2/4/08 CRM / Joyal / Quasi-categories J ①

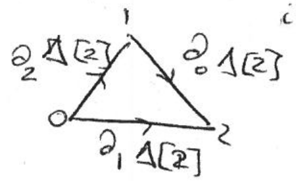
- History
  - Bordman - Vogt (1973)
  - Hieschovitz and Simpson (199~) - Segal categories
  - Toen and Vezzosi HAG (higher topoi)
  - Rezk (higher topoi; 1st)
  - Lurie (higher topoi)

The notion of q.c

$\mathcal{S} = [\Delta^o, \text{Set}]$ s. sets.

$\Delta[n]$, $\partial\Delta[n] = \bigcup_{i \leq n} \partial_i \Delta[n]$

$\Lambda^k[n] = \bigcup_{k \neq i} \partial_i \Delta[n]$



A Kan cx is a simplicial set X which satisfies the Kan ~~completion~~ condition:

$$\begin{tikzcd} \forall \quad \Lambda^i[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

(you can fill every horn)

A quasi-category is a s.s. X which satisfies the Boardman condition:

$$\begin{tikzcd} \forall \quad \Lambda^i[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

$0 < i < n$

(you can fill every inner horn)

Other names: . Boardman complex; weak Kan complex; inner Kan cx; infinity categories.

$QCat \subset \mathcal{S}$ The full subcategory of q-cat

- QCat is closed under products.
- $X \in QCat$ and $A \in \mathcal{S} \Rightarrow X^A \in QCat$ (Cordier and Porter ?)

2) J

The Nerve Functor

$N: Cat \longrightarrow S$ Cat is the category of small categories.

$\Delta \subset Cat$ $[n]$ is a Category (it is a poset).

$(N\mathcal{C})_n = Cat([n], \mathcal{C}) = \{ c_0 \xrightarrow{f_1} c_1 \rightarrow \cdots \xrightarrow{f_n} c_n \mid f_i \in \mathcal{C} \}$

N is fully faithful. We denote $N\mathcal{C}$ by $\mathcal{C}$ (same information). $N: Cat \subset S$

- The functor N has a left adjoint $\tau_1: S \longrightarrow Cat$

  $\tau_1 X$ is the <u>fundamental category</u> of a simplicial set X.

$\Pi_1 X = G(\tau_1 X)$ $G\mathcal{C} = \mathcal{C}^{-1}\mathcal{C}$ (groupoid generated by $\mathcal{C}$).

$\tau_1 X$ only depends on the 2-skeleton $Sk^2 X$ $X_0 \leftleftarrows X_1 \overset{\leftarrow}{\leftleftarrows} X_2$

- The functor τ_1 preserves finite products $\tau_1(X \times Y) \xrightarrow{\sim} \tau_1(X) \times \tau_1(Y)$

$$\begin{tikzcd}
Cat \arrow[r, phantom, "\subset"] & QCat \arrow[r, phantom, "\subset"] \arrow[l, bend right, "ho"'] & S \arrow[ll, bend left, "\tau_1"]
\end{tikzcd}$$

τ_1 induces ho. $ho\,X$ is the homotopy category of a quasi-category X.

$I = \Delta[1]$

$a, b \in X_0$, define $X(a,b)$ the simplicial set of arrows $a \rightarrow b$, as

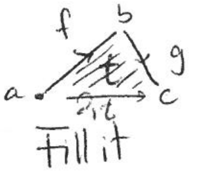
$$\begin{tikzcd}
X(a,b) \arrow[r] \arrow[d] & X^I \arrow[d, "{(s,t)}"] \\
1 \arrow[r, "{(a,b)}"'] & X \times X
\end{tikzcd}$$

$\{0,1\} \in I$

$X(a,b)$ is Kan if $X \in QCat$

$(ho\,X)(a,b) = \pi_0 X(a,b)$

$$\begin{tikzcd}
& b \arrow[dr, "{[g]}"] & \\
a \arrow[ur, "{[f]}"] \arrow[rr, "{[\partial_1 t]}"'] & & c
\end{tikzcd}$$



Fill it

$$\begin{tikzcd}
\Lambda^1[2] \arrow[r, "h"] \arrow[d, hook] & X \\
\Delta[2] \arrow[ur, dashed, "t"'] &
\end{tikzcd}$$

Define $[g] \cdot [f] = [\partial, t]$

<u>Thm</u> A q.c. X is a Kan complex if and only if ho X is a groupoid.
(Joyal)

$$\begin{tikzcd} \mathrm{Gpd} \arrow[r, phantom, "\subset"] \arrow[d, hook] & \mathrm{Cat} \arrow[d, hook] \\ \mathrm{Kan\ cx} \arrow[r, hook] \arrow[u, bend left=60, "\pi_1"] & \mathrm{QCat} \arrow[u, bend right=60, "\mathrm{ho}"'] \end{tikzcd}$$

$\mathrm{Cat} \cap \mathrm{Kan} = \mathrm{Gpd}$

$$\begin{tikzcd} \mathrm{Gpd} \arrow[r, hook] & \mathrm{Cat} \arrow[l, bend right=60, "G"'] \arrow[l, bend left=60, "J"] \end{tikzcd}$$

Left adjoint $G(\mathcal{C}) = \mathcal{C}^{-1}\mathcal{C}$

right adjoint $J(\mathcal{C})$ = the gd of isomorphisms in $\mathcal{C}$ (largest gpd contained as subcat).

$$\begin{tikzcd} \mathrm{Kan} \arrow[r, hook] & \mathrm{QCat} \arrow[l, bend left=60, "J"] \end{tikzcd}$$

$J(X)$ = largest sub-Kan cx of a quasi-cat.

$$\begin{tikzcd} JX \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & J(\mathrm{ho}X) \arrow[d, hook] \\ X \arrow[r] & \mathrm{ho}\,X \end{tikzcd}$$

This is the construction of $J(X)$.

(by adjointness)

The 2-cat S

$\tau_1(A,B) := \tau_1(B^A)$ This defines a category of arrows from A to B.

Composition $C^B \times B^A \xrightarrow{c} C^A$

$\tau_1(C^B) \times \tau(B^A) \longrightarrow \tau_1(C^A)$ (τ_1 preserves products).

Can define the notions of equivalence and of adjoints. (as in any 2-cat).

$A \underset{g}{\overset{f}{\rightleftarrows}} B$ $gf \simeq 1_A$, $fg \simeq 1_B$ <u>categorical equivalence</u> of s. sets.

An <u>equivalence</u> of quasi-categories (if A and B are quasi-categories).

A map $f: A \to B$ is an equivalence in QCat iff it is 1) fully faithful and 2) essentially surjective.

D^J 1) $A(a,b) \longrightarrow B(fa, fb)$ is a homotopy equivalence
(def of fully faithful)

2) The map $\tau_0(f) : \tau_0(A) \longrightarrow \tau_0(B)$
$\tau_0(A)$ = set of isomorphism class of objects in $\tau_1(A)$.

Def A map $u : A \longrightarrow B$ is a <u>weak categorical equivalence</u> if the map

$$\tau_0(u, X) : \tau_0(B, X) \longrightarrow \tau_0(A, X) \quad \text{is bijective } \forall X \in QCat.$$

$\tau_0(A,B) = \tau_0(B^A)$

Thm The category $\mathcal{S}$ admits a Quillen model structure in which the w.e are the ~~weak~~ weak cat. equiv and the cof are the mono.

The fibrations are called pseudo-fibrations.

2/5/08 // CRM // Joyal

Observation A model structure on a category is determined by its <u>cofibrations</u> and its <u>fibrant objects.</u>

Sketch of pf $\quad \mathcal{E} \longrightarrow Ho(\mathcal{E}) \qquad (\mathcal{C}, \mathcal{W}, \mathcal{F}) \qquad \mathcal{E}_f$ = subcategory of fibrant objects.

$$Ho(\mathcal{E})(A,B) = h(A,B) = \mathcal{E}(A,B) / \text{left htpy relation}$$ for A cofibrant, B fibrant

Cylinder
$$\begin{tikzcd} A \amalg A \arrow[rr, "\text{codiagonal}"] \arrow[dr, "\text{cofibration}"'] & & A \\ & IA \arrow[ur, "\text{w.e}"'] & \end{tikzcd}$$
$\in \mathcal{C}$ $\quad$ $\in \mathcal{C}^{\pitchfork}$ (could take an acyclic fibration)

We know $(\mathcal{C}, \mathcal{C}^{\pitchfork}) \quad \mathcal{C}^{\pitchfork} = \mathcal{W} \cap \mathcal{F}$

$$\begin{tikzcd} \cdot \arrow[r, "\forall"] \arrow[d, "u"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "\forall"'] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

$u \pitchfork f$ means u has left lifting property w.r.t f

$\mathcal{M}^{\pitchfork} = \{ f \mid \forall u \in \mathcal{M}, u \pitchfork f \}$; ${}^{\pitchfork}\mathcal{M} = \{ u \mid \forall f \in \mathcal{M}, u \pitchfork f \}$

With cylinder can construct left htpy relation. With $Ho(\mathcal{E})(A,B)$ can construct $\mathcal{W}$: A map $u : A \to A'$ between cofibrant objects is in $\mathcal{W}$ iff $\forall X$ fibrant the map $h(u,X) : h(A',X) \to h(A,X)$ is bijective.

J ⑤

Now take A, A' any objects.

$$\begin{tikzcd} QA \arrow[r, "a.f."] \arrow[d, "Qu"'] & A \arrow[d, "u"] \\ QA' \arrow[r, "a.f \in \mathcal{C}^{\pitchfork}"'] & A' \\ 0 \arrow[u, "cof."'] & \end{tikzcd}$$

Cofibrant resolution

2 of 3 $\Rightarrow$ $Qu \in \mathcal{W}$ iff $u \in \mathcal{W}$

Cisinski model structure on a topos $\mathcal{E}$

$(\mathcal{C}, \mathcal{W}, \mathcal{F})$ cofibrantly generated ; $\mathcal{C}$ = monomorphisms

- The classical model structure on $\mathcal{S} = [\Delta^{op}, Set]$ is determined by the subcategory $Kan \subset \mathcal{S}$ (fibrant objects).

  We may denote it $(\mathcal{S}, Kan)$.

  The fibration are the Kan fibrations.

  The weak equivalences are the weak homotopy equivalences

- ~~For qua~~ Another model structure given by $(\mathcal{S}, QCat)$

  The weak equivalences are the weak categorical equivalences

  The fibrations are the pseudo-fibrations.

- The natural model structure on Cat

  The weak equivalences are the equivalences of categories.

  The cofibrations are the functors monic on objects $(u: A \to B ;\ \underset{monic}{Ob(u): Ob(A) \to Ob(B)})$

  The fibrations are the pseudo-fibrations:

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$$

$\forall a \in Ob(A)$ $\forall$ iso $\beta: b \xrightarrow{\sim} f(a)$

$\exists$ $\alpha: c \xrightarrow{\sim} a$ s.t $f(\alpha) = \beta$

(every isomorphism lifts)

Other names: iso-fibrations, quasi-fibrations (not ~~recommended~~ recomended)

ⓘ J The notions of left, right and mid fibrations

$$\begin{tikzcd}
\Lambda^i[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$
Kan fibration

With $0 < i \leq n$ we call it right fibration.
$0 \leq i < n$ left fibration
$0 < i < n$ mid fibration

Prop A map $f : X \to Y$ between quasi-categories is a pseudo-fibration iff it is a mid fibration and $ho(f) : ho(X) \to ho(Y)$ is a pseudo-fibration.

Cisinski theory $\mathcal{E}$ a topos $\mathcal{C} = \text{mono}$, triv fib $= \mathcal{C}^{\pitchfork}$

Def A class of maps $W \subseteq \mathcal{E}$ is called a localizer if

- W has the "3 for 2" property
- $W \cap \mathcal{C}$ is saturated and accessible (ie $(W \cap \mathcal{C}, \mathcal{F})$ a weak factorization system w/ $\mathcal{F} = (W \cap \mathcal{C})^{\pitchfork}$)
- $W \supseteq$ trivial fibrations.

Thm (Cisinski) Any set of maps $\Sigma \subseteq \mathcal{E}$ is contained in a smallest localizer $W(\Sigma)$.

Given a localizer, $(\mathcal{C}, W, \mathcal{F})$ gives a Cisinski model structure. The converse is also true. We can order the model structures by inclusion of localizers.

There is a minimal localizer, taking $\Sigma = \emptyset$.

This poset gives a lattice.

The minimal structure $\mathcal{M}_0$ on a topos $\mathcal{E}$

I injective interval $\{0,1\} \overset{\iota}{\hookrightarrow} I \xrightarrow[\text{fibration}]{\text{trivial}} 1$

$\{0,1\} = 1+1$

$X \in \mathcal{E}$ is fibrant for $\mathcal{M}_0$ iff the projections $X^I \rightrightarrows X$ are trivial fibrations.

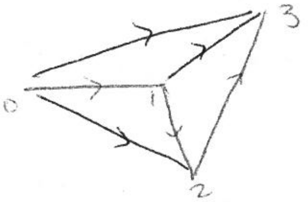
A mono $A \rightarrowtail B$ is a weak equivalence for $\mathcal{M}_0$ iff the map $X^B \to X^A$ is a trivial fibration $\forall X$ fibrant.

Another description of the quasi-cat model structure on $\mathcal{S}$.

The n-chain $I[n] \overset{\subset \Delta[n]}{}$ is the union of the segments $(i, i+1)$ for $0 \le i < n$

$\Delta[1] \xrightarrow{(i,i+1)} \Delta[n]$ is an edge.

$I[3]$
$\subset \Delta[3]$



$I[2]$
$\subset \Delta[2]$

We will call it the <u>spine</u> of $\Delta[n]$

<u>Thm</u> The localizer W_{cat} (weak categorical equivalences) is generated by the spine inclusions: $I[n] \hookrightarrow \Delta[n]$. $(n \ge 0)$.

- Cisinski
- Mark Web~~ber~~

2/6/08 // CRM // Joyal (Chapter 12)

<u>Higher Quasi-categories</u>

Cat $\quad$ n-Cat $\quad$ $n=2$

A <u>2-category</u> is a category enriched over Cat. Let A be a 2-cat.

- Set of objects $\mathrm{Ob}\,A$
- $\forall a, b \in A$ $\quad$ $A(a,b)$ a category of arrows (1-cells)

The arrows between arrows are called 2-cells.

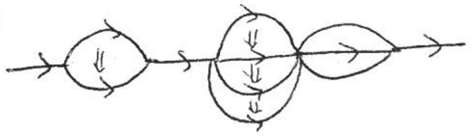
$$\begin{tikzcd} a \arrow[r, bend left=50, "f", ""{name=U, below}] \arrow[r, bend right=50, "g"', ""{name=D}] & b \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} \qquad \alpha : f \Rightarrow g : a \to b$$

$\Delta \subset \mathrm{Cat}$. We would like to have the same thing in n-Cat.

$\Theta_n \subset n$-Cat $\quad$ Θ_n is a category, the objects are n-categories.

③ $n=2$

$\mathrm{Ob}\,\Theta_2 = \Big\{$ diagrams of the type

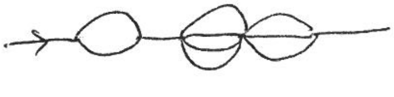


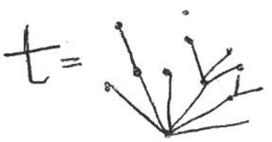
$\Big\}$ (may ~~intro~~ introduce bubbles, 2-cells) globular graphs (introduced by Batanin)

$$\mathrm{Ob}\,\Delta = \left\{ \overset{}{\underset{0}{\bullet}} \to \underset{1}{\bullet} \to \cdots \to \underset{n}{\bullet} \right\}$$

~~These~~ Each diagram generates freely a 2-category. Convenient to represent the diagram by a tree.

Order preserving map $\underline{\Theta} \to \underline{6}$





$t=$ planar tree of height 2.

~~[4]2 → [5]~~

~~{0,1,2,3,4}~~

$$\left\{\begin{array}{c} n_1 \\ \downarrow \\ n_2 \\ \downarrow \\ \vdots \\ \downarrow \\ n_k \end{array}\right\} = \mathrm{Ob}\,\Theta_k$$

$\Theta_2 \subset 2\text{-Cat}$ full subcategory of 2-cat, since each diagram generates a 2-category.

$[t] \in \mathrm{Ob}\,\Theta_2$ $t =$ tree of height 2.

$\Delta \xrightarrow{\quad} \mathrm{Top}$

$[2] \longmapsto$

$R: \Theta_2 \longrightarrow \mathrm{Top}$

$\mathcal{D}' = \Delta^{op} =$ category of finite strict intervals.

$\Theta_2^{op} =$ category of 2-disks

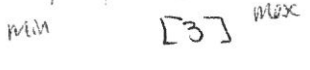
Interval : a linearly ordered set I with first and last elts denoted $\underset{\perp}{0} \in I$ and $\underset{\top}{1} \in I$.

Interval is <u>strict</u> if $0 \neq 1$.

A morphism of intervals $I \xrightarrow{f} J$ is an ordered preserving map, with $f(0)=0$, $f(1)=1$.

$[n] \in \Delta$, $[n]^* = \mathrm{Hom}([n],[1])$ $f \leq g$ linearly ordered, has top and bottom elt.

$[0]^* = [1]$ strict interval.



dual: set of cuts $[3]^*$

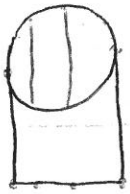
Can pull back the cut

$$[m] \longrightarrow [n] \longrightarrow [1]$$

$\uparrow$ cut

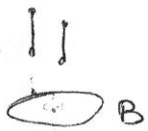
get $[n]^* \longrightarrow [m]^*$

$\Theta_2^{op} = \mathcal{D}^2$ = Category of finite 2-disks. (Batanin - Street - Makkai - Zawadowski - J
Berger) ⑨



projects onto an interval and each fiber is an interval except at the endpoints where it degenerates to a point :

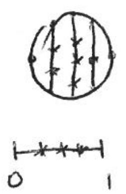
A bundle of intervals is an interval object in Set / B



$$\begin{tikzcd} E \arrow[d, "p"] \\ B \end{tikzcd}$$

A 2-disk is a bundle of intervals $E_1 \xleftarrow[p]{} E_2$ with E_1 a strict interval

s.t $p^{-1}(a)$ is degenerate iff $a \in \{0,1\} \subset E_1$



A map of 2-disks :

$$\begin{tikzcd} E_1 \arrow[d, "f_1"'] & E_2 \arrow[l, "p"'] \arrow[d, "f_2"] \arrow[dl, phantom, "\circlearrowleft"] \\ F_1 & F_2 \arrow[l, "q"] \end{tikzcd}$$

f_1 is a morphism of intervals

f_2: fiberwise a morphism of intervals.

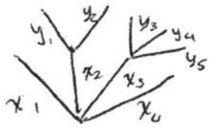
We will use this to define $R : (\mathcal{D}^2)^{op} \longrightarrow Top$

Eucledian disk

$$\left.\begin{array}{l} B^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\} \\ \downarrow \\ B^1 = \{-1 \leq x \leq 1\} \end{array}\right\} \mathbb{B}^2$$

$E \in \mathcal{D}^2$

$R(E) = Hom(E, \mathbb{B}^2) \subset \mathbb{R}^N$ compact convex , contravariant by construction..



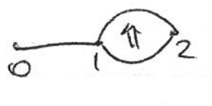
$[t] \in \Theta_2$ $\quad R([t])$

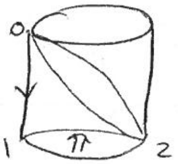
$-1 \leq y_1 \leq y_2 \leq 1 \quad -1 \leq y_3 \leq y_4 \leq y_5 \leq 1$

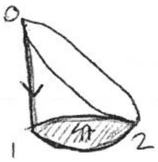
$-1 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1$

$x_2^2 + y_1^2 \leq 1 \quad x_2^2 + y_2^2 \leq 1 \quad x_3^2+y_3^2 \leq 1 \quad x_3^2+y_4^2 \leq 1 \quad x_3^2+y_5^2 \leq 1$

Ex







Θ - cell

$$\begin{tikzcd} \Theta_2 \arrow[r] \arrow[d, "yoneda"'] & Top \\ \hat{\Theta}_2 \arrow[ur, "\bar{R}"'] & \end{tikzcd}$$

R:

presheaves on Θ_2

The spine is the Batanian diagram.

J $S[t] \subset \Theta_n[t]$ Spine inclusion.

Cisinski : weak equivalences containing spine inclusions.

Fibrant objects = n-Q Cat need a combinatorial description

2/7/08 // CRM // Joyal

Equivalence with simplicial categories

$\mathcal{S}Cat$ = category of categories enriched over $\mathcal{S}$ (ie simplicial categories)

$Ob\,\mathcal{C}$, $\mathcal{C}(x,y) \in \mathcal{S}$ $\forall x, y \in Ob\,\mathcal{C}$

simplicial functor $f : \mathcal{C} \to \mathcal{D}$ $\mathcal{C}(x_1, x_2) \longrightarrow \mathcal{D}(fx_1, fx_2)$ maps in $\mathcal{S}$.

$ho : \mathcal{S}Cat \longrightarrow Cat$; $ho\,\mathcal{C}$ is the homotopy category of $\mathcal{C}$

$(ho\,\mathcal{C})(x,y) := \pi_0\,\mathcal{C}(x,y)$

A map $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a Dwyer-Kan equivalence if:

1) it is homotopy fully faithful : $\mathcal{C}(x,y) \to \mathcal{D}(fx, fy)$ is w.h.e $\forall x, y \in \mathcal{C}$.

2) htpy essentially surjective : $ho(f) : ho(\mathcal{C}) \to ho(\mathcal{D})$ is essentially surj.

A map $f : \mathcal{C} \to \mathcal{D}$ is a Dwyer-Kan fibration if:

1) the map $\mathcal{C}(x,y) \to \mathcal{D}(fx\, fy)$ is a Kan fibration $\forall x, y \in Ob\,\mathcal{C}$.

2) The functor $ho(f) : ho(\mathcal{C}) \to ho(\mathcal{D})$ is a pseudo fibration.

Thm (J. Bergner) The category $\mathcal{S}Cat$ admits a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ with $\mathcal{W} = \{DK\text{-}equiv\}$, $\mathcal{F} = \{DK\text{-}fib\}$

The coherent nerve functor

$$\begin{tikzcd} \mathcal{S}Cat \arrow[r, "\tilde{N}"] & \mathcal{S} \\ Cat \arrow[u, hook] \arrow[ur, "N"'] & \end{tikzcd}$$

$\tilde{N} : \mathcal{S}Cat \to \mathcal{S}$ (extends the usual nerve functor)

$C_* : \Delta \longrightarrow \mathcal{S}Cat$ $\tilde{N}(\mathcal{C})_n = \mathcal{S}Cat(C_*[n], \mathcal{C})$

Defined by Cordier.

Will define $C_* : Cat \longrightarrow SCat$ (recall $\Delta \subset Cat$)

$C_*(A)$ obtained from a cotriple.

$$\begin{tikzcd}(\text{reflexive})\ Graph \arrow[r, shift right, "F"'] & Cat \arrow[l, shift right, "u"'] \arrow[loop right, "C"]\end{tikzcd}$$ $C = Fu$ a Comonad

$$\begin{tikzcd}X_0 \arrow[r, "i" description] \arrow[r, bend right, "u"'] & X_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2]\end{tikzcd}$$

Get a simplicial resolution: $\underbrace{A \xleftarrow{\varepsilon} C(A) \leftleftarrows C^2(A) \overset{\leftarrow}{\leftleftarrows} \cdots}_{C_*(A)}$

$(C_*(A)(x,y))_n = C^{n+1}(A)(x,y)$

$C_*([n])$ has a simple description

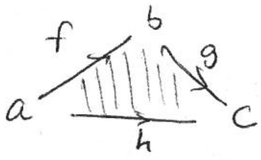
$Ob\, C_*[n] = \{0, 1, \ldots, n\}$ $\quad C_*[n](i,j) = \begin{cases} \emptyset & \text{if } i > j \\ \text{set of paths } i \to j \text{ in } [n] & \text{if } i \le j \end{cases}$

a path is given by $\{i,j\} \subseteq S \subseteq [i,j]$

It is a poset (by inclusion), so we take the nerve. (We get a simplicial cube).

