VIII HOMOLOGICAL ALGEBRA

Modern Classical Algebra

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In this chapter, some basic concepts of homological algebra will be defined and some of their elementary properties developed. The duality of the concepts "projective module" and "injective module" will be systematically employed and, where applicable, dual propositions will be stated.

We will need to modify some of the definitions concerning graded modules. We will mean by a graded module a sequence indexed on the integers rather than on the positive integers. If X and Y are graded modules, a morphism of degree n, $f: X \longrightarrow Y$, is a sequence of morphisms $f^i: X^i \longrightarrow Y^{i+n}$.

Λ will be a ring, not necessarily commutative. All modules will be assumed left modules unless otherwise specified.

1. Differential operators and resolutions.

<u>Definitions 1.1</u>: A differential operator, or complex, is a pair (X,d) where X is a graded module and $d: X \longrightarrow X$ is a morphism of degree $+1$ such that $d^{q+1}d^q = 0$ for all q. We define further the q^{th} cocycle, $Z^q(X)$, as $\ker(d^q)$, the q^{th} coboundary, $B^q(X)$, as $\operatorname{im}(d^{q-1})$, and the q^{th} cohomology as $Z^q(X)/B^q(X)$.

We introduce the convention $X_N = X^{-N}$, $d_N = d^{-N}$, $Z_N = Z^{-N}$, $B_N = B^{-N}$, $H_N = H^{-N}$. Thus, when (X,d) is written as a complex with subscripts, $d_q: X_q \longrightarrow X_{q-1}$ and $d_q d_{q+1} = 0$; in this case $Z_q(X)$ is called the q^{th} cycle, $B_q(X)$, the q^{th} boundary, $H_q(X)$ the q^{th} homology.

(X,d) is called a right complex if $X^N = 0$ for all $N < 0$; it is called a left complex if $X_N = 0$ for all $N < 0$. Thus a right

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1. Differential operators and resolutions.

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(X,d) is called a right complex if $X^N = 0$ for all $N < 0$; it is called a left complex if $X_N = 0$ for all $N < 0$. Thus a right

complex has the form $\ldots \longrightarrow 0 \longrightarrow 0 \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \ldots$,
a left complex $\ldots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} 0 \xrightarrow{d_{-1}} 0 \longrightarrow \ldots$.

For notational convenience, a complex (X,d) will hereafter be denoted simply by X .

Definitions 1.2: A left complex over a module A is a left complex X together with an epimorphism $\epsilon\colon X_0 \longrightarrow A$ such that $\ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$ is a 0-sequence (the composition of any two consecutive morphisms is zero). A left complex is called a left resolution if $\ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$ is exact. A projective (free) resolution of A is a left resolution such that for each q , X_q is projective (free).

The concepts of right complex over A , right resolution, and injective resolution are analogously defined. In this case, $0 \longrightarrow A \xrightarrow{\epsilon} X^1 \longrightarrow X^2 \longrightarrow \ldots$ is a zero (resp., exact) sequence.

Definitions 1.3: Let X and Y be complexes. A translation $f\colon X \longrightarrow Y$ is a morphism of degree 0 which commutes with the boundary maps, that is, for which

$$\begin{array}{ccc} X_q & \xrightarrow{f_q} & Y_q \\ {\scriptstyle d_q}\downarrow & & \downarrow{\scriptstyle d_q} \\ X_{q-1} & \xrightarrow{f_{q-1}} & Y_{q-1} \end{array}$$

is a commutative diagram for all q . If f is a translation, f induces morphisms $H_N(f)\colon H_N(X) \longrightarrow H_N(Y)$, since $f(Z_N(X)) \subset Z_N(Y)$ and $f(B_N(X)) \subset B_N(Y)$.

If $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Y$ are translations, a homotopy between f and g is a morphism $D\colon X \longrightarrow Y$ of degree -1 such that

complex has the form $\dots \to 0 \to C_0 \xrightarrow{d_1} C_1 \xrightarrow{d_2} C_2 \to \dots \xrightarrow{d_n} C_n \to \dots$,
a left complex $\dots \to X_n \xrightarrow{d_n} X_{n-1} \to \dots \to X_1 \xrightarrow{d_1} X_0 \to 0 \to 0 \to \dots$.

For notational convenience, a complex (X_n, d) will frequently be denoted simply by X.

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If $f: X \to Y$ and $g: X \to Y$ are transformations, a homotopy between f and g is a morphism $D: X \to Y$ of degree 1 such that

$f_q - g_q = d_{q+1}D_q + D_{q-1}d_q$. The relation of homotopy is an equivalence equation.

Proposition 1.4: If two translations of X into Y are homotopic, then the corresponding morphisms $H_N(X) \longrightarrow H_N(Y)$ coincide.

Proof: If f and g are the homotopic translations, then f - g is null homotopic (homotopic to zero). Let $x_N \varepsilon Z_N(X)$.

$$\begin{aligned}(f-g)(x_N) &= d_{n+1}D_N(x_N) + D_{N-1}d_N(x_N) \\ &= d_{N+1}D_N(x_N) \,\varepsilon\, B_N(Y) .\end{aligned}$$

Therefore f - g induces a zero map $H_N(X) \longrightarrow H_N(Y)$.

We now obtain a series of propositions concerning projective resolutions of modules. We will then give the dual results concerning injective resolutions.

Proposition 1.5: Every module has a projective resolution.

Proof: Given a module A , construct exact sequences

$$0 \longrightarrow Z_o \longrightarrow X_o \longrightarrow A \longrightarrow 0$$
$$0 \longrightarrow Z_1 \longrightarrow X_1 \longrightarrow Z_o \longrightarrow 0$$
$$0 \longrightarrow Z_N \longrightarrow X_N \longrightarrow Z_{N-1} \rightarrow 0 ,$$

where the X_i are projective. Define d_N as the composition

$$X_N \longrightarrow Z_{N-1} \longrightarrow X_{N-1} .$$

Then $\ldots \longrightarrow X_N \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_o \longrightarrow A \longrightarrow 0$ is a projective resolution of A .

Corollary 1.6: If Λ is left Noetherian and A is a finitely generated left module, then A has a free resolution X where each X_N is finitely generated.

Proof: X_o may be chosen free and finitely generated. Then Z_o is finitely generated, and X_1 may be chosen free with a finite base, etc.

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Proof: If f and g are the homotopic translations, then $f - g$ is null homotopic (homotopic to zero). Let $x_n \in Z_n(X)$.

$$(f-g)(x_n) = d_{n+1}D_n(x_n) + D_{n-1}d_n(x_n)$$
$$= d_{n+1}D_n(x_n) \in B_n(Y) ;$$

Therefore $f - g$ induces a zero map $H_n(X) \longrightarrow H_n(Y)$.

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where the X_i are projective. Define ∂_n as the composition

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Proof: X_0 may be chosen free and finitely generated. Then Z_0 is finitely generated, and X_1 may be chosen free with a finite base, etc.

<u>Proposition 1.7</u>: Let A and B be modules, let X be a projective resolution of A and Y a left resolution of B. If $f: A \longrightarrow B$ is a morphism, then there exists a translation $\tilde{f}: X \longrightarrow Y$ such that

$$\begin{array}{ccc} X_o & \xrightarrow{\tilde{f}_o} & Y_o \\ \epsilon \downarrow & & \downarrow \epsilon \\ A & \xrightarrow{f} & B \end{array}$$

commutes. $\tilde{f}$ is said to be over f .

Proof: Since X_o is projective, there exists $\tilde{f}_o: X_o \longrightarrow Y_o$ such that $\epsilon\tilde{f}_o = f\epsilon$. Since $\epsilon\tilde{f}_o d_1 = f\epsilon d_1 = 0$, $\tilde{f}_o d_1(X_1) \subset \ker(\epsilon) = \operatorname{im}(d_1)$, and there exists $\tilde{f}_1: X_1 \longrightarrow Y_1$ such that $\tilde{f}_o d_1 = d_1\tilde{f}_1$. Proceeding inductively, we obtain the proposition.

<u>Proposition 1.8</u>: Under the hypothesis of proposition 1.7, if $\tilde{f}$ and $\tilde{g}$ both lie over f , then $\tilde{f}$ and $\tilde{g}$ are homotopic.

Proof:

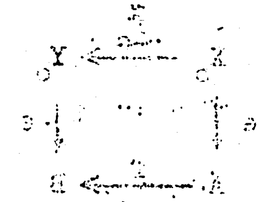
$$\begin{array}{ccc} X_1 & & Y_1 \\ d_1 \downarrow & & \downarrow d_1 \\ X_o & \xrightarrow{\tilde{f}_o, \tilde{g}_o} & Y_o \\ \epsilon \downarrow & & \downarrow \epsilon \\ A & \xrightarrow{f} & B \end{array}$$

$f\epsilon = \epsilon\tilde{f}_o = \epsilon\tilde{g}_o$, so $\epsilon(\tilde{f}_o - \tilde{g}_o) = 0$. Hence $\operatorname{im}(\tilde{f}_o - \tilde{g}_o) \subset \ker(\epsilon) = \operatorname{im}(d_1)$, and, since X_o is projective, there exists $D_o: X_o \longrightarrow Y_1$ such that $d_1 D_o = \tilde{f}_o - \tilde{g}_o$. Now consider $\tilde{f}_1 - \tilde{g}_1 - D_o d_1: X_1 \longrightarrow Y_1$,

$$\begin{array}{ccc} X_2 & & Y_2 \\ d_2 \downarrow & & \downarrow d_2 \\ X_1 & \xrightarrow{\tilde{f}_1 - \tilde{g}_1 - D_o d_1} & Y_1 \\ d_1 \downarrow & & \downarrow d_1 \\ X_o & & Y_o \end{array}$$

$d_1(\tilde{f}_1 - \tilde{g}_1 - D_o d_1) = \tilde{f}_o d_1 - \tilde{g}_o d_1 - d_1 D_o d_1 = 0$, so $\operatorname{im}(\tilde{f}_1 - \tilde{g}_1 - D_o d_1) \subset \ker(d_1) = \operatorname{im}(d_2)$ and there exists $D_1: X_1 \longrightarrow Y_2$ such that $d_2 D_1 = \tilde{f}_1 - \tilde{g}_1 - D_o d_1$.

Proposition 1.7: Let A and B be modules, let X be a projective resolution of A and Y a left resolution of B. If $f: A \longrightarrow B$ is a morphism, then there exists a translation $\tilde{f}: X \longrightarrow Y$ such that



commutes. $\tilde{f}$ is said to be over f.

