm{diag}(z_1, z_2, z_3)$ with $z_1 z_2 z_3 = 1$ form a maximal torus; if we write $z_j = e^{2\pi i d_j}$ and $\exp(d_1, d_2, d_3) = \mathrm{diag}(z_1, z_2, z_3)$ with $d_1 + d_2 + d_3 = 0$, the roots are $\pm(d_1 - d_2)$, $\pm(d_2 - d_3)$, $\pm(d_1 - d_3)$. The diagram is the plane $d_1 + d_2 + d_3 = 0$ shown in fig. 1 , with a fundamental domain for the torus shaded. It is not possible to choose a bi-invariant inner product in such a way that the integer lattice is identified with the lattice of weights, so we choose the integer points to represent the integer lattice (this has been done in the formula above for exp) and obtain the weight lattice using the inner product $\langle (d_1, d_2, d_3), (d_1', d_2', d_3') \rangle = d_1 d_1' + d_2 d_2' + d_3 d_3'$ induced from that of $U(3)$. The basic weights in the fundamental chamber are $(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$ and $(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ and these are the maximal weights of the irreducible representations λ , $\Lambda^2\lambda$ (λ denotes the standard representation $\lambda : SU(3) \longrightarrow U(3)$ as usual) .

λ , $\Lambda^2\lambda$: These both give the same function f, ie $\cos 2\pi d_1 + \cos 2\pi d_2 + \cos(2\pi d_1 + 2\pi d_2)$ (using $d_1 + d_2 + d_3 = 0$). Its critical points are indicated by crosses in fig. 2 . By lemma 1.1 of §1 , nondegeneracy may be checked by evaluating the Hessian of f at the critical points. The determinant of the Hessian turns out to be

$4\pi^2(\cos 2\pi d_1 \cos 2\pi d_2 + (\cos 2\pi(d_1 + d_2)) \cdot (\cos 2\pi d_1 + \cos 2\pi d_2)))$, which is nonzero at the critical points. So f satisfies I and II of §1. There are 4 points of type I (see §1) and 2 points of type II, the latter being the nontrivial elements of the centre.

$\lambda \otimes \Lambda^2\lambda$: A simple calculation shows that this gives 24 critical points on the torus, as shown in fig. 3. Again appealing to lemma 1.1, a calculation of the Hessian on the torus shows that the function is nondegenerate.

(Remark: $\lambda \otimes \Lambda^2\lambda$ generates the self-conjugate representations)

Ad(SU(3)): Tnis gives a function with the same critical point theory as $\lambda \otimes \Lambda^2\lambda$; in fact the representations are the "same", as $Ad(SU(3))^*_c = (\lambda \otimes \bar{\lambda})_0$ and $\bar{\lambda} \cong \Lambda^2\lambda$.

It is clear that none of the functions above give "minimal" Morse-Bott functions. The functions satisfy I and II of §1 ; we do not know of any characters which do not have this property, for SU(3).

The groups SO(4), Spin(4)

The matrices $\exp(d_1, d_2) = \mathrm{diag}(R_1, R_2)$, where R_j denotes the 2×2 matrix representing rotation through an angle $2\pi d_j$, form a maximal torus of SO(4); the roots are $\pm(d_1 \pm d_2)$. The diagram is the (d_1, d_2)-plane shown in fig. 4, with fundamental domains for the maximal tori of SO(4) and Spin (4) shaded. The roots and basic weights are portrayed according to the scheme given in §1. Using the natural bi-invariant inner product $\langle (d_1, d_2), (d_1', d_2') \rangle = d_1 d_1' + d_2 d_2'$

the weights of SO(4) are identified with the integer lattice of SO(4), ie the integral points. The weights of Spin(4) are the points (m_1, m_2) where $2m_1, 2m_2 \in \mathbb{Z}$, $m_1+m_2 \in \mathbb{Z}$. The centre of SO(4) is $\mathbb{Z}_2$ and that of Spin(4) is $\mathbb{Z}_2 \times \mathbb{Z}_2$ $(\pi_1 SO(4) \cong \mathbb{Z}_2)$.

Although Spin(4) is not simple, the usual results for representations of a simply connected simple group hold; the representation ring is polynomial on two generators Δ^+, Δ^- (the spin representations). Let $\lambda : SO(4) \to U(4)$ be the standard representation of SO(4) and $\tilde{\lambda}$ the corresponding one of Spin (4).

The maximal weights of Δ^+, Δ^- are the basic weights $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})$ and $\tilde{\lambda}$ (λ) has maximal weight (1,0). The adjoint representation is reducible and in fact $\Lambda^2\lambda = \lambda^+ \oplus \lambda^-$ $(\Lambda^2\tilde{\lambda} = \tilde{\lambda}^+ \oplus \tilde{\lambda}^-)$ where λ^+, λ^- have maximal weights (1,1), (1,-1). Finally, the third exterior power $\Lambda^3\lambda$ is isomorphic to λ.

The Weyl character formula gives the character of the representation with highest weight (m_1, m_2) $(m_1 \geq m_2,\ 2m_1, 2m_2 \in \mathbb{Z})$ as :

$$\chi(d_1, d_2) = \frac{\cos 2\pi((1+m_1)d_1 + (m_2)d_2) - \cos 2\pi((m_2)d_1 + (m_1+1)d_2)}{\cos 2\pi d_1 - \cos 2\pi d_2}$$

The dimension of this representation is $(1+m_1)^2 - (m_2)^2$. (As usual, we are referring to unitary representations, but of course all the representations of SO(4) and Spin(4) are real.)

If ρ is a representation of SO(4), and $\tilde{\rho}$ the corresponding one of Spin(4), the critical point theory of the function f is related to that of $\tilde{f}$ in an obvious way. We shall therefore deal only with representations of SO(4) when this is possible.

λ : This was examined by Frankel [12]; we exhibit the critical points in fig. 5. The function f is a Morse-Bott function with isolated maxima and minima at $I, -I$ (respectively), and a manifold of critical points isomorphic to $SO(4)/S^1 \times S^1 \times \mathbb{Z}_2$ (ie homeomorphic to $S^2 \times P_2\mathbb{R}$) of index 1. The $\mathbb{Z}_2$ Morse-Bott polynomial is $1 + t(1 + t + t^2 + t^3 + t^4) + t^6$ and the $\mathbb{Z}_2$ Poincaré polynomial is $(1+t)(1+t^2)(1+t^3)$ so the function f is "minimal".

$Ad(SO(4)) = \Lambda^2\lambda$: This is not irreducible; its character is $(1 + z_1 z_2 + z_1^{-1} z_2^{-1}) + (1 + z_1 z_2^{-1} + z_1^{-1} z_2)$ which gives $f(d_1, d_2) = 2 + 2\cos 2\pi(d_1 + d_2) + 2\cos 2\pi(d_1 - d_2)$. The critical points are shown in fig. 6. On the torus there are 4 points of type I and 4 points of type II, so there are two isolated points in SO(4) (maxima), one manifold $SO(4)/S^1 \times S^1 \times \mathbb{Z}_2$ (minima), and one manifold $SO(4)/U(2)$ (index 3). These indices are obtained by a simple calculation on the torus, as is the fact that the critical manifolds are all nondegenerate (the determinant of the Hessian is $256\pi^4\cos 2\pi(d_1 + d_2)\cos 2\pi(d_1 - d_2)$). Of course, lemma 1.1 is used here. The $\mathbb{Z}_2$ Morse-Bott polynomial is $(1+t+2t^2+t^3+t^4)+t^3(1+t^2)+2t^6$ so f is not "minimal".

Δ^+, Δ^- : The critical points (on the torus) of f are not isolated; they are given respectively by $\sin\pi(d_1 + d_2) = 0$, $\sin\pi(d_1 - d_2) = 0$. An argument similar to that used for the representation $\Lambda^2\lambda$ of U(2) shows that there are two critical manifolds, both isomorphic to Spin(3), and these are nondegenerate with a trivial negative bundle over the minima. We see by Morse theory that Spin(4) is homeomorphic to $S^3 \times \mathrm{Spin}(3)$ (ie $S^3 \times S^3$).

λ^+, λ^- : The critical point theory here is similar to the last example; the function f is respectively $1 + 2\cos 2\pi(d_1 + d_2)$, $1 + 2\cos 2\pi(d_1 - d_2)$. Both functions give the homeomorphism $SO(4) \cong SO(3) \times S^3$.

In fact one can deal with the irreducible representations quite generally:

Theorem 3.2 : All irreducible representations of SO(4) and Spin(4) satisfy condition I, ie they give nondegenerate functions. Those with maximal weights (m_1, m_2) satisfying $m_1 \neq m_2$, $m_1 \neq -m_2$ also satisfy condition II, ie they give isolated critical points on the torus.

Proof: The Weyl character formula can be written:

$$\chi(d_1, d_2) = \frac{\sin(\pi(1 + m_1 + m_2)(d_1 + d_2))}{\sin\pi(d_1 + d_2)} \cdot \frac{\sin(\pi(1 + m_1 - m_2)(d_1 - d_2))}{\sin\pi(d_1 - d_2)}$$

$$= \chi_{m_1+m_2}(b_1) \cdot \chi_{m_1-m_2}(b_2)$$

where $b_1 = d_1 + d_2$, $b_2 = d_1 - d_2$ and χ_n denotes the character of the irreducible representation ρ_n of S^3 with maximal weight $n/2$.

The critical points are given by $\chi'_{m_1+m_2}(b_1) = 0$, $\chi'_{m_1-m_2}(b_2) = 0$ and by $\chi_{m_1+m_2}(b_1) = 0$, $\chi_{m_1-m_2}(b_2) = 0$. From theorem 2.1 of §2 it is clear that when $m_1 = m_2$ or $m_1 = -m_2$ the critical points given by the first pair of equations are nondegenerate (although they are not isolated on the torus). The second pair of equations contradicts the assumption $m_1 = m_2$ or $m_1 = -m_2$ so it gives no critical points.

If $m_1 \neq m_2$ and $m_1 \neq -m_2$, then again appealing to theorem 2.1 for the points given by the first pair of equations, and by straightforward calculation for the others, we see that the critical points are isolated on the torus. To obtain nondegeneracy in this case, we must use lemma 1.1 and calculate the Hessian on the torus (with respect to the variable b_1 , b_2 , of course). This turns out to be $(\pi \quad)^4(1 - (1 + m_1 + m_2)^2)(1 - (1 + m_1 - m_2)^2)\, \chi^2_{m_1+m_2}(b_1)\, \chi^2_{m_1-m_2}(b_2)$ for the first kind of critical points, and $-\, \chi'^2_{m_1+m_2}(b_1)\, \chi'^2_{m_1-m_2}(b_2)$ for the second kind (using the calculation given in the proof of theorem 2.1); this is never zero as a simple calculation shows that one can never have $\chi_n = \chi'_n = 0$. This completes the proof.

The groups SO(5) , Spin(5) , Sp(2)

The matrices $\exp(d_1, d_2) = \mathrm{diag}(R_1, R_2, 1)$, where R_j is the 2×2 matrix representing rotation through an angle $2\pi_j d$, form a maximal torus of SO(5) ; the roots are $\pm(d_1 \pm d_2)$, $\pm d_1$, $\pm d_2$. The diagram is the (d_1, d_2)-plane shown in fig. 7 , with fundamental domains for the maximal tori of SO(5) and Spin(5) shaded. Using the same inner product as for the diagram of SO(4) , the weights of SO(5) are identified with the integer lattice of SO(5) , ie the integral points. The weights of Spin(5) are the points (m_1, m_2) where $2m_1, 2m_2 \in \mathbb{Z}$, $m_1+m_2 \in \mathbb{Z}$. The centre of SO(5) is 0, and that of Spin(5) is $\mathbb{Z}_2$ ($\pi_1 SO(5) \cong \mathbb{Z}_2$).

The groups Spin(5) and Sp(2) are isomorphic; a maximal torus for Sp(2) is given by the matrices $\exp(b_1, b_2) = \operatorname{diag}(z_1, z_2)$, with $z_j = e^{2\pi i b_j}$. The diagram of Sp(2) is the (b_1, b_2)-plane shown in fig. 8; an explicit isomorphism of the diagrams of Spin(5) and Sp(2) may be obtained by writing $b_1 = d_1 + d_2$, $b_2 = d_1 - d_2$. In view of this, we shall deal with Spin(5) instead of Sp(2) from now on.

The group Spin(5) is simply connected and simple; its representation ring is polynomial on the generators $\tilde{\lambda}$, Δ, where $\tilde{\lambda}$ denotes the representation corresponding to the standard representation $\lambda : SO(5) \longrightarrow U(5)$ of SO(5), and Δ denotes the spin representation. Their maximal weights are the basic weights (1,0) and $(\frac{1}{2}, \frac{1}{2})$ respectively. (In terms of Sp(2), they are the standard representation and its second exterior power.) The representation ring of SO(5) is also a polynomial algebra, on generators λ, $\Lambda^2\lambda$. The adjoint representation is $\Lambda^2\lambda$ for SO(5), and $\Lambda^2\tilde{\lambda}$ for Spin(5).

The Weyl character formula gives the character of the representation with highest weight (m_1, m_2) as:

$$\chi(d_1, d_2) = \frac{\sin 2\pi(m_1+\frac{3}{2})d_1 \sin 2\pi(m_2+\frac{1}{2})d_2 - \sin 2\pi(m_2+\frac{1}{2})d_1 \sin 2\pi(m_1+\frac{3}{2})d_2}{\sin 3\pi d_1 \quad \sin \pi d_2 \quad - \sin \pi d_1 \quad \sin 3\pi d_2}.$$

λ : See Frankel [12]; the critical points are shown in fig. 9. There are isolated maximum and minimum points (I and $-I$ respectively) and one manifold of points isomorphic to $SO(5)/S(O(2) \times O(3))$ (of index 3). The function f is a "minimal" Morse-Bott function.