$\tilde{N}(C)_0 = Ob\, C$

$\tilde{N}(C)_1 = Arr(C) = \coprod_{x,y \in Ob C} C(x,y)_0$



$\tilde{N}(C)_2$

$g \circ f \longrightarrow h$ homotopy in $C(a,c)$

Thm The functor $\tilde{N}$ has a left adjoint $C_! : S \longrightarrow SCat$ and the pair $(C_!, \tilde{N})$ is a Quillen equivalence $(S, QCat) \longleftrightarrow (SCat, Bergner)$ Lurie, Joyal

Cor (Cordier, Porter) The coherent nerve of a category enriched over Kan complexes is a Quasi-category.

Ex Kan = category of Kan complexes. $\tilde{N}(Kan) = \mathcal{U}$ - big quasi-cat of all homotopy types.
Archetype of an $\hat{u}$topos (higher topos or homotopos)
(uppertopos)

QCat is enriched over Kan cxs.
$Hom(X,Y) = J(Y^X)$ $\quad$ ($J(Z)$ is the Kan complex of isomorphisms of Z)

② $\exists$ $\tilde{N}(QCat) = \mathbb{U}_1$

$\mathbb{U}_0 \in \mathbb{U}_1 \subset \underline{\mathbb{U}_2 \subset \cdots}$ we will talk about this construction later.

$A \in Cat$, presheaves : $\hat{A} = [A^{op}, Set]$

We need an analogue for quasi-categories: $\mathbb{U}^{A^{op}} = \cancel{[A^{op}, \mathbb{U}]}$

You can localize a quasi-category X w.r.t. a set Σ of arrows in X.

J = groupoid generated by one isomorphism $0 \longrightarrow 1$ $\quad I = \Delta[1]$

$f \in X \Leftrightarrow f : I \longrightarrow X$
arrow

$$\begin{tikzcd}
I \times \Sigma \arrow[r] \arrow[d, hook] & X \arrow[d, "\underline{h}"'] \\
J \times \Sigma \arrow[r] & L(X, \Sigma)
\end{tikzcd}$$

homotopy pushout (take a pushout, then take a fibrant replacement to get a q-cat).

⊛ $\mathcal{E}$ a model category : $L(\mathcal{E}) := L(\mathcal{E}, W) \simeq \tilde{N}(\text{DK localization of } \mathcal{E})$

Joyal dreams of a simple construction for the localization of a model category.

$L(\mathcal{E})$ is complete and cocomplete if $\mathcal{E}$ is combinatorial (as a q-cat)

(Simpson)

2/8/08 // CRM // Joyal

Homotopy Factorization Systems

1) (Strict) factorization sys. on a category $\mathcal{E}$:

A pair $(\mathcal{A}, \mathcal{B})$ of subcategories s.t

a) $\mathcal{A}$ and $\mathcal{B}$ contain the isomorphism

b) $\forall$ $A \xrightarrow{f} B$ admits a factorization

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "u"'] & & B \\
& E \arrow[ur, "p"'] &
\end{tikzcd}$$

with $u \in \mathcal{A}$, $p \in \mathcal{B}$

and this factorization is unique up to unique isomorphism, ie

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "u"'] & & B \\
& E \arrow[ur, "p"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "u'"'] & & B \\
& E' \arrow[ur, "p'"'] &
\end{tikzcd}$$

$u, u' \in A$, $p, p' \in B$

$\Rightarrow \exists!$ iso $i : E \longrightarrow E'$ s.t

$$\begin{tikzcd}
& E \arrow[dr, "p"] \arrow[d, "\wr", "i"'] & \\
A \arrow[ur, "u"] \arrow[dr, "u'"'] & E' \arrow[r, "p'"'] & B
\end{tikzcd}$$

commutes.

Ex $\mathcal{E} = \mathrm{Set}$, $\mathcal{A} = \{\text{surjection}\}$, $\mathcal{B} = \{\text{mono}\}$

The relation $u \pitchfork f$ between two maps $u: A \to B$; $f: X \to Y$ means that every commutative square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r, "b"'] \arrow[ur, dashed, "d"] & Y \end{tikzcd}$$

has a diagonal filler d.
f has the RLP wrt u
u has the LLP wrt f

If the diagonal filler is unique, we write $u \perp f$ (orthogonal).

If $(\mathcal{A}, \mathcal{B})$ is a strict fact. sys. then $\mathcal{A} \perp \mathcal{B}$

$\mathcal{M}$ = class of maps.
$\mathcal{M}^{\pitchfork} = \{f \mid \mathcal{M} \pitchfork f\}$; ${}^{\pitchfork}\mathcal{M} = \{f \mid f \pitchfork \mathcal{M}\}$, $\mathcal{M}^{\perp} = \{f \mid \mathcal{M} \perp f\}$
${}^{\perp}\mathcal{M} = \{f \mid f \perp \mathcal{M}\}$

∘ If $(\mathcal{A}, \mathcal{B})$ is a strict fact. sys. then
a) $\mathcal{B} = \mathcal{A}^{\perp} = \mathcal{A}^{\pitchfork}$ and $\mathcal{A} = {}^{\perp}\mathcal{B} = {}^{\pitchfork}\mathcal{B}$
b) $\mathcal{A}$ has the right cancellation property and $\mathcal{B}$ has the left cancellation property.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[rr, bend right, "vu"'] & B \arrow[r, "v"] & C \end{tikzcd}$$

$vu \in \mathcal{A}$ and $u \in \mathcal{A} \Rightarrow v \in \mathcal{A}$ (RCP)
$vu \in \mathcal{B}$ and $v \in \mathcal{B} \Rightarrow u \in \mathcal{B}$ (LCP)

---

2) A weak factorization sys on a category $\mathcal{E}$ is a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps s.t
a) $\mathcal{B} = \mathcal{A}^{\pitchfork}$ and $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$
b) every map $f: A \to B$ admits a factorization $f = pu$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$.

Ex If $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure then $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems

Ex In Cat : $\mathcal{A}$ = class of final ~~maps~~ functors ; $\mathcal{B}$ = class of discrete fibrations. is a fact. sys.

⑭ J A functor $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ is called a *discrete* *fibration* if for all $e \in \mathrm{Ob}E$ and $f : b \to p(e)$ in B there exists a unique

arrow $g : a \to e$ s.t $p(g) = f$.

ie, if $I = \{0 \to 1\}$ $i_1 : 1 \to I$ as target, a ~~map~~ functor p is a discrete fibration iff

$i_1 \perp p$

A presheaf $X : B^{op} \longrightarrow \mathrm{Set}$ on B.

$el(X)$ $(= B/X)$ $\mathrm{Ob}(el(X)) = \coprod_{b \in \mathrm{Ob}B} X(b)$, ie pair (b, x) $b \in \mathrm{Ob}B$, $x \in X(b)$.

$(a, x) \xrightarrow{f} (b, y)$ is $a \xrightarrow{f} b$ s.t $X(f)(y) = x$

$$\begin{tikzcd} el(X) \arrow[d, "P_1"'] \\ B \end{tikzcd}$$

projection functor is a discrete functor.

$[B^{op}, \mathrm{Set}] \simeq \mathrm{Discrete\,Fib}/_B \subset \mathrm{Cat}/B$

presheaf cat

A functor $u : A \to B$ is *final* if the category $b \backslash A$ is connected for every object $b \in B$.

$$\begin{tikzcd} b \backslash A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "u"] \\ b \backslash B \arrow[r] & B \end{tikzcd}$$

$\mathrm{Ob}(b \backslash B) = \{b \to a\}$

objects under b

$A \xrightarrow{u} B$ induces $\hat{B} \xrightarrow{u^*} \hat{A}$,

$$\begin{tikzcd} \hat{A} \arrow[r, bend right, "u_!"'] & \hat{B} \arrow[l, "u^*"'] \end{tikzcd}$$

$u_!$ left adjoint

($u_!$ is the left Kan extension functor).

u is final iff $u_!(1) = 1$.

---

3) Homotopy fact. sys. in a model cat $\mathcal{E}$ $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

Take Cat with its natural model structure

$\mathcal{A}$ = class of essentially surjective functors.

$\mathcal{B}$ = class of fully faithful functors.

This is not even a weak fact. sys. We can indeed factor every functor, but there is no lifting property.

Two remarks ∘ The class $\mathcal{A}$ is homotopy replete, also $\mathcal{B}$.

$\mathcal{M}$ is homotopy replete if $\forall$

$$\begin{tikzcd} A \arrow[r, "W"'] \arrow[d, "u"'] & A' \arrow[d, "u'"] \\ B \arrow[r, "W"'] & B' \end{tikzcd}$$

$u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$

∘ $(\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F})$ is a weak factorization system.

Notation $\mathcal{E}_c$ full subcategory spanned by the cofibrant objects of $\mathcal{E}$

$\mathcal{E}_f$ " " " " fibrant " "

$$\mathcal{E}_{cf} = \mathcal{E}_c \cap \mathcal{E}_f$$

Def A pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a model category $\mathcal{E} = (\mathcal{E}, \mathcal{C}, \mathcal{W}, \mathcal{F})$ is a homotopy factorization system if the following conditions are satisfied:

1) $\mathcal{A}$ and $\mathcal{B}$ are homotopy replete.

2) $(\mathcal{A} \cap \mathcal{C} \cap \mathcal{E}_{cf}, \mathcal{B} \cap \mathcal{F} \cap \mathcal{E}_{cf})$ is a weak factorization system. in $\mathcal{E}_{cf}$.

3) $\mathcal{A}$ has the right cancellation property

4) $\mathcal{B}$ has the left cancelation property.

(You can prove $3 \Leftrightarrow 4$ if 1 and 2 are true)

A htpy fact sys $(\mathcal{A}, \mathcal{B})$ is uniform if the pair $(\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F}) = (\mathcal{A}', \mathcal{B}')$ is a weak factorization system.

$(\mathcal{A}, \mathcal{B})$ is called a Bousfield factorization system if $(\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F})$ is a weak fact. syst.

J ~~Remark~~ Ex In the model cat ($\mathcal{S}$, Qcat) there exists a Bousfield class (A', B') where B' = class of right fibrations and A' the class of right anodyne maps

---

2/11/08 // CRM // Joyal

- Connection with homotopical categories

"Homotopy limit functors and homotopical categories" (Dwyer-Hirschhorn-Kan-Smith)

Homotopical category $(\mathcal{C}, W)$ C: category, W: a subcategory of C

satisfying certain axioms.

For example $C \longrightarrow W^{-1}C$ ...

Looking as it is in $\mathcal{S}$. $J = \{0 \simeq 1\}$

$$\begin{tikzcd} W \times I \arrow[r] \arrow[d, hook] & C \arrow[d] \\ W \times J \arrow[r] & W^{-1}C \end{tikzcd}$$

pushout in Cat.

Pushout in $\mathcal{S}$: (Htpy po).

$$\begin{tikzcd} W \times I \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d] \\ W \times J \arrow[r] & L(C, W) \end{tikzcd}$$

Dwyer-Kan localization

Any quasi-category X is equivalent to a quasi-category $L(C, W)$ for a homotopical category (C, W).

ie the theory of htpical categories is more or less equivalent to the theory of quasi-categories.

X = a simplicial set, in part a presheaf.

$el(X)$ = cat of elements of X

An object in $el(X)$ is a map $\Delta[n] \longrightarrow X$

An arrow is a com triangle

$$\begin{tikzcd} \Delta[n] \arrow[r] \arrow[dr] & \Delta[m] \arrow[d] \\ & X \end{tikzcd}$$

$\Delta/X = el(X)$

There is a natural map:

$$\boxed{\lambda : el(X) \longrightarrow X}$$

(actually $N(el(X)) \longrightarrow X$).

$$\lambda_0 : el(X)_0 \longrightarrow X_0 \qquad 1 \xrightarrow{n} \Delta[n]$$

$$(\Delta[n] \xrightarrow{\alpha} X) \longmapsto \underset{\text{vertex}}{\lambda_0(\alpha)} = \alpha(n)$$ top vertex of the simplex.

$$\lambda_n : el(X)_n \longrightarrow X_n$$

$$\begin{tikzcd} 1 \arrow[d, "k_0"] & 1 \arrow[d, "k_1"] & & \\ \Delta[k_0] \arrow[r, "\alpha_1"] & \Delta[k_1] \arrow[r] & \cdots \arrow[r, "\alpha_n"] & \Delta[k_n] \arrow[d, "x"] \\ & & & X \end{tikzcd}$$

chain of simplices of length n $\quad \alpha = (\alpha_1, \ldots, \alpha_n; x) \longmapsto \underset{n\text{-simplex}}{\lambda_n(\alpha)}$

Push to $\Delta[k_n]$ the top elements of each simplex, so get a map $\Delta[n] \longrightarrow \Delta[k_n] \xrightarrow{x} X$ to get an n-simplex.

λ is a homotopy localization!

$M < \Delta$ subcategory $\quad f \in \Delta[m] \longrightarrow \Delta[n] \quad f \in M$ if $f(m) = n$

(preserves top element)

$$W \subset el(X) \qquad \begin{tikzcd} \Delta[m] \arrow[rr, "f"] \arrow[dr] & & \Delta[n] \arrow[dl] \\ & X & \end{tikzcd} \qquad f \in W \text{ iff } f \in M \text{ (as in } \Delta)$$

λ inverts the ~~elements~~ arrows of W (sending them to degenerate 1-simplices, ie vertices).

$$L(el(X), W) \xrightarrow{\sim} X \qquad \text{(if } X \text{ is a quasi-category)}.$$

Abstract ideal objects:
- Kan cxs
- Quasi-categories
- Segal categories

Concrete presentations of ideal objects:
- sSet
- double simplicial set
- homotopical categories
- ⋮

We need to work out the connection between these two things.

On the RHS things are more concrete but more complicated. In this course, Joyal is trying to describe LHS. We should work on both sides.

⑱ J <u>Left & right fibrations</u>

A map $f: X \longrightarrow Y$ is called a <u>right fibration</u> if it has the RLP wrt the horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$ $\quad 0 < k \leq n$

Also called <u>contravariant fibrations</u>

$I = \Delta[1] \quad i_1 : \{1\} \hookrightarrow I$

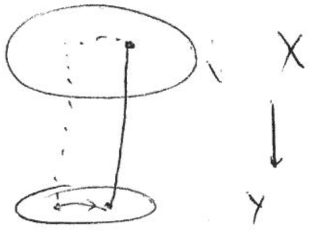
$$\begin{tikzcd} X^I \arrow[r, "X^{i_1}"] \arrow[d, "f^I"'] & X \arrow[d, "f"] \\ Y^I \arrow[r, "Y^{i_1}"] & Y \end{tikzcd}$$

get :

$\langle i_1, f \rangle : X^I \longrightarrow Y^I \times_Y X$

f is a right fibration iff $\langle i_1, f \rangle$ is a trivial fibration

Given (x, u) : $\quad y \xrightarrow{u} f(x)$ get lifting for y.



Is a condition of pulling back. So f can be called a contravariant fibration.

<u>Thm</u> For any simplicial set B, the category S/B admits a Cisinski model structure for which the fibrant objects are the right fibrations $X \longrightarrow B$.

This is called the <u>contravariant model structure.</u>

If B is a category, then there is a Quillen equivalence between simplicial presheaves over B $\quad [B^{op}, S]$ and S/B w/ contravariant model structure.

projective model structure

For any map $u : A \longrightarrow B$, the pair $u_! : S/A \rightleftarrows S/B : u^*$

is a Quillen adjunction.

<u>Def</u> u is <u>final</u> if $Lu_!$ (left derived functors) takes a terminal object to a terminal object.

$$Lu_! : Ho(S/A) \longleftrightarrow Ho(S/B) : Ru^*$$

<u>Def</u> u is <u>h. fully faithful</u> if $Lu_!$ is fully faithful.

Def u is dominant if RU^* is fully faithful.

Def u is a Morita equivalence if $LU_!$ is an equivalence.

A map $u: A \longrightarrow B$ is final iff it admits a factorization eu':
$$\begin{tikzcd} A \arrow[r, "u'"] \arrow[dr, "u"'] & B' \arrow[d, "e"] \\ & B \end{tikzcd}$$
where e is a weak cat. equivalence and u' a right anodyne map.

Recall $\exists$ a w fact. syst. $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B}$ = class of right fibrations and $\mathcal{A}$ = class of right anodyne maps. (it is the saturated class generated by the horns $\Lambda^k[n] \hookrightarrow \Delta[n]$ $0<k\leq n$).

Properties of $(\mathcal{A}, \mathcal{B})$ (This is a Bousfield fact sys on $[S, Qcat\})$

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] & A' \arrow[d, "u'"] \\ B \arrow[r, "b"] & B' \end{tikzcd}$$

$a, b \in W Cat$

- If u and u' are monic, then $u \in \mathcal{A} \Leftrightarrow u' \in \mathcal{A}$
- If u and u' are pseudo fibrations, then $u \in \mathcal{B} \Leftrightarrow u' \in \mathcal{B}$.

$A \xrightarrow{u} B \xrightarrow{v} C$

- if u and v are monic, then $vu \in \mathcal{A}$ & $u \in \mathcal{A} \Rightarrow v \in \mathcal{A}$ (right cancellation property)
- if u and v are pseudo fibrations, then $vu \in \mathcal{A}$ & $v \in \mathcal{B} \Rightarrow u \in \mathcal{B}$ (left c. prop).

2/12/08 // CRM // Joyal

Today: - Limits and colimits in a quasi-category (pg 159)
- Limit sketches
- Locally presentable quasi-categories
- Factorization systems in a quasi-category

Initial (and terminal) objects in a quasi-cat

Def An object a in a quasi-category is initial if every map $\partial\Delta[n] \xrightarrow{x} X$ with $n>0$ and $x(0)=a$ has a filler $\Delta[n] \xrightarrow{\bar{x}} X$

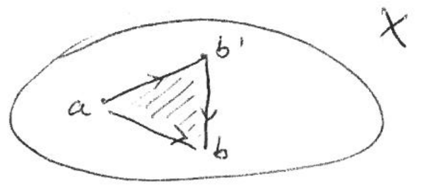
5 $n=1$ $\Delta[1] = I$, $\partial\Delta[1] = \{0,1\}$

A map $x : \{0,1\} \longrightarrow X$ such that $x(0) = a$ is just an object $b = x(1)$.

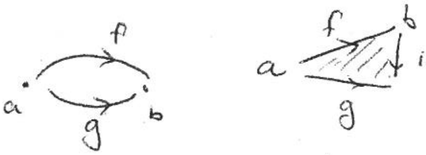
In this case, the condition means that for every object $b \in X_0$ there exists an arrow from a to b $(a \xrightarrow{f} b$, $f = \bar{x}$)

(You would like it to be unique up to homotopy).

$n=2$



In particular



ie, there is a htpy between any two arrows between a and b.
The uniqueness of this htpy is taken care by $n=3$, and so on.

Rmk When X is a category, this extends the notion of initial object.

Recall :
$$\begin{tikzcd} X(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X^I \arrow[d, "(0,1)"] \\ 1 \arrow[r, "(a,b)"] & X \times X \end{tikzcd}$$

Thm An object $a \in X$ is initial iff the simplicial set $X(a,b)$ is contractible for every $b \in X_0$.
(contractible : $X(a,b) \longrightarrow 1$ is a htpy equiv)

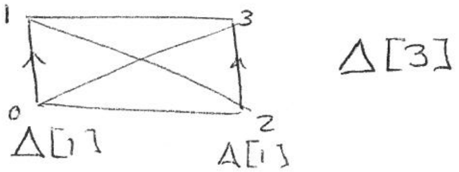
ie the arrow between a and b is htpy unique.

Thm A vertex $a \in X$ is initial iff the map $1 \xrightarrow{a} X$ is left adjoint to the map $X \longrightarrow 1$ (in the 2-category S^{τ_1}).

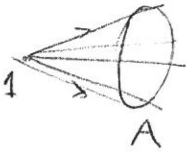
Cones

Based on the join $A * B$ of two simplicial sets.

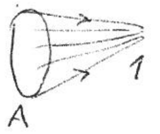
$\Delta[m] * \Delta[n] = \Delta[m+n+1]$



$1 * A$ = projective cone



$A * 1$ = inductive cone



Definition of join (maybe not the most useful one)

$i : \partial I \subset I$ (ie $\{0,1\} \subset I$)

$i^* : S/_I \rightleftarrows S/_{\partial I} : i_*$ (has a right adjoint)

$S/_{\partial I} \simeq S/_{\{0,1\}} \simeq S \times S$

So we have ~~i_*~~ : $$\begin{tikzcd} S \times S \arrow[r, "i_*"] \arrow[rr, bend right, "*"'] & S/_I \arrow[r, "\text{forget}"] & S \end{tikzcd}$$ is the composite of a right adjoint and a left adjoint.

It is associative : $(A * B) * C \simeq A * (B * C)$

Unit : $\phi * A \simeq A \simeq A * \phi$

It doesn't have a right adjoint. ($A * \phi \neq \phi$ initial object).

We have natural inclusions $A \subseteq A * B$ and $B \subseteq A * B$

$A \simeq A * \phi \longrightarrow A * B$

$(-) * B : S \longrightarrow B \backslash S$ This functor does have a right adjoint.

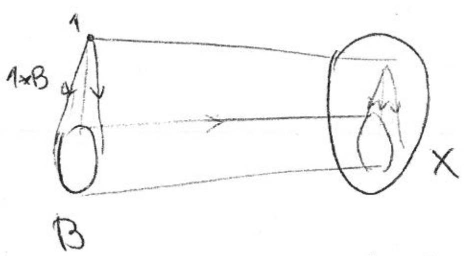
$\{B \longrightarrow X\}$

$A \longmapsto (B \longrightarrow A * B)$

The right adjoint associates to $b : B \longrightarrow X$ a simplicial set X/b.

A simplex $\Delta[n] \longrightarrow X/b$ is the same thing as a map $\Delta[n] * B \longrightarrow X$ (byadjointness) which extends the map $b : B \longrightarrow X$

A vertex ~~of~~ $1 \xrightarrow{a} X/b$ is a map $c : 1 * B \longrightarrow X$ which extends $b : B \longrightarrow X$

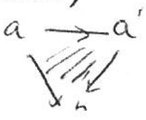


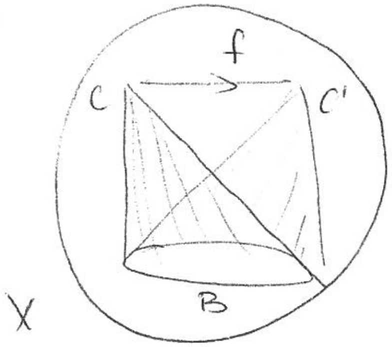
A vertex of X/b is a projective cone with base b in X.

If $B = *$. An object of X/b is an arrow ~~from~~ $a \longrightarrow b$ $(b : * \longrightarrow X)$

(This construction is a generalized slice)

The arrows in the case above are :



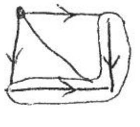


$c, c' \in X/b$

arrows $f : \Delta[1] \star B \to X$ extending $b : B \to X$

<u>Def</u> A projective cone $c : 1 \star B \longrightarrow X$ in X/b ($b : B \longrightarrow X$) is said to be <u>exact</u> if c is a terminal object of X/b.

The <u>limit</u> ($\varprojlim b$) is defined to be the appex $c(1)$ of an exact projective cone c with base b.

<u>Pullback square</u> $I \times I$ is a projective cone  $c : I \times I \longrightarrow X$ exact square.

<u>Thm</u> A quasi-category is Complete iff it admits products and pull backs.

<u>Limit sketches</u>

A <u>projective cone</u> in a simplicial set is a map $c : 1 \star K \longrightarrow A$.

A <u>limit sketch</u> is a pair (A, P) where A is a simplicial set and P is a set of projective cones in A.

A map $A \xrightarrow{f} X$ (X a category) is a <u>model</u> of (A, P) if f takes every cone in P to an exact cone in X.

ie, If $c : 1 \star K \longrightarrow A$ in P, then $fc : 1 \star K \longrightarrow X$ is exact.