Proof: Since X_0 is projective, there exists $\tilde{f}_0: X_0 \longrightarrow Y_0$ such that $\varepsilon\tilde{f}_0 = f\varepsilon$. Since $\varepsilon\tilde{f}_0 d_1 = f\varepsilon d_1 = 0$, $\tilde{f}_0 d_1(X_1) \subset \ker(\varepsilon) = \operatorname{im}(d_1)$, and there exists $\tilde{f}_1: X_1 \longrightarrow Y_1$ such that $\tilde{f}_0 d_1 = d_1\tilde{f}_1$. Proceeding inductively, we obtain the proposition.

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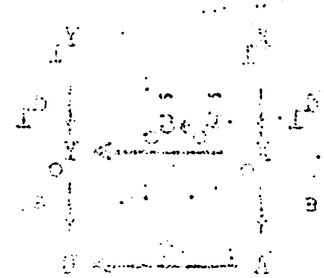

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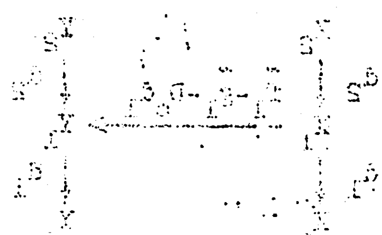

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Proceeding inductively, the result is obtained.

<u>Proposition 1.5'</u>: Every module has an injective resolution.

Proof: Given a module A, construct exact sequences

$$0 \longrightarrow A \longrightarrow Y^0 \longrightarrow Z^0 \longrightarrow 0$$
$$0 \longrightarrow Z^0 \longrightarrow Y' \longrightarrow Z' \longrightarrow 0$$
$$0 \longrightarrow Z^{N-1} \longrightarrow Y^N \longrightarrow Z^N \longrightarrow 0 ,$$

where the Y^i are injective (proposition 3.10 of ch.3). Define d^N as the composition $Y^N \longrightarrow Z^N \longrightarrow Y^{N+1}$. Then $0 \longrightarrow A \longrightarrow Y^0 \longrightarrow Y' \longrightarrow \ldots$ is an injective resolution of A.

Note that there is no statement dual to corollary 1.6.

<u>Proposition 1.7'</u>: Let A and B be modules, let X be a right resolution of A and Y an injective resolution of B. If $f: A \longrightarrow B$ is a morphism, then there exists a translation $\tilde{f}: A \longrightarrow B$ over f.

Proof: Since Y^0 is injective, there exists $\tilde{f}^0: X^0 \longrightarrow Y^0$ such that $\tilde{f}^0\epsilon = \epsilon f$. Since $d^0\tilde{f}^0\epsilon = d^0\epsilon f = 0$, $d^0\tilde{f}^0$ induces a morphism $X^0/\mathrm{im}(\epsilon) \longrightarrow Y^1$. Since $0 \longrightarrow X^0/\mathrm{im}(\epsilon) \xrightarrow{d^0} X^1$ is exact, there exists $\tilde{f}^1: X^1 \longrightarrow Y^1$ such that $\tilde{f}^1 d^0 = d^0\tilde{f}^0$. Proceeding inductively, we obtain the proposition.

<u>Proposition 1.8'</u>: Under the hypotheses of proposition 1.7', if $\tilde{f}$ and $\tilde{g}$ both lie over f, then $\tilde{f}$ and $\tilde{g}$ are homotopic.

Proof:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \epsilon \downarrow & & \downarrow \epsilon \\ X^0 & \xrightarrow[f^0,g^0]{} & Y^0 \\ d^0 \downarrow & & \downarrow d^0 \\ X' & & Y' \end{array}$$

$\epsilon f = \tilde{f}^0\epsilon = \tilde{g}^0\epsilon$, so $(\tilde{f}^0 - \tilde{g}^0)\epsilon = 0$. Hence $\tilde{f}^0 - \tilde{g}^0$ induces a morphism $X^0/\mathrm{im}(\epsilon) \longrightarrow Y^0$, and, since $0 \longrightarrow X^0/\mathrm{im}(\epsilon) \longrightarrow X^1$ is exact, there exists a morphism $D^1: X^1 \longrightarrow Y^0$ such that $D^1 d^0 = \tilde{f}^0 - \tilde{g}^0$.

Now consider $\tilde{f}^1 - \tilde{g}^1 - d^oD^1 \colon X^1 \longrightarrow Y^1$.

$$\begin{array}{ccc} X^o & & Y^o \\ {\scriptstyle d^o}\downarrow & & \downarrow{\scriptstyle d^o} \\ X^1 & \xrightarrow{\tilde{f}^1-\tilde{g}^1-d^oD^1} & Y^1 \\ {\scriptstyle d^1}\downarrow & & \downarrow{\scriptstyle d^1} \\ X^2 & & Y^2 \end{array}$$

$(\tilde{f}^1-\tilde{g}^1-d^oD^1)d^o = d^o\tilde{f}^o-d^o\tilde{g}^o-d^oD^1d^o = 0$. Hence $\tilde{f}^1-\tilde{g}^1-d^oD^1$ induces a a morphism $X^1/\mathrm{im}(d^o) \longrightarrow Y^1$, and, since $0 \longrightarrow X^1/\mathrm{im}(d^o) \xrightarrow{d^1} X^2$ is exact, there exists $D^2\colon X^2 \longrightarrow Y'$ such that $D^2d^1 = \tilde{f}^1-\tilde{g}^1-d^oD^1$. Proceeding inductively, the result is obtained.

2. Resolutions of sequences.

Here we obtain some results concerning translations of resolutions over the modules of short exact sequences. We will prove our statements only for left resolutions, since the method of proof for right resolutions is step-by-step dualization just as in the proofs given above.

We first obtain a general lemma that will be of great importance in later applications.

<u>Lemma 2.1</u>: Suppose $0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{j} X'' \longrightarrow 0$ is an exact sequence of complexes and translations. Then there is a canonical exact sequence

$$\cdots \longrightarrow H_q(X') \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(j)} H_q(X'') \xrightarrow{\delta} H_{q-1}(X') \longrightarrow \cdots .$$

δ is called the connecting morphism.

Proof: We are given a commutative diagram with exact rows and whose columns are 0-sequences:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X'_{N+1} & \xrightarrow{i} & X_{N+1} & \xrightarrow{j} & X''_{N+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\ 0 & \longrightarrow & X'_{N} & \xrightarrow{i} & X_{N} & \xrightarrow{j} & X''_{N} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\ 0 & \longrightarrow & X'_{N-1} & \xrightarrow{i} & X_{N-1} & \xrightarrow{j} & X''_{N-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

i) Definition of δ : Let $x'' \varepsilon Z_N(X'')$. Let $x \varepsilon X_N$ be such that $j(x) = x''$. $jd(x) = 0$, so $d(x) \varepsilon \operatorname{im}(i)$. Let $x' \varepsilon X_{N-1}$ be such that $i(x') = d(x)$. $di(x') = 0$, so $d(x') = 0$. Define $\delta(\overline{x''}) = \overline{x'}$. Now assume that $y'' \varepsilon Z_N(X'')$ is such that $y'' \equiv x'' \bmod B_N(X'')$. Choose any y and y' such that $j(y) = y''$, $i(y') = d(y)$. We must show that $x' \equiv y' \bmod B_{N-1}(X')$. $x''-y'' \varepsilon B_N(X'')$, say $x''-y'' = d(z'')$. Let $z \varepsilon X_{N+1}$ be such that $j(z) = z''$. $j(x-y-d(z)) = x''-y''-d(z'') = 0$, so $x-y-d(z) \varepsilon \operatorname{im}(i)$, say $i(z') = x-y-d(z)$, $z' \varepsilon X'_N$.

$$i(x'-y'-d(z')) = d(x)-d(y)-id(z')$$

$$= d(x-y-i(z')) = dd(z) = 0 .$$

Since i is a monomorphism, $x'-y'-d(z') = 0$, $x' \equiv y' \bmod B_{N-1}(X')$. δ is thus well-defined, and is clearly a morphism.

ii) $H_N(X') \longrightarrow H_N(X) \longrightarrow H_N(X'')$ is exact:

Clearly $\ker(H_N(j)) \supset \operatorname{im}(H_N(i))$ since $ji = 0$. Let $x \varepsilon Z_N(X)$ be such that $H_N(j)(\overline{x}) = 0$. $j(x) \varepsilon B_N(X'')$. Let $y \varepsilon X_{N+1}$ be such that $dj(y) = j(x)$. $j(x-d(y)) = j(x) - dj(y) = 0$, so $x-d(y) \varepsilon \operatorname{im}(i)$, say $i(z') = x-d(y)$. $di(z') = dx-ddy = 0$, so $d(z') = 0$. Thus $H_N(i)(\overline{z'}) = \overline{x}$, and $\ker(H_N(j)) \subset \operatorname{im}(H_N(i))$.

iii) $H_N(X) \longrightarrow H_N(X'') \longrightarrow H_{N-1}(X')$ is exact:

$x \varepsilon Z_N(X)$ implies $\delta H_N(j)(\overline{x}) = 0$ by construction of δ , so $\ker(\delta) \supset \operatorname{im}(H_N(j))$. Let $x'' \varepsilon Z_N(X'')$ and $\delta(\overline{x''}) = 0$. Let $j(x) = x''$, $i(x') = d(x)$. $x' \varepsilon B_{N-1}(X')$, say $x' = d(y')$. Now $d(x-i(y')) = d(x) - i(x') = 0$, so $x-i(y') \varepsilon Z_N(X)$. $H_N(j)(\overline{x-i(y')}) = H_N(j)(\overline{x}) = \overline{x''}$, and $\operatorname{im}(H_N(j)) \supset \ker(\delta)$.

i) Definition of δ: Let $x'' \in Z_N(X'')$. Let $x \in X_N$ be such that $j(x) = x''$. $jd(x) = 0$, so $d(x) \in \mathrm{im}(i)$. Let $x' \in X_{N-1}$ be such that $i(x') = d(x)$. $di(x') = 0$, so $d(x') = 0$. Define $\delta(\bar{x}'') = \bar{x}'$.

Now assume that $y'' \in Z_N(X'')$ is such that $y'' = x''$ mod $B_N(X'')$. Choose any y and y' such that $j(y) = y''$, $i(y') = d(y)$. We must show that $x' = y'$ mod $B_{N-1}(X')$. $x''-y'' \in B_N(X'')$, say $x''-y'' = d(z'')$. Let $z \in X_{N+1}$ be such that $j(z) = z''$. $j(x-y-d(z)) = x''-y''-d(z'') = 0$, so $x-y-d(z)$ is $\mathrm{im}(i)$, say $i(z') = x-y-d(z)$, $z' \in X'_N$.