$Ad(SO(5)) = \Lambda^2\lambda$: This is irreducible; the critical points are shown in fig. 10. There are 4 points of type I on the torus, and an orbit of 4 points of type II giving a critical manifold $SO(5)/U(2) \times I$. It is easily checked that f is nondegenerate using lemma 1.1.

Δ : This gives the function $f(d_1, d_2) = 2\cos\pi(d_1 + d_2) + 2\cos\pi(d_1 - d_2)$ on the maximal torus of Spin(5). (See fig. 11.)

§ 4. Some conjectures

(a) Conditions I and II of §1

In view of the examples of §2 and §3 one is lead to make the conjecture :

Preliminary conjecture: If G is a simple semisimple compact Lie group and ρ is an irreducible representation, the function f satisfies I, II.

The conditions are necessary; for a non-semisimple group condition I may fail (eg $\Lambda^2\lambda$ of U(2)), for a non-simple group condition I may fail (eg λ^+, λ^- for SO(4) or Δ^+, Δ^- for Spin(4)), and for a reducible representation II may fail (eg many examples for S^3). For the groups SO(5), Spin(5) an examination of the Weyl character formula suggests that the conjecture is true. However, for Spin(2n + 1) and $n > 2$, the spin representation (which has character $2^n \cos\pi d_1 \cos\pi d_2 \ldots \cos\pi d_n$) gives non-isolated critical points on the torus with some degenerate critical manifolds. So we modify the conjecture as follows:

Conjecture: For G = Sp(n), SO(n), $(n > 4)$, SU(n) the preliminary conjecture holds.

It is feasible to do some simple calculations for groups such as SU(3) and SU(4), eg with the representations λ, $\Lambda^2\lambda$, which confirm the conjecture in these cases.

(b) Location of critical points

We have seen that even for a group as simple as S^1, the locations of the critical points of the function f associated with the general irreducible representation are not of particular interest (ie solutions of an equation of the form $k \tan x = \tan kx$). It seems best therefore to concentrate attention on those representations of particular geometrical significance, eg Ad(G) and the representations $\Lambda^k \lambda$.

For the representation λ of $U(n)$, it is of interest to note that the critical points in the diagram correspond precisely to the maximal weight vectors of the irreducible representations $1, \lambda, \Lambda^2\lambda, \ldots, \Lambda^n\lambda$ (ie the points $(0,\ldots,0)$, $(1,0,\ldots,0)$, $\ldots,(1,\ldots,1)$). These representations generate the representation ring of $U(n)$ (although one must use the inverse of $\Lambda^n\lambda$), and on the other hand the characteristic classes of the bundle associated to the classifying bundle $EU(n) \to BU(n)$ by the representation λ generate the cohomology ring $H^* BU(n)$.

In general, one would hope for a relation between the weights of a representation and the critical points of the associated function f, via characteristic classes (of the associated vector bundle) and Morse theory (of the function f).

In particular, one would expect a relation between the weights and the location of critical points. In this direction, one has the following result:

<u>Proposition 4.1</u> : If G is a semisimple compact Lie group, the centre Z is critical for the function f associated with any character of G .

<u>Proof:</u> First, lemma 1.1 of chapter I shows that grad f is tangent not only to the fixed maximal torus but also to the circles in the torus given by the lines of the diagram through the origin. Here we use the fact that G has rank (G) simple roots, in order to obtain such lines. (This would be true for any class function f .) This means that such circles are flow lines in the sense of chapter I , ie away from critical points they can be reparametrised to give flow lines. Since points of the centre are precisely the points where all such circles intersect, they must be critical points, at least when rank (G) $\geqslant$ 2 . However, we have already dealt with the groups of rank 1 anyway.

Fig. 1 :   SU(3)

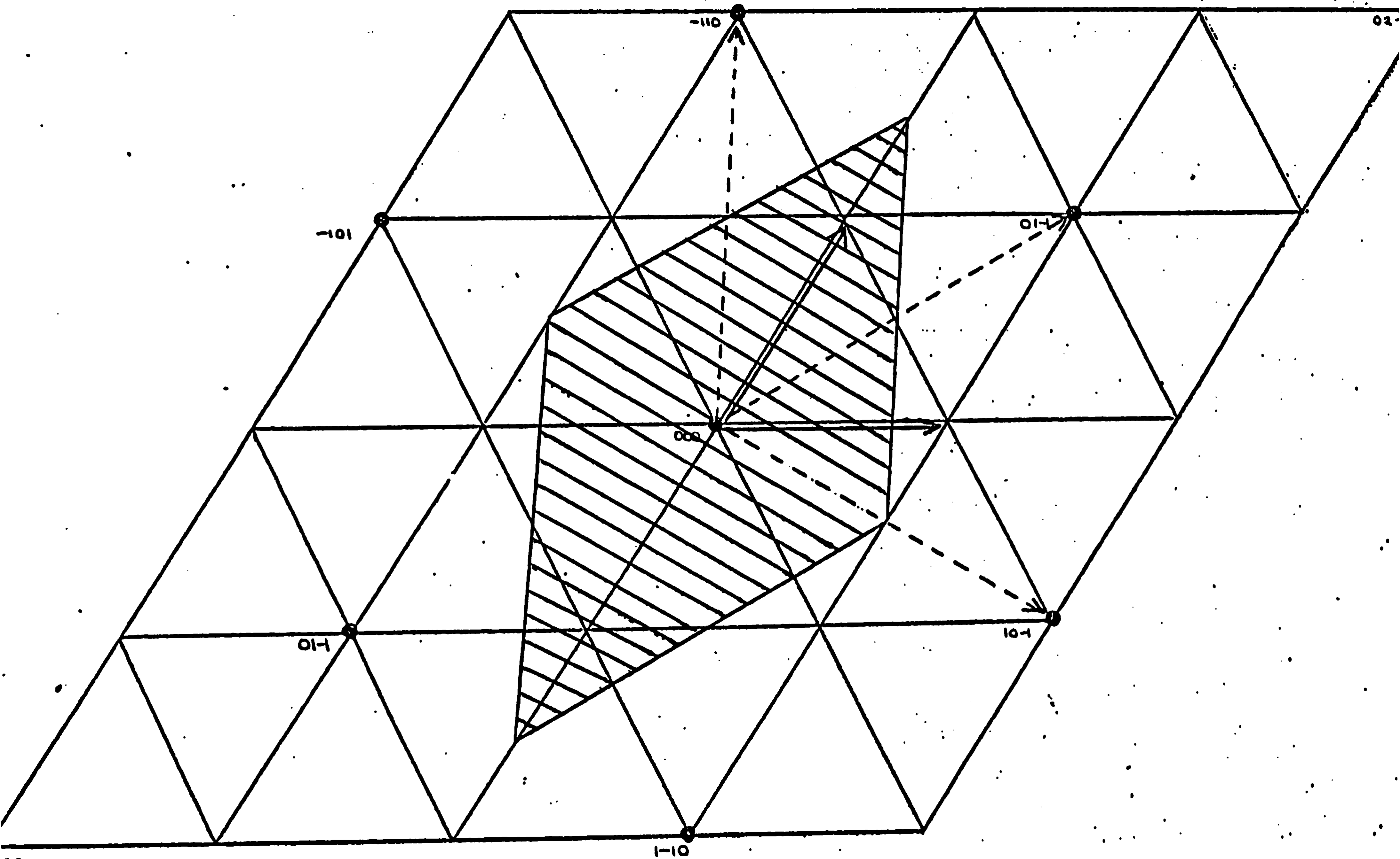


Fig. 3 : SU(3) Critical points of Ad $\cong \lambda \otimes \Lambda^2 \lambda$

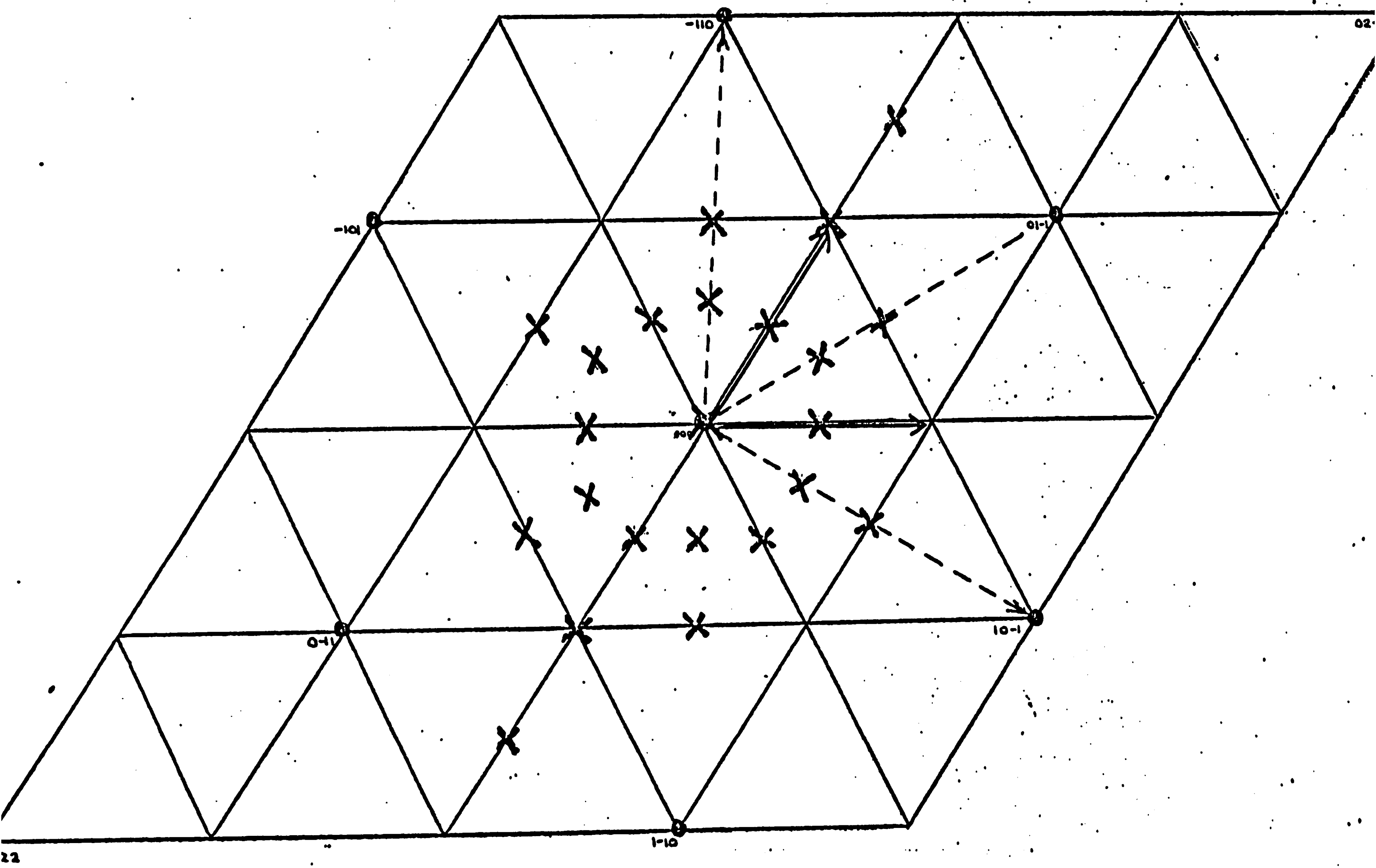


Fig. (3) ... itial point ... λ ... λ ...
λ, and $\lambda^2\lambda$.

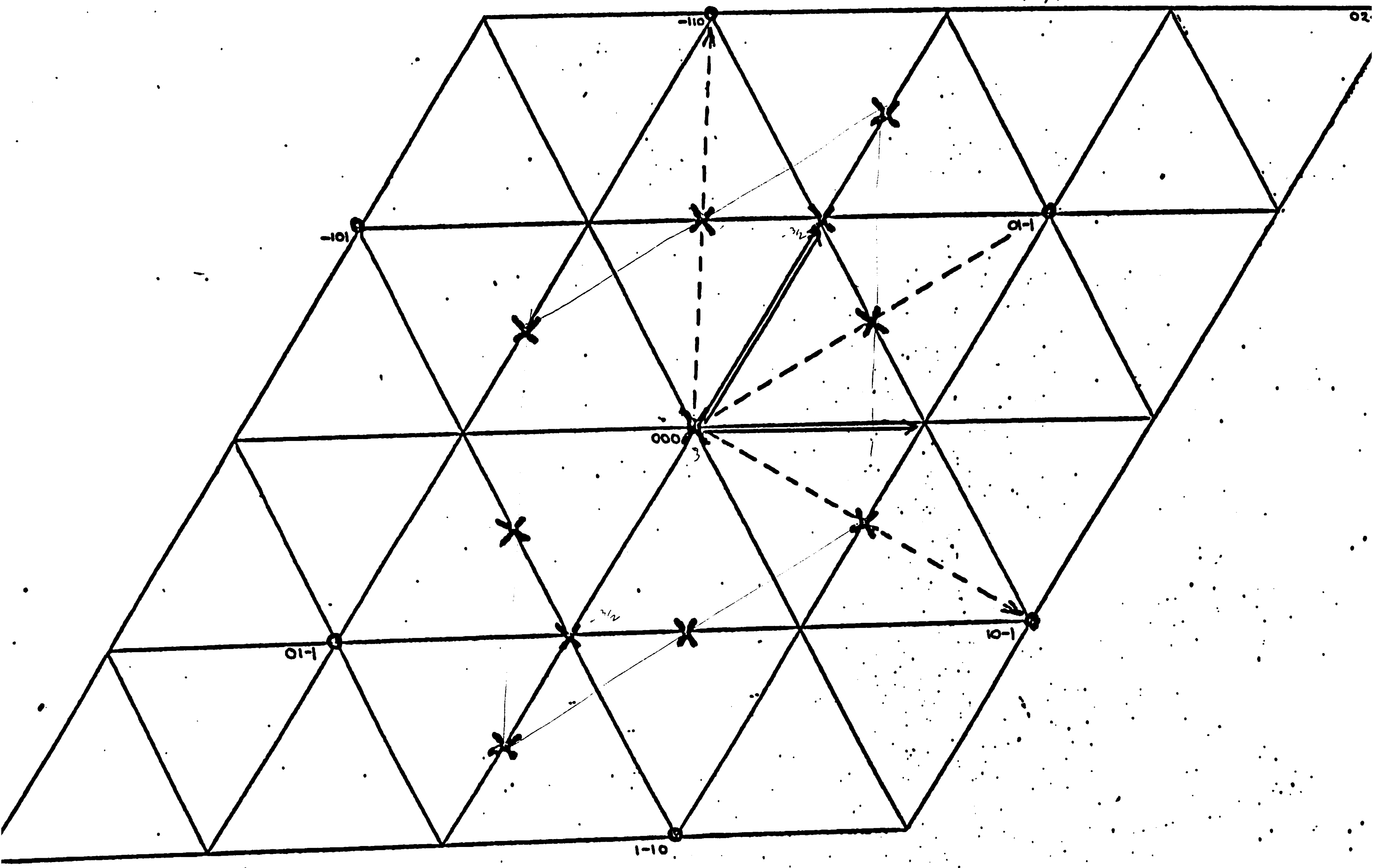


Fig. 4: SO(4) , Spin(4)

• = int. lattice O = int. lattice

≡ = fund. dom /// = fund. dom.