Most of the time, interested on models in $\mathcal{U}$ (the quasi-cat of Kan cxs).

$\mathcal{U}$ is complete and cocomplete.

$\mathrm{Mod}(A/P, X)$ the quasi-category of models $(A, P) \longrightarrow X$.

$\cap$ full simplicial subset spanned by the models.

X^A

$\mathrm{Mod}(A/P) = \mathrm{Mod}(A/P, \mathcal{U})$

<u>Def</u> A quasi-category X is <u>locally presentable</u> if it is equivalent to a quasi-category $\mathrm{Mod}(A/P)$ for some limit sketch (A, P).

2/13/08 // CRM // Joyal

- Algebraic theory
- Cartesian theory
- Categories in a quasi-category
- Factorization systems in a quasi-category
- Higher categories

<u>Lawvere (1963)</u>

- An <u>algebraic theory</u> is a category with finite products. A theory T is <u>unisorted</u> if $\mathrm{Ob}\,T = \{A^n \mid n \geq 0\}$ for some $A \in \mathrm{Ob}\,T$.
- A <u>model</u> of T is a functor (product preserving) $T \longrightarrow \mathrm{Set}$
- A model of T with values in a cat C (w/products) is a product preserving functor $T \to C$.

<u>In q-cat</u>

- An <u>algebraic theory</u> T is a quasi-category with finite products.
- A <u>model</u> for T w/ values in a quasi-category w/ finite products X is a map $f: T \longrightarrow X$ which preserves finite products.

- $\underset{\text{quasi-cat}}{\mathrm{Mod}(T, X)} \subset X^T$ is the full simplicial subset of X^T spanned by the models.

- <u>If $X = \mathbb{U}$,</u> $\mathrm{Mod}(T) := \mathrm{Mod}(T, \mathbb{U})$
  - $\mathrm{Mod}(T)$ is complete and cocomplete.
  - $\mathrm{Mod}(T) \subset \mathbb{U}^T$ admits a left adjoint.

AT the category of algebraic theories (in q-cat)

narrow: $T \xrightarrow{f} T'$ is a product preserving map.

$AT \underset{\text{subcat}}{\subset} S$.

The category AT is a 2-category (since S is a 2-category)

It is symmetric monoidal.

Mod(T, T') has finite products (product of two morphisms is pointwise).

This defines Hom(T, T') in AT.

There is a tensor $T \odot T'$

$$\begin{tikzcd} & T \odot T' \arrow[dr] & \\ T \times T' \arrow[ur] \arrow[rr] & & X \end{tikzcd}$$

universal property (unique up to equivalence)

product preserving in each variable.

For any quasi-category with finite products X, we have an equivalence of quasicategories $\mathrm{Mod}(T \odot T', X) \simeq \mathrm{Mod}(T, \mathrm{Mod}(T', X))$

In particular $\mathrm{Mod}(T \odot T') \simeq \mathrm{Mod}(T, \mathrm{Mod}(T'))$

<u>Ex</u> Mon : the algebraic theory of monoids. (in q-cat).

It is the same as the Lawvere theory of monoids (it is indeed a category).

$\mathrm{Mon} \odot \mathrm{Mon}$ is classically the theory of Abelian monoids.

$$\mathrm{Mod}(\mathrm{Mon} \odot \mathrm{Mon}, \mathrm{Set}) = \mathrm{Mod}(\mathrm{Mon}, \mathrm{Mod}(\mathrm{Mon}, \mathrm{Set}))$$

(get a monoid in the category of monoids, and you can prove this is the same as an abelian monoid).

<u>Conjecture</u>: $\mathrm{Mon} \odot \mathrm{Mom}$ (in q-cat) is the theory of braided monoids.

More generally, $\mathrm{Mon}^{\odot n}$ n-fold monoids should be the theory of En-spaces.

There is a simplicial category of En-spaces : $\mathbb{E}n$

$$\tilde{N}(\mathbb{E}_n) \overset{?}{\simeq} \mathrm{Mod}(\mathrm{Mon}^{\odot n})$$

AT has a unit OB for $\odot$.

OB is the theory of a naked object. (no algebraic structure)

$(\mathrm{OB})^{op} \simeq$ finite sets

$\mu : OB \longrightarrow Mon$ canonical map.

$$OB \xrightarrow{u} Mon \xrightarrow{u \odot Mon} Mon^2 \xrightarrow{u \odot Mon^2} Mon^3 \longrightarrow \cdots$$

The colimit should be the theory of E_∞ - spaces. (conjecture) $\sim Mon^\infty$.

Grp = algebraic theory of gps.
$(Grp)^{op}$ = category of finitely generated free groups.
$Grp^{\odot n}$: the conjecture should imply that this is the theory of n-fold loop spaces.

Cartesian theories

Def A cartesian theory T is a quasi-category with finite limits.

$Mod(T, X)$, $Mod(T) = Mod(T, \mathcal{U})$, ... etc

Also $\exists T \odot T'$, defined as a colimit, want $T \times T' \longrightarrow T \odot T'$ preserving finite limits in each variable.

A sketch (A, P) is finitary if every cone in P is finite:

$c \in P$ $\quad c : 1 * K \longrightarrow A$ $\quad$ K finte

$(A, P) \longrightarrow T(A/P)$ universal model, Cartesian theory defined by the sketch

$Mod(A/P) \simeq Mod(T(A/P))$ for a sketch.
It is the universal model

Ex TCat the theory of categories. It is the universal model for a sketch.
What is a category object in a quasi-category X?

Def A simplicial object is a map $C : \underset{(Nerve...)}{\Delta^{op}} \longrightarrow X$. It is called a category object if it satisfies the Segal conditions (Girand conditions)

$$\begin{tikzcd} {[0]} \arrow[r, "n"] \arrow[d, "0"] & {[n]} \arrow[d] \\ {[m]} \arrow[r] & {[n+m]} \end{tikzcd} \rightsquigarrow \begin{tikzcd} C([n+m]) \arrow[r] \arrow[d] & C([m]) \arrow[d] \\ C([n]) \arrow[r] & C([0]) \end{tikzcd}$$

cone $\quad$ pushout $\qquad$ pullback (exact cone) → it is a limit sketch

Mod(T Cat) = Cat(𝒰)
quasi-category of category objects in 𝒰

T Gpd defined similarly

$$\begin{tikzcd} i : T\mathrm{Cat} \arrow[r] & T\mathrm{Gpd} \end{tikzcd} \quad \leftrightsquigarrow$$

$$\begin{tikzcd} \mathrm{Mod}(T\mathrm{Cat}) \arrow[r, bend right, "J"'] & \mathrm{Mod}(T\mathrm{Gpd}) \arrow[l, "i^*"'] \end{tikzcd}$$

(take a gpd and consider it as a cat).

i^* has a right adjoint J : subgroupoid of isomorphisms (it has a left adjoint too)

We say that a category $\mathcal{C}$ satisfies the Rezk condition if the canonical map if the canonical map $Sk^0(\mathcal{C}_0) \longrightarrow J(\mathcal{C})$ is an isomorphism.

ie The only isos of $\mathcal{C}$ are the units.

2/13/08 #CRM # Joyal - "Problem Session"

- T Cat : is a quasi-cat w/ finite limits with the property that

$$\mathrm{Mod}(T\mathrm{Cat}, X) = \mathrm{Cat}(X) \subseteq X^{\Delta^{op}}$$

$$\overset{\shortparallel}{\{c : \Delta^{op} \longrightarrow X \mid \text{satisfying Segal conditions}\}}$$

full subsimplicial set.

$$C_n \xrightarrow{\sim} \underbrace{C_1 \underset{C_0}{\times} \cdots \underset{C_0}{\times} C_1}_{n}$$

- T Spec (theory of spectra)

<u>Def</u> A <u>pointed object</u> in X (a Cartesian quasi-category) is a morphism

$$1 \xrightarrow{u} a$$

where 1 is the terminal object.

Loop space:

$$\begin{tikzcd} \Omega_u(a) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, "u"] \\ 1 \arrow[r, "u"] & a \end{tikzcd}$$

A <u>prespectrum</u> in X is an infinite sequence of pointed objects $1 \xrightarrow{u_n} a_n$ $(n \geq 0)$, equipped with an infinite sequence of arrows

$$f_n : a_n \longrightarrow \Omega(a_{n+1})$$

It is a <u>spectrum if</u> $f_n : a_n \xrightarrow{\sim} \Omega(a_{n+1})$ is invertible.

It is given by a limit sketch:

$$\begin{tikzcd} & & 1 \arrow[dr] & \\ a_0 \arrow[r] & \Omega(a_1) \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\ulcorner", very near start] & & a_1 \\ & & 1 \arrow[ur] & \end{tikzcd} \quad \overset{\Longleftrightarrow}{\text{universality of pullback}} \quad \begin{tikzcd} & 1 \arrow[dr] & & 1 \arrow[dr] & & {} \\ a_0 \arrow[ur] \arrow[dr] & & a_1 \arrow[ur] \arrow[dr] & & a_2 \arrow[ur] \arrow[dr] & \cdots \\ & 1 \arrow[ur] & & 1 \arrow[ur] & & {} \end{tikzcd}$$

data of prespectrum

J The notion of a spectrum is defined by a finitary limit sketch (A, P)

$$T(A/P) = T\,Spec \quad \text{(universal model)}.$$

$$Mod(TSpec, X) = Spec(X)$$

quasi-category of Spectra in X.

<u>Example</u> $\tilde{N}(sComm) = X$ ~~$Spec(X)$~~ = ~~quasi~~

* $Spec(X/b)$ Spectrum with base b

A prespectrum in X/b, $b \in X_0$

$$\begin{tikzcd} a_0 \arrow[d, "p"] \\ b \arrow[u, bend left, "s"] \end{tikzcd}$$

a_0 a pointed object $/b$.

$ps = 1$

$$\begin{tikzcd}[column sep=small] & b \arrow[dr, "s_1"] & & b \arrow[dr, "s_2"] & & {} \\ a_0 \arrow[ur, "p_0"] \arrow[dr, "p_0"'] & & a_1 \arrow[ur, "p_1"] \arrow[dr, "p_1"'] & & a_2 \arrow[ur] \arrow[dr] & \cdots \\ & b \arrow[ur, "s_1"'] & & b \arrow[ur, "s_2"'] & & {} \end{tikzcd}$$

To be a spectrum, you want each square to be a pullback.

PSpec = theory of parametrized spectra (varying bases)

This morning, Toën was looking at $PSpec(X)$ where $X = \tilde{N}(sComm)$

<u>Thm</u> (G. Biedermann & J.)

If X is a higher topos (utopos) then

$$PSpec(X)$$

is also a utopos.

1) $X = \mathbb{U} = \tilde{N}(Kan)$ archetype of a utopos.

2) $\mathbb{U}^{A^{op}}$ also a utopos

3) A left exact reflection of $\mathbb{U}^{A^{op}}$ is a utopos.

Simpson, Rezk

Ie, a utopos is a subcategory

$$\begin{tikzcd} X \arrow[r, hook, shift left] & \mathbb{U}^{A^{op}} \arrow[l, shift left, "r"] \end{tikzcd}$$

r left exact reflection

To every prespectrum w/ associate a spectrum

$$\begin{tikzcd}
 & 1 \arrow[dr] & & 1 \arrow[dr] & & {} \\
a_0 \arrow[ur] \arrow[dr] \arrow[dddd] & & a_1 \arrow[ur] \arrow[dr] \arrow[dddd] & & a_2 \arrow[ur] \arrow[dr] \arrow[dddd] & \\
 & 1 \arrow[ur] & & 1 \arrow[ur] & & {} \\
 & 1 \arrow[dr] & & 1 \arrow[dr] & & {} \\
\Omega a_1 \arrow[ur] \arrow[dr] & & \Omega a_2 \arrow[ur] \arrow[dr] \arrow[dl] & & \Omega a_3 \arrow[ur] \arrow[dr] & \\
 & 1 & & 1 \arrow[ur] & & {}
\end{tikzcd}$$

Get a map of prespectra. Repeat the process, take colimit (stabilization process), get a spectrum. This construction preserves finite limits.

$$\begin{tikzcd}
\mathrm{Spec}(\mathbb{U}) \arrow[r, hook] & \mathrm{Spec}'(\mathbb{U}) \arrow[l, bend left, "r"]
\end{tikzcd}$$

prespectra

→ but this is not a topos (have a basepoint).

Fix: let basepoint vary (b), replace by parametrized construction and it works!

$$\begin{tikzcd}
\mathrm{PSpec}(\mathbb{U}) \arrow[r, hook] & \mathrm{PSpec}'(\mathbb{U}) \arrow[l, bend left, "r"]
\end{tikzcd} \simeq \mathbb{U}^D \text{ so a utopos}$$

category of diagrams =

G. Bredermann has an idea to relate to Goodwillie calculus. (Goodwillie tangent space)

2/4/08 / CRM / Toën / DAG

Motivations and objectives

Main problem in moduli theory $\quad F : \underset{\text{Cat of comm rings}}{\mathrm{Comm}} \longrightarrow \underset{\text{Cat of sets}}{\mathrm{Sets}}$ $\quad$ moduli functor

$A \in \mathrm{Comm}$, think of $F(A)$ as the sets of families of objects in F parametrized by $\mathrm{Spec}\,A$.

Main question: is F representable? ie $\exists$? a scheme X together with functorial bijection $F(A) = \mathrm{Hom}(\mathrm{Spec}\,A, X)$

When X exists, it is called a moduli space for F, or a solution to the moduli problem F. We also say, F is representable by X.

Rmks 1) Moduli spaces are a nice way to construct interesting schemes.

2) Understanding X "$\Longleftrightarrow$" solving the classification problem related to F.

M_g "$\Longleftrightarrow$" " " " " of alg curves.

Example: Γ = distinct gp (finitely presented).

Pb Classify linear f.d representations of Γ.

$$R(\Gamma) : \mathrm{Comm} \longrightarrow \mathrm{Sets}$$

$A \longmapsto R(\Gamma)(A)$ = set of isomorphism classes of $A[\Gamma]$-modules where the underlying A-module is projective of finite type.

= set of isomorphism classes of vector bundles on $\mathrm{Spec}\,A$ together with a Γ-action.

Q: Is $R(\Gamma)$ representable?

$R'_n(\Gamma) : \mathrm{Comm} \longrightarrow \mathrm{Sets}$ $\qquad$ Fix underlying A-module to be A^n.

$n \geq 0 \qquad A \longmapsto \mathrm{Hom}_{Gp}(\Gamma, GL_n(A))$

$$R'(\Gamma) = \coprod_{n \geq 0} R'_n(\Gamma) \xrightarrow{\pi} R(\Gamma)$$

$$(\Gamma \xrightarrow{f} GL_n(A)) \longmapsto (A^n \circlearrowleft_f \Gamma)$$

⊃ᵀ Strategy: 1) $R'_n(\Gamma)$ is representable

2) $R'_n(\Gamma) \longrightarrow R(\Gamma)$ is GL_n - equivariant

$$\begin{tikzcd}
R'_n(\Gamma) \arrow[rr] \arrow[dr] & & R(\Gamma) \\
 & R'_n(\Gamma)/GL_n \arrow[ur] &
\end{tikzcd}$$

<u>hope</u>: $\coprod_n R'_n(\Gamma)/_{GL_n} \xrightarrow{\sim} R(\Gamma)$

would like $R'_n(\Gamma)/GL_n$ to be representable.

1) $R'_n(\Gamma)$ is representable by an affine scheme.

$GL_n : Comm \longrightarrow Sets$

$A \longmapsto GL_n(A)$ presheaf of sets

is representable by $\operatorname{Spec} \mathbb{Z}[T_{ij}]_{\phi \le i,j \le n}[\det T_{ij}^{-1}]$ $n \times n$ invertible matrices.

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{Z}[T_{ij}][\det T_{ij}^{-1}] \\
\operatorname{Spec} A \arrow[u]
\end{tikzcd} \iff \begin{tikzcd}
\mathbb{Z}[T_{ij}][\det T_{ij}^{-1}] = C_n \arrow[d] \\
A
\end{tikzcd} \iff GL_n(A)$$

In general $\Gamma = \langle g_1, \dots, g_m \rangle / \langle r_1, \dots, r_p \rangle$,

$$\begin{tikzcd}
\operatorname{Hom}(\Gamma, GL_n(A)) \arrow[r] \arrow[d, "\Gamma"] & GL_n(A)^m \arrow[d] \\
\{1\} \arrow[r] & GL_n(A)^p
\end{tikzcd}$$

We have a fibred product of functors

$$\begin{tikzcd}
R'_n(\Gamma) \arrow[r] \arrow[d] & GL_n(\text{[illegible]})^m \arrow[d, "\varphi"] \\
\{1\} \arrow[r] & GL_n(\text{[illegible]})^p
\end{tikzcd}$$

but: affine schemes are stable under fibred products, ie

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\
\operatorname{Spec} B \arrow[r] & \operatorname{Spec} C
\end{tikzcd} \Rightarrow F \cong \operatorname{Spec}(A \otimes_C B)$$

$\Rightarrow R'_n(\Gamma) \simeq \operatorname{Spec}(C_n^{\otimes m} \otimes_{C_n^{\otimes p}} \mathbb{Z})$

<u>Observation</u> * $C_n^{\otimes m}$ is not flat / $C_n^{\otimes p}$ (in general)

* $R'_n(\Gamma)$ is the schematic intersection of the graph of φ, and $GL_n^m \times \{1\}$ inside $GL_n^m \times GL_n \times^p$ and the intersection can be <u>bad</u> (not transversal; not of the expected dimension).

$\Rightarrow R'_n(\Gamma)$ is badly behaved
- badly singular
- not of the expected dimension.

Philosophical point 1 : We should always replace with derived $\otimes$.
The "right" moduli space for $R'_n(\Gamma)$ is "Spec" $\left(C_n^{\otimes m} \overset{\amalg}{\underset{C_n^{op}}{\otimes}} \mathbb{Z}\right)$

Conclusion 1 We should replace Comm by another category for which $\overset{\amalg}{\otimes}$ makes sense (ie simplicial comm rings).

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2) $\coprod_n R'_n(\Gamma)/Gl_n \xrightarrow{p} R(\Gamma)$

Gl_n acts on $R'_n(\Gamma)$, ie for $A \in Comm$, $Gl_n(A)$ acts by conjugation on $Hom(\Gamma, Gl_n(A))$.

$R'_n(\Gamma) \longrightarrow R(\Gamma)$ is conjugation invariant

$$\begin{tikzcd}
R'_n(\Gamma) \arrow[rr] \arrow[dr] & & R(\Gamma) \\
& R'_n(\Gamma)/Gl_n \arrow[ur] &
\end{tikzcd}$$

It is a monomorphism, but it is not surjective.

p is close to be an isomorphism.
However it is locally surjective for the Zariski topology.
So p is a local isomorphism.

In general, if X is a scheme with a G-action (G a group scheme $_{\text{smooth, affine}}$), the quotient sheaf X/G is not representable.

eg $X = \mathbb{A}^1 = Spec\ \mathbb{Z}[t]$, $G = \mathbb{Z}/2$ $\circlearrowright t \mapsto -t$

$X/\mathbb{Z}/_2$ is not representable. If it were then $\mathbb{C}/\mathbb{Z}/_2$ would be representable as an analytic space.

$$\begin{tikzcd}
& \mathbb{C}/\mathbb{Z}/_2 \arrow[dl] \arrow[d, equal] \\
\mathbb{C} \arrow[r] & \mathbb{C} \\
z \arrow[r, mapsto] & z^2
\end{tikzcd}$$

In general when $X \circlearrowleft G$ has fixed points X/G is not representable.
Instead of the quotient (pre)sheaf X/G we should rather use the "groupoid quotient"

E is set $\circlearrowleft$ $H = gp$
$[E/H]$ = groupoid whose objects are elts of E, morphisms $e \to e' = \{h \in H \mid he = e'\}$

④ $^{T}\pi_0([E/H]) \simeq E/H$

$\mathrm{Aut}_{[E/H]}(e) = \{h \in H \mid he = e\}$ stabilizer

$[R_n'(\Gamma)/Gl_n] : \mathrm{Comm} \longrightarrow \mathrm{Groupoids} \subseteq \mathrm{SSets}$

Philosophical pt 2 quotienst should be derived

$\Rightarrow$ we need $\mathrm{Comm} \longrightarrow \mathrm{sSets}$

Conclusion In order to really understand $R(\Gamma)$ we need a notion of representability (of "schemes", "alg stacks") for functors

$$\underset{\text{simp. comm. rings}}{s\mathrm{Comm}} \longrightarrow s\,\mathrm{Sets}$$

2/5/08 / Toën

Simplicial presheaves and stacks

Reminder We need a notion of representability for functors $F : s\mathrm{Comm} \longrightarrow \mathrm{SSets}$.

We start to study $F : \mathrm{Comm} \longrightarrow s\mathrm{Sets}$

In this lecture: C is a Grothendieck site, $F : C^{op} \longrightarrow s\mathrm{Sets}$ — fix universes — "small presheaves" (see Choriny's talk)

(C, τ) is a Grothendieck site: - C is a category with finite limits ("small")

- τ a topology on C, ie $\forall X \in C$, we are given

a set of families $\{X_i \longrightarrow X\}_i$, called Cohen ~~fam~~ families + conditions:

- * $\{X_i \to X\}$ is a Cohen family and $\{$ ~~X~~ $Y_i \to X\}$, then $\{X_i \underset{X}{\times} Y_i \longrightarrow X\}$ is a Cohen family.
- * $\{X \xrightarrow{id} X\}$ is a Cohen family
- * $\{X_i \to X\}$ is a Cohen family, $\forall i\ \{X_{ij} \to X_i\}$ Cohen $\Rightarrow$ $\{X_{ij} \to X\}_{i;j}$ also Cohen.

$\mathrm{Pr}(C) =$ Cat of presheaves on $C = \mathrm{Fun}(C^{op}, \mathrm{Sets})$

$\cup\!|\ \downarrow$ left adj (sheafifying)

$\mathrm{Sh}(C) =$ Cat of sheaves $= \{F \in \mathrm{Pr}(C) \mid \forall \{X_i \to X\}$ cohen $(F(X) \to \prod_i F(X_i) \rightrightarrows \prod_{ij} F(X_i \underset{X}{\times} X_j))$ exact $\}$

$SPr(C)$ = Cat of simplicial presheaves on C = Fun$(C^{op}, SSets)$

Level wise (proj) model structure:

There is a model cat. structure on $SPr(C)$ for which fibrations (resp equiv) are $f: F \longrightarrow G$ s.t $\forall X \in C$, $F(X) \to G(X)$ is a fibration (resp an equiv) for the standard model structure on SSets.

- $F \in SPr(C)$ $\quad \pi_0^{Pr}(F) : C^{op} \longrightarrow Set$ $\quad$ presheaf of connected cpts.

$$X \longmapsto \pi_0(F(X))$$

$\pi_0(F)$ is the associated sheaf to $\pi_0^{Pr}(F)$.

$K \in SSets$,
$\pi_i(K, x) := \pi_i(|K|, x)$

$X \in C$, $s \in F(X)_0$ (a point, 0-simplex)

Define $\pi_i^{Pr}(F,s) : (C/X)^{op} \longrightarrow Sets$

$$(Y \xrightarrow{u} X) \longmapsto \pi_i(F(Y), u^*(s))$$

$\pi_i(F,s) :=$ associated sheaf to $\pi_i^{Pr}(F,s)$

<u>Def</u> Let $f: F \to G$ in $SPr(C)$ is a <u>local equivalence</u> if

- $*$ $\pi_0(F) \longrightarrow \pi_0(G)$ is an iso
- $*$ $\forall X \in C$, $\forall s \in F(X)_0$, $\forall i > 0$, $\pi_i(F,s) \longrightarrow \pi_i(G, f(s))$ is an iso.

(it induces weak equivalences fiberwise ...)