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Clearly $\ker(H_N(j)) \supseteq \mathrm{im}(H_N(i))$ since $ji = 0$. Let $\bar{x} \in H_N(X)$ be such that $H_N(j)(\bar{x}) = 0$. $j(x) \in B_N(X'')$. Let $y \in X_{N+1}$ be such that $dj(y) = j(x)$. $j(x-d(y)) = j(x) - dj(y) = 0$, so $x-d(y) \in \mathrm{im}(i)$, say $i(z') = x-d(y)$. $di(z') = dx-ddy = 0$, so $d(z') = 0$. Thus $H_N(i)(\bar{z}') = \bar{x}$, and $\ker(H_N(j)) \subseteq \mathrm{im}(H_N(i))$.

iii) $H_N(X) \longrightarrow H_N(X'') \longrightarrow H_{N-1}(X')$ is exact:

$x \in Z_N(X)$ implies $\delta H_N(j)(\bar{x}) = 0$ by construction of δ, so $\ker(\delta) \supseteq \mathrm{im}(H_N(j))$. Let $x'' \in Z_N(X'')$ and $\delta(\bar{x}'') = 0$. Let $j(x) = x''$, $i(x') = d(x)$. $x' \in B_{N-1}(X')$, say $x' = d(y')$. Now $d(x-i(y')) = d(x) - i(x') = 0$, so $x-i(y') \in Z_N(X)$. $H_N(j)(\overline{x-i(y')}) = H_N(j)(\bar{x}) = \bar{x}''$, and $\mathrm{im}(H_N(j)) \supseteq \ker(\delta)$.

iv) $H_N(X'') \longrightarrow H_{N-1}(X') \longrightarrow H_{N-1}(X)$ is exact:

If $x' \varepsilon Z_{N-1}(X')$ and $\bar{x}' = \delta(\bar{x}'')$, then $i(x') = d(x)$ for some $x \varepsilon X_N$, and $H_N(i)(\bar{x}') = 0$, so $\ker(H_N(i)) \supset \mathrm{im}(\delta)$. Let $x' \varepsilon Z_{N-1}(X')$ and $H_N(i)(\bar{x}') = 0$. $i(x') \varepsilon B_{N-1}(X)$, say $d(x) = i(x')$. Let $x'' = j(x)$. $d(x'') = dj(x) = jd(x) = ji(x') = 0$. $\delta(\bar{x}'') = \bar{x}'$ by construction of δ, and $\mathrm{im}(\delta) \supset \ker(H_N(i))$. This completes the proof.

<u>Corollary 2.2</u>: If $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is an exact sequence of complexes and any two of X', X, X'' are exact, then so is the third.

<u>Definitions 2.3</u>: Let $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$ be an exact sequence. An exact sequence $0 \longrightarrow X' \xrightarrow{\tilde{i}} X \xrightarrow{\tilde{j}} X'' \longrightarrow 0$ of complexes, where X', X, X'' are left complexes (or left resolutions, etc.), over A', A, A'' and $\tilde{i}$, $\tilde{j}$ are morphisms over i, j is called a left complex (or left resolution, etc.), over $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$. Right complexes over exact sequences are analogously defined.

<u>Proposition 2.4</u>: If $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is a left complex over $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ and if X' and X'' are projective complexes, then so is X.

Proof: For all N $0 \longrightarrow X'_N \longrightarrow X_N \longrightarrow X''_N \longrightarrow 0$ is split exact, X_N is isomorphic $X'_N \oplus X''_N$, hence is a direct summand of a free module.

<u>Corollary 2.5</u>: If X' and X'' are projective resolutions of A' and A'', then X is a projective resolution of A.

<u>Proposition 2.6</u>: Let $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$ be an exact sequence. Let X' be a left resolution of A', X'' a projective complex over A''. Then there exists a left complex X over A and maps $\tilde{i}$, $\tilde{j}$ over i, j such that $0 \longrightarrow X' \xrightarrow{\tilde{i}} X \xrightarrow{\tilde{j}} X'' \longrightarrow 0$ is a left complex over $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$.

Proof: Set $X_N = X'_N \oplus X''_N$; let $\tilde{i}_N: X'_N \longrightarrow X_N$ and $\tilde{j}_N: X_N \longrightarrow X''_N$ be the canonical injection and projection. We must define a differential operator on X such that the desired commutativity relations are satisfied.

i) Consider degree 0 .

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X'_o & \xrightarrow{\tilde{i}} & X'_o \oplus X''_o & \xrightarrow{\tilde{j}} & X''_o & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon'} & & \vdots\,{\scriptstyle \epsilon} & \swarrow{\scriptstyle f_o} & \downarrow{\scriptstyle \epsilon''} & & \\
0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{j} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since X''_o is projective, there exists $f_o: X''_o \longrightarrow A$ such that $jf_o = \epsilon''$. Define $\epsilon: X'_o \oplus X''_o \longrightarrow A$ by $\epsilon(x',x'') = i\epsilon'(x') + f_o(x'')$. Then $\epsilon\tilde{i}(x') = i\epsilon'(x')$, $j\epsilon(x',x'') = jf_o(x'') = \epsilon''\tilde{j}(x',x'')$, and the diagram commutes. We must show that ϵ is an epimorphism. Let $x \varepsilon A$. Let $x'' \epsilon X''_o$ be such that $j(x) = \epsilon''(x'')$. Let $y = x - f_o(x'')$. $j(y) = j(x) - jf_o(x'') = j(x) - \epsilon''(x'') = 0$, so $y \varepsilon \operatorname{im}(i)$. Let $x' \varepsilon X'_o$ be such that $i\epsilon'(x') = y$. Then

$$\epsilon(x',x'') = i\epsilon'(x') + f_o(x'') = y + f_o(x'') = x \ .$$

ii) For $N \geq 1$, let $f_N: X''_N \longrightarrow X'_{N-1}$ be, for the moment, arbitrary morphisms. Define $d_N(x',x'') = (d'_N x' + f_N x'', d''_N x'')$. Clearly

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X'_N & \longrightarrow & X_N & \longrightarrow & X''_N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d'_N} & & \downarrow{\scriptstyle d_N} & & \downarrow{\scriptstyle d''_N} & & \\
0 & \longrightarrow & X'_{N-1} & \longrightarrow & X_{N-1} & \longrightarrow & X''_{N-1} & \longrightarrow & 0
\end{array}$$

is commutative for all $N \geq 1$. We will define the f_N so that $\epsilon d_1 = 0$, $d_{N-1}d_N = 0$ for $N > 1$.

$$\epsilon d_1 = \epsilon(d'_1 + f_1, d''_1) = i\epsilon' d'_1 + i\epsilon' f_1 + f_o d''_1 = i\epsilon' f_1 + f_o d''_1 .$$

$$d_{N-1} d_N = d_{N-1}(d'_N + f_N, d''_N) = (d'_{N-1} d'_N + d'_{N-1} f_N + f_{N-1} d''_N, d''_{N-1} d''_N)$$
$$= (d'_{N-1} f_N + f_{N-1} d''_N, 0) .$$

Thus we wish $-f_o d''_1 = i\epsilon' f_1$, $-f_{N-1} d''_N = d'_{N-1} f_N$. Now

$$\begin{array}{ccccc} & & X''_1 & & \\ & {\scriptstyle -f_o d''_1} & \downarrow & \searrow {\scriptstyle -\epsilon'' d''_1 = 0} & \\ X'_o & \xrightarrow{i\epsilon'} & A & \xrightarrow{j} & A'' \end{array}$$

commutes, giving $\operatorname{im}(-f_o d''_1) \subset \ker(j)$, and the row is exact. Hence, since X''_1 is projective, there exists $f_1 : X''_1 \longrightarrow X'_o$ such that $i\epsilon' f_1 = -f_o d''_1$. Again

$$\begin{array}{ccccc} & & X''_2 & & \\ & {\scriptstyle -f_1 d''_2} & \downarrow & \searrow {\scriptstyle +f_o d''_1 d''_2 = 0} & \\ X'_1 & \xrightarrow{d'_1} & X'_o & \xrightarrow{i\epsilon'} & A \end{array}$$

commutes and the row is exact, so there exists $f_2 : X''_2 \longrightarrow X'_1$ such that $d'_1 f_2 = -f_1 d''_2$. Finally,

$$\begin{array}{ccccc} & & X''_{N+1} & & \\ & {\scriptstyle -f_N d''_{N+1}} & \downarrow & \searrow {\scriptstyle +f_{N-1} d''_N d''_{N+1} = 0} & \\ X'_N & \xrightarrow{d'_N} & X'_{N-1} & \xrightarrow{d'_{N-1}} & X'_{N-2} \end{array}$$

commutes, by induction, and the row is exact by hypothesis, so there exists $f_{N+1} : X''_{N+1} \longrightarrow X'_N$ such that $d'_N f_{N+1} = -f_N d''_{N+1}$. This completes the proof.

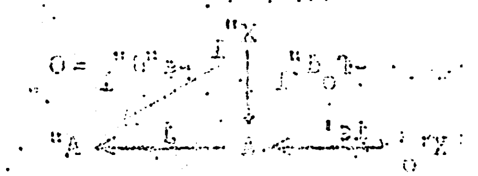
<u>Corollary 2.7</u>: In the proposition above, if X' and X'' are projective resolutions of A' and A'' , then X will necessarily be a projective resolution of A .

$ed_1 = e(d'_1 + f_0 d''_1) = ie'd'_1 + ie'f_1 + f_0 d''_1 \cdot d''_1 + f_0 d''_1$.

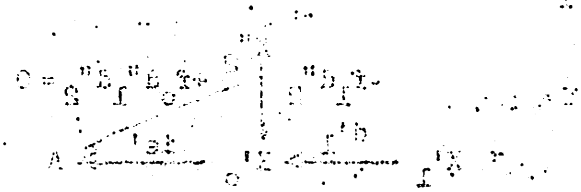
$d_{N-1}d_N = d_{N-1}(d'_N + f_N d''_N) = (d'_{N-1}d'_N + d'_{N-1}f_N + f_{N-1}d''_N,\ [illegible]\, d''_{N-1}d''_N)$

$= (d'_{N-1}f_N + f_{N-1}d''_N, 0)$.

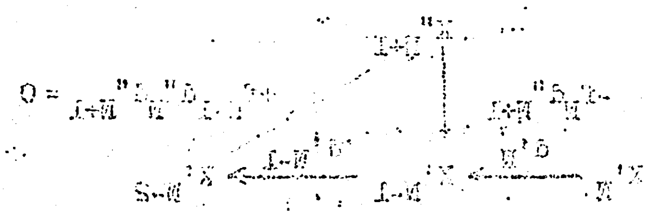
Thus we wish $-f_0 d''_1 = ie'f_1$, $-f_{N-1}d''_N = d'_{N-1}f_N$. Now



commutes, giving $\operatorname{Im}(-f_0 d''_1) \subset \ker(j)$, and the row is exact. Hence, since X''_1 is projective, there exists $f_1 : X''_1 \longrightarrow X'_0$ such that $ie'f_1 = -f_0 d''_1$. Again



commutes and the row is exact, so there exists $f_2 : X''_2 \longrightarrow X'_1$ such that $d'_1 f_2 = -f_1 d''_2$. Finally,



commutes, by induction, and the row is exact by hypothesis, so there exists $f_{N+1} : X''_{N+1} \longrightarrow X'_N$ such that $d'_N f_{N+1} = -f_N d''_{N+1}$. This completes the proof.