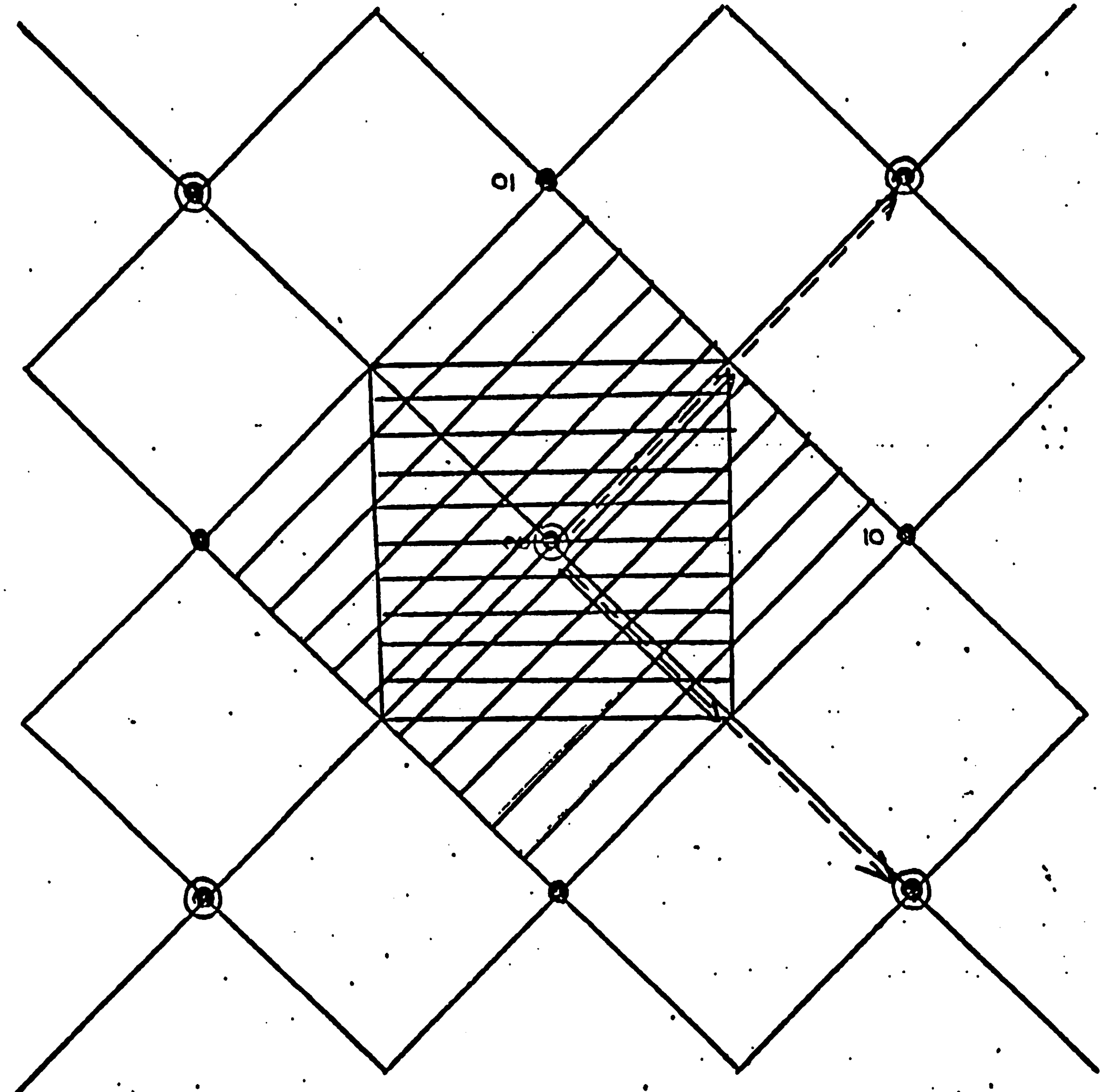


Fig. 5 : SO(4) , Spin(4) Critical points of λ

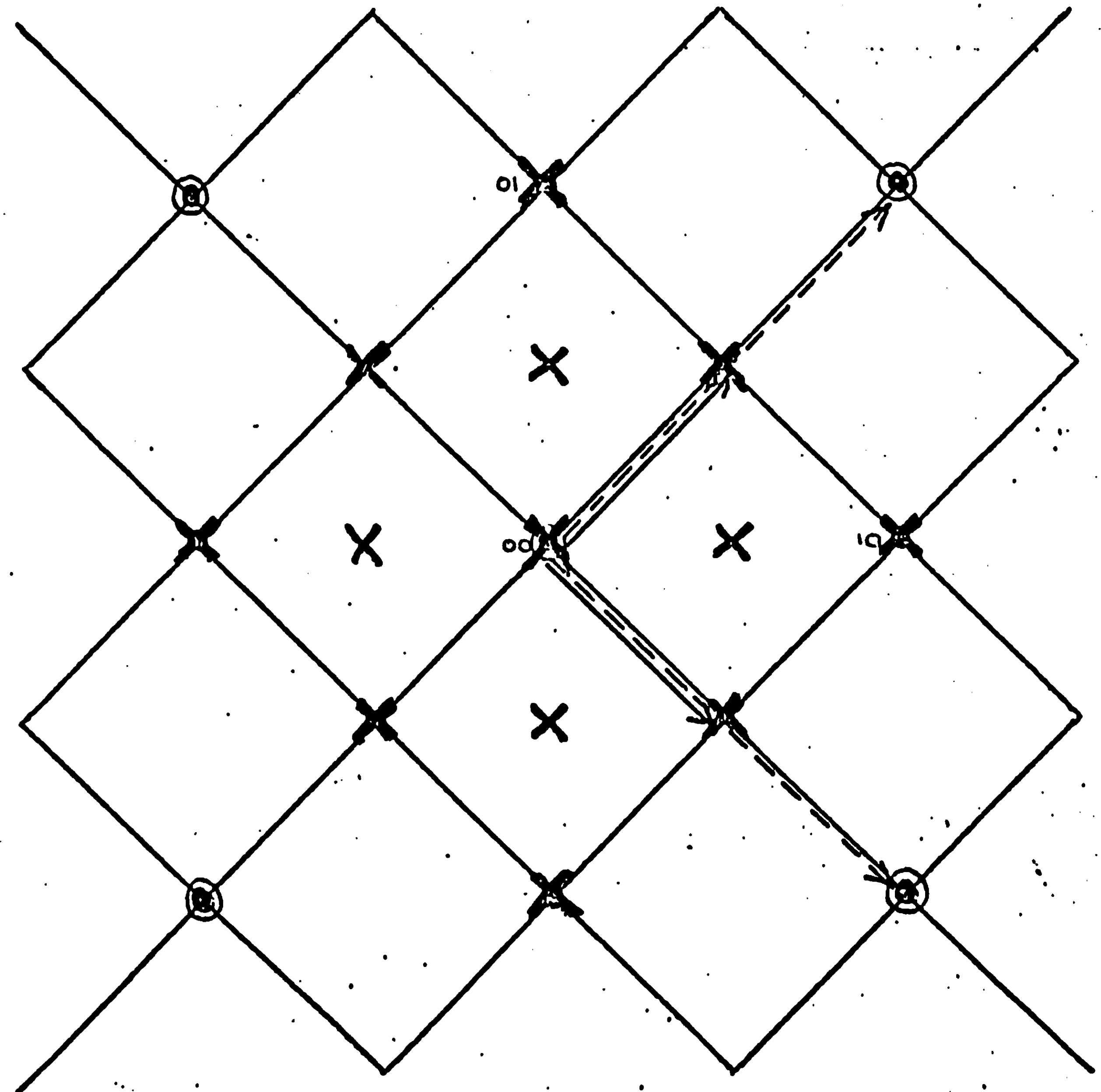


Fig. 6 : SO(4) , Spin(4) Critical points of Ad $= \Lambda^2\lambda$

O = unit. lattice ⊙ = unit. lattice

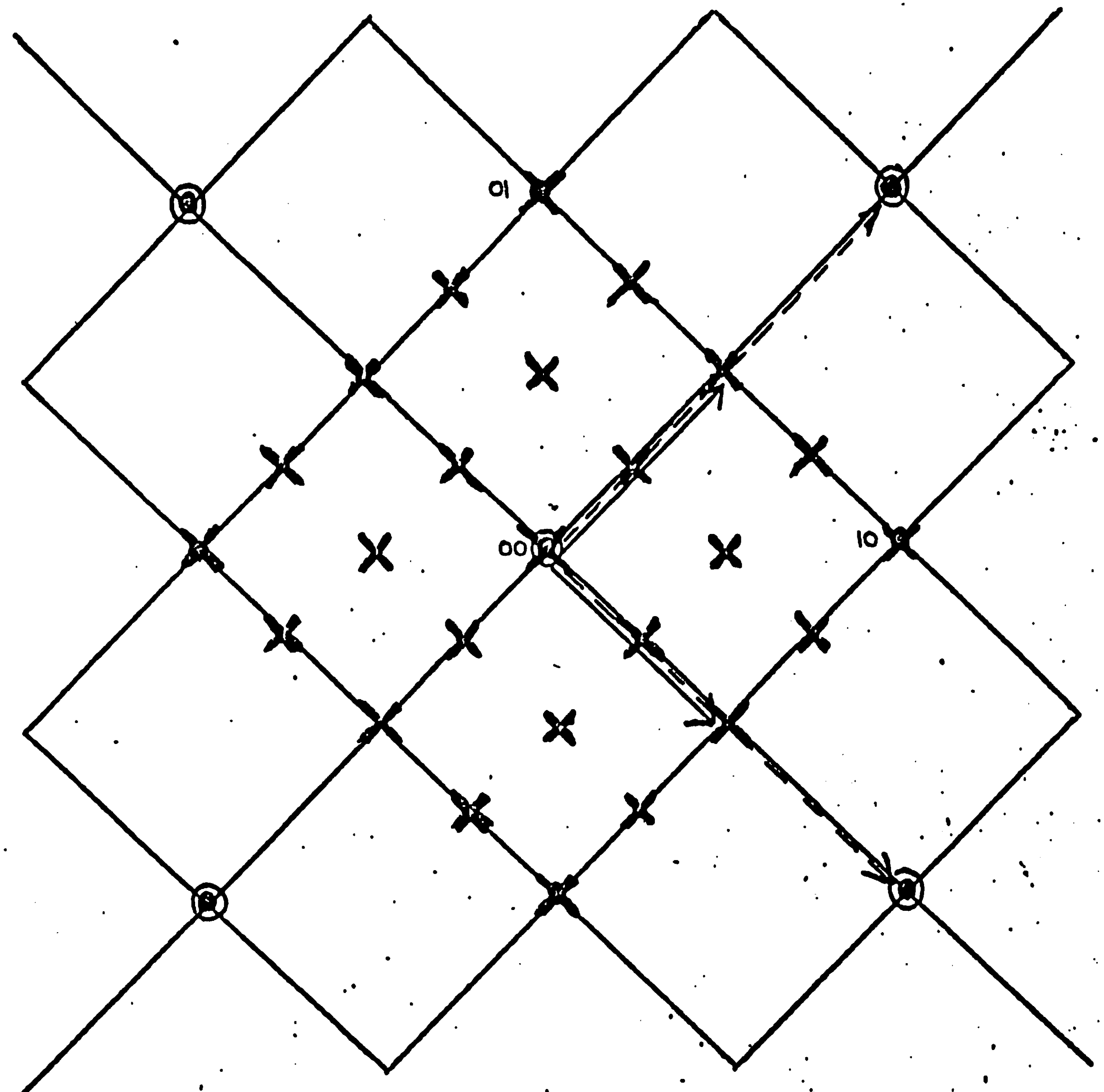


Fig. 7 : SO(5) , Spin(5)