<u>Thm</u> (Joyal - Jardine - Blander)
The notion of local equivalences and cofibrations (ie the ones defined in model structure above) define a model category structure on $SPr(C)$.
projective fibrations

(Local model structure)

$X \in C$ $\quad H \in SPr(C)$ $\quad H \longrightarrow X$ in $SPr(C)$ $\quad$ ~~(X as constant simp presheaf)~~

We say H is a <u>hypercovering</u> if :

$$\begin{cases} H_n \simeq \coprod_\alpha X_{n,\alpha} \ ; \ X_{n,\alpha} \in C \\ H_n \simeq Hom(\Delta^n, H) \longrightarrow Hom(\partial\Delta^n, H) \times_{Hom(\partial\Delta^n, X)} Hom(\Delta^n, X) \end{cases}$$

is locally surjective

The second condition means

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & H(Y) \arrow[r] \arrow[d] & H(Y_i) \arrow[d] \\ \Delta^n \arrow[r] \arrow[rru, dashed] & X(Y) \arrow[r] & X(Y_i) \end{tikzcd}$$

$Y_i \to Y \in C$

H is contractible as an object over X.

⓪T <u>Thm</u> (Dugger - Isakson - Hollander) $F \in SPr(C)$ is fibrant $\Leftrightarrow$

$* \ \forall X \in C, \ F(X)$ is fibrant

$** \ \forall X \in C, \ \forall H \to X$ hypercovering

$$F(X) \xrightarrow[\text{w. equiv}]{} \operatorname{holim}_{\Delta} \left(F(H_0) \rightrightarrows F(H_1) \overset{\rightarrow}{\rightrightarrows} \cdots \right)$$

$$F(H_n) = \underline{Hom}(H_n, F) \simeq \prod_{\alpha} F(X_{n,\alpha})$$

<u>Def</u> A <u>stack</u> on C is an object $F \in SPr(C)$ satisfying condition $**$.

<u>Def</u> The <u>homotopy category of stacks</u> is $Ho(SPr(C))$ (for the local model structure)

<u>Rmk</u> $Ho^{lev}(SPr(C))$ = homotopy cat of $SPr(C)$ for the levelwise model str.

$$\begin{tikzcd} \cup\!\text{l} \arrow[d, "\text{left adjoint (associated stack function)}"] \\ {} \end{tikzcd}$$

$Ho(SPr(C)) \cong \{F \in Ho^{lev}(SPr(C)) \mid F \text{ is a stack}\}$ full subcat.

Morphisms in $Ho(SPr(C))$ are denoted by $[-,-]$.

$F \to H \leftarrow G$ diagram of stacks, $F \overset{h}{\underset{H}{\times}} G$ is the htpy fiber product of this diagram

<u>Basic examples</u>: $Pr(C) \hookrightarrow SPr(C)$ (constant simplicial set)

A presheaf of sets F is a stack $\Longleftrightarrow$ F is a sheaf.

$Sh(C) \hookrightarrow Pr(C) \hookrightarrow SPr(C) \longrightarrow Ho(SPr(C))$, the induced functor is fully faithful.

The ess image of this functor consists of F s.t. $\forall X, \forall s \in F(X)_0, \forall i > 0, \pi_i(F, s) = 0$.

<u>Classifying stacks</u> •

$$\begin{tikzcd} G: C^{op} \arrow[r] \arrow[dr, "BG"'] & SGp \arrow[d, "B"] \\ & SSets \end{tikzcd}$$

SGp = simplicial group

$K(G,1) = BG$

$\cdot \ A : C^{op} \longrightarrow SAb \xrightarrow{B} SAb \xrightarrow{B} \cdots \longrightarrow SSets$.

$\Rightarrow K(A;n) = B(K(A;n-1)) \quad , \quad K(A;0) = A$

$X \in C$, G is a sheaf of gps.

$[X, K(G,1)] \cong H^1(X, G)$ set of isomorphism classes of G-torsors on X.

$X \in C$, A is a sheaf of ab gps $[X, K(A,n)] \simeq H^n(X, A)$

Internal homs $Ho(SPr(C))$ is Cartesian closed.

$\forall F, G \quad \exists\, \mathbb{R}Hom(F,G) \in Ho(SPr(C))$ s.t

$[H \times F, G] \simeq [H, \mathbb{R}Hom(F,G)]$ functorial on $H \in Ho(SPr(C))$

This is a very good way to construct new stacks from old ones.

2/6/08 // CRM // Toën

Algebraic (n)-Stacks

$C =$ Affine schemes $= Comm^{op}$ $A \in Comm$ $Spec A \in Aff$

$A \to B$ is étale if
- $A \to B$ is flat ie $Spec B \to Spec A$ is flat
- B is finitely presented as an A-algebra $\left(B \simeq \frac{A[X_1, \dots X_n]}{(P_1, \dots, P_m)}\right)$
- B is flat as $B \otimes_A B$-algebra ie. $Spec B \to Spec B \times_{Spec A} Spec B$ is an open immersion

$\Updownarrow$ (unramified)

"local isomorphisms"

$\{A \to A_i\}$ is an étale covering if:
- $A \to A_i$ is étale
- $\{A\text{-mod} \xrightarrow{-\otimes_A A_i} A_i\text{-mod}\}$ is a conservative family (reflects isomorphism)

This defines a topology on Aff

$Spec B \to Spec A$ is a Zariski open immersion if it is étale and a monomorphism in Aff.

$SPr(Aff)$ + local equiv.

$\Downarrow$

$Ho(SPr(Aff))$ homotopy category of stacks.

$\cup\!\!\mid$ $\simeq$

$Sh(Aff) \supseteq Aff$ (Yoneda + descent : representable presheaves are sheaves).

Schemes

Def 1) $F \xrightarrow{i} Spec A$ in $Ho(SPr(Aff))$ is Zariski open if

a) F is a sheaf (0-truncated, $\pi_i(F_r) = 0 \ \forall i > 0$)

b) $\exists$ a family of Zariski open $\{Spec A_i \to Spec A\}$ s.t

i is a mono whose image $=$ image $\coprod Spec A_i \to Spec A$

2) $F \to F'$ in $Ho(SPr(Aff))$ is a Zariski open if $\forall X = Spec A \in Aff$, $\forall X \to F'$ the induced morphism $F \overset{h}{\underset{F'}{\times}} X \to X$ is a Zariski open in the sense of 1).

⑧^T 3) A stack F is a <u>scheme</u> if $\exists\, X_i = \mathrm{Spec}\, A_i \in \mathrm{Aff}$ + $X_i \longrightarrow F$ s.t

a) $X_i \longrightarrow F$ is a Zariski open

b) $\coprod X_i \longrightarrow F$ is an ~~epimorphism~~ surjective of stacks ($F \to F'$ is ~~epi~~ surjective if $\pi_0(F) \to \pi_0(F')$ is an epimorphism of sheaves). (ess surjective maybe?)

Rmk *) A scheme F is a sheaf.

*) The full subcategory of schemes $\subseteq \mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}))$ is equivalent to the category of Grothendieck's schemes.

*) Schemes are stable by $\coprod$ and $-\overset{h}{\underset{-}{\times}}-$.

Def $A \longrightarrow B$ is <u>smooth</u> if it is flat, finitely presented, B is of finite ~~Tor~~ dimension over $B \otimes B$

Spec $B \to$ Spec A is smooth $\Leftrightarrow$ "submersion"

Def $X \to Y$ of schemes is smooth (étale) if $\exists\ \{X_i \to X\}$ & $\{Y_i \to Y\}$ atlas

$$\begin{tikzcd} X \arrow[r] & Y \\ X_i \arrow[u] \arrow[r] & Y_i \arrow[u] \end{tikzcd}$$

and $X_i \longrightarrow Y_i$ are smooth (étale)

△ Need to check definitions are local so that it makes sense for affine too.

Def 1) A stack F is <u>0-algebraic</u> if it is a scheme.

2) A morphism $F \to F'$ of stacks is <u>0-representable</u> (or 0-algebraic) if $\forall X \in \mathrm{Aff}$, $\forall X \to F'$, $F \overset{h}{\underset{F'}{\times}} X$ is a scheme (0-algebraic).

3) A morphism $F \to F'$ of stacks is <u>0-algebraic and smooth</u> if $\forall X \in \mathrm{Aff}$, $\forall X \to F'$, $F \overset{h}{\underset{F'}{\times}} X \to X$ is a smooth morphism of schemes.

We assume we know the notions of $(n-1)$-algebraic stacks $(n>0)$, $(n-1)$-algebraic morphisms and $(n-1)$-alg and smooth morphism:

4) A stack F is n-algebraic if : $\begin{cases} \circ\ F \to F \times F \text{ is } (n-1)\text{-algebraic} \\ \circ\ \exists X_i \in \mathrm{Aff},\ X_i \to F \text{ smooth st } \coprod X_i \to F \text{ is surjective in stacks.} \end{cases}$

(the stack of morphisms between 2 pts is $(n-1)$-rep)

(smooth atlas) $\to$

$\Downarrow$ ($X_i \to F$ is automatically $(n-1)$-algebraic)

$\Leftrightarrow$ "F is the quotient of $\coprod X_i$ by the action of a <u>smooth $(n-1)$-algebraic groupoid</u>":

$Y_1 \underset{t}{\overset{s}{\rightrightarrows}} Y_0 = \coprod X_i$ $\quad Y_i$ is an $(n-1)$ alg stack and s, t are smooth

5) A morphism $f: F \to F'$ is <u>n-algebraic</u> if $\forall X \in Aff$, $\forall X \to F'$, $F \overset{h}{\underset{F'}{\times}} X$ is n-algebraic

6) A morphism $f: F \to F'$ is <u>n-algebraic and smooth</u> if $\forall X \in Aff$, $\forall X \to F$ $\exists X_i \in Aff$ $\{X_i \to F \overset{h}{\underset{F'}{\times}} X\}$ smooth atlas s.t

$$\begin{tikzcd} X_i \arrow[r] \arrow[rr, bend right] & F \overset{h}{\underset{F'}{\times}} X \arrow[r] & X \end{tikzcd}$$ is smooth.

7) A stack F is <u>algebraic</u> if it is algebraic for some n.

<u>Rmk</u> *) F n-algebraic $\Rightarrow$ n-truncated: $\forall X \in Aff$, $\forall s \in F(X)$, $\pi_i(F, s) = 0$ $\forall i > n$.

*) $\exists$ an equivalence between the full subcategory of algebraic stacks which are sheaves and the category of algebraic spaces (Artin).

*) $\exists$ an equivalence between the full subcategory of algebraic stacks which are 1-truncated ($\pi_i(F, s) = 0$ $\forall i > 1$) and Artin's algebraic stacks.

*) The full subcategory of algebraic stacks is closed under $\coprod$, homotopy fibered products, smooth quotients.

<u>Example</u> (M. Anel) (2-Stack) Stack of abelian categories.

$\mathcal{A}b$: $Comm \to$ ~~SSets~~ Cat.

$Ab(A)$ = category of A-linear abelian categories which are equivalent to R-mod for R an associative A-algebra projective of finite type/A.

+ A-linear equivalences between them.

$A \to B$ in $Comm$ $\quad Ab(A) \to Ab(B)$

$\mathcal{C} \mapsto B\text{-mod}(\mathcal{C}) = \mathcal{C}^{B/A} := Fun(B, \mathcal{C})$

$$\mathcal{A}b : Comm \xrightarrow{Ab} Cat \xrightarrow{1 \cdot 1} SSets$$

<u>Thm</u> $\mathcal{A}b$ is an algebraic stack which is 2-truncated.

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Some results (not in the notes)

1) Dévissage
2) Numerical invariants
$\wr$
invariants of C^∞-manifolds (TQFT)
3) Rational and padic htpy theory

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1) Dévissage

F algebraic stack of finite type / Spec k, k a Noetherian ring

e.g * $\exists$ an affine scheme X finite type / k and $X \twoheadrightarrow F$ smooth atlas

$$\begin{tikzcd} & F \arrow[r] & F^{\partial\Delta^n} \\ Y \arrow[r, "\text{smooth atlas}"] & G \arrow[u] \arrow[r, "L"] & X \text{ affine} \arrow[u] \end{tikzcd}$$

* $(\forall n \geq 1)$ are quasi-compact, ie $n=1$ $F \to F \times F$ diagonal

Y affine

Thm $\exists\ \phi = F_0 \hookrightarrow F_1 \hookrightarrow F_2 \hookrightarrow \cdots \hookrightarrow F_n = F$ of closed substacks, $\exists$ schemes $\{M_i\}_{1 \leq i \leq n}$

together w/ $F_i - F_{i-1} \longrightarrow M_i$ s.t

a) $\pi_0(F_i - F_{i-1}) \xrightarrow{flat} M_i$ iso of sheaves (gerbe condition)

b) $\forall$ affine X, $\forall\ X \xrightarrow{s} F_i - F_{i-1}$, $\pi_m(F_i - F_{i-1}, s)$ is a flat group scheme / X $\forall m > 0$

Consequences * Any algebraic stack F of f. type has a "formal" moduli space "ΣM_i".
(can't glue them together, but useful to compute some things).

* If k is a field, s: Spec $k \to F$, then the sheaf $\pi_n(F, s)$ is always an alg. group over k.

"ΣM_i" is not right : "$\Sigma M_i \prod_{m \geq 0} \pi_m(F_i - F_{i-1})^{(-1)^m}$"

---

2) Numerical invariants k field

For varieties : $K_0(Var/k) = \mathbb{Z}[\text{alg var}/k] \Big/ X = U + Y$ if $Y \hookrightarrow X$ closed
$U = X - Y$

Ring for $X \times_k Y$.

A (multiplicative) numerical invariant for varieties is a ring morphism

$$Z : K_0(Var/k) \longrightarrow R = \text{comm. ring}.$$

Example a) $k = \mathbb{F}_q$, $K_0(\mathrm{Var}/k) \longrightarrow \mathbb{Z}$

$$X \longmapsto \# X(\mathbb{F}_q)$$

b) $k = \mathbb{C}$ $\quad K_0(\mathrm{Var}/k) \longrightarrow \mathbb{Z}$ $\quad$ Euler char

$$X \longmapsto \sum (-1)^i h_c^i(X, \mathbb{Q}) = \chi_c(X) \quad \text{(Betti numbers)}$$

c) $k = \mathbb{C}$ $\quad K_0(\mathrm{Var}/k) \longrightarrow \mathbb{Z}[u,v]$

$$X \xmapsto{\ \rho_H\ } \sum_{p,q} (-1)^{p+q} h^{p,q}(X)\, u^p v^q$$

where X is smooth projective ; $h^{p,q}(X) = \dim H^p(X, \Omega^q)$

$u = v = 1$ get $\rho_H(X)(1,1) = \chi_c(X)$

d) ℓ-adic Euler characteristic

$$K_0(\mathrm{Var}/\mathbb{F}_q) \longrightarrow \mathbb{Z}[W_q] \qquad \underset{\overline{\mathbb{Q}}^*}{\underset{\cap}{W_q}} = \left\{ x \in \overline{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall i : \overline{\mathbb{Q}} \to \mathbb{C} \\ |i(x)| = q^{j/2} \\ \quad j \in \mathbb{Z} \end{array} \right\}$$

$$X \longmapsto \sum (-1)^i H_c^i(\overline{X}, \mathbb{Q}_\ell)$$

$\overline{X} = X \underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}_q}$

Def An algebraic stack F is special if ~~for every field k~~ it is of finite type $/k$ and for every field K, $s : \mathrm{Spec}\, K \longrightarrow F$:

- $\pi_1(F, s)$ is a linear algebraic group.
- $\pi_m(F, s)$ is a unipotent linear algebraic gp $/k$ $\forall m > 1$.

$K_0(St^{sp}) = \mathbb{Z}[\text{special alg stacks}] \big/ \text{relations}$

relations: - $F_0 \hookrightarrow F$ closed, $F = F_0 + F_1$ $\quad \Rightarrow F_1 = F - F_0$

- F_0 is a given special stack, $F' \to F$ which is a "Zariski local" fibration w/ fibers F_0, then $F' = F \times F_0$

(In the Grothendieck gp of varieties the 2nd relation is implied by the first).

Thm $\mathbb{L} = \mathbb{A}^1 \in K_0(\mathrm{Var})$

$$\begin{tikzcd} K_0(\mathrm{Var})[\mathbb{L}^{-1}, (\mathbb{L}^i - 1)^{-1}]_{i>0} \arrow[d, "\wr"] \\ K_0(St^{sp})[\mathbb{L}^{-1}, (\mathbb{L}^i - 1)^{-1}]_{i>0} \end{tikzcd}$$

⑫ T **Corollary** Any numerical invariant of alg varieties $Z : K_0(\mathrm{Var}) \longrightarrow R$ s.t $Z(\mathbb{L})$ and $Z(\mathbb{L}^i - 1)$ are invertible, extends uniquely to a numerical invariant of special stacks.

All examples above except the topological Euler characteristic can be extended.
$(Z(\mathbb{L}-1)=0)$

Examples : ∘

$$\begin{tikzcd}
K_0(\mathrm{Var}/\mathbb{C}) \arrow[r] \arrow[d] & \mathbb{Z}[u,v] \arrow[r] & \mathbb{Z}[u,v][(uv)^{-1},((uv)^i-1)^{-1}] \arrow[r] & \mathbb{Z}[[u,v]][(uv)^{-1}] \\
\mathbb{L} \longmapsto uv & & & \\
K_0(St^{sp}) \arrow[uurr, bend right, "P_H"'] & & &
\end{tikzcd}$$

$\rightsquigarrow$ Hodge numbers are defined.

∘

$$\begin{tikzcd}
K_0(\mathrm{Var}/\mathbb{F}_q) \arrow[r] \arrow[dr] & \mathbb{Z} \arrow[r] & \mathbb{Q} \\
& K_0(St^{sp}) \arrow[ur] &
\end{tikzcd}$$

\# of rational pts

$\#(-)(\mathbb{F}_q) : K_0(\mathrm{Var}/\mathbb{F}_q) \longrightarrow \mathbb{Q}$

$F(\mathbb{F}_q)$ is a finite htpy type $\left(\begin{array}{l}\pi_0 \text{ is finite} \\ \pi_i \text{ are finite } \forall i, \forall \text{base pt}\end{array}\right)$.

$$\#F(\mathbb{F}_q) = \sum_{\substack{\text{connected} \\ \text{cpts} \\ F(\mathbb{F}_q) \\ x}} \prod_{m \geq 1} |\pi_m(F(\mathbb{F}_q), x)|^{(-1)^m}$$

If $\pi_i = 0$ for $i > 1$ $\quad \sum_{x \in \pi_0} \frac{1}{|\pi_1|}$ = "orbifold number of pts".

2b) TQFT's (not very interesting ones)

M a compact C^∞-mfld. F a special alg stack

Fix $Z : K_0(St^{sp}) \longrightarrow R$ numerical invariant.

$\mathrm{Map}(M,F) := \mathbb{R}\,\underline{\mathrm{Hom}}(S(M), F)$ stack of morphisms from the simplicial set of simplices in M to F.

$S(M)$ is the constant simplicial presheaf on affine schemes ~~ov~~ w/ étale top.

$\mathrm{Map}(M,F)$ is again a special alg stack.

$Z_F(M) := Z(\mathrm{Map}(M,F)) \in R$

These Z_F come (in general) from TQFT.

T ⑬

$Z = \#(-)(\mathbb{F}_q) : K(St^{sp}) \longrightarrow \mathbb{Q}$.

n-Cob = cat of cobordisms of dim n[illegible] : ob = $\overset{(n-1)}{\cancel{n}}$-dim mflds
arrows = (n[illegible])-dim cobordisms /iso

symmetric monoidal cat.

monoidal functor $\quad n\text{-Cob} \longrightarrow \mathbb{Q}\text{-vect}$

$$\emptyset \longmapsto \mathbb{Q}$$

$$\underset{\substack{\text{closed}\\ n\text{-dim}}}{\emptyset \circlearrowright M} \longmapsto \text{auto of } \mathbb{Q} \in \mathbb{Q}$$

$N \in \text{Ob}\, n\text{-Cob} \rightsquigarrow \text{Map}(N,F) \rightsquigarrow \mathbb{Q}(\text{Map}(N,F)(\mathbb{F}_q))$ = locally constant functions on $\text{Map}(N,F)(\mathbb{F}_q)$ to $\mathbb{Q}$.

(n-1)-mfld

$$\begin{tikzcd}
\text{Map}(M,F) \arrow[r, "q"'] \arrow[d, "p"'] & \text{Map}(N_1,F) \\
\text{Map}(N_0,F) &
\end{tikzcd}$$

$q_! p^* : \mathbb{Q}(\text{Map}(N_0,F)(\mathbb{F}_q)) \longrightarrow \mathbb{Q}(\text{Map}(N_1,F)(\mathbb{F}_q))$ is a tensor functor

which gives rise to the TQFT above

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Simplicial Commutative Rings

$$\begin{tikzcd}
\text{Comm} \arrow[r] \arrow[d] \arrow[rd, "\text{alg stacks}"] & \text{Sets} \arrow[d] \\
\text{sComm} \arrow[r, "\substack{\text{derived (alg)}\\ \text{stacks}}"'] & \text{sSets}
\end{tikzcd}$$

$\text{sComm} = \text{Comm}^{\Delta^{op}}$ = category of simplicial commutative rings.

$A \in \text{sComm} \qquad \pi_*(A) := \bigoplus_{n \geq 0} \pi_n(A) \qquad$ graded abelian group.

(A is a simp Ab gp : $\pi_n(A) := \pi_n(A, 0)$)

$\pi_*(A)$ is a commutative-graded ring (ie comm w/ sign rules in graded-sense)

$\alpha : S^n \to A \quad , \beta : S^m \longrightarrow \cancel{A}$

$$\begin{tikzcd}
S^n \times S^m \arrow[r] \arrow[d] & A \times A \arrow[r, "\mu"] & A \\
S^n \wedge S^m \simeq S^{n+m} \arrow[urr, bend right, "\alpha \cdot \beta"'] & &
\end{tikzcd}$$

(factors through)

sComm $\longrightarrow$ sSet forgetful functor.

Quillen = We define fibrations (and equivalences) as morphisms in sComm whose images in sSets are fibrations and equivalences

$f: A \to B$ equiv $\Leftrightarrow \pi_*(A) \xrightarrow{\sim} \pi_*(B)$

Thm (We) This endows sComm with a (simplicial) model cat structure.

$A \in$ sComm sA-Mod = [illegible] = cat of simplicial A-modules

= simplicial $\overset{M}{\text{ab gps}}$ + map $A \otimes M \longrightarrow M$ + conditions.

sA-Mod $\to$ sSet forgetful. This defines fibrations & equiv in sA-mod

Thm Get a model cat structure on sA-Mod.

$f: A \to B$ in sComm get sA-Mod $\underset{\text{forgetful}}{\overset{-\otimes_A B}{\rightleftarrows}}$ sB-Mod. This is a Quillen adjunction

($\otimes$ is the level wise tensor product)

Get an induced adjunction : Ho(sA-Mod) $\underset{\text{forgetful}}{\overset{-\overset{\mathbb{L}}{\underset{A}{\otimes}}B}{\rightleftarrows}}$ Ho(sB-Mod)

($-\overset{\mathbb{L}}{\underset{A}{\otimes}} B$ is the derived tensor product)

If f is an equivalence then $(-\underset{A}{\otimes} B)$ is a Quillen equivalence and

$M \longmapsto M \overset{\mathbb{L}}{\underset{A}{\otimes}} B$ is an equivalence of categories.

(The converse is also true).

Comm $\hookrightarrow$ sComm constant simplicial object.

It has a left adjoint : sComm $\longrightarrow$ Comm

$A \longmapsto \pi_0(A)$

$\rightsquigarrow$ Comm $\underset{\pi_0}{\overset{}{\rightleftarrows}}$ Ho(sComm) is fully faithful.

~~[illegible]~~ $A \in s\mathrm{Comm}$

$$\begin{tikzcd} \pi_0(A)\text{-mod} \arrow[r, hook] & \mathrm{Ho}(sA\text{-Mod}) \arrow[l, bend left, "\pi_0"] \end{tikzcd}$$ is fully faithful.

Cotangent complexes (André - Quillen cohomology)

- Non-simplicial situation: $A \in \mathrm{Comm}$, $M \in A\text{-Mod}$

  $A \oplus M$ = trivial square zero extension of A by M

  underlying ab gp of $A \oplus M$ is $A \oplus M$.