Corollary 2.7: In the proposition above, if X' and X'' are projective resolutions of A' and A'', then X will necessarily be a projective resolution of A .

<u>Proposition 2.8</u>: Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{j} & A'' & \longrightarrow & 0 \\ & & \downarrow g' & & \downarrow g & & \downarrow g'' & & \\ 0 & \longrightarrow & B' & \xrightarrow{k} & B & \xrightarrow{\ell} & B'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram with exact rows. Let $0 \longrightarrow X' \xrightarrow{\tilde{i}} X \xrightarrow{\tilde{j}} X'' \longrightarrow 0$ be a split exact left complex over $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$ and $0 \longrightarrow Y' \xrightarrow{\tilde{k}} Y \xrightarrow{\tilde{\ell}} Y'' \longrightarrow 0$ be a split exact left complex over $0 \longrightarrow B' \xrightarrow{k} B \xrightarrow{\ell} B'' \longrightarrow 0$. Further, let X'' be a projective complex and let Y' be exact. Then if $\tilde{g}': X' \longrightarrow Y'$ and $\tilde{g}'': X'' \longrightarrow Y''$ are translations over g' and g'', there exists a translation $\tilde{g}: X \longrightarrow Y$ over g such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{\tilde{i}} & X & \xrightarrow{\tilde{j}} & X'' & \longrightarrow & 0 \\ & & \downarrow \tilde{g}' & & \downarrow \tilde{g} & & \downarrow \tilde{g}'' & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\tilde{k}} & Y & \xrightarrow{\tilde{\ell}} & Y'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of complexes and translations.

Proof: Writing X_N as $X'_N \oplus X''_N$ and Y_N as $Y'_N \oplus Y''_N$; we see that $\tilde{g}_N(x',x'') = (\tilde{g}'_N(x'_N) + q_N(x''_N), \tilde{g}''_N(x''_N))$ is necessary for

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X'_N & \longrightarrow & X_N & \longrightarrow & X''_N & \longrightarrow & 0 \\ & & \downarrow \tilde{g}'_N & & \downarrow \tilde{g}_N & & \downarrow \tilde{g}''_N & & \\ 0 & \longrightarrow & Y'_N & \longrightarrow & Y_N & \longrightarrow & Y''_N & \longrightarrow & 0 \end{array}$$

to commute, where $q_N: X''_N \longrightarrow Y'_N$ is to be determined. The problem is to choose the q_N such that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{Y} \\ \epsilon \downarrow & & \downarrow \epsilon \\ A & \xrightarrow{g} & B \end{array}$$

commutes. We write d',d,d'',e',e,e'' and δ',δ,δ'', ϵ', ϵ, ϵ'' for the differentiation and augmentation morphisms of X',X,X'' and Y',Y,Y''.

Proposition 2.6: Let

$$\begin{array}{ccccccccc} 0 & \to & A' & \xrightarrow{i} & A & \xrightarrow{j} & A'' & \to & 0 \\ & & \downarrow g' & & \downarrow g & & \downarrow g'' & & \\ 0 & \to & B' & \xrightarrow{k} & B & \xrightarrow{l} & B'' & \to & 0 \end{array}$$

be a commutative diagram with exact rows. Let $0 \to X' \xrightarrow{\tilde{i}} X \xrightarrow{\tilde{j}} X'' \to 0$ be a split exact left complex over $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ and $0 \to Y' \xrightarrow{\tilde{k}} Y \xrightarrow{\tilde{l}} Y'' \to 0$ be a split exact left complex over $0 \to B' \xrightarrow{k} B \xrightarrow{l} B'' \to 0$. Further, let X'' be a projective complex and let Y' be exact. Then if $\tilde{g}': X' \to Y'$ and $\tilde{g}'': X'' \to Y''$ are translations over g' and g'', there exists a translation $\tilde{g}: X \to Y$ over g such that

$$\begin{array}{ccccccccc} 0 & \to & X' & \xrightarrow{\tilde{i}} & X & \xrightarrow{\tilde{j}} & X'' & \to & 0 \\ & & \downarrow \tilde{g}' & & \downarrow \tilde{g} & & \downarrow \tilde{g}'' & & \\ 0 & \to & Y' & \xrightarrow{\tilde{k}} & Y & \xrightarrow{\tilde{l}} & Y'' & \to & 0 \end{array}$$

is a commutative diagram of complexes and translations.

Proof: Writing X_N as $X'_N \oplus X''_N$ and Y_N as $Y'_N \oplus Y''_N$, we see that $\tilde{g}_N(x',x'') = (\tilde{g}'_N(x'_N) + q_N(x''_N), \tilde{g}''_N(x''_N))$ is necessary for

$$\begin{array}{ccccccccc} 0 & \to & X'_N & \to & X_N & \to & X''_N & \to & 0 \\ & & \downarrow \tilde{g}'_N & & \downarrow \tilde{g}_N & & \downarrow \tilde{g}''_N & & \\ 0 & \to & Y'_N & \to & Y_N & \to & Y''_N & \to & 0 \end{array}$$

to commute, where $q_N: X''_N \to Y'_N$ is to be determined. The problem is to choose the q_N such that

$$\begin{array}{ccc} Y & \xleftarrow{\tilde{g}} & X \\ e \downarrow & & \downarrow e \\ B & \xleftarrow{g} & A \end{array}$$

commutes. We write d', d, d'', e', e, e'' and $\delta', \delta, \delta''$, e', e, e'' for the differentiation and augmentation morphisms of X', X, X'' and Y', Y, Y''.

Let $f_o: X''_o \longrightarrow A$, $f_N: X''_N \longrightarrow X'_{N-1}$ and $\varphi_o: Y''_o \longrightarrow B$, $\varphi_N: Y''_N \longrightarrow Y'_{N-1}$ be morphisms as in the proof of the previous proposition. (These necessarily exist, since the sequences of complexes are split exact.)

By hypothesis, then, we have relations:

$e = ie' + f_o,\ ie'f_1 + f_o d''_1 = 0\ ,\ \ d'_{N-1}f_N + f_{N-1}d''_N = 0\ ,$

$\epsilon = k\epsilon' + \varphi_o,\ k\epsilon'\varphi_1 + \varphi_o\delta''_1 = 0\ ,\ \ \delta'_{N-1}\varphi_N + \varphi_{N-1}\delta''_N = 0$ and

$g'e' = \epsilon'\tilde{g}'_o,\ \tilde{g}'_{N-1}d'_N = \delta'_N\tilde{g}'_N\ ,\ \ g''e'' = \epsilon''\tilde{g}''_o,\ \ \tilde{g}_{N-1}d''_N = \delta''_N\tilde{g}''_N$.

We wish to obtain $ge = \epsilon\tilde{g}_o$, $\tilde{g}_{N-1}d_N = \delta_N\tilde{g}_N$. These relations then take the forms

i) $k\epsilon' q_o = -\varphi_o\tilde{g}''_o + gf_o$

$(\epsilon\tilde{g}_o = \epsilon(\tilde{g}'_o + q_o, \tilde{g}''_o) = k\epsilon'\tilde{g}'_o + k\epsilon' q_o + \varphi_o\tilde{g}''_o$;

$ge = gie' + gf_o = kg'e' + gf_o = k\epsilon'\tilde{g}'_o + gf_o$) .

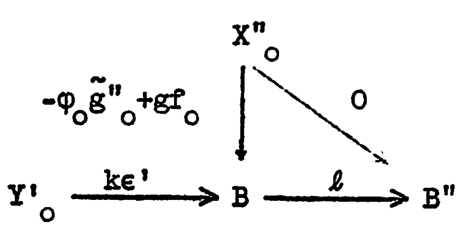
ii) $\delta'_N q_N = \tilde{g}'_{N-1}f_N + q_{N-1}d''_N - \varphi_N\tilde{g}''_N$

$(\tilde{g}_{N-1}d_N = \tilde{g}_{N-1}(d'_N + f_N, d''_N) = (\tilde{g}'_{N-1}d'_N + \tilde{g}'_{N-1}f_N + q_{N-1}d''_N, \tilde{g}''_{N-1}d''_N)$;

$\delta_N\tilde{g}_N = \delta_N(\tilde{g}'_N + q_N, \tilde{g}''_N) = (\delta'_N\tilde{g}'_N + \delta'_N q_N + \varphi_N\tilde{g}''_N\ ,\ \delta''_N\tilde{g}''_N)$;

$\delta''_N\tilde{g}''_N = \tilde{g}''_{N-1}d''_N\ ,\ \ \delta'_N\tilde{g}'_N = \tilde{g}'_{N-1}d'_N$)

Finally, then, we observe that in the following diagrams, the rows are exact and the diagrams commute, so that, by the projectivity of X'' , the desired morphisms are obtained:



$: -\ell\varphi_o\tilde{g}''_o + \ell gf_o = -\epsilon''\tilde{g}''_o + g''jf_o = -\epsilon''\tilde{g}''_o + g''e'' = 0$

Let $f_o : X''_o \longrightarrow A$, $f_N : X''_N \longrightarrow Y'_{N-1}$ and $\varphi_o : Y''_o \longrightarrow B$, $\varphi_N : Y''_N \longrightarrow Y'_{N-1}$ be morphisms as in the proof of the previous proposition. (These necessarily exist, since the sequences of complexes are split exact.)