≡ = fund dom ∴ • = int. lattice | /// = fund. dom ; O = int. lattice

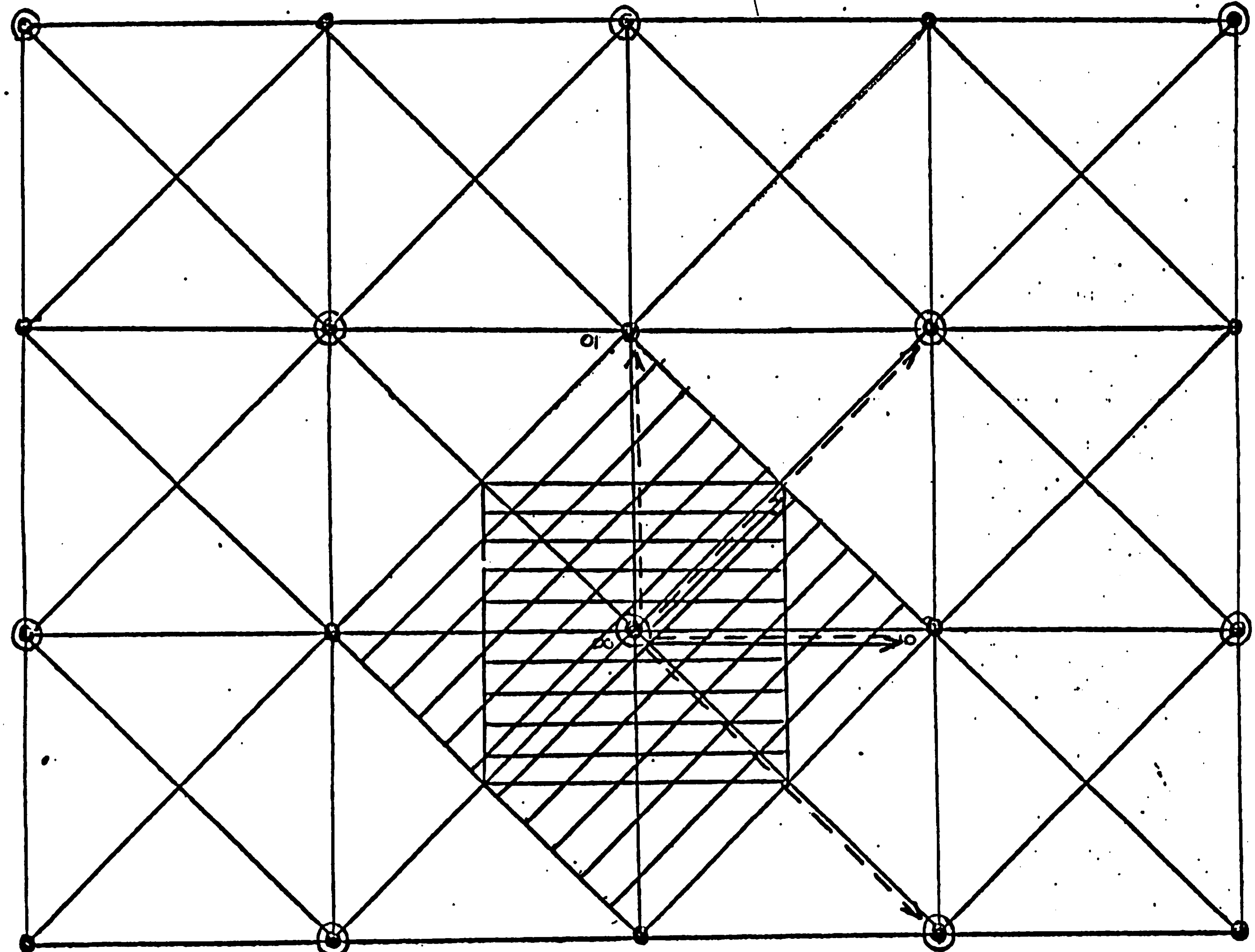


Fig. 8 : Sp(2)