  Ring structure : $(a,m) \cdot (a', m') := (aa', am' + a'm)$

$$\begin{tikzcd} a \arrow[r, mapsto] & (a,0) & \\ A \arrow[r, "\text{zero-section}"'] & A \oplus M \arrow[r] & A \\ & (a,m) \arrow[r, mapsto] & a \end{tikzcd}$$

$\mathrm{Sect}(A \oplus M \to A) \simeq \mathrm{Der}(A,M)$

$$\rightsquigarrow \begin{array}{rcl} A\text{-Mod} & \longrightarrow & \mathrm{Set} \\ M & \longmapsto & \mathrm{Der}(A,M) \end{array}$$ functorial in M. It is corepresentable, ie

$\exists\, \Omega^1_A \in A\text{-Mod}$ s.t. $\mathrm{Hom}(\Omega^1_A, M) \simeq \mathrm{Der}(A,M)$ functorially in M.

Ω^1_A is called the "cotangent space" of A.

- Simplicial situation: $A \in s\mathrm{Comm}$, $M \in sA\text{-Mod}$.

  $A \oplus M \in s\mathrm{Comm}$ $\quad ((A \oplus M)_n = A_n \oplus M_n)$

  $A \oplus M \to A$ in $s\mathrm{Comm}$

  $A \oplus M \in s\mathrm{Comm}/A \rightsquigarrow A \oplus M \in \mathrm{Ho}(s\mathrm{Comm}/A)$

Def $\mathbb{R}\mathrm{Der}(A,M) := \mathbb{R}\underline{\mathrm{Hom}}_{s\mathrm{Comm}/A}(A, A \oplus M)$; $\underline{\mathrm{Hom}}$ are the simplicial Hom sets,

$\mathbb{R}\underline{\mathrm{Hom}}$ is the derived version: $\mathbb{R}\underline{\mathrm{Hom}}(X,Y) = \underline{\mathrm{Hom}}(Q(X), R(Y))$

We ~~called~~ call it the space of derived derivations of A with coeffs in M.

Thm (Quillen) The functor $$\begin{array}{rcl} sA\text{-Mod} & \longrightarrow & \mathrm{Ho}(s\mathrm{Sets}) \\ M & \longmapsto & \mathbb{R}\mathrm{Der}(A,M) \end{array}$$

is corepresentable by $\mathbb{L}_A \in \mathrm{Ho}(sA\text{-Mod})$. $\mathbb{L}_A$ is called the cotangent cx of A.

⑯ T <u>Rmk</u> * $A \in s\mathrm{Comm}$, $Q(A) \xrightarrow{\sim} A$ cofibrant model

$$\begin{tikzcd} \Omega^1_{Q(A)} \arrow[d, phantom, "\wr"] & \in \mathrm{Ho}\,(sQ(A)\text{-Mod}) \arrow[d, phantom, "\wr"] \\ \mathbb{L}_A & \mathrm{Ho}\,(sA\text{-Mod}) \end{tikzcd}$$

* $\pi_0(\mathbb{L}_A) \simeq \Omega^1_{\pi_0(A)}$

* When A is a non-simplicial comm ring $\mathbb{L}_A \not\simeq \Omega^1_A$ (in general)

* $A \longrightarrow B$ in $s\mathrm{Comm}$ $\rightsquigarrow$ $\mathbb{L}_A \overset{\mathbb{L}}{\underset{A}{\otimes}} B \longrightarrow \mathbb{L}_B \underset{\text{cofiber}}{\longrightarrow} \mathbb{L}_{B/A} \in \mathrm{Ho}\,(sB\text{-Mod})$

relative cotangent cx of B/A

<u>Smooth and étale morphisms</u>

Def $A \xrightarrow{f} B$ in $\mathrm{Ho}(s\mathrm{Comm})$.

1) f is <u>homotopically finitely presented</u> if $\forall$ filtered system (C_α) of objects in $A/s\mathrm{Comm}$ the natural map $\operatorname{colim}_\alpha \mathbb{R}\underset{A/s\mathrm{Comm}}{\underline{\mathrm{Hom}}}(B, C_\alpha) \longrightarrow \mathbb{R}\underset{A/s\mathrm{Comm}}{\underline{\mathrm{Hom}}}(B, \operatorname{colim}_\alpha C_\alpha)$ is an equivalence

$\mapsto B$

2) f is <u>flat</u> if $\mathrm{Ho}\,(sA\text{-mod}) \longrightarrow \mathrm{Ho}(sB\text{ Mod})$ is <u>homotopically left exact</u> (ie commutes with homot. fibered products).

3) f is <u>formally étale</u> if $\mathbb{L}_{B/A} \sim 0$

4) f is <u>formally smooth</u> if $\forall M \in \mathrm{Ho}\,(sB\text{-mod})$ with $\pi_0(M) = 0$,

$[\mathbb{L}_{B/A}, M] = 0$ ($\mathbb{L}_{B/A}$ is of proj. dim ≤ 0).

5) f is <u>smooth</u>, (<u>étale</u>) if it is formally smooth (étale) and homotopically finitely presented (in the sense of 1))

6) f is a <u>Zariski open</u> if it is flat, homotopically finitely presented and $B \overset{\mathbb{L}}{\underset{A}{\otimes}} B \xrightarrow{\sim} B$ in $\mathrm{Ho}\,(sB\text{-Mod})$.

All these properties are stable under composition, base change.

<u>Thm</u> $A \xrightarrow{f} B$ flat, smooth, étale or Zariski open iff

$$\begin{cases} \cdot\ \pi_0(A) \longrightarrow \pi_0(B) \text{ is flat, smooth, étale or Zariski open in the non-simp. case.} \\ \cdot\ \pi_*(A) \underset{\pi_0(A)}{\otimes} \pi_0(B) \xrightarrow{\sim} \pi_*(B) \end{cases}$$

<u>Cor 1</u> ~~A,~~ smooth. étale are flat.

<u>Cor 2</u>. A non-simplicial comm ring, $A \longrightarrow B$ flat in sComm

$\Rightarrow B \xrightarrow{\sim} \pi_0(B)$

<u>Cor 3</u> $\exists$ a Groth topology on $\mathrm{Ho}(s\mathrm{Comm})^{\mathrm{op}}$ with covering families

$\{A \longrightarrow A_i\}$ with $\begin{cases} A \to A_i \text{ étale } \forall i \\ \coprod_i \mathrm{Spec}\, \pi_0(A_i) \longrightarrow \mathrm{Spec}\, \pi_0(A) \text{ surjective.} \end{cases}$

~~Thus topology~~

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Derived stacks and derived algebraic stacks

<u>Reminders</u>.

- $\overset{M^{\mathrm{op}}}{\overset{\parallel}{s\mathrm{Comm}}}$ / model category of simplicial comm $\mathbb{Z}$-algebras.
- smooth, flat, étale, Zariski open maps in $\mathrm{Ho}(s\mathrm{Comm})$
- $\exists$ topology on $\mathrm{Ho}(s\mathrm{Comm})^{\mathrm{op}}$ (étale) w/ covering families $\{A \to A_i\}$ w/

$$\begin{cases} A \to A_i \text{ étale} \\ \coprod_i \mathrm{Spec}\, \pi_0(A_i) \to \mathrm{Spec}\, \pi_0(A) \text{ surjective.} \end{cases}$$

<u>Derived stacks</u> M a model category together w/ a Groth top $\overset{\tau}{}$ on $\mathrm{Ho}(M)^{\mathrm{op}}$.

We want a model cat of $\{M^{\mathrm{op}} \to s\mathrm{Sets}\}$ involving the model category structure on M and the topology τ.

$SPr(M^{\mathrm{op}})$ = model cat of simplicial presheaves on M, with the projective levelwise model structure (fib, equiv are defined levelwise)

Set theory problem solved using universes or small presheaves.

⑬ T $M \underset{\text{(Yoneda)}}{\xhookrightarrow{h}} SPr(M)$ $M^\wedge :=$ left Bousfield localization of $SPr(M)$

$x \longmapsto h_x = Hom(-,x)$ along $\{h_x \xrightarrow{h_u} h_y\}_{u \in W \subseteq M}$

$F: M^{op} \longrightarrow sSet$ is fibrant in $M^\wedge$ iff $\begin{cases} F(x) \text{ is fibrant } \forall x \in M \\ F(u): F(y) \to F(x) \text{ is an equiv } \forall u: x \to y \text{ in } W \subseteq M. \end{cases}$

$Ho(M^\wedge) \simeq \{F \in Ho(SPr(M)) \mid F \text{ is equiv preserving}\} \subseteq Ho(SPr(M))$

<u>Rmk</u> $M^\wedge$ is a model for the quasi-category $\mathcal{U}^{L(M)^{op}}$ $\mathcal{U} = L(SSet_{Kan})$

$$L(M^\wedge) \simeq \mathcal{U}^{L(M)^{op}}$$

$M^\wedge$ is called the model category of prestacks on M.

$F \in Ho(M^\wedge)$, $F: M^{op} \longrightarrow SSet$ equiv preserving (ie take fibrant replacement).

$\pi_0^{Pr}(F): M^{op} \longrightarrow Sets$ factors through (since it is equiv preserving)

$$\begin{tikzcd} M^{op} \arrow[r] \arrow[d] & Sets \\ Ho(M)^{op} \arrow[ur, "\pi_0^{Pr}(F)"'] & \end{tikzcd}$$

Sheaf associated to $\pi_0^{Pr}(F)$ is denoted by $\pi_0(F)$. It is a sheaf on $Ho(M)$ for the topology τ.

$x \in M, s \in F(x)$, $\pi_i(F,s) = ?$ Sheaf on $Ho(M/x)^{op}$. Assume x fibrant.

$\pi_i^{Pr}(F,s) : (M/x)^{op} \longrightarrow Sets$

$(y \xrightarrow{u} x) \longmapsto$ ~~[illegible]~~ $\pi_i(F(y), u^*s)$

If F is equiv preserving, factors through $Ho(M/x)^{op}$

<u>Observation</u>: τ induces a topology on $Ho(M/x)^{op}$ by the forgetful functor

$$Ho(M/x) \longrightarrow \underset{\text{comma topology}}{Ho(M)/x}$$

Take $\pi_i(F,s)$ to be the associated sheaf to $\pi_i^{Pr}(F,s)$. Is ~~[illegible]~~ a sheaf on $Ho(M/x)^{op}$.

Def $f: F \longrightarrow F'$ between equiv preserving simp presheaves is a <u>local equiv</u> if

$$\begin{cases} * \ \pi_0(F) \xrightarrow{\sim} \pi_0(F') \\ * \ \forall x \in M^{fib}, \ \cancel{\forall} s \in F(x), \ \pi_i(F, s) \xrightarrow{\sim} \pi_i(F', f(s)) \quad \forall i > 0 \end{cases}$$

Thm There is a model category structure on $SPr(M)$ where equivalences are the local equivalences; cofibrations are the projective cofibrations: denoted $M^{\sim, \tau}$.

---

* $\exists$ a Yoneda embedding $h: Ho(M) \hookrightarrow Ho(M^\wedge)$
fully faithful (Thm)

$$\begin{tikzcd} & M^\wedge \arrow[d, phantom, "\shortparallel"] & \\ M \arrow[r, hook, "h"] \arrow[dr] & SPr(M) \arrow[r] & Ho(M^\wedge) \\ & Ho(M) \arrow[ur] & \end{tikzcd}$$

* $Ho(M^{\sim, \tau}) \cong \{ F \in Ho(M^\wedge) \mid \text{satisfying descent for hyper covering} \} \subseteq Ho(M^\wedge)$

Def τ is "<u>sub-canonical</u>" if h factors : $Ho(M) \hookrightarrow Ho(M^{\sim, \tau}) \subseteq Ho(M^\wedge)$

$M = sComm^{op}$, τ = étale topology.

Def A <u>derived stack</u> is an object in $Ho((sComm^{op})^{\sim, ét})$

The homotopy category of derived stacks is $Ho((sComm^{op})^{\sim, ét})$

* $dAff = sComm^{op}$ $\quad Ho(dAff) \hookrightarrow Ho(dAff^{\sim, ét})$

Fact: The étale topology is subcanonical.

$$\begin{tikzcd} Comm \arrow[r, hook, shift right, "j"'] & sComm \arrow[l, shift right, "\pi_0"'] \end{tikzcd} \qquad j_! : \begin{tikzcd} Aff^{\sim ét} \arrow[r, shift left] & dAff^{\sim ét} \arrow[l, shift left] \end{tikzcd} : j^* \qquad (Aff = Comm^{op})$$

Quillen pair $\Longrightarrow$ Quillen pair

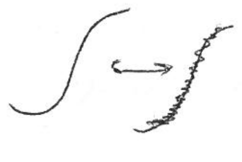
$$i = \mathbb{L} j_! : Ho(Aff^{\sim ét}) \begin{tikzcd} {} \arrow[r, hook, shift left] & {} \arrow[l, shift left] \end{tikzcd} H_o(dAff^{\sim ét}) ; j^* = h^0 \quad (j^* \text{ exact already})$$

Fact: i is fully faithful.

$\Rightarrow$ ~~[illegible]~~ The homotopy category of stacks is a full subcategory of the htpy category of derived algebraic stacks.

$F \in Ho(dAff^{\sim ét})$, $h^0(F)$ is called the classical part of F.

$h^0(F) \longrightarrow F$ "F is some thickening of $h^0(F)$"



⑳ T There are higher truncation functors $h^1, h^2, \ldots$

<u>Def</u> F_0 is a stack. A <u>derived extension</u> of F_0 is a derived stack F together with

$$F_0 \xrightarrow{\sim} h^0(F).$$

Any stack has a trivial extension $i(F_0)$. For moduli problems the "natural" extensions are not trivial.

<u>Def</u> A derived stack F is <u>algebraic</u> if . . . (same definition as nonderived case, check notes).

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Reminder:

$$\begin{tikzcd}
\{Aff = Comm^{op}\} \arrow[r, hook] \arrow[d, "\mathrm{Spec}\ \genfrac{}{}{0pt}{}{A}{\overset{\downarrow}{Hom(A,-)}}"] & \{dAff = sComm^{op}\} \arrow[l, bend right, "\pi_0"'] \arrow[d, "\underline{\mathbb{R}Spec}\ \genfrac{}{}{0pt}{}{A}{\overset{\downarrow}{\mathbb{R}Hom(A,-)}}"] & \leftarrow \text{stable by homotopy limit} \\
\{Alg\ stacks\} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \{derived\ alg\ stacks\} \arrow[l, bend right, "h^0"'] \arrow[d] & \leftarrow \text{stable by finite htpy limit.} \\
\{Stacks\} \arrow[r, "i"] & \{derived\ stacks\} \arrow[l, bend right, "h^0"'] & \boxed{\leftarrow \text{Cartesian!}}
\end{tikzcd}$$

i is <u>not</u> "left exact": doesn't commute w/ homotopy fibered products.

<u>$\mathbb{R}Hom$</u> = derived simplicial Hom's of sComm.

$$\mathrm{Spec}\,A \times_{\mathrm{Spec}\,C} \mathrm{Spec}\,B = \mathrm{Spec}(A \otimes_C B) \text{ as stacks}$$

$$= \underline{\mathbb{R}Spec}\,(A \overset{L}{\otimes}_C B) \text{ as derived stacks.}$$

$$\begin{tikzcd}
F_1 \arrow[dr] & & F_2 \arrow[dl] \\
& F_3 &
\end{tikzcd}$$
of stacks $\quad i(F_1 \overset{h}{\times}_{F_3} F_2) \longrightarrow i(F_1) \overset{h}{\times}_{i(F_3)} i(F_2)$

$$h^0\left(i(F_1) \overset{h}{\times}_{i(F_3)} i(F_2) \right) = F_1 \overset{h}{\times}_{F_3} F_2$$

In other words, $i(F_1) \overset{h}{\times}_{i(F_3)} i(F_2)$ is a derived extension of $F_1 \overset{h}{\times}_{F_3} F_2$

<u>Cotangent Complexes and Obstruction Theory</u>

F is a derived <u>algebraic</u> stack.

k ring. $\quad x : X = \mathrm{Spec}\,k \longrightarrow F$

$M \in D^{\leq 0}(k)$ $\qquad$ $k \oplus M \in s\mathrm{Comm}$

$\wr$

$Ho(s\text{-}k\text{-mod})$ $\qquad$ $X[M] := \mathbb{R}\underline{Spec}\,(k \oplus M)$

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d] & F \\ X[M] \arrow[ur, dotted, "?"', "\text{derivations}"] & \end{tikzcd}$$

<u>Def</u> The space of derivations of F at x with coefficients in M is

$$\mathbb{D}er_x(F, M) := \mathbb{R}\underline{Hom}_{X/dAff^{\sim,et}}(X[M], F) \simeq hofiber_x\big(F(k\oplus M) \to F(k)\big)$$

$dAff^{\sim,et}$ = model category of derived stacks

$\mathbb{R}\underline{Hom}$ = derived simplicial Hom's.

This is functorial in M

$$D^{\leq 0}(k) \longrightarrow Ho(SSets)$$

<u>Thm</u> $\exists\ \mathbb{L}_{F,x} \in D(k)$ s.t $\quad \mathbb{D}er_x(F, M) \simeq \mathbb{R}\underline{Hom}(\mathbb{L}_{F,x}, M)$ functorially in M.

⚠ $\mathbb{L}_{F,x} \notin D^{\leq 0}(k)$ in general because $h^0(F)$ is a stack, not 0-truncated.

<u>Def</u> 1) $\mathbb{L}_{F,x}$ is called the <u>cotangent cx</u> of F at x

2) We say <u>the dimension of F at x is defined</u> if $\mathbb{L}_{F,x}$ is <u>perfect</u> (ie q-isom to a bounded cx of projective mod/k of finite rank).

3) If it is so $\dim_x F := [\mathbb{L}_{F,x}] \in K_0(k)$

When k is a local ring $\rightsquigarrow K_0(k) \longrightarrow \mathbb{Z}$ $\quad \dim_x F \in \mathbb{Z}$.

$\dim_x F$ is called the "<u>virtual dimension</u>".

In general $\dim_x F \neq$ usual dimension of $h^0(F)$ at x

<u>Rmk</u> F is a scheme, then $\mathbb{L}_{F,x} \simeq \mathbb{L}_{x^*}(\mathbb{L}_F)$ where $\mathbb{L}_F$ is usual cotangent cx.

<u>Obstruction theory</u>

$I \to k \to k_0$ square zero ext of comm rings.

$I^2 = 0$ $\quad$ I kernel.

(22) T

$$\begin{tikzcd} X_0 := \mathrm{Spec}\, k_0 \arrow[r, "x"] \arrow[d] & F \\ X := \mathrm{Spec}\, k \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

F derived alg stack

Thm $\exists$ a class $O(x) \in \mathrm{Ext}^1(\mathbb{L}_{F,x}, I[1])$ s.t

$O(x) = 0$ iff a lift exists

When $O(x) = 0$, the space of lifts $\underline{\mathbb{R}\mathrm{Hom}}_{X_0/d\mathrm{Aff}^{\sim,et}}(X, F)$ is a torsor

under the simplicial gp $\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{L}_{F,x}, I)$

In particular it is isomorphic to $\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{L}_{F,x}, I)$

Idea of pf $\exists$ a htpy cart. square in sComm

$$\begin{tikzcd} k \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \sim"] & k_0 \arrow[d, "x"] \\ k_0 \arrow[r, "d"'] & k_0 \oplus S(I) \end{tikzcd}$$

$S(I)$ is the simplicial k_0-module

$\pi_i(S(I)) = 0 \quad i \neq 1$

$\pi_1(S(I)) = I$.

$$\begin{tikzcd} \mathbb{R}\underline{\mathrm{Spec}}(k_0 \oplus S(I)) = X_0[S(I)] \arrow[r] \arrow[d] \arrow[rr, bend left, "O(x)"] & X_0 \arrow[r, "x"] \arrow[d] \arrow[l, bend left] & F \\ X_0 \arrow[r] & X \arrow[ur, dashed, "x'"'] & \end{tikzcd}$$

is homotopy CoCartesian in the category of $\underline{\text{derived alg stacks}}$ (not as a square of stacks)

$o(x) \in \pi_0 \mathrm{Der}_x(F, S(I)) \simeq [\mathbb{L}_{F,x}, S(I)] = [\mathbb{L}_{F,x}, I[1]]$

x' exists iff $o(x) = 0$

Prop $O(x)$ is functorial in F, ie

$$\begin{tikzcd} X_0 \arrow[r, "x"] \arrow[d] & F \arrow[r, "f"] & F' \\ X & & \end{tikzcd}$$

$$\mathbb{L}_{F', f(x)} \xrightarrow{f^*} \mathbb{L}_{F,x}$$

$$\Rightarrow [\mathbb{L}_{F,x}, I[1]] \xrightarrow{f^*} [\mathbb{L}_{F', f(x)}, I[1]]$$

$$f^*(o(x)) = o(f(x))$$

In practice: when F_0 is a non derived moduli problem, F its "natural derived extension"

$$x \in F_0(k) \longrightarrow F(k)$$

$\mathbb{L}_{F_0,x}$ is in general unknown (no description in terms of the moduli problem)

<u>But</u> $\mathbb{L}_{F,x}$ is very easy to describe! (in terms of the moduli problem).

(see an example of $Rep(\Gamma)$).

$$\begin{tikzcd} F \arrow[r, "f"] & F' \\ F_x \arrow[u] \arrow[r] \arrow[ur, phantom, "{}^{L}\ \sim"] & \operatorname{Spec} k = X \arrow[u, "x"'] \arrow[l, bend left, "y"] \end{tikzcd}$$

$\mathbb{L}_{F', f(x)} \longrightarrow \mathbb{L}_{F,x} \rightarrow \mathbb{L}_{F_x, y}$ is a htpy cofiber sequence

This is not true in the nonderived setting.

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<u>Further Results</u>

1) <u>**Artin's** representability thm</u> :

<u>Thm</u> (Lurie) F a derived stack is algebraic and locally of finite presentation iff

a) $h^0(F)$ is n-truncated for some n.

b) $h^0(F)$ is locally of finite presentation : $\forall$ filtered diagram A_α of rings

$$\operatorname{colim}_\alpha h^0(F)(A_\alpha) \xrightarrow{\sim} h^0(F)(\operatorname{Colim}_\alpha A_\alpha) \Leftrightarrow \operatorname{colim}_\alpha F(A_\alpha) \xrightarrow{\sim} F(\operatorname{Colim}_\alpha A_\alpha)$$

c) Formal deformations are algebraizable : $\forall$ complete local noetherian ring

$$\hat{A} = \lim_n \hat{A}/m^n \quad , \quad F(\hat{A}) \xrightarrow{\sim} \operatorname{holim}_n F(\hat{A}/m^n)$$

d) F is Postnikov complete : $F(A) \xrightarrow{\sim} \operatorname{holim}_k F(\tau_{\leq k} A)$

$\forall A \in sComm$, $\tau_{\leq k} A \rightarrow \tau_{\leq k-1} A \rightarrow \cdots$ Postnikov tower.

e) F has a good infinitesimal theory :

* $\forall$ $X = \mathbb{R}\operatorname{Spec} A \xrightarrow{x} F$, $\mathbb{D}er_x(F,-) : Ho(sA\text{-mod}) \longrightarrow Ho(SSets)$

is corepresentable by $\mathbb{L}_{F,x} \in D(N(A))$ which is compact. $\left([\mathbb{L}_{F,x}, \oplus -] \simeq \oplus [\mathbb{L}_{F,x}, -] \right)$

** $\forall A_\bullet \in sComm$, $\forall M \in sA_\bullet\text{-mod}$

$$\begin{tikzcd} \mathbb{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A_\bullet \arrow[d, "0"] \\ A_\bullet \arrow[r, "\alpha"'] & A_\bullet \oplus S(M) \end{tikzcd} \qquad \begin{tikzcd} F(\mathbb{B}) \arrow[r] \arrow[d] & F(A_\bullet) \arrow[d] \\ F(A_\bullet) \arrow[r] & F(A_\bullet \oplus S(M)) \end{tikzcd}$$

homotopy Cartesian

(24) T **Cor** X is a flat proper scheme (DM stack) over Spec k.
Y is a derived algebraic stack of finite presentation over Spec k.

Then $\mathbb{R}\underline{\mathrm{Hom}}(X, Y)$ is a derived algebraic stack.
derived Hom of $\mathrm{Ho}(dAff^{\sim, et})$

2) Derived categories

F a derived (algebraic) stack. We want to define $D_{q\text{-}coh}(F)$.