By hypothesis, then, we have relations:

$$e = ke' + f_o, \quad ke'f_1 + f_o d''_1 = 0, \quad d'_{N-1}f_N + f_{N-1}d''_N = 0,$$

$$\varepsilon = k\varepsilon' + \varphi_o, \quad k\varepsilon'\varphi_1 + \varphi_o\delta''_1 = 0, \quad \delta'_{N-1}\varphi_N + \varphi_{N-1}\delta''_N = 0 \quad \text{and}$$

$$\tilde{g}e' = \varepsilon\tilde{g}'_o, \quad \tilde{g}'_{N-1}d'_N = \delta'_N\tilde{g}'_N, \quad \tilde{g}e'' = \varepsilon\tilde{g}''_o, \quad \tilde{g}_{N-1}d''_N = \delta''_N\tilde{g}''_N.$$

We wish to obtain $ge = \varepsilon\tilde{g}_o$, $\tilde{g}_{N-1}d_N = \delta_N\tilde{g}_N$. These relations then take the form:

i) $ke'\varphi_o = -\varphi_o\tilde{g}''_o + ef_o$

$(\varepsilon\tilde{g}_o = \varepsilon(\tilde{g}'_o + \varphi_o\tilde{g}''_o) = k\varepsilon'\tilde{g}'_o + k\varepsilon'\varphi_o + \varphi_o\tilde{g}''_o$;

$ge = gke' + gf_o = kg'e' + gf_o = k\varepsilon'\tilde{g}'_o + gf_o)$;

ii) $\delta'_N\varphi_N = \tilde{g}'_{N-1}f_N + \varphi_{N-1}d''_N - \varphi_N\tilde{g}''_N$

$(\tilde{g}_{N-1}d_N = \tilde{g}_{N-1}(d'_N + f_N d''_N) = (\tilde{g}'_{N-1}d'_N + \tilde{g}'_{N-1}f_N + \varphi_{N-1}d''_N\tilde{g}''_{N-1}d''_N)$;

$\delta_N\tilde{g}_N = \delta_N(\tilde{g}'_N + \varphi_N\tilde{g}''_N) = (\delta'_N\tilde{g}'_N + \delta'_N\varphi_N + \varphi_N\delta''_N\tilde{g}''_N)$;

[illegible]

Finally, then, we observe that in the following diagrams, the rows are exact and the diagrams commute, so that, by the projectivity of X''_o, the desired morphisms are obtained:

$-\varphi_o\tilde{g}''_o + ef_o$; $\quad ke'\varphi_o + \ldots$ [illegible]

$Y'_o \xrightarrow{k\varepsilon'} B \longrightarrow B''$

$$\begin{array}{ccccc} & & X''_1 & & \\ \tilde{g}'_o f_1 + q_o d''_1 - \varphi_1 \tilde{g}''_1 & & \big\downarrow & \searrow^{0} & \\ Y'_1 & \xrightarrow{\delta'_1} & Y'_o & \xrightarrow{k\epsilon'} & B \end{array}$$

$$\begin{aligned} &: k\epsilon'\tilde{g}'_o f_1 + k\epsilon' q_o d''_1 - k\epsilon'\varphi_1 \tilde{g}''_1 \\ &= kg'e'f_1 - \varphi_o \tilde{g}''_o d''_1 + gf_o d''_1 + \varphi_o \delta''_1 \tilde{g}''_1 \\ &= g(ie'f_1 + f_o d''_1) + \varphi_o(\delta''_1 \tilde{g}''_1 - \tilde{g}''_o d''_1) = 0 \end{aligned}$$

$$\begin{array}{ccccc} & & X''_N & & \\ g'_{N-1} f_N + q_{n-1} d''_N - \varphi_N g''_N & & \big\downarrow & \searrow^{0} & \\ Y'_N & \xrightarrow{\delta'_N} & Y'_{N-1} & \xrightarrow{\delta'_{N-1}} & Y'_{N-2} \end{array} \quad :$$

$$\begin{aligned} &\delta'_{N-1}\tilde{g}'_{N-1} f_N + \delta'_{N-1} q_{N-1} d''_N - \delta'_{N-1}\varphi_N \tilde{g}''_N \\ &= \tilde{g}'_{N-2} d'_{N-1} f_N + (\tilde{g}'_{N-2} f_{N-1} + q_{N-2} d''_{N-1} - \varphi_{N-1}\tilde{g}''_{N-1}) d''_N - \delta'_{N-1}\varphi_N \tilde{g}''_N \\ &= \tilde{g}'_{N-2}(d'_{N-1} f_N + f_{N-1} d''_N) - (\varphi_{N-1}\delta''_N + \delta'_{N-1}\varphi_N)\tilde{g}''_N = 0 \end{aligned}$$

Proposition 2.4': If $0 \to Y' \to Y \to Y'' \to 0$ is a right complex over the exact sequence $0 \to A' \to A \to A'' \to 0$ and if Y' and Y'' are injective complexes, then so is Y.

Corollary 2.5': If Y' and Y'' are injective resolutions of A' and A'', then Y is an injective resolution of A.

Proposition 2.6': Let $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ be an exact sequence. Let Y' be an injective complex over A', Y'' a right resolution of A''. Then there exists a right complex Y over A and maps $\tilde{i}, \tilde{j}$ over i, j such that $0 \to Y' \xrightarrow{\tilde{i}} Y \xrightarrow{\tilde{j}} Y'' \to 0$ is a right complex over $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$.

Corollary 2.7': In the proposition above, if Y' and Y'' are injective resolutions of A' and A'', then Y will necessarily be an injective resolution of A.

Proposition 2.8': Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{j} & A'' & \longrightarrow & 0 \\ & & \downarrow{g'} & & \downarrow{g} & & \downarrow{g''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{k} & B & \xrightarrow{\ell} & B'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram with exact rows. Let $0 \longrightarrow X' \xrightarrow{\tilde{i}} X \xrightarrow{\tilde{j}} X'' \longrightarrow 0$ be a split exact right complex over $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$ and $0 \longrightarrow Y' \xrightarrow{\tilde{k}''} Y \xrightarrow{\tilde{\ell}''} Y'' \longrightarrow 0$ be a split exact right complex over $0 \longrightarrow B' \xrightarrow{k} B \xrightarrow{\ell} B'' \longrightarrow 0$. Further, let X'' be exact and let Y' be an injective complex. Then if $\tilde{g}': X' \longrightarrow Y'$ and $\tilde{g}'': X'' \longrightarrow Y''$ are translations over g' and g'', there exists a translation $\tilde{g}: X \longrightarrow Y$ over g such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{\tilde{i}} & X & \xrightarrow{\tilde{j}} & X'' & \longrightarrow & 0 \\ & & \downarrow{\tilde{g}'} & & \downarrow{\tilde{g}} & & \downarrow{\tilde{g}''} & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\tilde{k}} & Y & \xrightarrow{\tilde{\ell}} & Y'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of complexes and translations.

3. Construction of Tor(A,B)

Let A be a right Λ-module, B a left Λ module. Then $A \otimes B$ is an Abelian group (see ex. 10, 11 ch. 3). Recall that if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact, then $A' \otimes B \longrightarrow A \otimes B \longrightarrow A'' \otimes B \longrightarrow 0$ is exact, and if $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is exact, then so is $A \otimes B' \longrightarrow A \otimes B \longrightarrow A \otimes B'' \longrightarrow 0$. In this section we will construct objects by means of which the behavior of tensored exact sequences on the left may be studied.

Let X be a complex of right Λ-modules, Y a complex of left Λ-modules. $X \otimes Y$ is a graded module with $(X \otimes Y)_N = \bigoplus_{i+j=N} X_i \otimes Y_j$. For notational convenience, we write $X_i \otimes Y_j = X \overset{i,j}{\otimes} Y$.

<u>Proposition 2.6</u>: Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{j} & A'' & \longrightarrow & 0 \\ & & \downarrow g' & & \downarrow g & & \downarrow g'' & & \\ 0 & \longrightarrow & B' & \xrightarrow{k} & B & \xrightarrow{\ell} & B'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram with exact rows. Let

$0 \longrightarrow X' \xrightarrow{\bar{i}} X \xrightarrow{\bar{j}} X'' \longrightarrow 0$ be a split exact right complex over $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$ and $0 \longrightarrow Y' \xrightarrow{\bar{k}} Y \xrightarrow{\bar{\ell}} Y'' \longrightarrow 0$ be a split exact right complex over $0 \longrightarrow B' \xrightarrow{k} B \xrightarrow{\ell} B'' \longrightarrow 0$. Further, let X'' be exact and let Y' be an injective complex. Then if $\bar{g}': X' \longrightarrow Y'$ and $\bar{g}'': X'' \longrightarrow Y''$ are translations over g' and g'', there exists a translation $\bar{g}: X \longrightarrow Y$ over g such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{\bar{i}} & X & \xrightarrow{\bar{j}} & X'' & \longrightarrow & 0 \\ & & \downarrow \bar{g}' & & \downarrow \bar{g} & & \downarrow \bar{g}'' & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\bar{k}} & Y & \xrightarrow{\bar{\ell}} & Y'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of complexes and translations.

3. Construction of Tor(A,B)

Let A be a right Λ-module, B a left Λ-module. Then $A \otimes B$ is an Abelian group (see ex. 10, II ch. 3). Recall that if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact, then $A' \otimes B \longrightarrow A \otimes B \longrightarrow A'' \otimes B \longrightarrow 0$ is exact, and if $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is exact, then so is $A \otimes B' \longrightarrow A \otimes B \longrightarrow A \otimes B'' \longrightarrow 0$. In this section we will construct objects by means of which the behavior of tensored exact sequences on the left may be studied.

Let X be a complex of right Λ-modules, Y a complex of left Λ-modules. $X \otimes Y$ is a graded module with $(X \otimes Y)_N = \bigoplus_{i+j=N} X_i \otimes Y_j$. For notational convenience, we write $X_i \otimes Y_j = X^i \otimes Y^j$.

Now let X', X'' be further complexes of right Λ-modules, Y' and Y'' of left Λ-modules. Let $f: X \longrightarrow X'$ be a morphism of degree p, $g: Y \longrightarrow Y'$ be a morphism of degree q. Define
$f \overset{i,j}{\otimes} g = (-1)^{iq} f_i \otimes g_j$ and $(f \otimes g)_N = \oplus_{i+j=N} f \overset{i,j}{\otimes} g$. Suppose $f': X' \longrightarrow X''$ and $g': Y' \longrightarrow Y''$ are morphisms of degrees p' and q'. Then $f'f$ and $g'g$ are morphisms of degree $p+p'$ and $q+q'$. Further, we have

$$\begin{aligned} f'f \overset{i,j}{\otimes} g'g &= (-1)^{i(q+q')} f'_{i-p} f_i \otimes g'_{j-q} g_j \\ &= (-1)^{i(q+q')} (f'_{i-p} \otimes g'_{j-q})(f_i \otimes g_j) \\ &= (-1)^{i(q+q')} (-1)^{-(i-p)q'} (f' \overset{i-p,j-q}{\otimes} g')(-1)^{-iq} (f \overset{i,j}{\otimes} g) \\ &= (-1)^{pq'} (f' \overset{i-p,j-q}{\otimes} g')(f \overset{i,j}{\otimes} g) \quad . \end{aligned}$$

We now return to the consideration of $X \otimes Y$. We will define a differential operator on $X \otimes Y$ by $d = d_x \otimes i_y + i_x \otimes d_y$ where d_x and d_y are the differential operators on X and Y and i_x and i_y are the identities of X and Y. d is of degree 1, and $dd = 0$ since

$$\begin{aligned} dd &= d_x d_x \otimes i_y i_y + i_x d_x \otimes d_y i_y + d_x i_x \otimes i_y d_y + i_x i_x \otimes d_y d_y \\ &= i_x d_x \otimes d_y i_y + d_x i_x \otimes i_y d_y \\ &= (-1)(i_x \otimes d_y)(d_x \otimes i_y) + (d_x \otimes i_y)(i_x \otimes d_y) = 0 \ . \end{aligned}$$

Thus $X \otimes Y$ is given the structure of a complex.