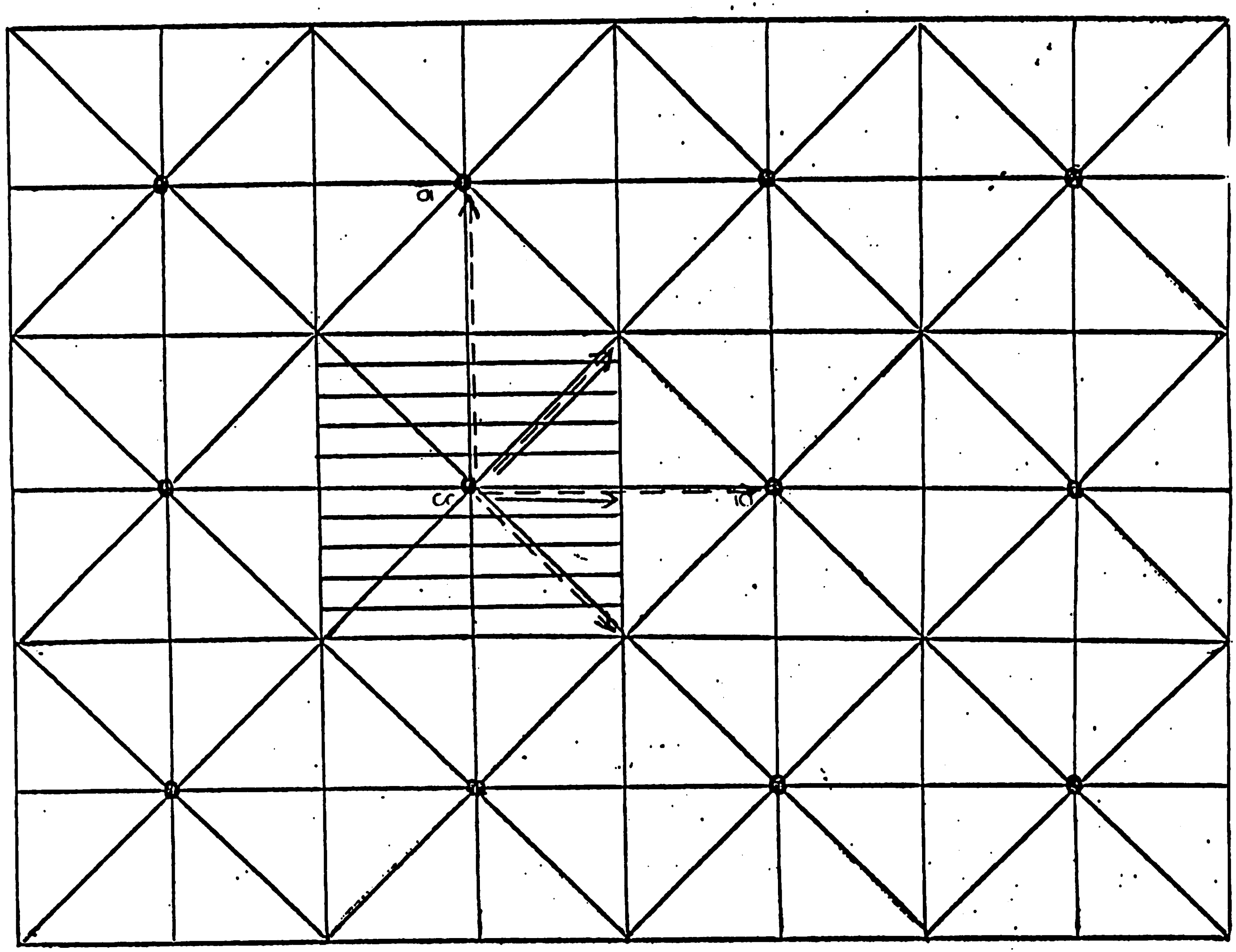


Fig. 9 : SO(5) , Spin(5) Critical points of Λ

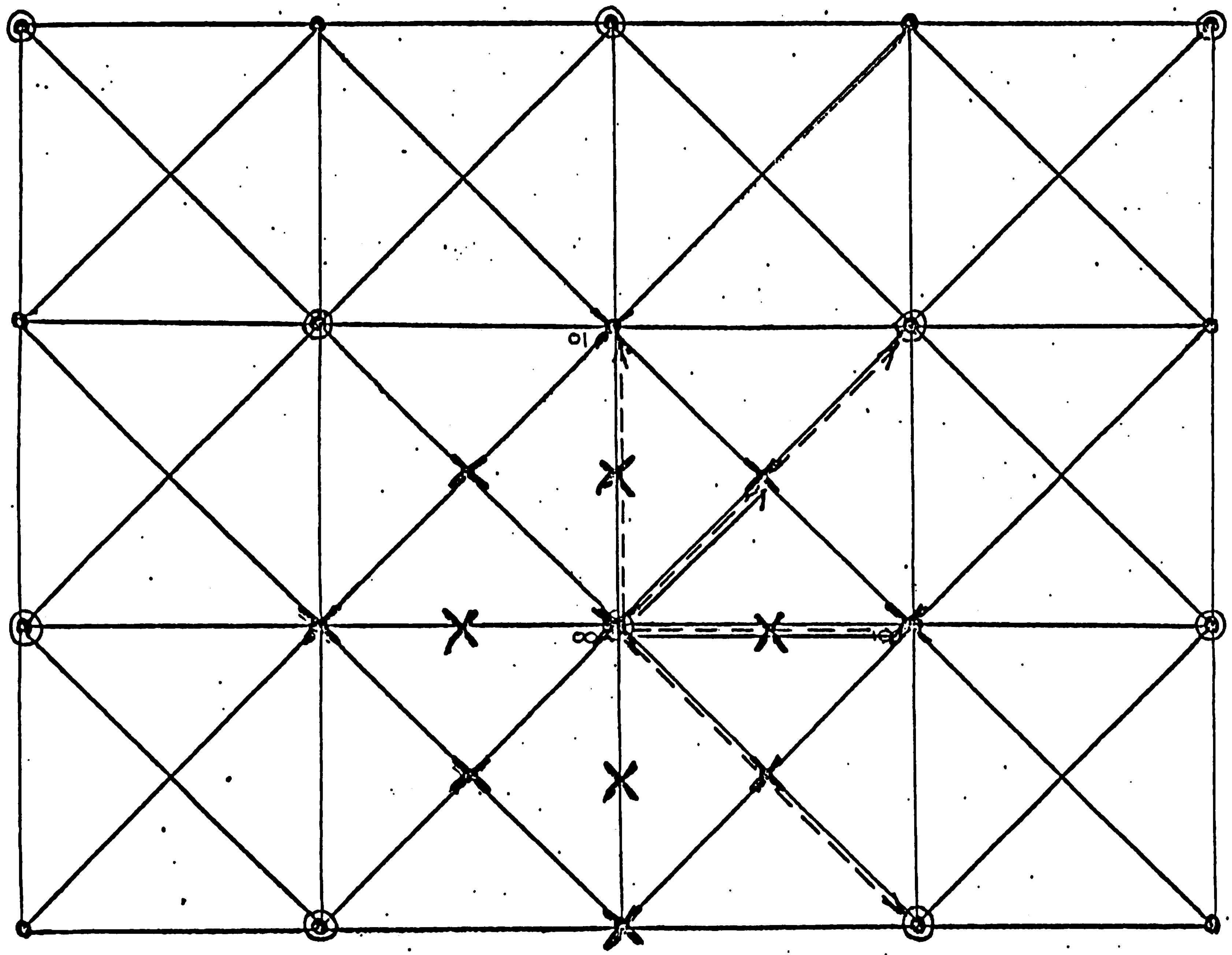


Fig. 10 : SO(5) , Spin(5) Critical points of Ad = $\Lambda^2\lambda$

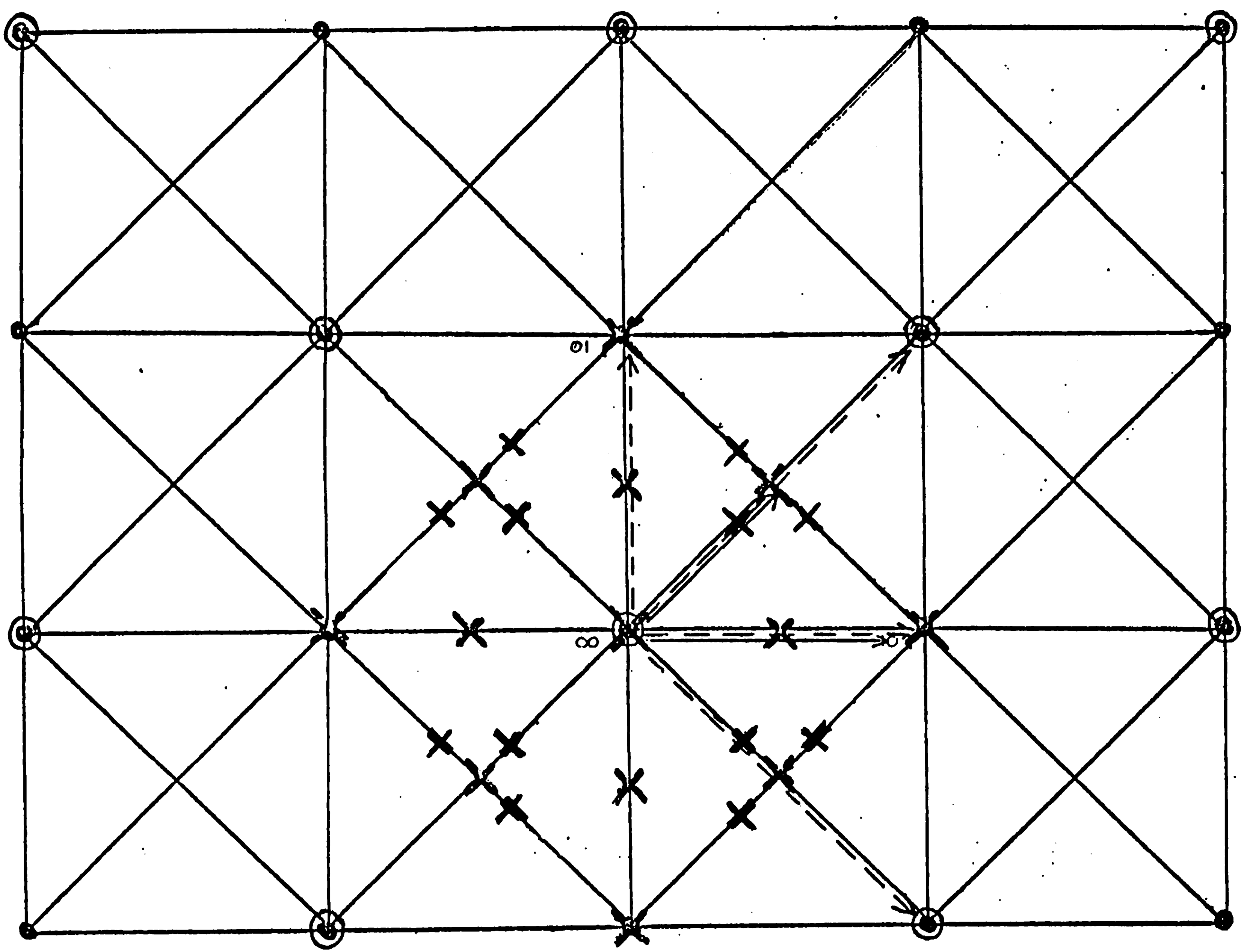
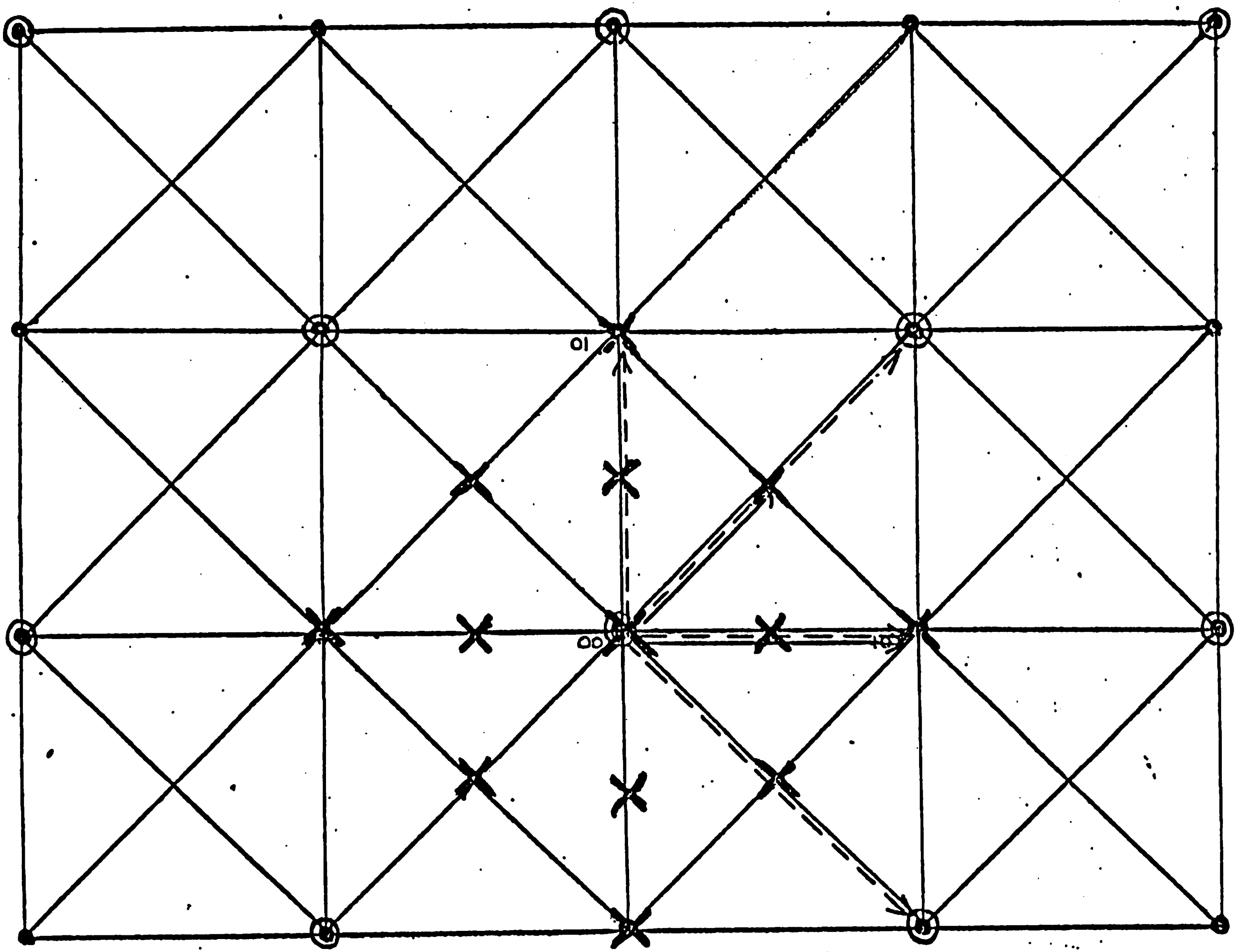


Fig. 11 : SO(5) , Spin(5) Critical points of Δ



Chapter III

In this chapter we turn our attention to $\Omega^k G$ for $k>0$, The case $k=1$ is well known from the work of Bott [5] and we state the main result in §1. In §2 we point out various applications of this result, mostly well known, but expressed in terms of Clifford algebras. It is with generalisations of §1 and §2 that we shall be concerned in chapter IV. In §3 we point out some difficulties which have prevented a generalisation of Bott's result to the case $k>1$. The basic problem is that there does not exist a Morse theory for the space $\Omega^k G$ $(k>1)$ which would relate the critical points of a functional to a cellular decomposition of the space of maps. Our contribution to this area will consist of two examples, discussed in chapters IV and V respectively, which give definite partial results. We calculate the space of absolute minima (normally the calculation of critical points is a difficult matter), and show that the inclusion of the minima in the whole space induces an isomorphism of homotopy groups in the first few dimensions.

§1. The energy functional on the loop space ΩG.

§2. Applications.

§3. Functionals on the iterated loop space $\Omega^k G$: harmonic maps.

§1. The energy functional on the loop space ΩG

Notation: ΩG = {Continuous basepoint preserving maps $S^1 \to G$}

$(\Omega G)_{P,Q}$ = {Continuous maps $[0,1] \to G$ such that $0 \mapsto P$, $1 \mapsto Q$}

These two spaces are homotopic; the first is the usual loop space of homotopy theory, while the second will be more useful for geometrical purposes. When we do not wish to be specific, we shall just write Ω . Thus, Ω_* denotes the $*$ component of Ω , $S\Omega$ denotes the smooth maps in Ω, and $PS\Omega$ denotes the piecewise smooth maps in Ω . As usual, G is a compact Lie group.

We shall now fix P = identity, and denote $(\Omega G)_{P,Q}$ by $(\Omega G)_Q$. We have:

Theorem 1.1 (Bott [5]): Consider the energy functional $E:(PS\Omega_* G)_Q \to R$, ie $E: f \mapsto \int \| df \|^2$. Then E is a Morse-Bott functional, and;

1) The critical points of E are the geodesics in G from I to Q .

2) The connected critical manifold containing a geodesic s is obtained by conjugating s by elements of G which fix Q, and is therefore a finite-dimensional homogeneous manifold.

3) The connected critical manifolds are indexed by Λ_*/W, where Λ_* is the sublattice (in the diagram of G) of the lattice $\exp^{-1}(Q)$ corresponding to the component Ω_* of Ω, and where W is the Weyl group of G . For a lattice point Q', the critical manifold is homeomorphic to the Adjoint orbit (for G) of a generic point on the line joining Q' to the origin 0 . (s is the image under

exp of the line segment $0Q'$.)

4) The index of the critical manifold containing s is twice the number of planes of the diagram which lie in the interior of the segment $0Q'$.

5) The Morse-Bott inequalities are equations, by the "lacunary principle".

The energy functional on Ω is so amenable to investigation because its critical points have a characterisation in terms of known objects, ie geodesics. We shall see that the absolute minima (taken with suitable basepoints) are of particular interest; they are the <u>planar</u> geodesics discussed in chapter I . We shall generalise this fact in chapter IV. Observe that geodesics are automatically smooth maps, so the precise differentiability requirements on Ω are not very important.

If, in the theorem, we take Q to be a regular point, then W does not act on Λ_* , so E has isolated critical points corresponding to the points of the lattice Λ_* . At the other extreme, if Q is the most singular type of point, ie Q is in the centre of the group, we obtain a Morse-Bott functional with the largest possible critical manifolds. The critical manifolds correspond to the points in a fundamental Weyl chamber which are in the appropriate sublattice.

There are various advantages in taking Q to be an element in the centre, but not equal to I . First, each geodesic has a unique parametrisation, whereas in $(\Omega G)_I$ the geodesics $\gamma(t)$ and $\gamma(-t)$ would in general be distinct. Second, the 0-component of $(\Omega G)_I$ clearly has an isolated minimum, whereas it will turn out to be more interesting to have larger manifolds of minima. It is always possible to choose such a Q (eg $Q = -I$) for the groups D_k, D_k^c (but not for any classical group, eg $O(n)$ with n odd).

§2. Applications

As motivation for chapter IV, we indicate below how theorem 1.1 is used to prove the periodicity theorem. Theorem 1.1 extends word for word to compact symmetric spaces, except that in (4) the index is just the number of planes crossed, (instead of twice the number), and so (5) no longer follows immediately (although it is still true).

Theorem 2.1: Consider $E: PS\Omega(E_k)_{e_{k+1},-e_{k+1}} \longrightarrow R$ where E_k is the space of §3, chapter I. The absolute minima of E are the conjugates of the geodesic $\theta \longmapsto e_{k+1}\cos\theta + e_k \sin\theta$ by elements of $\phi^2(D_{k-1})$, and so they form a manifold diffeomorphic to E_{k-1}.

Proof: Take $P = e_{k+1}$, $Q = -e_{k+1}$ in the symmetric space version of theorem 1.1. By the theorem, it will be sufficient to find just one minimal geodesic between $\pm e_{k+1}$, the others will be obtained by conjugating by $\phi^2(D_{k-1})$. It is clear that the path $\theta \longmapsto e_{k+1}\cos\theta + e_k \sin\theta$ is a minimal geodesic, being part of a great circle in the sphere S in $\mathcal{E}_{k+1}$ (in which E_k is embedded) – see the discussion in §4 of chapter I.

Corollary 2.2: $\pi_i\Omega E_k \cong \pi_i E_{k-1}$ for $0<i<n_k-1$ where n_k is the lowest index of any non-minimal geodesic in $PS\overline{\Omega}(E_k)_{e_{k+1},-e_{k+1}}$.

Proof: By Morse theory (see [18]), $PS\Omega(E_k)_{e_{k+1},-e_{k+1}}$ admits a decomposition consisting of the space of minima E_{k-1} with cells of dimension $\geq n_k$

attached. Hence the result.

<u>Theorem 2.3 (The Periodicity Theorem, Bott [5]):</u>

$$\pi_i E_k(\infty) = \pi_{i+8} E_k(\infty)$$

$$\pi_i E_k^c(\infty) = \pi_{i+2} E_k^c(\infty)$$

<u>Proof:</u> Recall the definitions of the stabilised spaces $E_k(n)$ from §3, chapter I. A simple calculation (see [5]) shows that for each space E_k the index n_k has the property that $n_k \to \infty$ as $k \to \infty$. We therefore have:

$$\begin{aligned} \pi_{i+8} E_k(\infty) &= \lim_{n\to\infty} \pi_{i+8} E_k(16^n) \\ &= \lim_{n\to\infty} \pi_{i+8} E_{k+8n} && \text{(by page 33)} \\ &= \lim_{n\to\infty} \pi_i E_{k+8n-8} && \text{(by 2.2, repeated 8 times)} \\ &= \lim_{n\to\infty} \pi_i E_k(16^{n-i}) \\ &= \pi_i E_k(\infty) \end{aligned}$$

The argument for E_k^c is similar.

Recall that the critical points of the "distance function" $f_{e_{k+1}}$ on E_k (see §4, chapter I) were characterised as being the <u>square roots of I</u> (with $Q(E_k) \leq D_k$, ie using the quadratic embedding). Remarkably, there is a similar characterisation of the critical points of the energy functional on ΩE_k:

<u>Proposition 2.4</u>: The minimal geodesics in $(\Omega E_k)_{e_{k+1}, -e_{k+1}}$ can be identified wi[th] components of the <u>square roots of -I</u> in E_k (considering $Q(E_k) \leq D_k$ as usual).

Proof: Taking embeddings in C_{k+1}, we have $\phi(Q(E_k)) = e_{k+1}E_{k-1}$. The minimal geodesic $\theta \longmapsto e_{k+1}\cos\theta + x\sin\theta$ becomes $\theta \longmapsto -(I\cos\theta - e_{k+1}x\sin\theta)$ and we identify the geodesic with its midpoint $y = e_{k+1}x$, so that the space of minimal geodesics is identified with $e_{k+1}E_{k-1}$. Note that x is in the $\phi^{2}(D_{k-1})$ orbit of e_k, so it has square $-I$ and it anticommutes with e_{k+1}. So we have $e_{k+1}E_{k-1} \subseteq \sqrt{-I} \cap e_{k+1}E_k$. Conversely, take $y = e_{k+1}x \in e_{k+1}E_k$ with $y^2 = -I$. Then $e_{k+1}xe_{k+1} = -x^{-1} = x$, so x anticommutes with e_{k+1}, hence the result by proposition 3.8.