$F = \mathbb{R}\mathrm{Spec}\, A$, then $D_{q\text{-}coh}(F) = D(N(A))$ $N(A) \in$ dg alg. (normalized ring)

$$\begin{tikzcd} A \arrow[r, mapsto] \arrow[d] & N(A)\text{-dg mod} \arrow[d, "- \otimes_{N(A)} N(B)"] \\ B & N(B)\text{-dg mod} \end{tikzcd}$$

(a model cat)
W.e: quasi-iso
fib: epi

$\rightsquigarrow$ left Quillen

This is a left Quillen presheaf on $sComm^{op}$.

$$\begin{tikzcd} A \arrow[r, mapsto] \arrow[d] & (N(A)\text{-dgmod}^{cf}, W) \arrow[r, mapsto, "L"] \arrow[d, "- \otimes_{N(A)} N(B)"] & L_W(N(A)\text{-dgmod}^{cf}) \arrow[d] \\ B \arrow[r, mapsto] & (N(B)\text{-dgmod}^{cf}, W) \arrow[r, mapsto, "L"] & L_W(N(B)\text{-dgmod}^{cf}) \end{tikzcd}$$

Where $L_W(N(A)\text{-dg}^{cf})$ is the quasi-category obtained by localizing.

$\underline{Qcoh}$: $dAff^{op} = sComm \longrightarrow QCat$ - Ssets + Joyal model structure

$\rightarrow Q\underline{Coh} \in \mathrm{Ho}(QCat^{sComm})$

Thm $Q\underline{Coh}$ is a derived stack in quasi-categories, ie

a) $Q\underline{coh}$ sends equiv to equiv

b) $Q\underline{Coh}$ has descent for the étale topology:

$$Q\underline{Coh}(A) \xrightarrow{\sim} \underset{n \in \Delta}{\mathrm{holim}}\, Q\underline{Coh}(B_n)$$

$A \longrightarrow B_*$ étale hypercovering in sComm

$\mathrm{Ho}(\text{derived stacks})$
$\cap$ $\nearrow$ (it is a localization)

the ones that factor through Kan cxs $\subseteq$ QCat.

SSet
$\downarrow \uparrow$
QCat

$Q\underline{Coh} \in \mathrm{Ho}(\text{Derived Quasi stacks})$
$\wr\wr$
$\{F : sComm \to QCat \mid a) \,\&\, b)\} \subseteq \mathrm{Ho}(QCat^{sComm})$

Def The quasi-category of quasi-coherent cxs on F is

$$\mathbb{R}\underline{\mathrm{Hom}}\,(F, Q\underline{\mathrm{Coh}}) =: L_{QCoh}(F)$$

$\uparrow$
derived Hom w/ values in $Ho(QCat)$

The derived category of F is $D_{qcoh}(F) := \tau_{\le 1}(L_{qcoh}(F))$

Rmk - This is a stable quasi-category, ie: it has finite limits and

$$\Omega_* : \tau_{\le 1}(L_{qcoh} F)) \xrightarrow{\sim} \tau_{\le 1}(L_{qcoh}(F))$$
$$x \longmapsto * \times_x *$$

$\Rightarrow$ $D_{qcoh}(F) = \tau_{\le 1}(L_{qcoh}(F))$ has a natural triangulated structure.

Triangles = images of fibration sequences.

$$\Omega Y \longrightarrow F \to X \to Y$$

- $D_{qcoh}(\mathbb{R}\,\mathrm{Spec} A) \simeq D(N(A)) \Rightarrow D_{qcoh}(X) \simeq D_{qcoh}(\mathcal{O}_X)$, X scheme.

$$\begin{tikzcd} F \arrow[r, mapsto] \arrow[d, "f"] & D_{qcoh}(F) \arrow[d, shift right, "f_*"'] \\ F' & D_{qcoh}(F') \arrow[u, shift right, "f^*"'] \end{tikzcd}$$

is functorial

adjunction

Thm
$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner"] & Y' \arrow[d, "p"] \\ X \arrow[r, "q"'] & Y \end{tikzcd}$$
fiber product of derived schemes (quasi-compact, separated, or DM stacks in char 0).

$$p^* q_* : D_{qcoh}(X) \longrightarrow D_{qcoh}(Y')$$
$$\wr\!\cong$$
$$g_* f^*$$

No flatness assumption!

Ex Donaldson-Thomas invariant

X smooth proj $/\mathbb{C}$ CY $(\omega_X = \mathcal{O}_X)$ dim 3.

$$\begin{tikzcd}
T_{\underline{Coh}_0(X)} \arrow[r, hook] \arrow[d] & \underline{Coh}(X) \arrow[d] \\
R\underline{Coh}_0(X) \arrow[r, hook] & \mathbb{R}\underline{Coh}(X)
\end{tikzcd}$$

(26) $\underline{Coh}(X)$ an algebraic stack of coherent sheaves on X

$\mathbb{R}\underline{Coh}(X)$ derived algebraic stack of coherent sheaves on X.

substack of coherent sheaves with $c_1 = 0$" It is the fiber at the trivial object of the map $\underline{Coh}(X) \longrightarrow Pic(X)$

$$\begin{tikzcd}
\underline{Coh}^{st,\nu}(X) \arrow[r, phantom, "\overset{open}{\subseteq}"] \arrow[d] & \underline{Coh}(X) \arrow[r, hook] & \underline{Coh}(X) \arrow[d] \\
\mathbb{R}\underline{Coh}^{st,\nu}(X) \arrow[r, phantom, "\subseteq"] & \mathbb{R}\underline{Coh}_0(X) \arrow[r, hook] & \mathbb{R}\underline{Coh}(X)
\end{tikzcd}$$

$\shortparallel$ coherent sheaves, stable + fixed numerical invariants ν

In good cases $\mathbb{R}\underline{Coh}^{st,\nu}(X)$ is a proper derived DM stack / $\mathbb{C}$ ...

$H^0(\mathbb{T}_E) \simeq Ext_0^1(E,E) = $ trace $\circ$ ext.

$H^1(\mathbb{T}_E) \simeq Ext_0^2(E,E)$

$H^i(\mathbb{T}_E) = 0 \quad i \neq 0,1$

$\Rightarrow D_{qcoh}(\mathbb{R}\underline{Coh}_0^{st,\nu}(X)) \xrightarrow{p_*} D_{qcoh}(\mathbb{C}) = D(\mathbb{C})$ preserves bounded coh cx's.

$p : \mathbb{R}\underline{Coh}_0^{st,\nu}(X) \longrightarrow Spec\,\mathbb{C}$

$p_* : D^b_{qcoh}(\mathbb{R}\underline{Coh}_0^{st,\nu}(X)) \longrightarrow D^b_{coh}(\mathbb{C})$

$DT_\nu(X) := \chi(p_*(\mathcal{O}_{\mathbb{R}\underline{Coh}\ldots})) \in \mathbb{Z}.$

Rmk *) $\dim_E \mathbb{R}\underline{Coh}_0^{st,\nu}(X) = 0$ "virtually it is a finite # of pts"

$\Rightarrow$ Euler characteristic $\chi(p_*(\mathcal{O}_{\mathbb{R}\underline{Coh}}))$ counts the virtual # of pts in $\mathbb{R}\underline{Coh}_0^{st,\nu}(X)$.

**) Base change $\Rightarrow$ $X \to S$ smooth proj. family of CY var of dim 3,

$s \in S \longmapsto DT_\nu(X_s)$ is <u>locally constant</u>.

2/4/08 / CRM / Dendroidal Sets

Operads

An operad P (in topological spaces) consists of spaces $P(n)$, $n \geq 0$, and structure maps.

Think of $P(n)$ as space of operations of n variables

- * special point $1 \in P(1)$
- * substitution $P(n) \times P(k_1) \times \dots \times P(k_n) \longrightarrow P(k)$ $\quad k = k_1 + \dots + k_n$

$$(p, q_1, \dots, q_n) \longmapsto p(q_1, \dots, q_n)$$

- * "permutation of variables" : Σ_n acts from the right on $P(n)$.

The axioms express that these operations have their intended meaning.

For emphasis, we sometimes speak of symmetric operads.

A non-symmetric operad is one without the permutations.

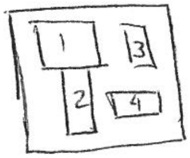
An algebra for an operad P is a space A together with an action by the operations : $P(n) \times A^n \longrightarrow A$ $\quad (p, a_1, \dots, a_n) \longmapsto p(a_1, \dots, a_n)$

compatible with the structure maps.

Ex 1) $P(n) = *$ for all n. A P-algebra is the same as a commutative monoid. This P is denoted Comm.

2) $P = Ass$, $Ass(n) = \Sigma n$, substitution like substitution of block matrices.
The algebras are monoids (in spaces).

3) $P(n) = D_2(n)$ = configuration space of n numbered little rectangles in the unit square.



$\in D_2(4)$. The algebras "are" double loop spaces (up to homotopy).

2) Rmk Every non-Σ operad "generates" a symmetric operad with the same algebras.

Eg. The non-Σ operad P with $P(n) = *$ has as algebras the monoids, and Ass is its associated symmetric operad.

Rmk We can replace the "ambient" category of top spaces by any symmetric monoidal category.

Colored operads

A colored operad $\mathbb{P} = (C, P)$ is a pair consisting of a set C of colors and for each sequence $c_1, \dots, c_n; c$ a space (or set, or object of some symm mon cat)

$$P(c_1, \dots, c_n; c)$$

of operations which take inputs of "types" (or colors) $c_1, \dots, c_n$ to an output of type c.

Similar structure maps:

- $1_c \in P(c; c)$ for every $c \in \mathcal{C}$
- Substitution

$$P(c_1, \dots, c_n; c) \times P(d_1^1, \dots, d_{k_1}^1; c_1) \times \cdots \times P(d_1^n, \dots, d_{k_n}^n; c_n) \longrightarrow P(d_1^1, \dots, d_{k_1}^1, d_1^2, \dots, d_{k_n}^n; c)$$

- The symmetric group Σ_n acts: any $\sigma \in \Sigma_n$ gives a map

$$\sigma^*: P(c_1, \dots, c_n; c) \longrightarrow P(c_{\sigma(1)}, \dots, c_{\sigma(n)}; c)$$

A P-algebra A consists of spaces (or...) A_c, $c \in C$ together with action

$$P(c_1, \dots, c_n; c) \times A_{c_1} \times \cdots \times A_{c_n} \longrightarrow A_c$$

Rmk There is a corresponding notion of non-Σ colored operad (and an associated notion of algebra). Every non-Σ colored operad generates a (symmetric) Colored operad with the same algebra.

A <u>map</u> of colored operads from $\mathbb{P} = (C, P)$ to $\mathbb{Q} = (D, Q)$ is given by a map $f : C \longrightarrow D$ and maps $\varphi = \varphi_{c_1,\dots,c_n;c}$ $P(c_1,\dots, c_n; c) \longrightarrow Q(fc_1, \dots fc_n; fc)$ compatible with all the structure maps.

Examples

1) Excercise : write down an operad on two colors whose algebras are pairs (A, M), A a monoid and M a space on which acts (or rings and modules)

2) For given set S, there is a <u>non-Σ operad</u> Cat_S (or "Ass_S") whose algebras are categories with the set S as objects.
$S \times S$ is the set of colors.

$$Cat_S\left((s_1, s_1'), \dots, (s_n, s_n'); (s, s')\right) = \begin{cases} pt & s_1' = s_2 ;\ s_2' = s_3, \text{etc}\quad \begin{matrix} s_1 = s \\ s_n' = s' \end{matrix} \\ \emptyset & \text{ow} \end{cases}$$

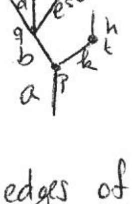
$$Cat_S\left(-; (s, s')\right) = \begin{cases} pt & s = s' \\ \emptyset & \text{ow} \end{cases} \qquad n = 0$$

For a Cat_S-algebra A, the set $A_{(s,s')}$ is the set of arrows from S to S'.

Every planar tree T defines a non-Σ operad $\Omega_p(T)$

Every tree T " " symmetric operad $\Omega(T)$

A <u>tree</u> has input edges and an output edge and in the non-planar case we number the input edges.



- input edges
— out put edges
-- internal edges

The colors of $\Omega_p(T)$ are <u>the edges</u> of the tree. (same for $\Omega(T)$)
The operations of $\Omega_p(T)$ (resp $\Omega(T)$) are generated by the vertices of the tree.

Eg $p \in \Omega_p(T)(b, k; a)$ $r \in \Omega_p(T)(-; d)$ $t \in \Omega_p(T)(h; k)$ generators.

④M induced operations are $1_x \in \Omega_p(T)(x;x)$ for $x \in \{a, b, \dots; k\}$

$$p \circ_1 q = p(q; 1_k) \in \Omega_p(T)\ (c, d, e, k; a)$$

In $\Omega(T)$, we also have operations like $(p \cdot \sigma) \in \Omega(T)(k, b; a)$

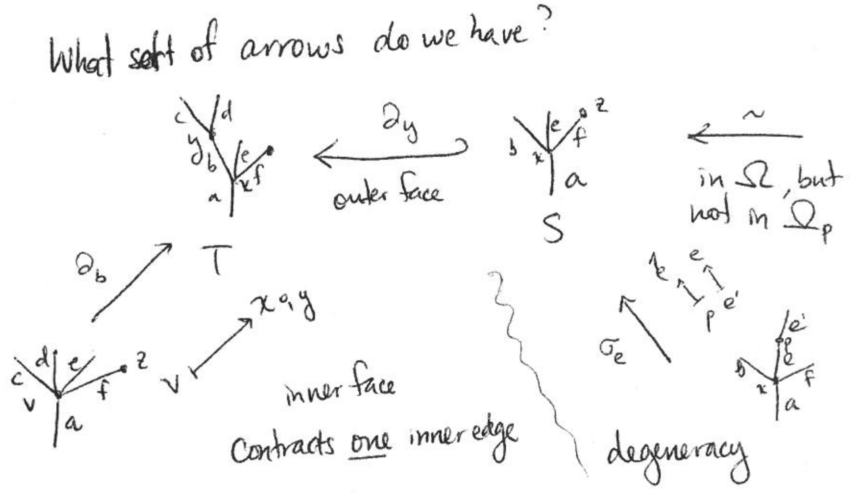
In the nonplanar case these two trees are the same :  and the previous.

For a non-planar tree, the choice of a planar structure amounts to the choice of specifig generators for $\Omega(T)$.

Def Ω is the full subcategory of the category of colored operads whose objects are of the form $\Omega(T)$.

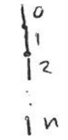
Ω_p " " " " " " non-Σ " " " $\Omega_p(T)$.

What sort of arrows do we have?



$k \mapsto f$
$i \mapsto a$
$j \mapsto b$
$l \mapsto e$
$w \mapsto x\sigma$ for the appropriate $\sigma \in \Sigma_3$.

$$\Delta \xhookrightarrow{i} \Omega$$

$[n] \longmapsto$ 

inner faces, outer faces and degeneracies correspond.

2/5/08 // CRM // Dendroidal Sets

$$\begin{tikzcd}
\Omega \\
\Omega_p \arrow[u, "u"] \\
\Delta \arrow[u, "v"] \arrow[uu, bend right, "i"']
\end{tikzcd}$$

faces, iso's, degeneracies

faces & degeneracies

Presheaf categories

$\hat{\Omega}$ dendroidal sets

$\hat{\Omega}_p$ planar dendroidal sets

$\hat{\Delta}$ simplicial sets

$$\begin{tikzcd}
\text{dsets} \arrow[d, "u^*"] \arrow[dd, bend left, "i^*"] \\
\text{pd sets} \arrow[d, leftrightarrow, "v^*"] \\
\text{s Sets}
\end{tikzcd}$$

There are evident restriction functors i^*, u^*, v^*, which all have left adjoints $i_!, v_!, u_!$

$$\begin{tikzcd}
\text{s Sets} \arrow[r, "v_!"] \arrow[dr, "i_!"'] & \text{pdsets} \arrow[d, "u_!"] \\
& \text{dsets}
\end{tikzcd}$$

$v_!$, $i_!$ are fully faithful.

The representable object in dsets given by a tree T will be denoted by $\Omega[T]$.

(just like $\Delta[n]$ in sSets) $i_!(\Delta[n]) = \Omega[i[n]]$

In general, a dendroidal set X will have <u>dendrices</u> $x \in X(T) = X_T$ of "dimension" or "shape" T.

The representible object $\Omega[1] = i_! \Delta[0]$ will be denoted U, which will be a unit in a monoidal structure on dsets. Note $U \leq 1$ (the map to the terminal object is mono)

$$\begin{tikzcd}
\text{ssets} = \text{dsets}/U \arrow[d, "i_!"'] & \\
\text{dsets} & \arrow[l, "\text{forget}"]
\end{tikzcd}$$

(sset s is an open subtopos of dsets)

For a tree T, let $P(T)$ be the set of planar structures on T, (it is contravariant), and

This an object of dendroidal sets

$$\underset{\substack{\Omega \\ \text{category of elts}}}{\int} P = \Omega_p$$

$$\begin{tikzcd}
\text{pdsets} = \text{dsets}/P \arrow[d, "u_!"'] & \\
\text{dsets} & \arrow[l, "\text{forget}"]
\end{tikzcd}$$

ssets $\to$ pdsets $\to$ dsets

dsets/U $\to$ dsets/P $\to$ dsets

canonical maps

unique map $U \to P$

⑥ M

$$\begin{tikzcd}
\text{ssets} \arrow[r, shift left, "i_!"] \arrow[d, shift right, "\tau_1"'] & \text{dsets} \arrow[l, shift left, "i^*"] \arrow[d, shift right, "\tau_d"'] \\
\text{Cat} \arrow[r, shift left, "j_!"] \arrow[u, shift right, "N"'] & \text{Operads} \arrow[l, shift left, "j^*"] \arrow[u, shift right, "N_d"']
\end{tikzcd}$$

N_d = dendroidal nerve
τ_d = its left adjoint
j^* = just take the unary operation
$j_!$ = its left adjoint; it views a category as an operad w/ unary operations only.

$$N_d(P)_T = \text{Hom}_{\text{operad}}(\Omega(T), P) \quad ; \text{eg} \quad \underbrace{N_d(\underbrace{\Omega(S)}_{\text{operad}})}_{\text{dset}} = \Omega[S]$$

$i_!, j_!, N, N_d$ are fully faithful.

Not everything commutes! (Even though there is a lot of commutation).

eg $j^* \tau_d \neq \tau i^*$

There is also a column in the middle, relating pdsets & non-Σ operad.

Monoidal structure

ssets : cartesian closed $\times$

pdsets : simplicially enriched w/ tensors and cotensors

dsets : closed symmetric monoidal $\otimes$

$i_!(X \times Y) = i_!(X) \otimes i_!(Y)$

Boardman - Vogt tensor product of operads for $\mathbb{P} = (C, P)$, $\mathbb{Q} = (D, \mathbb{Q})$,

$\mathbb{P} \otimes_{BV} \mathbb{Q}$ is an ~~tensor~~ operad whose set of colors is $C \times D$ and whose operations

are generated by

$p \otimes d \in P \otimes Q((c_1, d), \ldots, (c_n, d); (c, d))$ for $p \in P(c_1, \ldots, c_n; c)$, $d \in D$

$c \otimes q \in P \otimes Q((c, d_1), \ldots, (c, d_m); (c, d))$ for $c \in C$, $q \in Q(d_1, \ldots, d_m; d)$

Relations: These two operations distribute over each other. (using symmetries ...)

* (NB) We don't need symmetries if either p or q is unary.

Tensor product of dendroidal sets is defined by:

$$\begin{cases} X \otimes Y \text{ preserves colimits in each variable} \\ \Omega[S] \otimes \Omega[T] := N_d(\underbrace{\Omega(S) \otimes_{BV} \Omega(T)}_{\text{operad, but not representable by a tree.}}) \end{cases}$$

M ⑦

This will automatically be closed

$$\underline{\mathrm{Hom}}(X, Y)_T = \mathrm{Hom}_{dsets}(\Omega[T] \otimes X, Y)$$

By ⊛, we can tensor non-Σ operads with operads having unitary operations only.

$$\begin{tikzcd} \text{ssets} \arrow[r, "v_!"] & \text{pdsets} \\ S & X, Y \end{tikzcd}$$

~~$X \otimes Y$~~ doesn't make sense

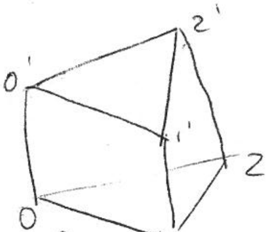
but $v_!(S) \otimes Y$ does make sense.

So pdset is a simplicial category, with $\underline{\mathrm{Hom}}(X, Y)_n = \mathrm{Hom}_{pdsets}(v_!(\Delta[n]) \otimes X, Y)$

and has cotensors $(Y^S)_T = \mathrm{Hom}_{pdsets}(\Omega_p[T] \otimes v_!(S), Y)$

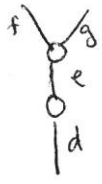
$\tau_d(X \otimes Y) = \tau_d(X) \otimes_{BV} \tau_d(Y)$, X, Y dsets

What does $\Omega[S] \otimes \Omega[T]$ look like?

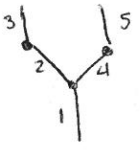


pecial
ase)

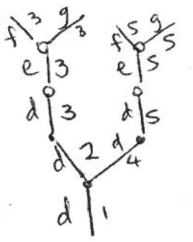
$$\Delta[n] \times \Delta[m] = \bigcup_{\substack{n,m \\ \text{shuffles}}} \Delta[n+m]$$

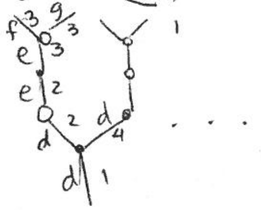


S



T





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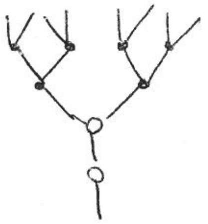
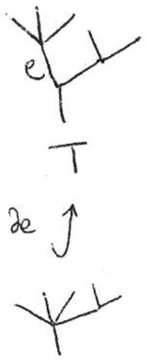


union of 14 dendrices, We can see how the glue together when they have faces in common.

2/6/08 // CRM // Dendroidal sets



$\Omega[T] \in$ dsets

$\partial\Omega[T]$ = union of all the faces

$\Lambda^e[T] \rightarrowtail \Omega[T]$ union of all faces except the one given by contracting the inner edge e.

inner horn

in this case, inner horns correspond to those in ssets.

③ᴹ <u>Rmk</u> A dendroidal set X is the nerve of an operad $(X \cong N_d(P))$ iff every inner horn has a unique filler

$$\begin{tikzcd} \Lambda^e[T] \arrow[r] \arrow[d, tail] & X \\ \Omega[T] \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

<u>Def</u> A dendroidal set X which has this extension property (but not necessarily uniquely) will be called, alternatively, <u>quasi-operad</u>, (dendroidal) <u>inner Kan cxs</u>, <u>infinity operad</u>.

Let X be a dendroidal set, $x \in X_T$. x is called <u>degenerate</u> if $\exists\, T \underset{\text{degeneracy}}{\overset{\sigma}{\twoheadrightarrow}} S$ and $y \in X_S$ with $x = \sigma^* y$.

A non-degenerate $x \in X_T$ is called <u>normal</u> if the only $\alpha \in \mathrm{Aut}(T)$ which fixes x is the identity.

$$\left(\alpha : T \to T \ ; \ \alpha^* : \begin{array}{l} X_T \longrightarrow X_T \\ x \longmapsto \alpha^T(x) \end{array} \right)$$

A monomorphism $X \rightarrowtail Y$ is called <u>normal</u> if all the non-deg dendrices in $Y-$ in X are normal.

Y is <u>normal</u> if $\emptyset \rightarrowtail Y$ is normal.

We'll see that normal mono's / objects are cofibrations / cofibrant objects.