Before defining $\mathrm{Tor}(A,B)$, we prove the

<u>Lemma 3.1</u>: Suppose X and Y are left complexes, X_q is flat for all q and Y is exact. Then $X \otimes Y$ is exact.

Proof: i) Suppose $Y_q \neq 0$ for $q=s$ and $q=s+1$ only. Then $0 \longrightarrow Y_{s+1} \xrightarrow{d} Y_s \longrightarrow 0$ is exact. Let $f_s: Y_s \longrightarrow Y_{s+1}$ be the inverse isomorphism to d_{s+1}, $f_q = 0$, $q \neq s$. Then $df + fd = i_y$. Define $D: X \otimes Y \longrightarrow X \otimes Y$ by $D = i_x \otimes f$.

Now let X', X'' be further complexes of right A-modules, Y' and Y'' of left A-modules. Let $f: X \to X'$ be a morphism of degree p, $g: Y \to Y'$ be a morphism of degree q. Define $(f \otimes g)_{i,j} = (-1)^{qi} f_i \otimes g_j$, and $(f \otimes g)_n = \oplus_{i+j=n} (f \otimes g)_{i,j}$. Suppose $f': X' \to X''$ and $g': Y' \to Y''$ are morphisms of degrees p' and q'. Then $f'f$ and $g'g$ are morphisms of degree $p+p'$ and $q+q'$. Further, we have

$$f'f \otimes g'g = (-1)^{i(q+q')} f'_{i+p} f_i \otimes g'_{j+q} g_j$$

$$= (-1)^{i(q+q')} (f'_{i+p} \otimes g'_{j+q})(f_i \otimes g_j)$$

$$= (-1)^{i(q+q')} (-1)^{-(i+p)q'} (f' \otimes g')_{i+p,j+q} (-1)^{-qi} (f \otimes g)_{i,j}$$

$$= (-1)^{pq'} (f' \otimes g')_{i+p,j+q} (f \otimes g)_{i,j} .$$

We now return to the consideration of $X \otimes Y$. We will define a differential operator on $X \otimes Y$ by $d = d_X \otimes 1_Y + 1_X \otimes d_Y$ where d_X and d_Y are the differential operators on X and Y and 1_X and 1_Y are the identities of X and Y. d is of degree 1, and $dd = 0$ since

$$dd = d_X d_X \otimes 1_Y 1_Y + d_X 1_X \otimes 1_Y d_Y + 1_X d_X \otimes d_Y 1_Y + 1_X 1_X \otimes d_Y d_Y$$

$$= 1_X d_X \otimes d_Y 1_Y + d_X 1_X \otimes 1_Y d_Y$$

$$= (-1)(1_X \otimes d_Y)(d_X \otimes 1_Y) + (d_X \otimes 1_Y)(1_X \otimes d_Y) = 0 .$$

Thus $X \otimes Y$ is given the structure of a complex.

Before defining $\mathrm{Tor}(A,B)$, we prove the

Lemma 3.1: Suppose X and Y are left complexes, X_q is flat for all q and Y is exact. Then $X \otimes Y$ is exact.

Proof: i) Suppose $Y_q \neq 0$ for $q=s$ and $q=s+1$ only. Then $0 \to Y_{s+1} \xrightarrow{d} Y_s \to 0$ is exact. Let $f_s: Y_s \to Y_{s+1}$ be the inverse isomorphism to d_{s+1}, $f_q = 0$, $q \neq s$. Then $df + fd = 1_Y$. Define $D: X \otimes Y \to X \otimes Y$ by $D = 1_X \otimes f$.

Then $dD = d_x i_x \otimes i_y f + i_x i_x \otimes d_y f$

$Dd = i_x d_x \otimes fi_y + i_x i_x \otimes fd_y$

$dD + Dd = i_x i_x \otimes d_y f + i_x i_x \otimes fd_y$

$= i_x \otimes (d_y f + fd_y) = i_x \otimes i_y = i_{x\otimes y}$.

Hence the identity of $X \otimes Y$ is homotpic to the zero map and $H_N(X \otimes Y) = 0$ for all N .

II) Proceeding inductively, assume $Y_q = 0$ for $q < r$ and for $q > N+1 > r$. Define Y' by $Y'_q = 0$ for $q \neq N, N+1$, $Y'_N = B_N(Y)$, $Y'_{N+1} = Y_{N+1}$; $H_q(X \otimes Y') = 0$ for all q by step i). Define Y'' such that $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ is exact. Since $H_q(Y') = H_q(Y) = 0$ for all q, $H_q(Y'') = 0$ for all q . $Y''_q = 0$ if $q < r$ and if $q > N$, so, by induction, $H_q(X \otimes Y'') = 0$ for all q . Since X is flat, $0 \longrightarrow X \otimes Y' \longrightarrow X \otimes Y \longrightarrow X \otimes Y'' \longrightarrow 0$ is exact (and is a sequence of translations), and $H_q(X \otimes Y) = 0$ for all q .

iii) Now assume the original hypotheses. For $s \geq 0$, $(X \otimes Y)_s = \bigoplus_{i=0}^{s} X_i \otimes Y_{s-i}$. Define Y' by $Y'_q = Y_q$ if $q > s+1$, $Y'_{s+1} = B_{s+1}(Y)$, $Y'_q = 0$ if $q < s+1$. Define Y'' such that $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ is exact. $H_q(Y') = 0$ for all q by construction since $H_q(Y) = 0$ for all q . Hence $H_q(Y'') = 0$ for all q . $Y''_q = 0$ for $q < 0$, $Y''_q = 0$ for $q > s+1$, so by step ii) $H_q(X \otimes Y'') = 0$ for all q . $H_s(X \otimes Y') = 0$ by construction, hence $0 \longrightarrow H_s(X \otimes Y) \longrightarrow 0$ is exact, $H_s(X \otimes Y) = 0$. Since s was arbitrary, $H_q(X \otimes Y) = 0$ for all q .

Note that the hypotheses and proof of the lemma are symmetric in X and Y: If X is exact and Y is flat, $H_q(X \otimes Y) = 0$ for all q .

Definition 3.2: Let A be a right Λ-module, B a left Λ-module, X a projective resolution of A, and Y a projective resolution of B. Define $\mathrm{Tor}_N(A \otimes B) = H_N(X \otimes Y)$. We must prove that $\mathrm{Tor}_N(A \otimes B)$ is independent of the choice of the projective resolutions X and Y. Instead of proving this directly, we first prove

Proposition 3.3: $H_q(A \otimes Y)$, $H_q(X \otimes Y)$, and $H_q(X \otimes B)$ are isomorphic for all q, where A and B are regarded as complexes concentrated in degree 0 with $d = 0$.

Proof: $Y \xrightarrow{\epsilon} B \longrightarrow 0$ may be regarded as a translation ($\epsilon_q = 0$ for $q \neq 0$). Define Y' by $Y'_q = Y_q$ for $q \neq 0$, $Y'_o = \ker(\epsilon_o)$. Then $0 \longrightarrow Y' \longrightarrow Y \longrightarrow B \longrightarrow 0$ is an exact sequence of translations. $H_q(Y') = 0$ for all q since $\ker(\epsilon_o) = B_o(Y) = Y'_o$. By lemma 3.1, $h_q(X \otimes Y') = 0$ for all q. By lemma 2.1, there is an exact sequence

$$\cdots \longrightarrow 0 = H_q(X\otimes Y') \longrightarrow H_q(X\otimes Y) \longrightarrow H_q(X\otimes B) \longrightarrow H_{q-1}(X\otimes Y') = 0 \longrightarrow \cdots$$

so $H_q(X \otimes Y)$ is isomorphic to $H_q(X \otimes B)$ for all q. Similarly, $H_q(X \otimes Y)$ is isomorphic to $H_q(A \otimes Y)$ for all q.

Proposition 3.4: $\mathrm{Tor}_q(A,B)$ is independent of the choice of X and Y.

Proof: Let X and X' be projective resolutions of A. By proposition 1.5, there exists $f: X \longrightarrow X'$ and $g: X' \longrightarrow X$ lying over the identity i_A of $A \longrightarrow A$. By proposition 1.8, $gf: X \longrightarrow X$ lying over i_A is homotopic to $i_X: X \longrightarrow X$, say $dD + Dd = i_X - gf$. Then

$$\begin{aligned} i_X \otimes i_B - gf \otimes i_B &= (i_X - gf) \otimes i_B \\ &= (dD + Dd) \otimes i_B \\ &= dD \otimes i_B + Dd \otimes i_B \\ &= (d \otimes i_B)(D \otimes i_B) + (D \otimes i_B)(d \otimes i_B), \end{aligned}$$

and,

Definition 3.3: Let A be a right A-module, B a left A-module, X a projective resolution of A, and Y a projective resolution of B. Define $Tor_n(A \otimes B) = H_n(X \otimes Y)$. We must prove that $Tor_n(A \otimes B)$ is independent of the choice of the projective resolutions X and Y. Instead of proving this directly, we first prove

Proposition 3.3: $H_q(A \otimes Y)$, $H_q(X \otimes Y)$, and $H_q(X \otimes B)$ are isomorphic for all q (where A and B are regarded as complexes concentrated in degree 0 with d = 0).

Proof: $Y \xrightarrow{\varepsilon} B \to 0$ may be regarded as a translation ($\varepsilon_q = 0$ for $q \neq 0$). Define Y' by $Y'_q = Y_q$ for $q \neq 0$, $Y'_0 = \ker(\varepsilon_0)$. Then $0 \to Y' \to Y \to B \to 0$ is an exact sequence of translations. $H_q(Y') = 0$ for all q since $\ker(\varepsilon_0) = B_0(Y) = Y'_0$. By lemma 3.1, $H_q(X \otimes Y') = 0$ for all q. By lemma 2.1, there is an exact sequence

$$\dots \to 0 = H_q(X\otimes Y') \to H_q(X\otimes Y) \to H_q(X\otimes B) \to H_{q-1}(X\otimes Y') = 0 \to \dots$$

so $H_q(X \otimes Y)$ is isomorphic to $H_q(X \otimes B)$ for all q. Similarly, $H_q(X \otimes Y)$ is isomorphic to $H_q(A \otimes Y)$ for all q.

Proposition 3.4: $Tor_q(A,B)$ is independent of the choice of X and Y.

Proof: Let X and X' be projective resolutions of A. By proposition 1.5, there exists $f: X \to X'$ and $g: X' \to X$ lying over the identity 1_A of $A \to A$. By proposition 1.6, $gf: X \to X$ lying over 1_A is homotopic to $1_X: X \to X$, say $dD + Dd = 1_X - gf$.