Remark: When $k \equiv 1,5\ (8)$, D_k contains an element in its centre of square $-I$ (ie $e_1 \dots e_k$), so $\sqrt{I}$ and $\sqrt{-I}$ are isomorphic (as collections of symmetric spaces). In general, however, $\sqrt{I}$ and $\sqrt{-I}$ are not related.

Remark: R. Wood [24] has given an interesting proof of the periodicity theorem, entirely avoiding the use of Morse theory, essentially by proving directly that

$$\pi_i\,\Omega\,(e_{k+1}E_k)(\infty) \cong \pi_i(\sqrt{-I} \cap e_{k+1}E_k)(\infty)$$

for all i (the notation here is obvious; if $X \subseteq E_k$ then $X(\infty)$ denotes the corresponding subset of $E_k(\infty)$).

§ 3. Functionals on the iterated loop space $\Omega^k G$: harmonic maps

If M and N are compact riemannian manifolds with their Levi-Civita connections, and ϕ is an appropriately differentiable map from M to N, we can consider $d\phi$ as a section of the bundle $T^*(M)\otimes\phi^*TN$, and the second fundamental form of ϕ is the covariant derivative $\nabla(d\phi) \in \Gamma(S^2T^*(M)\otimes\phi^*TN)$. The map ϕ is totally geodesic iff $\nabla(d\phi)=0$, (ie ϕ preserves geodesics).

Definition: The map ϕ is harmonic iff the "tension field" $\tau(\phi) = \mathrm{tr}\,\nabla(d\phi)$ vanishes. (tr denotes contraction on the first two factors here).

Clearly, a totally geodesic map is harmonic, and the converse holds when M is 1-dimensional, ie when ϕ is a path in N.

If h denotes the metric on N (so h is a section of S^2T^*N), ϕ^*h is the first fundamental form, and the energy density $e(\phi)$ is defined to be $\frac{1}{2}\mathrm{tr}\,\phi^*h$. We have $e(\phi)(x) = \frac{1}{2}\| d\phi(x)\|^2$.

Definition: The energy of the map is $E(\phi) = \int_M e(\phi)(x)\,dx$.

Theorem 3.1 : (Eells and Sampson): The map ϕ is harmonic iff it is an extremum of E. ("Appropriately differentiable" here shall be "C^2".)

For more information on this result, and on harmonic maps in general, see the survey article [9]. We shall be interested in the case $M = S^k$ and N = compact riemannian symmetric space, both of these spaces having natural riemannian metrics from their structures as homogeneous spaces. Unfortunately, most known results on harmonic maps do not give information in this case, as both manifolds have nonnegative sectional curvatures; indeed, even the question of existence of harmonic maps is a difficult one.

Example 1 : Harmonic maps from spheres to spheres

This is a case of evident interest for topologists, but harmonic representatives of homotopy classes are known to exist in only a few special cases. First, the Hopf fibrations $S^{2d+1} \longrightarrow S^d$, $d = 1,3,7$ are harmonic. More generally, an orthogonal map $R^m \times R^m \longrightarrow R^n$, $(x,y) \longmapsto f(x,y)$ gives by the Hopf construction a map $S^{2m-1} \longrightarrow S^n$ which is harmonic.

Polynomial maps are a good source of examples. If $f: R^n \longrightarrow R^m$ is a polynomial map which restricts to a map $S^{n-1} \longrightarrow S^{m-1}$, and if the components of f are harmonic functions (ie satisfy $\Delta f_i = 0$) , then the restriction of f is a harmonic map. (The components of f are eigenfunctions of the Laplacian of the sphere S^{n-1} .) Of course, the Hopf fibrations occur in this way.

A theorem of R.T. Smith (see section 8.7 of [9]) gives conditions for the join of two homotopy classes to have a harmonic representative, given that each of the two classes have such a representative. Using this, one obtains harmonic representatives of all elements of the groups $\pi_n S^n$ for $n \leq 7$. (One takes the join of the identity map $S^p \longrightarrow S^p$, with a map $S^1 \longrightarrow S^1$ of arbitrary degree; the conditions of the theorem only give harmonic representatives if $p \leqslant 5$.) Similarly one obtains a harmonic representative of the generator of $\pi_{n+1}S^n$ for $3 \leqslant n \leqslant 8$. These maps are not necessarily polynomial maps however; for example, a theorem of R. Wood [23] asserts that all polynomial maps $S^4 \to S^3$ are constant.

The Morse indices of several of these examples have been calculated [9]. For example, the index of the identity map $S^m \to S^m$ is zero if $m = 1$ or 2, but if $m > 2$ the index is $m + 1$ (ie the identity map is "non-minimal"). The situation for $m = 1$ or 2 is quite straightforward; for $m = 1$ the critical points are the geodesics and the identity map gives the essentially unique minimal geodesic, and for $m = 2$ the critical points are the rational functions (considering S^2 as $\mathbb{C} \cup \infty$) and these all turn out to be absolute minima (see [9], and also chapter V), including the identity map. For more information on the case $m > 2$, see section 3.11 of [9]. It is known that the energy functional has infimum zero in this case, so in particular E never takes its "minimum value" except on the null-homotopic component.

Example 2: Holomorphic maps

A map between almost-Kähler manifolds which is holomorphic or anti-holomorphic ("$\pm$ holomorphic") is known to be harmonic (see [9] section 9.11), and if the domain is compact such a map must also be an absolute minimum of the energy. Under certain conditions the converse is true – we shall discuss in detail an example of this in chapter V.

Example 3 : Totally geodesic maps

A totally geodesic map is harmonic, as has already been pointed out. Moreover, the composite fog of a harmonic g followed by a totally geodesic map f is also harmonic. (see [9] section 4.2) . In chapter IV we shall discuss examples of totally geodesic maps from spheres into compact symmetric spaces; these maps may then be combined with the maps of example 1 above to give new harmonic maps.

In view of the remarks following example 1 , it is not surprising that attempts to generalise the theorem of Bott described in §1 have in general met with little success. One requires a "Morse theory" for the space of maps from M to N . The work of Bott, which involved approximation methods due to Morse [18], has been rewritten by R. Palais and S. Smale in a manner suitable for generalisation, in that they make a direct assault on the space of maps considered as a Hilbert manifold [20] . One obtains a cell decomposition of this space providing (a) the functional under consideration is nondegenerate, and (b) the functional satisifies "condition (C) " , the latter requirement being a substitution for the noncompactness of the space of maps. When $M = S^1$, and the functional is the energy functional, these conditions are satisfied (eg Morse's procedure of approximating the space of paths by finite dimensional compact manifolds ensures (b) holds).

It is by now reluctantly admitted that the energy functional does not in general satisfy condition (C) , and moreover even the calculation of the critical points is a difficult problem. One may proceed in two ways:

1) Replace the energy functional by a different functional.

2) Alter the space of maps under consideration.

Even if a complete extrapolation of Morse theory is not possible, one may still attempt to prove some partial results. We shall do this in chapters IV and V, where we prove (for certain spaces of maps) a version of the following result of "ordinary" Morse theory:

Proposition 3.2: If M is a compact manifold, and $f:M \longrightarrow R$ is a function whose critical manifolds are nondegenerate, then

$$\pi_i M \cong \pi_i \{\text{minima of } f\} \qquad \text{for } i \leq N-2$$

where any non-minimal critical point of f has Morse index $\geq N$.

Proof: M is homeomorphic to a space obtained by attaching cells of dimension $\geq N$ to the manifold of minimum points [18], hence the result.

We shall also make alterations of the kinds (1) and (2) described above; we shall replace E by the "k-th order energy functional", $E_{(k)}$, or by the "volume functional", V, (see below for the definitions), and we shall impose significant basepoint conditions on the spaces of maps involved.

If $\phi:M \longrightarrow N$ is a map, its volume is defined by the formula:

$$V(\phi) = \int_M \sqrt{|\det(\phi^* h)|}\ dx$$

where as usual h denotes the metric on N and ϕ is assumed sufficiently smooth for the integral to exist. The significant property of V is that it is "parametric", ie its value is independent of reparametrisation of the domain M.

Example: If $m = 1$, V is the length functional on paths. By a well known result in differential geometry, a path of minimal length can be reparametrised to give a geodesic (the choice of parameter is unique up to linear transformations). Conversely, a path which minimises E is easily seen to be a geodesic (see §12 of [18] or the discussion below). In fact, the critical points of E are precisely the geodesics.

It is often possible to use geometrical considerations to calculate some of the critical points of V – this is a classical problem. For $m > 2$ however, the relation between V and E and their critical points becomes obscure. One method is to use the following general (and trivial) result and we shall do this in chapter IV.

Proposition 3.3: If F is a functional on the space of sufficiently smooth maps from M to N, and if

1) $V(\phi) \leq F(\phi)$ for all maps ϕ

2) There exists a map f, minimal for V, such that $V(f) = F(f)$;

then a minimum for F is a minimum for V.

<u>Proof</u>: Let ϕ be a minimum for F. Then :

$$\begin{aligned} V(\phi) &\leqslant F(\phi) && \text{by condition (1)} \\ &\leqslant F(f) && \text{since } \phi \text{ is a minimum for } F \\ &= V(f) && \text{by condition (2)} \\ &\leqslant V(\phi) && \text{by condition (2)} \end{aligned}$$

Hence $V(\phi) = V(f)$, ie ϕ is a minimum for V.

As the energy functional does not in general satisfy condition (1), we shall replace it by the <u>k-th order energy</u> (for some suitable k), which is defined by the formula:

$$E_{(k)}(\phi) = \int \left(\frac{|\operatorname{tr} \phi^* h|}{k} \right)^{k/2} dx = \left(\tfrac{1}{k}\right)^{k/2} \int \| d\phi \|^k \, dx$$

When $k = m$, Cauchy's inequality $\prod_{i=1}^{m} x_i \leqslant \left(\sum_{i=1}^{m} x_i/m \right)^m$ shows that $V(\phi) \leqslant E_{(m)}(\phi)$, with equality iff ϕ is weakly conformal (ie iff there is a nonnegative function λ on M such that $\phi^* h = \lambda g$, where h, g are the metrics on N, M respectively). In chapter IV we shall show that under certain circumstances the (absolute) minima of V and of $E_{(m)}$ coincide (up to parameter). This will be a direct generalisation of the example above. In the example, one may apply proposition 3.3 with $F = \sqrt{E}$ to show that a minimum of E is a minimum of L; to prove the converse one must appeal to the well known characterisation of minima of L. In chapter IV we take $F = E_{(m)}$ and use proposition 3.3 to show a minimum of E_H is a minimum of V ; for the converse we need a characterisation of the minima of V which is proved essentially by induction on m , thus reducing the problem to the case of geodesics.

We have explained at the beginning of this chapter how Morse – theoretical results can be obtained in the case m = 1 , and pointed out that when m > 1 this is in general no longer possible. It is well known that V does not satisfy condition (C) , and in general neither does $E_{(H)}$. In chapters IV and V we shall discuss two very different examples where it is nevertheless possible to prove the analogue of proposition 3.2 .. In chapter V we shall take m = 2 , and the maps f satisfying condition (2) of proposition 3.3 will be the holomorphic maps. In chapter IV we shall take m>2 and the maps f will be certain linear embeddings of spheres into symmetric spaces. We discuss the latter first, as this is more elementary and a direct generalisation of the case m = 1 described in this chapter.

We begin with the calculation of the absolute minima of $E_{\omega}: \Omega^k X \longrightarrow R$ in a well known situation, ie when X is a classical symmetric space and k is in the stable range. In general the higher index (ie non-minimal) critical points are not known, but the minima have a simple characterisation. We find that the minima form a   submanifold, provided that an appropriate "boundary condition" on the elements of $\Omega^k X$ is imposed.

The calculation itself is very simple, being essentially a rewording of Bott's original proof by Morse theory of the periodicity theorem [5]. Bott always discussed geodesics, but it has been observed by A. T. Fomenko [11] that one may describe the Bott isomorphism as induced by a map of the form $U(n) \rightarrow \Omega^2 U(2n)$, or, in the real case, $O(n) \rightarrow \Omega^8 O(16n)$, by "amalgamating" the geodesics to form spheres. In §1 we describe a generalisation of Fomenko's work, which hopefully makes his proof clearer. Moreover, we discuss the spaces $\Omega^j E_k$ entirely in terms of Clifford algebras, using the notation developed in chapter I, and we thereby avoid an unsightly computation with matrices. In §2 we show that the inclusion of the minima into the whole space of maps induces the periodicity isomorphism. The minimum energy maps themselves are of some topological interest, as they represent (in appropriate dimensions) generators of the stable homotopy groups of the symmetric spaces E_k ; this agrees with the well known description of representatives of these generators as being the symbols of the Dirac operators. Finally, §4 describes a <u>Lie-algebraic</u> interpretation of the minima, in the spirit of chapter I. This provides a new interpretation of the isomorphisms $R\,Spin(2n)/R\,Spin(2n+1) \cong K(S^{2n})$, $RO\,Spin(8n)/RO\,Spin(8n+1) \cong KO(S^{8n})$

of Atiyah, Bott and Shapiro [2].

§1. The minima of $E_{(k)}: \Omega^k X \to R$

Let us first recall that Bott's result on geodesics described in Chapter III (theorem 2.1) has the following description in terms of the spaces E_k :

<u>Proposition:</u> Consider paths in E_k from e_{k+1} to $-e_{k+1}$, and restrict attention to that component which contains the paths of shortest length. Then the minimal geodesics are precisely the conjugates under $\phi^2(D_{k-1})$ of the "standard geodesic" $\theta \mapsto e_{k+1}\cos\theta + e_k\sin\theta$.