The normal mono's can be obtained by saturating the class of boundary inclusions $\partial\Omega[T] \hookrightarrow \Omega[T]$ or alternatively by saturating the class of images under u_1 of monos in pdsets

$$\begin{tikzcd} \text{dsets}/p \arrow[d, equal] \arrow[dr, "\text{forget}"] & \\ \text{pdsets} \arrow[r, "u_1"'] & \text{dsets} \end{tikzcd}$$

The saturation of the class of inner horn inclusions $\Lambda^e[T] \hookrightarrow \Omega[T]$ will be referred to as the class of <u>(inner) anodyne extensions.</u>

The maps with the RLP wrt these will be referred to as <u>inner Kan fibrations.</u>

Prop If $A \hookrightarrow B$ is normal & $C \hookrightarrow D$ is inner anodyne then

$$A\otimes D \underset{A\otimes C}{\cup} B\otimes D \hookrightarrow B\otimes D$$

is inner anodyne.

This follows formally from the case $\partial\Omega[S]\otimes\Omega[T] \underset{\partial\Omega[S]\otimes\Lambda^e[T]}{\cup} \Omega[S]\otimes\Lambda^e[T] \hookrightarrow \Omega[S]\otimes\Omega[T]$

This case is proved by hand by analyzing the shuffles.

There are various formal corollaries obtained by playing around with

$$\begin{tikzcd} A\otimes D \cup B\otimes C \arrow[r] \arrow[d] & Y \arrow[d] & & A \arrow[r] \arrow[d] & \underline{Hom}(D,Y) \arrow[d] \\ B\otimes D \arrow[r] \arrow[ur, dashed, "?"] & X \arrow[r, phantom, "\Longleftrightarrow"] & & B \arrow[r] & \underline{Hom}(C,Y) \underset{\underline{Hom}(C,X)}{\times} \underline{Hom}(D,X) \end{tikzcd}$$

Such as: B normal, Y inner Kan $\Rightarrow \underline{Hom}(B,Y)$ inner Kan.

Diagrams of simplicial sets

$\mathbb{S}^{op} \xrightarrow{X} \text{sSets}$ ($\mathbb{S}$ small indexing category).

$\underset{\mathbb{S}}{\text{hocolim}}\, X$, the homotopy colimit. It can be constructed in many ways, eg

first $\underset{\substack{\mathbb{S} \\ \text{simplicial cat}}}{\int} X$, then $N \underset{\substack{\mathbb{S} \\ \text{bisset}}}{\int} X$ then $\text{diag}(N\int_{\mathbb{S}} X)$

Recall if $\mathbb{S}^{op} \xrightarrow{A} \text{Sets}$ then $\int_{\mathbb{S}} A = \mathbb{S}/A$ has as objects (s,a), $s \in \mathbb{S}$, $a \in A(s)$ & arrows $(s,a) \longrightarrow (t,b)$ are $f: s \to t$ in $\mathbb{S}$ with $\underset{A(f)(b)}{f^*(b)} = a$

If X came from a diagram of categories
$$\begin{tikzcd} \mathbb{S}^{op} \arrow[r, "F"] \arrow[dr, "N\circ F = X"'] & Cat \arrow[d, "N"] \\ & \text{SSets} \end{tikzcd}$$
then

$$\underset{\mathbb{S}}{\text{hocolim}}\, X \overset{\text{whe}}{\simeq} N\Big(\underbrace{\int_{\mathbb{S}} F}_{\text{category}}\Big)$$

$\int_{\mathbb{S}} F$ has as objects (s,x), $s \in \mathbb{S}$, $x \in F(s)$ object, arrows $(s,x) \longrightarrow (t,y)$ $\underset{\text{in } \mathbb{S}}{f: s \to t}$; $\underset{\text{in } F(s)}{\varphi: x \longrightarrow f^*(y)}$

⑩ M **Question** is there is a construction $\int_{\mathcal{S}} X \in sSet$ for $X : \mathcal{S}^{op} \longrightarrow sSets$,

for which $\int_{\mathcal{S}} N \circ F \cong N\left(\int_{\mathcal{S}} F\right)$?

Yes! An n-simplex of $\int_{\mathcal{S}} X$ is a pair $(s_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} s_n, x)$ where $s_0 \to \dots \to s_n \in N(\mathcal{S})_n$, and x assigns to each $u : [k] \to [n]$ in Δ an elt $x_u \in X(s_{u(0)})_k$, in a natural way.

$$\begin{tikzcd}
{[k]} \arrow[r, "u"] & {[n]} \\
{[l]} \arrow[u, "v"] \arrow[ur, "w"'] &
\end{tikzcd}$$

$s_0 \to \cdot\cdot\; s_{u(0)} \to \cdots s_{w(0)} \quad s_n$

there is a composite of f_i's, called $f_{w,u} : s_{u(0)} \to s_{w(0)}$

$f^*_{w,u}(x_w) = v^*(x_u)$ equality of elts of $X(s_{u(0)})_l$

__Guess__: this is also a model for $\operatorname{hocolim}_{\mathcal{S}} X$

__Fact__: If each of the $X(S)$ is Kan (inner), then so is $\int_{\mathcal{S}} X$

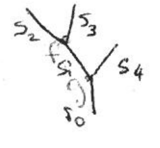
Diagrams of dendroidal sets

$\mathcal{S}^{op} \xrightarrow{X}$ ~~d~~sets Assume $\mathcal{S}$ has finite products.

10k Any symmetric monoidal category $\mathcal{E}$ can be viewed as a colored operad $\mathcal{E}$, where colores are the objects of $\mathcal{E}$, and $\mathcal{E}(c_1, \dots, c_n; c) = \operatorname{Hom}_{\mathcal{E}}(c_1 \otimes \cdots \otimes c_n, c)$ hence it determines a dendroidal set, denoted $N_d(\mathcal{E})$.

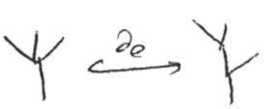
__Example__: for our $\mathcal{S}$ and a tree T

an element of $N_d(\mathcal{S})_T$ is given by a labelling



$T =$ 

$f : s_2 \times s_3 \to s_1 \quad g : s_1 \times$ ~~s_4~~ $\to s_0$



∂_e of this elt is $\quad s_2 \times s_3 \times s_4 \longrightarrow s_0$

$g \circ (f \times s_4)$

We define a dendroidal set $\int_{\mathcal{S}} X$, whose dendrices of shape T are pairs (t, x) where $t \in N_d(\mathcal{S})_T$ and x assigns to each $u : S \to T$ an elt of $X(\operatorname{in}(u^* t))_S$

determines an object in $(t) \in \mathcal{S}$

$u: S \to T$ induces a map from $in(t) \to \overset{in(u^*t)}{\cancel{in(s)}}$ in $\$$

Compatible in the following way

in Ω

$$\begin{tikzcd} S \arrow[r, "u"] & T \\ R \arrow[u, "v"] \arrow[ur, "w"'] & \end{tikzcd}$$

$x_u \in X(in(u^*t))_S$

gives $v^*(x_u) \in X(in(w^*t))_R$

should be equal to $f^*_{w,u}(x_w)$ where $x_w \in X(in(w^*t))_R$

$$\begin{tikzcd} in(w^*t) = in(u^*v^*t) \\ in(v^*t) \arrow[u, "f_{w,u}"'] \end{tikzcd}$$

If each $X(\circ)$ is inner Kan then so is $\int_{\$} X \in$ dsets.

2/7/08

From $\$ \xrightarrow{X}$ sSets construct $\int_{\$} X$

Artin-Mazur
Zisman
Bullejo-Cegarra (2-cat)

(i) $\int_{\$} X$ is a model for $\operatorname{hocolim}_{\$} X$

(ii) If X is a diagram of (nerves of) categories; more precisely $\int_{\$} X \to N\$$ is a strict inner Kan fib.

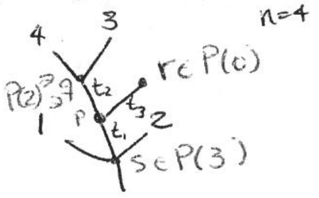
(iii) " " " " quasi-categories, " " " " inner Kan fib.

(iv) " " " " Kan cxs, then $\int_{\$} X \to N\$$ is a Kan fib.

Operads in Top (References: ~~C~~ Berger–M: Topology 45; Street Festschrift)

to start with, unicolored.

Boardman & Vogt construct for any such operad P a "resolution" $W(P) \to P$



whose algebras are "homotopy-P-algebras".

(eg if $P = Ass \Rightarrow W(P)$-algebras are A_∞-algebras)

Elts of $W(P)(n)$ are represented by planar trees with labels

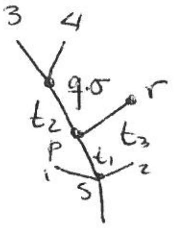
- inputs numbered $1, \dots, n$
- vertices labelled by elts of P
- inner edges labelled by a length $t \in [0,1] \subseteq \mathbb{R}$

⑫ M **Relations**: a) permutations

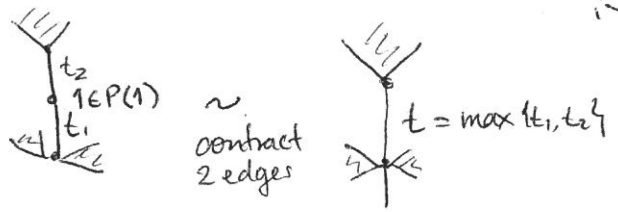
b) for "identity-vertices"

c) for length 0.

a) the tree above is equivalent



b)



c)

Contract edges of length 0 using substitution of P

W(P) is an operad · unit : |

· composition is by grafting, assigning length 1 to the newly arisen internal edges.

· Σ_n acts by permuting the labels on the input edges.

Ex For P = Ass, W(P)(n) is a subdivided Stasheff polytope K_n.

There is a closed model structure on Operads in Top for which $P \to Q$ is a we / fib iff $\forall n \; P(n) \to Q(n)$ is.

For suitable P, the map $W(P) \xrightarrow{\varepsilon} P$ is a cofibrant resolution of P in this model category. (Suitable = "Σ-cofibrant")

Moreover ε^* (P-algebras) $\longrightarrow$ (W(P)-algebras) is part of a Quillen equivalence.

Variations

1) Relative case for $Q \rightarrowtail P$, get

$$\begin{tikzcd} Q \arrow[r, tail] \arrow[dr] & W(P;Q) \arrow[d, "\wr"] \\ & P \end{tikzcd}$$

2) Colored version (just label the edges (also) by the colors of the operad)

3) Replace (Top, x, [0,1]) by any monoidal category $(\mathcal{E}, \otimes)$ with interval H equipped with a "max : $H \otimes H \xrightarrow{\vee} H$" $0 \vee x = x = x \vee 0$, $1 \vee x = 1 = x \vee 1$

eg. $\mathcal{E} = s\,Sets$, $H = \Delta[1]$; $\mathcal{E} = Cat$ $H = J = 0 \overset{\sim}{\leftrightarrows} 1$ or $I = 0 \to 1$

$\mathcal{C}$ a small category. View $\mathcal{C}$ as a colored operad in Top; colors are objects of $\mathcal{C}$, all operations are unary, the spaces $\mathcal{C}(c;d)$ are discrete.

What is $W(\mathcal{C})$? It is a category enriched in Top.

An arrow $c \overset{\text{in } W(\mathcal{C})}{\longrightarrow} d$ is a sequence $c=c_0 \xrightarrow{f_1} c_1 \to \cdots \to c_{n-1} \xrightarrow{f_n} c_n = d$ of arrows in $\mathcal{C}$ with waiting times $t_1 \ldots t_{n-1}$ lengths

(same description as this morning but w/ Top instead of sSets).

The Hom sets are cubes.

$\mathcal{D}$ a topological category $\rightsquigarrow$ homotopy coherent nerve $hcN(\mathcal{D}) = \tilde{N}(\mathcal{D})$

defined by $hcN(\mathcal{D})_n = \mathrm{Hom}_{TopCat}(W([n]), \mathcal{D})$

P a topological operad $\rightsquigarrow$ htpy coherent dendroidal nerve $hcN_d(P)$

$$hcN_d(P)_T = \mathrm{Hom}_{TopOperads}(\underbrace{W(\Omega(T))}_{\text{also denoted } W(T)}, P)$$

$$\mathrm{Top\,Op} \underset{\underset{\text{automatically (by Kan ext)}}{\xleftarrow{L}}}{\xrightarrow{hcN_d}} d\,\mathrm{Sets} \qquad L \text{ closely related to } W.$$

Eg for a discrete colored operad P: $L(N_d(P)) = W(P)$

Hope: this is part of a Quillen equivalence.

Prop $hcN_d(P)$ is a dendroidal inner Kan cx.

More generally, if P is an operad in a monoidal model category $\mathcal{E}$ with interval H and $P(c_1, \ldots, c_n; c)$ is a fibrant object, then $\underset{\text{w.r.t. } H}{\underline{hcN_d(P)}}$ is inner Kan.

Ex Let $P = Cat$ (this a cartesian symm monoidal category, hence can be used as an operad whose colors are small categories)

So we have an inner Kan cx $hcN_d(Cat)$

Let Cat_S be the operad whose algebras are Categories with S as objects (eg if $S=*$, $Cat_S = Ass$), then $\underline{\mathrm{Hom}}(\overset{N_d}{Cat_S}, hcN(Cat))$; this is a dendroidal set whose vertices are bicategories w/ S as objects, ?

⑭ M Take $P = \underline{Top}$, then similarly $\underline{Hom}(N_d(Ass), hcN_d(\underline{Top}))$ is a dendroidal set whose vertices are A_∞-spaces.

$\underline{Hom}$ = internal hom of dsets, These both are dendroidal inner Kan.
(adjoint to $\otimes$)

2/11/03 // CRM // Dendroidal Sets

<u>Non rigid Reedy categories</u> (Joint w/ C. Berger)

1 <u>Reedy category</u> is a small category $\mathbb{R}$, together with 2 subcategories $\mathbb{R}_+$ and $\mathbb{R}_-$ (on the same objects) and a degree function d

$$d : Ob(\mathbb{R}) \longrightarrow \mathbb{N}$$

s.t.

- every non-identity arrow in $\mathbb{R}_+$ raises degree $\quad r \xrightarrow{\overset{\in \mathbb{R}_+}{f}} s \quad d(s) > d(r)$
- every non-identity arrow in $\mathbb{R}_-$ lowers degree $\quad d(s) < d(r)$
- every arrow $r \xrightarrow{f} s$ in $\mathbb{R}$ factors uniquely as $r \xrightarrow{h} t \xrightarrow{g} s$ w/ $h \in \mathbb{R}_-$ & $g \in \mathbb{R}_+$

<u>Examples</u> 1 Δ $\quad d([n]) = n$, $\quad \Delta_+ =$ monos (comp of faces) $\quad \Delta_- =$ epis (comp of degeneracies)

In general if $\mathbb{R}$ is a Reedy category, so is $\mathbb{R}^{op}$ (switch roles of $\mathbb{R}_+$ & $\mathbb{R}_-$).

2. Δ^{op}

3 $\mathbb{N}$ $\quad 0 \to 1 \to 2 \to \cdots$

4. $$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \end{tikzcd} \qquad \rightrightarrows$$

<u>Thm</u> If $\mathcal{E}$ is a model category, $\mathbb{R}$ Reedy, then $\mathcal{E}^{\mathbb{R}}$ has a so-called "Reedy model category structure".

- Weak equivalences: pointwise i.e $X \to Y$ is a w.e iff $X_r \to Y_r$ is a weak equiv. in $\mathcal{E}$ $\forall r \in \mathbb{R}$.

- fibrations and cofibrations are described explicitely in term of <u>latching</u> & <u>matching</u> objects.

$X \in \mathcal{E}^{\mathbb{R}}$, $r \in \mathbb{R}$

$$\varinjlim_{\substack{s \to r \\ \cap \\ \mathbb{R}_+ \\ \text{non-identity}}} X_s =: L_r(X) \quad \text{latching object}$$

The colimit is over the $\overset{\text{full}}{\text{subcategory}}$ $\mathbb{R}_+/r$ given by the nonidentity arrows $s \to r$.

$$M_r(X) := \varprojlim_{\substack{r \to s \\ \cap \\ \mathbb{R}_- \\ \text{non-identity}}} X_s$$

$$\begin{tikzcd} L_r(X) \arrow[d] \\ X_r \arrow[d] \\ M_r(X) \end{tikzcd}$$

Limit over full subcat $^{r/}\mathbb{R}_-$ (excluding ident)

A map $X \to Y$ is a cofibration if, $\forall r$, the map

$$X_r \underset{L_r(X)}{\cup} L_r(Y) \longrightarrow Y_r \quad \text{is a cofibration}$$

Dually for fibrations, in terms of $X_r \longrightarrow M_r(X) \underset{M_r(Y)}{\times} Y_r$

(Hovey's or Hirschhorn's book).

In a Reedy category, every automorphism is an identity.
These limits and colimits are rather small (ob^s w/ $d(s)<d(r)$, there are not that many).

Another example is Ω_p = planar trees (for which $\text{sets}^{\Omega_p^{op}}$ =: pd sets).

- <u>Non-examples</u>: Λ (cyclic category)
  - $\mathbb{F}\text{in}_*$ = ptd finite sets
  - $\Gamma = \mathbb{F}\text{in}_*^{op}$
  - $\mathbb{F}\text{in}$ = finite sets
  - Ω = non-planar sets.

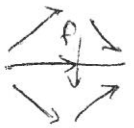
⑯M Recall: If G is a finite group, ~~these~~ $\mathcal{E}$ a model category, then there is a model category on $\mathcal{E}^G$ where w.e. (fibrations) are def pointwise ie $X \to Y$ is w.e (fib) iff underlying map in $\mathcal{E}$ is one.

Def A nr Reedy category is a category $\mathbb{R}$ together w/ subcategories $\mathbb{R}_+$, $\mathbb{R}_-$ (on the same objects) and a degree function $d : Ob(\mathbb{R}) \to \mathbb{N}$ s.t

(i). every non iso on $\mathbb{R}_+$ raises deg

(ii) every non iso on $\mathbb{R}_-$ lowers deg

(iii) Iso's $\subseteq \mathbb{R}_+ \cap \mathbb{R}_-$ (converse ⊇ follows from previous two).



(iv). every f factors $f = gh$ w/ $g \in \mathbb{R}_+$ & $h \in \mathbb{R}_-$; unique up to iso.

·self ual) (v). Iso's think the maps in $\mathbb{R}_-$ are epi: if $\theta f = f$ with $f \in \mathbb{R}_-$, θ iso, then $\theta = id$ $\quad r \xrightarrow[-]{f} s \circlearrowleft \theta$

(the last one implies that the iso in (iv) is unique).

ⓃⒷ Condition (v) is not self dual

All the non-examples above are examples of nr Reedy cat. (their duals too)

Other examples

- G : any finite groups.
- $\mathcal{O}_G$: orbit category (G/H); as is $\mathcal{O}_G^{op}$, G finite

If G is compact Lie, $\mathcal{O}_G$ is example but $\mathcal{O}_G^{op}$ is not

Define $L_r(X)$ & $M_r(X)$ excluding iso's too; ie $L_r(X) := \varinjlim_{\substack{s \to r \\ \mathbb{R}_+ \\ \text{non-iso}}} X_s$; $\boxed{\mathbb{R}_+ // r}$

$M_r(X) := \varprojlim_{\substack{r \to s \\ \mathbb{R}_- \\ \text{non-iso}}} X_s$

$$\begin{tikzcd} L_r(X) \arrow[d] \\ X_r \arrow[d] \\ M_r \end{tikzcd}$$

Aut(r) acts everywhere $(X_r, M_r(X), L_r(X))$ and the maps are equivariant under this action.

Thm If $\mathbb{R}$ is a nr Reedy category, $\mathcal{E}$ a model category, then $\mathcal{E}^{\mathbb{R}}$ has a "nr Reedy-model structure" in which

- $X \to Y$ w. equiv iff $X_r \to Y_r$ is one in $\mathcal{E}^{Aut(r)}$ $\forall r$
- " " cofibration iff $X_r \cup_{L_r(X)} L_r(Y) \to Y_r$ is one in $\mathcal{E}^{Aut(r)}$ $\forall r$
- " " fibration iff $X_r \to M_r(X) \cup_{M_r(Y)} Y_r$ is one in $\mathcal{E}^{Aut(r)}$ $\forall r$

(for first and third can work in $\mathcal{E}$ instead because of the way we defined model structure on $\mathcal{E}^{G}$).

The proof treats all r at the same time & uses

$$\mathbb{R} \xleftarrow{\delta_n} \mathbb{R}^{+}/\!/_n \xrightarrow{c_n} \underbrace{G_n(\mathbb{R})}_{\text{groupoid of iso's in } \mathbb{R}} \xhookrightarrow{i_n} \mathbb{R}$$

$L_n(X) = c_{n!}\, \delta_n^{*}(X) \in \mathcal{E}^{G_n(\mathbb{R})}$ is a description of the latching objects $L_r(X)$ for all r of deg n, and the action of $Aut(r)$.

Motivating example $\mathcal{E}^{\Omega^{op}} = d\mathcal{E} =$ dendroidal objects in $\mathcal{E}$.

(Λ)

$$\begin{tikzcd}
{[n]} \arrow[r, "f \in \Delta"] \arrow[d, dotted, "f^*(\sigma)"'] & {[m]} \arrow[d, "\sigma"] & \text{cyclic} \in C_{m+1} \\
{[n]} \arrow[r, "\sigma_*(f)"'] & {[m]} &
\end{tikzcd}$$

of factors uniquely as indicated

In Λ, arrows are pairs (f, τ)

$$(g, \sigma) \circ (f, \tau) = (g\,\sigma_*(f),\ f^*(\sigma)\tau)$$

$$\begin{tikzcd}
{[n]} \arrow[d, "\tau"'] & & \\
{[n]} \arrow[r, "f"] \arrow[d, dotted, "f^*(\sigma)"'] & {[m]} \arrow[d, "\sigma"] & \\
{[n]} \arrow[r, dashed, "\sigma_*(f)"'] & {[m]} \arrow[r, "g"] & {[l]}
\end{tikzcd}$$

$\Omega_p = \Omega/P$

$T \longmapsto Aut(T)$ is like a crossed group on Ω_p (instead of Δ)

$Sets^{\Lambda^{op}}/\Lambda[1] = ssets$. $\quad Sets^{\Omega_p^{op}} = Sets^{\Omega^{op}}/P$

(18) M 2/12/08 // CRM // Dendroidal sets

Corrections about Λ

Arrows are pairs (f, σ)

$$\begin{tikzcd} & {[n]} \arrow[d, "C_{n+1} \ni \sigma"'] & \\ & {[n]} \arrow[r, "f"] & {[m]} \\ & & \end{tikzcd}$$

$\overset{\Uparrow}{\Delta}$

$\Lambda \xrightarrow{\text{forget}} \Delta$ is not faithful

$[0]$ is not a terminal object $\qquad \mathrm{Hom}([n], [0]) \simeq C_{n+1}$

$\Lambda / \Lambda[0] \simeq \Delta \qquad Sets^{\Delta^{op}} \simeq Sets^{\Lambda^{op}} / \Lambda[0] \qquad \Lambda[0]$ generic circle

$\Delta \xhookrightarrow{i} \Lambda \rightsquigarrow Sets^{\Delta^{op}} \xrightarrow{i} Sets^{\Lambda^{op}} \qquad i^* \Lambda[0] = \Delta[1] / \dot{\Delta}[1]$

Strong parallel: $\Omega_p \hookrightarrow \Omega$

$\Delta \hookrightarrow \Lambda$

$$\begin{tikzcd} \Delta \arrow[r, hook, "v"] \arrow[dr, hook, "i"'] & \Omega_p \arrow[d, hook, "u"] \\ & \Omega \end{tikzcd}$$

$$\begin{tikzcd} sSets \arrow[r, "v_!"] \arrow[dr, "i_!"'] & pdsets \arrow[l, bend left, "v_*"] \arrow[d, "u_!"] \\ & dsets \arrow[ul, "i^*"] \arrow[u, bend right, "u^*"'] \end{tikzcd}$$

Cartesian closed

simplicial

SMC

$v_!, i_!$ f.f.

$$\begin{tikzcd} dsets/U \arrow[r] \arrow[dr] & dsets/P \arrow[d] \\ & dsets \end{tikzcd}$$

$=$

U unit for $\otimes$ in dsets

Inner horns $\Lambda^e(T) \longrightarrow \Omega(T)$ $\qquad$ (e internal edge)

Inner Kan cx : RLP wrt inner horns.