Then $1_X \otimes 1_B - gf \otimes 1_B = (1_X - gf) \otimes 1_B$

$$= (dD + Dd) \otimes 1_B$$

$$= dD \otimes 1_B + Dd \otimes 1_B$$

$$= (d \otimes 1_B)(D \otimes 1_B) + (D \otimes 1_B)(d \otimes 1_B) \text{ , and,}$$

since $d\otimes i_B$ is the differential operator of $X\otimes B$, $D\otimes i_B$ is a homotopy between $i_X\otimes i_B$ and $gf\otimes i_B$. $gf\otimes i_B = (g\otimes i_B)(f\otimes i_B)$, so $H_q(gf\otimes i_B) = H_q(g\otimes i_B)H_q(f\otimes i_B)$, and, arguing similarly for fg , since $H_q(i_X\otimes i_B)$ is the identity on $H_q(X\otimes B)$ and $H_q(i_{X'}\otimes i_B)$ is the identity on $H_q(X'\otimes B)$, we have that $H_q(g\otimes i_B)$ and $H_q(f\otimes i_B)$ are inverse isomorphisms. Thus $H_q(X\otimes B)$ and $H_q(X'\otimes B)$ are canonically isomorphic for all q . Arguing similarly, $\mathrm{Tor}_q(A,B)$ is independent of the choice of Y .

Proposition 3.5: Let $0 \to A' \xrightarrow{k} A \xrightarrow{\ell} A'' \to 0$ be an exact sequence of right Λ-modules and B a left Λ-module. Then there exists a canonical exact sequence

$$\cdots \to \mathrm{Tor}_N(A'B) \to \mathrm{Tor}_N(A,B) \to \mathrm{Tor}_N(A'',B) \to \mathrm{Tor}_{N-1}(A',B) \to \cdots$$
$$\to \mathrm{Tor}_0(A',B) \to \mathrm{Tor}_0(A,B) \to \mathrm{Tor}_0(A''B) \to 0 .$$

Proof: Let X' and X'' be projective resolutions of A' and A'' . By proposition 2.6 and corollary 2.7, there exists a projective resolution X of A and morphisms $\tilde{k}$ and $\tilde{\ell}$ over k and ℓ such that $0 \to X' \xrightarrow{\tilde{k}} X \xrightarrow{\tilde{\ell}} X'' \to 0$ is a projective resolution of $0 \to A' \xrightarrow{k} A \xrightarrow{\ell} A'' \to 0$. $0 \to X'\otimes B \to X\otimes B \to X''\otimes B \to 0$ is exact, so by lemma 2.1 there is an exact sequence i)

$$\cdots \to H_N(X'\otimes B) \to H_N(X\otimes B) \to H_N(X''\otimes B) \xrightarrow{\delta(x)} H_{N-1}(X'\otimes B) \to \cdots$$
$$\to H_0(X'\otimes B) \to H_0(X\otimes B) \to H_0(X''\otimes B) \to 0 .$$

To complete the proof, it suffices to show that the connecting morphism δ is independent of the choice of X. Suppose $0 \to X' \to Y \to X'' \to 0$ also lies over $0 \to A' \to A \to A'' \to 0$. Then by proposition 2.8, lying over the commutative diagram since $d'\otimes 1_B$ is the differential operator of $X'\otimes B$, $D\otimes 1_B$ is a homotopy between $1_{X'}\otimes 1_B$ and $gf\otimes 1_B$. $gf\otimes 1_B = (g\otimes 1_B)(f\otimes 1_B)$, so $H_q(gf\otimes 1_B) = H_q(g\otimes 1_B)H_q(f\otimes 1_B)$, and, arguing similarly for fg, since $H_q(1_{X'}\otimes 1_B)$ is the identity on $H_q(X'\otimes B)$ and $H_q(1_X\otimes 1_B)$ is the identity on $H_q(X\otimes B)$, we have that $H_q(g\otimes 1_B)$ and $H_q(f\otimes 1_B)$ are inverse isomorphisms. Thus $H_q(X\otimes B)$ and $H_q(X'\otimes B)$ are canonically isomorphic for all q. Arguing similarly, $Tor_q(A,B)$ is independent of the choice of X.

<u>Proposition 3.5</u>: Let $0 \to A' \xrightarrow{\chi} A \xrightarrow{\psi} A'' \to 0$ be an exact sequence of right Λ-modules and B a left Λ-module. Then there exists a canonical exact sequence

$$\cdots \to Tor_n(A',B) \to Tor_n(A,B) \to Tor_n(A'',B) \to Tor_{n-1}(A',B) \to \cdots$$
$$\to Tor_0(A',B) \to Tor_0(A,B) \to Tor_0(A'',B) \to 0 .$$

Proof: Let X' and X'' be projective resolutions of A' and A''. By proposition 2.6 and corollary 2.7, there exists a projective resolution X of A and morphisms κ over χ and λ over ψ such that $0 \to X' \xrightarrow{\kappa} X \xrightarrow{\lambda} X'' \to 0$ is a projective resolution of $0 \to A' \xrightarrow{\chi} A \xrightarrow{\psi} A'' \to 0$. $0 \to X'\otimes B \to X\otimes B \to X''\otimes B \to 0$ is exact, so by lemma 2.1 there is an exact sequence (1)

$$\cdots \to H_n(X'\otimes B) \to H_n(X\otimes B) \to H_n(X''\otimes B) \xrightarrow{\delta(X)} H_{n-1}(X'\otimes B) \to \cdots$$
$$\to H_0(X'\otimes B) \to H_0(X\otimes B) \to H_0(X''\otimes B) \to 0 .$$

To complete the proof, it suffices to show that the connecting morphism δ is independent of the choice of X. Suppose $0 \to X' \to Y \to X'' \to 0$ also lies over $0 \to A' \to A \to A'' \to 0$. Then by proposition 2.8, lying over the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_{A'}} & & \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_{A''}} & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_{X'}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i_{X''}} & & \\ 0 & \longrightarrow & X' & \longrightarrow & Y & \longrightarrow & X'' & \longrightarrow & 0 \end{array}$$

The latter diagram induces a translation of the sequence i).

Thus

$$\begin{array}{ccc} H_q(X''\otimes B) & \xrightarrow{\quad \delta(X) \quad} & H_{q-1}(X'\otimes B) \\ \downarrow{\scriptstyle H_N(i_{X''}\otimes i_B)} & & \downarrow{\scriptstyle H_{N-1}(i_{X'}\otimes i_B)} \\ H_q(X\otimes B) & \xrightarrow{\quad \delta(Y) \quad} & H_{q-1}(X'\otimes B) \end{array}$$

is a commutative diagram whose columns are identity morphisms, and δ is independent of the choice of X .

Note that the proof holds equally well for an exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ of left Λ-modules and a right Λ-module A , yielding a canonical exact sequence

$$\ldots \longrightarrow \mathrm{Tor}_N(A,B') \longrightarrow \mathrm{Tor}_N(A,B) \longrightarrow \mathrm{Tor}_N(A,B'') \longrightarrow \mathrm{Tor}_{N-1}(A,B') \longrightarrow \ldots$$
$$\longrightarrow \mathrm{Tor}_o(A,B') \longrightarrow \mathrm{Tor}_o(A,B) \longrightarrow \mathrm{Tor}_o(A,B'') \longrightarrow 0 \quad .$$

<u>Proposition 3.6</u>: $\mathrm{Tor}_o(A,B)$ is isomorphic to $A\otimes B$.

Proof: Let X be a projective resolution of A . $X_1\otimes B \xrightarrow{d_1\otimes i_B} X_o\otimes B \xrightarrow{\epsilon\otimes i_B} A\otimes B \longrightarrow 0$ is exact, so $H_o(X\otimes B)=X_o\otimes B/\mathrm{im}(d_1\otimes i_B)$ is isomorphic to $A\otimes B$.

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & \downarrow f_{A'} & & \downarrow f_A & & \downarrow f_{A''} & & \\ 0 & \to & A' & \to & A & \to & A'' & \to & 0 \end{array}$$

is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X' & \to & X & \to & X'' & \to & 0 \\ & & \downarrow f_{X'} & & \downarrow f & & \downarrow f_{X''} & & \\ 0 & \to & Y' & \to & Y & \to & Y'' & \to & 0 \end{array}$$

The latter diagram induces a translation of the sequence 1).

Thus

$$\begin{array}{ccc} H_p(X''\otimes B) & \xrightarrow{\delta(X)} & H_{p-1}(X'\otimes B) \\ \downarrow H_p(f_{X''}\otimes 1_B) & & \downarrow H_{p-1}(f_{X'}\otimes 1_B) \\ H_p(Y''\otimes B) & \xrightarrow{\delta(Y)} & H_{p-1}(Y'\otimes B) \end{array}$$

is a commutative diagram whose columns are identity morphisms, and δ is independent of the choice of X.

Note that the proof holds equally well for an exact sequence $0 \to B' \to B \to B'' \to 0$ of left Λ-modules and a right Λ-module A, yielding a canonical exact sequence

$$\ldots \to Tor_n(A,B') \to Tor_n(A,B) \to Tor_n(A,B'') \to Tor_{n-1}(A,B') \to \ldots$$

$$\to Tor_0(A,B') \to Tor_0(A,B) \to Tor_0(A,B'') \to 0 .$$

<u>Proposition 5.6</u>: $Tor_0(A,B)$ is isomorphic to $A\otimes B$.

Proof: Let X be a projective resolution of A.

$X_1\otimes B \xrightarrow{d_1\otimes 1_B} X_0\otimes B \xrightarrow{\varepsilon\otimes 1_B} A\otimes B \to 0$ is exact, so $H_0(X\otimes B) = X_0\otimes B/Im(d_1\otimes 1_B)$ is isomorphic to $A\otimes B$.

Proposition 3.7: The following are equivalent:

i) A is flat

ii) $\mathrm{Tor}_1(A,B) = 0$ for all B .

iii) $\mathrm{Tor}_q(A,B) = 0$ for all $q \geq 1$ and for all B .

iv) $\mathrm{Tor}_q(A,B) = 0$ for all $q \geq 1$ and for all finitely generated B .

v) $\mathrm{Tor}_q(A,B) = 0$ for all $q \geq 1$ and for all cyclic B .

vi) $\mathrm{Tor}_q(A,\Lambda/I) = 0$ for all $q \geq 1$ and for all ideals I .