<u>Corollary:</u> The space of minimal geodesics is homeomorphic to E_{k-1}.

We shall now generalise this to the case of maps $S^j \to E_k$, where $j = 0,1,\ldots,k$. Define the <u>standard i-sphere</u> in E_k to be the map:

$$\gamma_i : S^i \to E_k \quad (x_1, \ldots, x_{i+1}) \mapsto \sum_{t=1}^{i+1} x_t e_{t+k-i}$$

For example, the standard 0-sphere in E_k is the pair of points $\pm e_{k+1}$, and the standard 1-sphere is the "standard geodesic" $\theta \mapsto e_{k+1}\cos\theta + e_k\sin\theta$.

We shall be interested in the following two spaces of maps (which are obviously homotopic spaces, since γ_{j-1} is contractible):

$\Omega^j E_k$ = Continuous basepoint preserving maps $S^j \to E_k$ ("$x_{i+1} = 1$" $\mapsto e_{k+1}$)

$\bar{\Omega}^j E_k$ = Maps $f: D^j \longrightarrow E_k$, where D^j denotes the hemisphere "$x_1 \geq 0$" in S^j, satisfying the conditions

1) $f\,|_{S^{j-1}} = \gamma_{j-1}$ (S^{j-1} is the equator "$x_1 = 0$" of S^j)

2) f is "sufficiently smooth" (so that $E_{(j)}(f)$ exists)

Recall the definitions of the volume functional V and the nth order energy function $E_{(n)}$ from chapter III ; we take $n = j$. During this chapter we shall always deal with $E_{(j)}$ and never with E .

Proposition 1.1 : Any minimum of V is given by the intersection with S (the unit sphere in E_{k+1}) of a $(j+1)$dimensional half plane through the origin of E_{k+1}.

Proof: First, it is clear that all such half planes give minima, since a half plane whose boundary is the plane spanned by $e_{k+1}, \ldots, e_{k-j+2}$ must a) be minimal and b) be such that its intersection with S lies completely in E_k . We claim a minimum of V on $\bar{\Omega}^j E_k$ must also be a minimum of the volume functional on the following space:

$$\{ \text{maps } f: D^j \longrightarrow S \mid f\,|_{S^{j-1}} = S(e_{k+1}, \ldots, e_{k-j+2}) \subseteq S \}$$

That the solutions to this problem are half planes is ensured by theorem 4.5, page 397 , of [7]. To prove the claim, let $f: D^j \longrightarrow E_k$ be of minimal volume

(with $f|_{S^{j-1}}$ fixed as above). Then $V(f) \leq V(\gamma)$ where $\gamma = \gamma_j|_{D^j}$. If $m: D^j \to E_k$ is any map, we denote by $\bar{m}$ the map obtained by composition with $E_k \to S$. Since $E_k \to S$ is an isometric embedding, $V(m) = V(\bar{m})$. We therefore have $V(\bar{f}) \leq V(\tilde{\gamma})$. By the remarks above [7] we must have $V(\tilde{\gamma}) \leq V(\bar{f})$, so $V(\tilde{\gamma}) = V(\bar{f})$ and $\bar{f}$ is of minimal volume (subject to $f|_{S^{j-1}}$ fixed as usual) as required.

<u>Proposition 1.2</u>: On the space $\bar{\Omega}^j E_k$, the minima of $E_{(j)}$ and V are the same (up to reparametrisation of the disc D^j).

<u>Proof</u>: Consider $f = \gamma_j|_{D^j}$. This map embeds the disc D^j as the intersection of the half plane containing $e_{k+1}, \dots, e_{k-j+1}$ with the unit sphere S in E_{k+1}. We clearly have 1) f is a minimum of V on $\bar{\Omega}^j E_k$

2) $V(f) = E_{(j)}(f)$ (since $f: D^j \to M$ is isometric, and in particular conformal, as remarked in the proof of 1.1 above). Hence, by proposition 3.3 of chapter III, any minimum of $E_{(j)}$ is a minimum for V. Conversely, proposition 1.1 above shows that a minimum of V – with the obvious parametrisation – is a minimum of $E_{(j)}$.

After all this preparation (c/f chapter I) it is now easy to state and prove the main theorem concerning the space $\bar{\Omega}^j E_k$. Let $\bar{\Omega}^j_* E_k$ be the component of $\bar{\Omega}^j E_k$ which contains the map $\gamma_j|_{D^j}$.

<u>Theorem 1.3</u>: Consider the functional $E_{(j)}: \bar{\Omega}^j_* E_k \to R$. The absolute minima are the map $\gamma_j|_{D^j}$ and its conjugates by the group $\phi^{j+1}(D_{k-j})$.

<u>Proof</u>: The map $\gamma_j|_{D^j}$ and its conjugates are given by the intersections

of half planes in $\mathcal{E}_{k+1}$ with the sphere $S \subseteq \mathcal{E}_{k+1}$, so they are certainly minima for $E_{(j)}$, as was remarked in the proof of proposition 1.2 above.

Conversely, let $f: D^j \longrightarrow E_k$ be a minimum for $E_{(j)}$; by propositions 1.1, 1.2 it is a minimum of V and therefore given by a half plane in $\mathcal{E}_{k+1}$. The plane must contain the vectors $e_{k+1}, \ldots, e_{k-j+2}$ because of the boundary condition; call the unit vector in the direction perpendicular to these elements x. Observe that x can be considered as the mid-point of a planar geodesic in E_k passing through $\pm e_{k-j+2}$. Another such geodesic is $e_{k-j+2} \cos\theta + e_{k-j+1} \sin\theta$, so we know by theorem 2.1 of chapter III that x is conjugate to e_{k-j+1} by an element of D_{k-j} (ie $\phi^{j+1} D_{k-j}$). Since the action of D_{k-j} preserves the plane spanned by $e_{k+1}, \ldots, e_{k-j+2}$, the whole disc $f(D^j)$ can be conjugated by this element of D_{k-j} to the "standard disc" $\gamma_j \big|_{D^j}$. This completes the proof.

<u>Corollary 1.4</u>: The space of absolute minima is homeomorphic to E_{k-j}.

We thus obtain an inclusion $\theta : E_{k-j} \longrightarrow \overline{\Omega}^j_x E_k (\simeq \Omega^j_x E_k)$; if it were possible to extrapolate the Morse theory from the case $k = 1$ to the case $k > 1$, we would be able to deduce a homotopy equivalence $\pi_i E_{k-j} \cong \pi_{i+j} E_k$ up to some dimension (depending on the index of the next critical point in $\overline{\Omega}^j_x E_k$). While such an extrapolation does not at the moment exist, it is nevertheless possible to show that the inclusion does induce the well known periodicity isomorphism as described by Bott. This we shall do in §2.

§2. Bott Periodicity

In this short section we verify that the map induced in homotopy by the inclusion $\theta : E_{k-j} \to \overline{\Omega}^j_k E_k$ is an isomorphism when $j \equiv 0 \ (8)$, by showing that it induces the same map as the appropriate composite of maps obtained from the "one-stage" inclusions of the form $E_\ell \longrightarrow \overline{\Omega}_k E_{\ell+1}$. The latter are known to be isomorphisms, by Morse theory [5]. It would be interesting to prove that θ induces an isomorphism without appealing to Bott's original proof, for example by generalising the method of R. Wood [24] (see the remarks following 2.4 of chapter III).

We shall write down an explicit formula for the map of loop spaces induced by θ, ie a map $\Omega^i E_{k-j} \to \Omega^{i+j} E_k$. First, let us note that θ can be written explicitly as follows:

$$\theta : E_{k-j} \to \overline{\Omega}^j_k E_k \qquad x \longmapsto \left[(s_0, \ldots, s_j) \mapsto \sum_{\nu=0}^{j-1} (s_\nu \, e_{k+1-\nu}) + s_j \, i(x) \right]$$

where $i : E_{k-j} \to E_k$ is the inclusion described in proposition 3.8 of chapter I. (NB: We have not previously given a name to this inclusion i. It is <u>not</u> the same as ϕ !)

<u>Definition:</u> $\Theta : \Omega^i E_{k-j} \to \Omega^{i+j} E_k$ is the map obtained by taking $f : S^i \to E_{k-j}$, and defining $\Theta(f) : S^{i+j} \to E_k$ by:

1) On $S^i \times D^j$: $(r,s) \longmapsto \theta(f(r))(s)$

2) On $D^{i+1} \times S^{j-1}$: $(r,s) \longmapsto Y_{j-1}(s)$

Here we consider D^{i+1} as the hemisphere $r_{i+1} \geq 0$ of S^{i+1} ($\|r\| = 1$), so that S^i is given by $r_{i+1} = 0$. Similarly D^j is the hemisphere $s_j \geq 0$ of S^j ($\|s\| = 1$), and S^{j-1} is given by $s_j = 0$. We have $S^i \times D^j \subseteq D^{i+1} \times D^j$ (ie $r_{i+1} = 0$, $s_j \geq 0$) and $D^{i+1} \times S^{j-1} \subseteq D^{i+1} \times D^j$ (ie $r_{i+1} \geq 0$, $s_j = 0$) and then $S^i \times D^j \cap D^{i+1} \times S^{j-1}$ $S^i \times S^{j-1}$ (ie $r_{i+1} = 0$, $s_j = 0$). The union $S^i \times D^j \cup D^{i+1} \times S^{j-1}$ is the boundary of $D^{i+1} \times D^j$ and is homeomorphic to a sphere S^{i+j} (this is the standard toral decomposition of S^{i+j}).

The map $\Theta(f)$ is well defined since both parts of the definition agree on their intersection. As the formula (2) is independent of r, we may choose coordinates t for S^{i+j} as follows:

$$t_i = \begin{cases} s_i & i = 0, \ldots, j-1 \\ s_j r_{i-j} & i = j, \ldots, i+j+1 \end{cases}$$

Note that $t_{i+j+1} = 0$, since one of s_j, r_{i+1} is always zero (so we consider $S^{i+j} \subsetneq D^{i+j} \subsetneq S^{i+j+1}$ in the usual way, with S^{i+j+1} given by $\|t\| = 1$). As basepoints we take $s_0 = 1$ in S^i, $t_0 = 1$ in S^{i+j}, and e_{j+1} in E_j. With these choices, $\Theta(f)$ is basepoint preserving when f is.

This formula has some similarity with the <u>Samelson product</u> construction, which is apparent when the following alternative description is used.

Observe that Θ is the map induced on $E_{k-j} = D_{k-j} / D_{k-j-1}$ by the map

$$D_{k-j} \longrightarrow \bar{\Omega}^j E_k \qquad x \longmapsto \phi(x)\,\gamma_j\Big|_{D^j} (\phi(x))^{-1}.$$

The map ϕ here is from D_{k-j} to D_{k+1}, and its image commutes with the elements $e_{k+1}, \ldots, e_{k-j+2}$ of D_{k+1} and hence with the restriction of $\gamma_j : D^j \longrightarrow E_k$ to its boundary $S^{j-1} = S(e_{k+1}, \ldots, e_{k-j+2})$. Given a map $S^i \longrightarrow E_{k-j}$, we obtain an induced map $S^i \times D^j$ which is independent of the first variable when restricted to ∂D^j, and hence a map of S^{i+j}.

Remark: If γ_j were actually constant on ∂D^j (instead of being the map γ_{j-1}) this construction would give a map in the homotopy class $\langle [f], [\gamma_j] \rangle$ (Samelson product). The class of $\Theta(f)$ is therefore a type of generalised Samelson product of the classes of f and of γ_j.

Example: take $E_1^c = U(1)$, $E_3^c = U(2)$. The map $i: U(1) \longrightarrow U(2)$ is $z \longmapsto \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}$ and $\Theta : U(1) \longrightarrow \Omega^2 U(2)$ is given by

$$z \longmapsto \left[(s_0, s_1, s_2) \mapsto s_0 e_4 + s_1 e_3 + s_2\, i(z) \right]$$

ie, taking $e_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$,

$$z \longmapsto \left[(s_0, s_1, s_2) \mapsto \begin{pmatrix} s_0 + is_1 & s_2 z \\ -s_2 \bar{z} & s_0 - is_1 \end{pmatrix} \right].$$

Note that we do have $e_i^2 = -I$ and $e_i e_j = -e_j e_i$ $(i \neq j)$; the multiplication is that of the Clifford algebra C_4^c and is induced via the embedding $E_3^c \longrightarrow C_4^c$ given by $X \longmapsto \begin{pmatrix} 0 & X \\ -\bar{X}^t & 0 \end{pmatrix}$.

Writing $s_2 = \sin\theta$ for $\theta \in [0, \pi]$, $w = s_0 + is_1 / \| s_0 + is_1 \|$, we have:

$$\left[f : S^i \longrightarrow U(1)\right] \longmapsto \left[\Theta(f) : S^{i+2} \longrightarrow U(2)\right] \quad \Theta(f)(\theta, w) = \begin{pmatrix} w\cos\theta & f(z)\sin\theta \\ -\bar{f}(z)\sin\theta & \bar{w}\cos\theta \end{pmatrix}$$

(Of course f is nullhomotopic except when $i = 1$, but the formula illustrates that which holds in higher dimensions.)

Lemma 2.1 : With $\Theta : \Omega^k E_k \longrightarrow \Omega^{k+j} E_{k+j}$ we have $[\Theta(\gamma_k)] = [\gamma_{k+j}]$.

Proof: Trivial (using the transformation to the t-coordinates for S^{k+j}, as described above). Two such transformations give:

Proposition 2.2: With $\Theta_j : \Omega^i E_k \longrightarrow \Omega^{i+j} E_{k+j}$,
$\Theta_1 : \Omega^{i+j} E_{k+j} \longrightarrow \Omega^{i+j+1} E_{k+j+1}$ and $\Theta_{j+1} : \Omega^i E_k \longrightarrow \Omega^{i+j+1} E_{k+j+1}$ we have:
$\Theta_1 \circ \Theta_j \simeq \Theta_{j+1}$.