- In ssets : quasi-categories
- In dsets : quasi-operads $\qquad$ (quasi-multicategory)
- In pdsets : quasi-non Σ operads

Thm There is a simplicial closed model category structure on pdSets, in which

- the cofibrations are the monos
- the weak equivalences are maps $A \longrightarrow B$ for which, for any Kan cx F,

$\tau \, \underbrace{\underline{\mathrm{Hom}}(B,F)}_{\text{quasi-cat}} \longrightarrow \tau \, \underbrace{\underline{\mathrm{Hom}}(A,F)}_{\text{q-cat}}$ is an equivalence of categories.

- the fibrant objects are exactly the inner Kan cxs.

Corollary There is an induced closed model structure on pdsets $/ \mathcal{U} =$ sSets
This in fact the Joyal model structure. (for quasi-categories).

Model structure on dsets

Recall that a monomorphism $X \longrightarrow Y$ is normal if the stabilizer $\mathrm{Aut}(T)_y$ of any nondegenerate $y \in Y(T) - X(T)$ is trivial. Y is normal if $\emptyset \longrightarrow Y$ is normal.
The normal mono's are generated by $\partial\Omega(T) \hookrightarrow \Omega(T)$ for all T.

Rmks
1. Every object $X \in$ dSets has a "normalization" $X_n \xrightarrow{p} X$, given by a normal object X_n and a map p which has the RLP w.r.t all normal monos (usual small object argument using $\{\partial\Omega(T) \hookrightarrow \Omega(T) \mid T \in \Omega\}$)
In dSets we have the natural interval $i_!(J)$, & we can use it to define homotopies $i_!(J) \otimes A \xrightarrow{H} B$.
The normalization is unique up to homotopy in this sense.
2. If $Y \longrightarrow X$ is any map and X is normal, then so is Y.
Related fact Ω/T has no automorphisms.

$$\begin{tikzcd} S \arrow[rr, "\theta"', "\sim"] \arrow[dr, "u"'] & & S \arrow[dl, "u"] \\ & T & \end{tikzcd}$$

factor u as
$$\begin{tikzcd} S \arrow[r, "\sim", "\tau"'] & S \arrow[r, two heads, "\sigma", "\text{comp of degeneracies}"'] & R \arrow[r, hook, "\delta", "\text{comp of faces}"'] & T \end{tikzcd}$$

$\delta\sigma\tau\theta = \delta\sigma\tau$

$\sigma\tau\theta = \sigma\tau$

$\sigma(\tau\theta\tau^{-1}) = \sigma$

So $\tau\theta\tau^{-1}$ permutes fibers of a degeneracy, but these are linear so $\tau\theta\tau^{-1} = id$
$\Rightarrow$ a tree T ~~has~~ is normal.

Thm There is a closed model structure on dsets in which:
- the cofibrations are the normal monomorphisms
- $A \longrightarrow B$ is a weak equiv iff for normalizations $B_n \longrightarrow B$ and $A_n \longrightarrow A$

⑳ M fitting into a square (you can do this)

$$\begin{tikzcd} A_n \arrow[r] \arrow[d] & B_n \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

the induced map

$$\tau\left(i^* \underline{\mathrm{Hom}}(B_n, F)\right) \longrightarrow \tau\left(i^* \underline{\mathrm{Hom}}(A_n, F)\right)$$

is an equivalence of categories, for inner Kan cxs F.

($\underline{\mathrm{Hom}}(A_n, F)$ is a quasi-operad, $i^*(\underline{\mathrm{Hom}}(A_n, F))$ is a quasi-cat)

- Any fibrant object is an inner Kan cx (the converse might not be true).
  presumably false

Question: Does this make dSets into a monoidal model category?

Rmk If we slice the model structure to get one on dsets/$\mathcal{U}$ = ssets we recover the Joyal model structure.

Question. Do we get the early model structure on pdSets when we slice by P?

About the proofs

The case of pdSets is relatively standard and the difficulty is concentrated in showing that there is a set of generators for the trivial cofibrations.

It is going to be a set of trivial cofibrations between countable pdsets.

- A map $X \xrightarrow{f} Y$ is a trivial cofibration iff the induced map $X_\infty \rightarrowtail Y_\infty$ is a deformation retract, where $X \rightarrowtail X_\infty$ is an anodyne ext into an inner Kan cx.
- If X is countable, X_∞ can be taken to be countable.
- If $A \subseteq X_\infty$ is countable, $\exists\, \underset{\text{count}}{Y} \subset X$ with $A \subseteq Y_\infty$.

$$\begin{tikzcd} X \supseteq A_0 \arrow[d] \arrow[r, phantom, "\subseteq"] & U_1 \arrow[d, "\simeq"] \arrow[r, phantom, "\subseteq"] & (A_1)_\infty \arrow[d] \arrow[r, phantom, "\subseteq"] & U_2 \arrow[d, "\simeq"] \arrow[r, phantom, "\subseteq"] & \cdots & X_\infty \arrow[d, "\simeq"] \\ Y \supseteq B_0 \arrow[r, phantom, "\subseteq"] & V_1 \arrow[u, bend right] \arrow[r, phantom, "\subseteq"] & (B_1)_\infty \arrow[r, phantom, "\subseteq"] & V_2 \arrow[u, bend right] \arrow[r, phantom, "\subseteq"] & \cdots & Y_\infty \arrow[u, bend right] \end{tikzcd}$$

f

countable

arbitrary countable subsets

$$\begin{tikzcd} \bigcup U_n \arrow[d, "\simeq"] \\ \bigcup V_n \arrow[u, bend right] \end{tikzcd}$$

def retract

equal to $U(A_n)_\infty = (UA_n)_\infty$

$$\downarrow$$

$U(B_n)_\infty = (UB_n)_\infty$

Conclusion: there are countable $\bar{A} \supset A_0$, $\bar{B} \supset B_0$ for which $f: \bar{A} \to \bar{B}$ induces a def retraction on $\bar{A}_\infty \to \bar{B}_\infty$

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$$\begin{tikzcd} \text{sSets} \arrow[r, shift left, "i_!"] \arrow[d, shift right, "\tau"'] & \text{dsets} \arrow[l, shift left, "i^*"] \arrow[d, shift right, "\tau_d"'] \\ \text{Cat} \arrow[u, shift right, "N"'] \arrow[r, shift left, "j_!"] & \text{Operads} \arrow[l, shift left, "j^*"] \arrow[u, shift right, "N_d"'] \end{tikzcd}$$

$\tau_d \circ N_d(P) = P$ (N_d is fully faithful)

$\mathcal{E}$ symmetric monoidal category.

P operad in Sets

$\underline{\mathrm{Hom}}(N_d(P), N_d(\mathcal{E}))$ is the dendroidal set whose vertices are P-algebras in $\mathcal{E}$.

Eg · $P = \mathrm{Ass}$ we get monoids in $\mathcal{E}$.

· $P = \mathrm{Ass}_S$ (the operad with $S \times S$ as set of colors whose algebras are categories w/ S as set of objects)

$\underline{\mathrm{Hom}}(N_d(P), N_d(\mathcal{E}))$ has vertices the $\mathcal{E}$-enriched categories on S.

$$\left.\begin{array}{l} X \overset{\text{(strict)}}{} \text{ inner Kan cx} \\ A \text{ normal} \end{array}\right\} \quad \underline{\mathrm{Hom}}(A, X) \text{ is (strict) inner Kan.}$$

$\underline{\mathrm{Hom}}(N_d(P), N_d(\mathcal{E}))$ is again the nerve of an operad.

If X is an inner Kan dendroidal set, then

$$\underline{\mathrm{Hom}}(N_d(\mathrm{Ass}_S), X) \text{ is again inner Kan.}$$

Its vertices are "categories on S enriched in X".

Vertices of $\underline{\mathrm{Hom}}(N_d(P), N_d(\mathcal{E}))$ are P-algebras in $\mathcal{E}$.

Dendrices of shape ϕ are given by two such & a map between them:

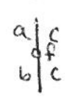
$$i(\Omega(\phi)) = \Omega[\phi] \longrightarrow \underline{\mathrm{Hom}}(N_d(P), N_d(\mathcal{E})) \iff \Omega[\phi] \otimes N_d(P) \longrightarrow N_d(\mathcal{E})$$
$$\iff \tau_d(\Omega[\phi] \otimes N_d(P)) \longrightarrow \mathcal{E}$$

㉒ M Since $\tau_d(X \otimes Y) = \tau_d(X) \otimes_{BV} \tau_d(Y)$, the last line $\Leftrightarrow$

$$\Omega(\,|\,) \otimes_{BV} P \longrightarrow \mathcal{E}$$

ie a $(\Omega(\,|\,) \otimes_{BV} P)$-algebra in $\mathcal{E}$. If P hast just one color c,

$\Omega\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\Big|\right) \otimes_{BV} P$ has two colors (a,c), (b,c) and one new operation



commuting w/ the operations from P :

What happens if instead of $N_d(\mathcal{E})$ we put a non-strict inner Kan cx,

eg $hcN_d(Top)$?

$\underline{Hom}(N_d(Ass), hcN_d(Top))$ is an inner Kan cx, whose vertices are A_∞-spaces.

Such a vertex is a map $N_d(Ass) \xrightarrow{F} hcN_d(Top)$

F is a natural transformation

$$F_1 : \underset{\overset{\shortparallel}{pt}}{N_d(Ass)_1} \longrightarrow \underset{\text{the "set" of spaces (of objects of } Top)}{\overset{}{hcN_d(Top)}}$$

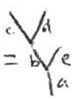
F_1 picks a space X.

$F_Y : N_d(Ass)_{{}_bY^c_a}$ = a set of two points "b·c", "c·b"

$hcN_d(Top)_Y$ = all diagrams given by three spaces & map $U \times V \longrightarrow W$

Since it is ~~functorial~~ natural F_Y gives $X \times X \xrightarrow{\mu_2} X$, together with it's twist $\mu_2 \circ \tau$.

Similarly F_{C_k} defines $\mu_k : X^{\times k} \longrightarrow X$ where C_k is the corolla

T =


F_T; $hcN_d(Top)_T$ its elements are given by $X_a, \ldots, X_e$ spaces;

maps $X_c \times X_d \xrightarrow{f} X_b$, $X_b \times X_e \xrightarrow{g} X_a$, composing to

$$\begin{tikzcd} X_c \times X_d \times X_e \arrow[d] \\ X_a \end{tikzcd}$$

and for each $t \in [0,1]$, a map $h_t : X_c \times X_d \times X_e \longrightarrow X_a$

$\lambda a- \quad -\lambda c- \lambda \; ; \; +\Sigma y = \mu_2$

By naturality : $\cdot\, h_t$ is a homotopy from $(t=0)$ $\mu_3 : X \times X \times X \longrightarrow X$

to $(t=1)$ $(X \times X) \times X \xrightarrow{\mu_2 \times X} X \times X \xrightarrow{\mu_2} X$.

The mirror tree Y gives a homotopy k_t from μ_3 to $\mu_2 \circ (X \times \mu_2)$.

So on, & so forth.

Dendrices of shape ϕ in $\underline{\mathrm{Hom}}\,(N_d(\mathrm{Ass}), \mathrm{h}c N_d(\mathrm{Top}))$ are given by two such A_∞-spaces X, Y and a map $f : X \to Y$, and dot's of explicitly given htpies saying f respects the multiplication up to homotopy.

What comes in is the BV-resolution of operads of the form $\phi_P \otimes Y_{m_2}$ or

$\phi \otimes Y$

Similarly if we replace $(\mathrm{Top}, \times, [0,1])$ w/ $(\underline{\mathrm{Cat}}, \times, 0 \overset{\sim}{\underset{J}{\leftrightarrows}} 1)$, we find that $\underline{\mathrm{Hom}}\,(N_d(\mathrm{Ass}), \mathrm{h}c N_d(\underline{\mathrm{Cat}}))$ is the inner Kan cx (in dendroidal sets) of monoidal categories.

$$\begin{tikzcd}
(C \otimes D) \otimes E \arrow[rr, dotted, dash] & & C \otimes (D \otimes E) \\
& C \otimes D \otimes E \arrow[ul, Rightarrow, "\wr"] \arrow[ur, Rightarrow, "\wr"'] &
\end{tikzcd}$$

Don't have this, but

three-fold tensor product.

A map $N_d(\mathrm{Ass}) \longrightarrow \underline{\mathrm{Hom}}\,(N_d(\mathrm{Ass}), \mathrm{h}c N_d(\underline{\mathrm{Cat}}))$ $\overset{\text{by adjunction}}{\Longleftrightarrow}$

$N_d(\mathrm{Ass}) \otimes N_d(\mathrm{Ass}) \longrightarrow \mathrm{h}c N_d(\underline{\mathrm{Cat}})$ ⊛

(to be distinguished from a map $N_d(\mathrm{Ass}) \otimes N_d(\mathrm{Ass}) \to N_d(\underline{\mathrm{Cat}})$ which is the same as $\underset{\substack{\| \\ \mathrm{Comm}}}{\mathrm{Ass} \otimes_{BV} \mathrm{Ass}} = \tau_d(N_d((\mathrm{Ass}) \otimes N_d(\mathrm{Ass})) \longrightarrow \underline{\mathrm{Cat}}$)

⊛ is a category w/ ~~a structure~~ two monoidal structure distributing over each other up to higher coherences. (Known to be essentially the same as a braided monoidal category - Joyal-Street).

㉔ M The map $Nd(Ass) \otimes N_d(Ass)$ is very far from being an equivalence.

$$\begin{tikzcd}
N_d(Ass) \otimes N_d(Ass) \arrow[d] \\
N_d(Ass \otimes_{BV} Ass) \arrow[d, phantom, "\shortparallel"] \\
N_d(Comm)
\end{tikzcd}$$

(~~Comm~~ symmetric monoidal ≠ braided monoidal)

2/14/08 // CRM // Dendroidal Sets

1. Geom realization / class topoi
2. Relation dsets ⟷ top operads
3. Weak n-categories

Is there an informative "geometric" realization of dendroidal sets?

$$\begin{tikzcd}
\text{ssets} \arrow[r, "i_1"] \arrow[d, "|\text{-}|"'] & \text{dsets} \arrow[dl, dashed, "?"', "R"] \\
\text{Top} &
\end{tikzcd}$$

i_1 fully faithful

$i_1(X \times Y) = i_1(X) \otimes i_1(Y)$
$i_1(1) = U$ (unit)

a. Would like extension R of $|\text{-}|$ via i_1

b. R sends $\text{-}\otimes\text{-}$ to $\text{-}\times\text{-}$.

c. "R doesn't lose too much info."

($(|\text{-}| \cdot i^*)$ satisfies the first two, but definitely not the third.)

Some simple-minded functors satisfying a, b but not c:

- $RX = |i^*(X)|$ badly fails c.

- $\Omega \longrightarrow$ posets $\quad T \longmapsto$ "T is the poset of its edges" $=: E(T)$

Compose w/ classifying space and "Kan extend".

$$\begin{tikzcd}
\Omega \arrow[r, "\text{Yoneda}"] \arrow[d, "r"'] & \text{dsets} \arrow[dl, dashed, "R"] \\
\text{Top} &
\end{tikzcd}$$

(Kan extension)

$r(T) = BE(T)$

$r(Y) =$ 

still doesn't satisfy c.

R maps $\partial\Omega(T) \hookrightarrow \Omega(T)$ to an iso for some rather small trees already ()

<u>Variation of the question</u>: replace Top by a monoidal model category which we know.

Compare the situation to that of cyclic sets

$$\begin{tikzcd} \text{ssets} \arrow[r, hook, "w_!"'] & \text{cyclic sets} \arrow[l, bend right, "w^*"'] \\ \Delta \arrow[r, "w"'] & \Lambda \end{tikzcd}$$

If X is a cyclic set, we ~~can~~ could realize it as $|w^*(X)|$, but in fact this is an S^1-space.

Cyclic sets is a classifying topos for "arbitrary circles".
So if $\mathcal{E}$ is any category with sufficiently nice exactness properties and C is an abstract circle in $\mathcal{E}$, we find a realization

$$\begin{tikzcd} \text{ciclic sets} \arrow[r] & \mathcal{E} \\ \Lambda[0] \arrow[r, mapsto] & C \end{tikzcd}$$

Eg. $\mathcal{E} = S^1\text{-Top}$, & $C = S^1$ with its standard action by translation.

<u>Related fact</u>: $B\Lambda = BS^1$

So to find $R: \text{dsets} \longrightarrow \mathcal{E}$, it could help to find

(i) what does dsets "classify"? either via

- topos theoretic interpration

or

- something similar, replacing × by ⊗.

Related question: what is multilinear order? $\dfrac{?}{\text{multicategories}} = \dfrac{\text{linear order}}{\text{categories}}$

(ii) What is the homotopy type of $B\Omega$ (or $B\Omega_p$)?

About weak n-categories

<u>Recall</u> • If A is normal and X is a (strict) inner Kan, then <u>Hom</u> (A, X) is again (strict) inner Kan.

(26) M

- If $\mathcal{S}$ is a category with finite products & $X : \mathcal{S}^{op} \longrightarrow dSets$ is a diagram of (strict) inner Kan cxs, then

$$\int_{\mathcal{S}} X \qquad \text{is again (strict) inner Kan.}$$

<u>Cat</u> is a symm mon cat (of small categories).

Define a strict inner Kan cx <u>n-Cat</u> by induction

$$1\text{-}\underline{Cat} = N_d(\underline{Cat})$$

$$(n+1)\text{-}\underline{Cat} = \int_{\mathcal{S}} \underbrace{\underline{Hom}(N Ass_{\mathcal{S}}, n\text{-}\underline{Cat})}_{\text{vertices here are categories on } S \text{ enriched in } n\text{-}\underline{Cat}} \qquad \mathcal{S} = Sets$$

<u>Cat</u> has a nice interval $J = 0 \overset{\sim}{\rightleftarrows} 1$, so instead of $N_d(\underline{Cat})$, we could take

$$hc N_d(\underline{Cat}) =: W, 1\text{-}\underline{Cat}$$

$$W, (n+1)\text{-}\underline{Cat} = \int_{\mathcal{S}} \underline{Hom}(Ass_{\mathcal{S}}, (W, n\text{-}\underline{Cat}) \qquad \text{inner Kan cx.}$$

<u>Eg</u>. The vertices of $(W, 2\text{-}\underline{Cat})$ are the usual bicategories. (unbiased notion)

Questions:

1) Would you be so kind to explore this definition in low degrees?

2) "Strictification hypothesis": For any reasonable definition of weak n-category and equivalence between two such any weak n-category is equivalent to one arising as a vertex of the inner Kan cx $(W, n\text{-}\underline{Cat})$.

3) Baez-Dolan stabilization hypothesis: (From the thesis of Ittay Weiss)

$$W\underline{Cat}^n_k = \underline{Hom}((N Ass)^{\otimes k}, W, n\text{-}\underline{Cat})$$

"k-fold monoidal weak n-category"

$$W\underline{Cat}^n_k = W\underline{Cat}^n_{k+1} \quad \text{as soon as } k \geq n+2$$

In this language, the hypothesis would be a consequence of

Conjecture: $w\underline{Cat}^n_n$ is strict inner Kan

(ie as soon as you have as many monoidal structures as there is "room", everything is strict: strict composition)

$$w\underline{Cat}^n_{k+\ell} = \underline{Hom}\left(N_d(Ass)^{\otimes \ell}, w\underline{Cat}^n_k\right)$$

If $w\underline{Cat}^n_n$ is a strict inner Kan, it is $N_d(P)$ for an operad P, use adjunction and $\tau_d(N_d(Ass)^{\otimes\ell}) = Ass^{\otimes_{BV}^{\ell}} = Comm \quad \ell \geq 2$.

$\boxed{\text{dsets} \overset{?}{=} \text{"topological operads"}}$

Facts:

① For a fixed set C, the C-colored operads in a s.m.c. $\mathcal{E}$ (Top, sets, Ch) carry a model structure (Berger-M; Street volume, Contemp Math) & the BV-resolution $W(P) \xrightarrow{\varepsilon} P$ is a cofibrant resolution under some conditions on P. (P-alg) is a model category.

Moreover, for such P, $\varepsilon^*: (P\text{-alg}) \longrightarrow (W(P)\text{-alg})$ is part of a Quillen equiv (w/ left adjoint $\varepsilon_!$) "strictification"!

By the way, for a fixed C, there is a nice operad O_C with algebras C-colored operads. (& the model structure on operads becomes a special case of these on algebras).

We can assemble the categories $(O_C\text{-alg})$ of C-colored operads for various C into one big fibered category of colored operads, & get a model structure on these. (in which the weak equiv are fiberwise, ie induce iso's on the sets of colors).

② The "naive" model structure on $\underline{Cat}$ in which the weak equiv are equivalences of categories extends to a model structure on the category of colored operads in Sets.

㉓ M A map $\mathbb{P} = (C, P) \xrightarrow{(f,\varphi)} (D, Q) = \mathbb{Q}$ is a w.e. iff it is fully faithful ($P(c_1, \dots, c_n; c) \xrightarrow{\sim} Q(fc_1, \dots, f_n; fc)$, $\forall c_1, \dots, c_n, c$) and essentially surjective (ie $j^*(\mathbb{P}) \longrightarrow j^*(\mathbb{Q})$ is an equivalence of categories).

Question: Is there a model structure on topological (simplicial) colored operads that mixes these two? In the sense that

$\mathbb{P} \longrightarrow \mathbb{Q}$ is a w.e. iff it is a w.e in the sense of ①

($P(c_1, \dots, c_n; c) \xrightarrow{\sim} Q(fc_1, \dots, fc_n; fc)$ w.e. of spaces) and

$\pi_0(\mathbb{P}) \longrightarrow \pi_0(\mathbb{Q})$ is a w.e. in the sense of ②.

[This would be an extension of the "DK model structure" due to Julia Bergner].

The Boardman-Vogt resolution $T \longmapsto W(\Omega(T)) =: W(T)$

$\Omega \longrightarrow$ (Top operads) induces an adjunction

$$\begin{tikzcd} \text{dsets} \arrow[r, bend left, "hc\tau_d"] & \text{topological operads} \arrow[l, bend left, "hcN_d"] \end{tikzcd}$$

Question: Is this part of a Quillen equivalence? (W.r.t the alleged Bergner structure above on the right hand side and the model structure of the lectures on the left).

Might want to do the planar case first.

There is a derived tensor on Top operads : $P \overset{L}{\otimes} Q = W(P) \underset{BV}{\otimes} W(Q)$ and

$hc\tau_d$ hopefully preserves this.

$hc\tau_d(Ass) = W(Ass) = A_\infty$

If $hc\tau_d$ behaves well w.r.t tensor products, this should show that

$hc\tau_d(N_d(Ass)^{\otimes k})$ is an E_k operad.

M ㉙

Any dendroidal inner Kan cx X defines an "operad up to homotopy", whose colors are the vertices of X (dendrices of shape $|$), denoted $o(X)$

q-cat?

$$\begin{tikzcd} o(X)(c_1,\dots,c_n;c_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \underline{\mathrm{Hom}}(C_n, X) \arrow[d] \\ | \arrow[r, "c_0,\dots,c_n"] & X^{n+1} \end{tikzcd}$$

It is a dendroidal set over a unit, so it is a simplicial ~~set~~ (q-cat)

$| \hookrightarrow$ 

$\coprod_{i=0}^{n} | \longrightarrow$ 

<u>Guess</u>: this $O(X)$ is a $W(\Theta_{\text{vertices}(X)})$-algebra

(recall Θ_C-algebras are C-colored operads).

Using the Quillen equivalence

$$\Theta_{\text{vertices}(X)}\text{-alg} \longrightarrow W(\Theta_{\text{vert}(X)}\text{-alg})$$

the operad-up-to-htpy $o(X)$ is w.e. to a "strict" (ie true) operad on the same set of colors $o(X)^{\text{Str.}}$

<u>Guess</u>: $o(X)^{\text{str}} \simeq hc\tau_d(X)$.