Corollary 2.3: Θ_8 induces the Bott periodicity isomorphism in the stable range.

Proof: Each of the maps Θ_1 was shown to induce an isomorphism in the stable range by Bott [5], and by 2.2 we have $\Theta_8 = \Theta_1 \circ \cdots \circ \Theta_1$ (8 times).

§3. Representatives of the stable generators

The minimal maps in $\Omega^k X$ can be thought of as "canonical" representatives of their homotopy classes ; for example if $k = 1$ one obtains geodesic loops. The maps described in §1 do indeed have such an interpretation; in $\Omega^i E_k$ the map γ_i is obviously contractible unless $i = k$, but in the latter case we have:

Proposition 3.1: The sphere $S(e_1, \dots, e_{2k+2})$ represents a generator of $\pi_{2k+1} E^c_{2k+1}$ $(= \pi_{2k+1} U(2^k) \cong \mathbb{Z})$.

Proof: When $k = 0$ this is obviously true, as $E^c_1 = U(1) = S(e_1, e_2)$.
The general case now follows by induction, because:

a) $[\circledcirc]: \pi_{2k-1} E^c_{2k-1} \longrightarrow \pi_{2k+1} E^c_{2k+1}$ is an isomorphism, by the proposition of §2 .

b) $[\circledcirc(\gamma_{2k-1})] = [\gamma_{2k+1}]$ by the lemma of §2 .

The map $\gamma_{2k+1} : S^{2k+1} \longrightarrow E^c_{2k+1}$ has an alternative interpretation, as the symbol of the Dirac operator $D = \sum_{i=1}^{2k+2} e_i(\partial/\partial x_i)$ on R^{2k+2}. (Here we consider $R^{2k+2} \subseteq C^c_{2k+2}$ as usual, and $S(e_1, \dots, e_{2k+2})$ is the unit sphere in R^{2k+2}.) Recall from chapter I that Δ^+_{2k+1} and Δ^-_{2k+1} denote the two basic ungraded modules for the algebra C^c_{2k+1} (considered as embedded in C^c_{2k+2}); each element of $S(e_1, \dots, e_{2k+2})$ gives by Clifford multiplication an isomorphism of Δ^+_{2k+1} with Δ^-_{2k+1}, and this map $S^{2k+1} \longrightarrow \text{Isom}(\Delta^+_{2k+1}, \Delta^-_{2k+1})$ is the symbol of D ("Isom" denotes unitary isomorphism here). We can now identify $\text{Isom}(\Delta^+_{2k+1}, \Delta^-_{2k+1})$ with E^c_{2k+1} .

Remark: It is proved in [2] that the vector bundle over S^{2k+2} with clutching function

given by $S(e_1, \ldots, e_{2k+2}) \longrightarrow \mathrm{Isom}(\Delta^+_{2k+1}, \Delta^-_{2k+1})$ gives the element $1 \in K(S^{2k+2})$ $(\cong \pi_{2k+1}U)$.

With the identification $E^c_{2i+1} \cong \mathrm{Isom}(\Delta^+_{2i+1}, \Delta^-_{2i+1})$ the periodicity map $\Omega^i E^c_{2k-1} \to \Omega^{i+2} E^c_{2k+1}$ has the following description.

We have: $\Delta^+_{2k+1} \cong \Delta^+_{2k-1} \otimes \Delta^+_1 \oplus \Delta^-_{2k-1} \otimes \Delta^-_1$

$\Delta^-_{2k+1} \cong \Delta^+_{2k-1} \otimes \Delta^-_1 \oplus \Delta^-_{2k-1} \otimes \Delta^+_1$ (because of the "periodicity" $C^c_{2k+2} \cong C^c_{2k} \otimes C^c_2$)

and if $f: S^i \longrightarrow \mathrm{Isom}(\Delta^+_{2k-1}, \Delta^-_{2k-1})$ we have $f^*: S^i \longrightarrow \mathrm{Isom}(\Delta^-_{2k-1}, \Delta^+_{2k-1})$. The periodicity map now has the form:

$$f \longmapsto \left(\begin{array}{c|c} I \otimes \gamma_2 & f \otimes I \\ \hline -f^* \otimes I & I \otimes \gamma_2^* \end{array}\right)$$

This construction has been described by Atiyah [1], and is easily seen to be a special case of the construction $f \mapsto \Theta(f)$ described in §2.

One can obtain a totally geodesic representative of the generator of the groups $\pi_{2i+1}U(2^k)$ by composing the generator of $\pi_{2i+1}U(2^i)$ described above with the stability map $s: U(2^i) \longrightarrow U(2^k)$. In this way we obtain the generator of $\pi_{2i+1}U(2^k)$ for $i = 0, 1, \ldots, k$.

<u>Remark:</u> By composing the above generators with other maps one obtains <u>multiples</u> of the generators. For example, using the map $\phi: U(2^i) \longrightarrow U(2^k)$ one obtains totally geodesic representatives of the elements $1, 2, 2^2, \ldots, 2^{k-i}$ in

$\pi_{8i+1} U(2^k)$. By using homomorphisms of the form $X \mapsto \mathrm{diag}(X,\ldots X, 1,\ldots 1)$ one obtains all elements between 1 and 2^{k-i}. The totally geodesic maps appearing in this way are not in any way a complete list; for example any element of $\pi_1 U(2^k)$ has a totally geodesic representative.

A similar procedure works in the case of the real Clifford algebras. One should observe however that the group $\pi_k E_k$ is only stable for $k > 8$, although for $k \leqslant 8$ the groups all have the stable value (ie $\mathbb{Z}$) except for $k = 7$. In this case we have $E_7 = 0(8)$, and $\pi_7 0(8) \cong \mathbb{Z} \oplus \mathbb{Z}$. The sphere $S(e_1, \ldots, e_8)$ is one of the generators, and on applying the maps $\oplus$ and s we obtain the usual stable generators.

Example: The real orthogonal group $0(8.16^k) = E_{8k+7}$.

The stable homotopy groups are $\pi_0 0 = \pi_1 0 = \mathbb{Z}_2$, $\pi_3 0 = \pi_7 0 = \mathbb{Z}$, and totally geodesic (hence harmonic) representatives of some of these can be obtained in the following way.

The generator of $\pi_{8i+7} 0(8.16^k)$ $(i = 1,2,\ldots,k)$ is obtained by stabilising the sphere $S(e_1, \ldots, e_{j+1}) \subseteq E_j = 0(8.16^i)$ $(8i + 7 = j)$. By composing with maps of the form $X \mapsto \mathrm{diag}(X,\ldots X, 1,\ldots 1)$ one obtains totally geodesic representatives of the elements $1,2,\ldots,16^{k-i}$ in $\pi_{8i+7} 0(8.16^k)$.

The elements $j, 2j, \ldots, 16^{k-i} j$ $(j = 1,2)$ of $\pi_{8i+3}\ 0(8.16^k)$ are obtained by composing representatives of the elements $1,2,\ldots,16^{k-i}$ of $\pi_{8i+3} Sp(16\ )$

with either of the two natural homomorphisms $Sp(16^k) \longrightarrow O(8.16^k)$. (The elements $1,2,\ldots,16^{k-i}$ of $\pi_{8i+3}\, Sp(16^k) = \pi_{8i+3} E_{8k+3}$ are constucted as for $\pi_{8i+7} E_{8k+7}$.)

Observe that in both these cases the elements in question all come from generators of $\pi_3\, S^3 (= \pi_3 E_3)$ and $\pi_7\, O(8)$ $(= \pi_7 E_7)$ by application of the maps ⊛ and various natural homomorphisms of groups. To obtain representatives of the nonzero elements of <u>$\pi_{8i+1}\ O(8.16^k)$</u> and <u>$\pi_{8i}\ O(8.16^k)$</u> one uses the same procedure on the generators of $\pi_0\, S^0 (= \text{"} \pi_0 E_0 \text{"})$ and $\pi_1\, S^1 (= \pi_1 E_1)$.

§4. A Lie-algebraic characterisation of the minima

The map β of chapter I admits the following generalisation:

$$\beta : \quad \{\text{Cartan subalgebras of } \mathfrak{o}_{k-j+1}\} \longrightarrow \left\{\begin{matrix}\text{submanifolds of minimal } j\text{-spheres} \\ \text{in } E_k \text{ with equator } S(e_{k+1},\ldots,e_{k-j+2})\end{matrix}\right\}$$

Here, $\mathfrak{o}_{k-j+1}$ is embedded in $\mathfrak{o}_{k+1}$ in the usual way, and we select a fixed Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{o}_{k-j+1}$ which contains the element $e_{k-j+2}\, e_{k-j+1} \in \mathfrak{o}_{k-j+1}$. Given an arbitrary Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{o}_{k-j+1}$, $\beta(\mathfrak{h})$ is obtained as follows. There exists (a collection of) elements $d \in D_{k-j}$ with $d\,\mathfrak{h}_0\, d^{-1} = \mathfrak{h}$. Then $\beta(\mathfrak{h})$ consists of the spheres $d\ \gamma_j\ d^{-1}$ where d ranges over all possibilities satisfying $d\,\mathfrak{h}_0\, d^{-1} = \mathfrak{h}$.

The following analogue of proposition 4.4 of chapter I holds:

<u>Proposition 4.1</u> : 1) Given any minimal j-sphere in E_k with equator γ_{j-1}, there exists some Cartan subalgebra of $\mathfrak{o}_{k-j+1}$ whose image under β contains the sphere.

2) The set of Cartan subalgebras of $\mathfrak{o}_{k-j+1}$ whose image under β contains the standard sphere γ_j is in 1:1 correspondence with the set of all Cartan subalgebras of $\mathfrak{o}_{k-j}$.

<u>Proof:</u> 1) Given the sphere $S(e_{k+1}, \ldots, e_{k-j+2}, e)$, one can just take any Cartan subalgebra of $\mathfrak{o}_{k-j+1}$ containing $e.e_{k-j+1}$ (note that $e \in e_{k-j+2}\, \mathfrak{o}_{k-j+1}$).

2) If $\beta(\mathfrak{h}) \ni \gamma_j$, then $\mathfrak{h} = d\mathfrak{h}_0 d^{-1}$, where $d e_{k-j+1} d^{-1} = e_{k-j+1}$.

As $d \in \phi(D_{k-j})$, the condition $d e_{k-j+1} d^{-1} = e_{k-j+1}$ means that in fact $d \in \phi(D_{k-j-1})$.

This should be compared with the following theorem of Atiyah, Bott, and Shapiro (see [2], theorem 11.5), in which $M(C_k)$ denotes the algebra of finite-dimensional C_k -modules:

<u>Theorem 4.2:</u> The map $a : M(C_k) \longrightarrow KO(S^k)$, defined by taking the induced bundle from the principal bundle $Spin(k) \longrightarrow Spin(k+1) \longrightarrow S^k$, has the properties:

1) a is a surjective homomorphism
2) $\ker a \cong M(C_{k+1})$.

Of course, there are results corresponding to 4.1 and 4.2 in the complex case. Let us note the following special case of 4.2, where ROG denotes the real representation ring of the group G:

<u>Corollary 4.3:</u> There is an isomorphism

$$RO\,Spin(j+1) / RO\,Spin(j+2) \cong \pi_j\, E_k\,,\ k \equiv 7(8)$$

In view of 4.1 and 4.3, one expects a relation between

a) Cartan subalgebras of $\mathcal{E}_{k-j+1}$

and b) C_{j+1} modules.

For example, when $j = k = 8\ell + 7$ we have $C_{j+1} = M(16^{\ell+1}, R)$ and there is a basic Clifford module Δ_{j+1} of dimension $16^{\ell+1}$. This corresponds to the unique Cartan subalgebra of $\mathcal{E}_1 = R$ (and they both correspond to the totally

geodesic sphere $S^k \subseteq E_k \subseteq C_{k+1}$).

We describe briefly such a relation, which shows that the maps α and β are essentially the same. For simplicity we treat the case of complex Clifford algebras, since we have already written down an explicit description of the modules in §3 of chapter I. We assume $k \equiv j \equiv 1(2)$. We have $\Delta_{j+1} \otimes \Delta_{k-j} \cong \Delta_{k+1}$ where Δ_i is the irreducible (ungraded) C_i^c module (for i even). The module Δ_{k-j} is embedded in C_{k-j}^c (see §3 of Chapter I) as

$$\bigcap_{r=1}^{(k-j)/2} \{(-1)\text{-eigenspace of } R_{ie_{2r-1}e_{2r}}\}.$$

Consider now the vector subspace V of C_{k-j+2}^c spanned by elements of the form $e_{k-j+1}x$, where x is a product (possibly the empty product) of elements of the form $e_{2r-1}e_{2r}$ $(r = 1,2,\ldots,(k-j)/2)$. Then $V \subseteq e_{k-j+2}\mathcal{L}_{k-j+1}^c$ as $V \subseteq (C_{k-j+2}^c)^{odd}$ and every $v \in V$ anticommutes with e_{k-j+2}. But V is clearly commutative, so it is contained in a Cartan subalgebra of $e_{k-j+2}\mathcal{L}_{k-j+1}^c$. For dimensional reasons, it must be equal to this Cartan subalgebra.

This establishes a direct construction of a Cartan subalgebra of $\mathcal{L}_{k-j+1}^c$ from the Clifford module Δ_{j+1}. It is clear that the image under β of this Cartan subalgebra contains the standard j-sphere $S(e_{k+1},\ldots,e_{k-j+1})$, since the subalgebra contains the element e_{k-j+1}.

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