Proof: i)==>iii) Let B be a module and X a projective resolution of B . $\ldots \longrightarrow A\otimes X_N \longrightarrow \ldots \longrightarrow A\otimes X_o \longrightarrow A\otimes B \longrightarrow 0$ is exact.

iii)==>ii) is immediate.

ii)==>i) Let $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ be exact.
$0 = \mathrm{Tor}_1(A,B'') \longrightarrow A\otimes B' \longrightarrow A\otimes B \longrightarrow A\otimes B'' \longrightarrow 0$ is exact.

iii) ==> vi) is obvious.

iv) ==> iii): Let B be a module. Let the finitely generated submodules of B be indexed by I where we define $i \leq j$ if $B_i \subset B_j$. Let $\mathbb{B}$ denote the direct system of the finitely generated submodules of B so obtained. Then $\varinjlim \mathbb{B} = B$. $\varinjlim(A\otimes\mathbb{B}) = A\otimes \lim \mathbb{B} = A\otimes B$ by proposition 5.12 of chapter 3. If X is a projective resolution of A , $\ldots \longrightarrow X_N\otimes\mathbb{B} \longrightarrow \ldots \longrightarrow X_o\otimes\mathbb{B} \longrightarrow 0$ is an exact sequence of direct systems, so that $\ldots \longrightarrow X_N\otimes B \longrightarrow \ldots \longrightarrow X_o\otimes B \longrightarrow 0$ is exact, by proposition 5.16 of chapter 3.

v) ==> iv): Let B have N generators, and assume the result for modules with N-1 generators. Let $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ be exact, where B'' has one generator, B' has N-1 generators. Then $\ldots \longrightarrow 0 = \mathrm{Tor}_q(A,B') \longrightarrow \mathrm{Tor}_q(A,B) \longrightarrow \mathrm{Tor}_q(A,B'') = 0 \longrightarrow \ldots$ is exact, $\mathrm{Tor}_q(A,B) = 0$ for $q \geq 1$.

<u>Proposition 3.7</u>: The following are equivalent:

i) A is flat

ii) $Tor_1(A,B) = 0$ for all B.

iii) $Tor_q(A,B) = 0$ for all $q \geq 1$ and for all B.

iv) $Tor_q(A,B) = 0$ for all $q \geq 1$ and for all finitely generated B.

v) $Tor_q(A,B) = 0$ for all $q \geq 1$ and for all cyclic B.

vi) $Tor_q(A,A/I) = 0$ for all $q \geq 1$ and for all ideals I.

Proof: i)$\Rightarrow$iii) Let B be a module and X a projective resolution of B, $\ldots \to A\otimes X_n \to \ldots \to A\otimes X_0 \to A\otimes B \to 0$ is exact.

iii)$\Rightarrow$ii) is immediate.

ii)$\Rightarrow$i) Let $0 \to B' \to B \to B'' \to 0$ be exact.

$0 = Tor_1(A,B'') \to A\otimes B' \to A\otimes B \to A\otimes B'' \to 0$ is exact.

iii) $\Rightarrow$ vi) is obvious.

iv) $\Rightarrow$ iii): Let B be a module. Let the finitely generated submodules of B be indexed by I where we define $i \leq j$ if $B_i \subset B_j$. Let B' denote the direct system of the finitely generated submodules of B so obtained. Then $\varinjlim B' = B$, $\varinjlim(A\otimes B') = A\otimes \varinjlim B' = A\otimes B$ by proposition 7.12 of chapter 3. If X is a projective resolution of A, $\ldots \to X_n\otimes B' \to \ldots \to X_0\otimes B' \to 0$ is an exact sequence of direct systems, so that $\ldots \to X_n\otimes B \to \ldots \to X_0\otimes B \to 0$ is exact, by proposition 5.16 of chapter 3.

v) $\Rightarrow$ iv): Let B have N generators, and assume the result for modules with $N-1$ generators. Let $0 \to B' \to B \to B'' \to 0$ be exact, where B'' has one generator, B' has $N-1$ generators. Then $\ldots \to 0 = Tor_q(A,B') \to Tor_q(A,B) \to Tor_q(A,B'') = 0 \to \ldots$ is exact, $Tor_q(A,B) = 0$ for $q \geq 1$.

vi) ==> v): Let B have one generator. Then there exists I such that $0 \to I \to \Lambda \to B \to 0$ is exact, B is isomorphic to Λ/I. Of course, the proposition holds also if the roles of the first and second variables in Tor(A,B) are interchanged.

Remark 3.8: If Λ is a commutative ring, then $A \otimes B$ and $X \otimes Y$ are Λ-modules, so that $\mathrm{Tor}_q(A \otimes B)$ will be a Λ-module.

For further use, we note the following:

Proposition 3.9: If A is a finitely generated module over a commutative Noetherian local ring Λ with maximal ideal M, the following are equivalent:

i) A is free

ii) A is projective

ii) A is flat

iv) $\mathrm{Tor}_1(A, \Lambda/M) = 0$.

Proof: i) ==> ii) ==> iii) ==> iv) are clear.

iv) ==> i) $A/MA = \Lambda/M \otimes A$ is a finitely generated vector space over Λ/M. Choose $x_1, \dots, x_N \varepsilon A$ such that $\bar{x}_1, \dots, \bar{x}_N$ generate A/MA. Let F be free with N generators $e_1, \dots, e_N$. Define $f\colon F \to A$ by $f(e_i) = x_i$. Since $\Lambda/M \otimes F \xrightarrow{i_{\Lambda/M} \otimes f} \Lambda/M \otimes A$ is an epimorphism, so is $F \xrightarrow{f} A$ by proposition 4.2 of chapter 5. Let $B = \ker(f)$. Since Λ is Noetherian, B is finitely generated.

$0 \to B \to F \to A \to 0$ is exact, hence

$0 = \mathrm{Tor}_1(A_1, \Lambda/M) \to B \otimes \Lambda/M \to F \otimes \Lambda/M \to A \otimes \Lambda/M \to 0$ is exact. Thus $B \otimes \Lambda/M = 0$; $B = 0$ by proposition 4.1, chapter 5. Thus A is isomorphic to F.

d) ⇒ v): Let B have one generator. Then there exists I such that $0 \to I \to A \to B \to 0$ is exact. B is isomorphic to A/I. Of course, the proposition holds also if the roles of the first and second variables in $\mathrm{Tor}(A,B)$ are interchanged.

<u>Remark 3.8</u>: If A is a commutative ring, then $M \otimes B$ and $N \otimes M$ are A-modules, so that $\mathrm{Tor}_n(M,B)$ will be a A-module.

For further use, we note the following:

<u>Proposition 3.9</u>: If A is a finitely generated module over a commutative Noetherian local ring A with maximal ideal M, the following are equivalent:

i) A is free

ii) A is projective

iii) A is flat

iv) $\mathrm{Tor}_1(A, A/M) = 0$.

Proof: i) ⇒ ii) ⇒ iii) ⇒ iv) are clear.

iv) ⇒ i) $A/MA = A/M \otimes A$ is a finitely generated vector space over A/M. Choose $x_1, \dots, x_n \in A$ such that $\bar{x}_1, \dots, \bar{x}_n$ generate A/MA. Let F be free with n generators $e_1, \dots, e_n$. Define $f: F \to A$ by $f(e_i) = x_i$. Since $F/MF \xrightarrow{f \otimes A/M} A/MA$ is an epimorphism, so is $F \xrightarrow{f} A$ by proposition [illegible] of chapter [illegible]. Let $B = \ker(f)$. Since A is Noetherian, B is finitely generated. $0 \to B \to F \to A \to 0$ is exact, hence:

$0 = \mathrm{Tor}_1(A, A/M) \to B \otimes A/M \to F \otimes A/M \to A \otimes A/M \to 0$ is exact. Thus $B \otimes A/M = 0$; $B = 0$ by proposition 4.1, chapter 3. Thus A is isomorphic to F.

4. Construction of Ext(A,B) .

In this section all modules will be assumed to be left Λ-modules. If A and B are modules, $\mathrm{Hom}(A,B)$ is an Abelian group. Recall that if $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ are exact sequences, then $0 \to \mathrm{Hom}(A'',B) \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(A',B)$ and $0 \to \mathrm{Hom}(A,B') \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B'')$ are exact. Here we will construct objects by means of which we may study the behavior of the latter sequences on the right. The constructions of this section will closely parallel those of the previous section, and most proofs will be outlined only.

Let X and Y be complexes, X written with subscripts, Y with superscripts. $\mathrm{Hom}(X,Y)$ is a graded module with $\mathrm{Hom}(X,Y)^N = \prod_{i+j=N} \mathrm{Hom}(X_i,Y^j)$. Observe that if $f \varepsilon \mathrm{Hom}(X,Y)^N$, then $f_i : X^{-i} = X_i \to Y^{N-i}$; that is, f is a morphism of degree N . For notational convenience we write $\overset{i,j}{\mathrm{Hom}}(X,Y) = \mathrm{Hom}(X_i,Y^j)$.

Now let X',X'',Y',Y'' also be complexes. Let $f: X \to X'$ and $g: Y \to Y'$ be morphisms of degrees p and q . Define

$$\overset{i,j}{\mathrm{Hom}}(f,g) = (-1)^{iq}\, \mathrm{Hom}(f_{i+p},g^j) \quad (\overset{i,j}{\mathrm{Hom}}(f,g):\mathrm{Hom}(X'_i,Y^j) \to \mathrm{Hom}(X_{i+p},Y'^{j+q}))\ ,$$

and define $\mathrm{Hom}(f,g)^N = \prod_{i+j=N} \overset{i,j}{\mathrm{Hom}}(f,g)$. Thus $\mathrm{Hom}(f,g)$ is a morphism of degree $p+q$. Suppose $f': X' \to X''$ and $g': Y' \to Y''$ are morphisms of degrees p' and q' . $f'f$ and $g'g$ are morphisms of degrees $p+p'$ and $q+q'$. We have

$$\overset{i,j}{\mathrm{Hom}}(f'f,g'g) = (-1)^{i(q+q')}\, \mathrm{Hom}(f'_{i+p}\, f_{i+p'+p}, g'^{j+q}\, g^{j})$$
$$= (-1)^{i(q+q')}\, \mathrm{Hom}(f_{i+p'+p}, g'^{j+q})\, \mathrm{Hom}(f'_{i+p'}, g^{j})$$
$$= (-1)^{i(q+q')}(-1)^{-(i+p')q'} \overset{i+p',j+q}{\mathrm{Hom}}(f,g')(-1)^{-iq} \overset{i,j}{\mathrm{Hom}}(f',g)$$
$$= (-1)^{p'q'} \overset{i+p',j+q}{\mathrm{Hom}}(f,g')\, \overset{i,j}{\mathrm{Hom}}(f',g)\ .$$

We will now define a differential operator on $\mathrm{Hom}(X,Y)$ by $d = \mathrm{Hom}(d_x,i_y) + \mathrm{Hom}(i_x,d_y)$.

$$dd = \mathrm{Hom}(d_x,i_y)\,\mathrm{Hom}(d_x,i_y) + \mathrm{Hom}(d_x,i_y)\,\mathrm{Hom}(i_x,d_y)$$
$$+ \mathrm{Hom}(i_x,d_y)\,\mathrm{Hom}(d_x,i_y) + \mathrm{Hom}(i_x,d_y)\,\mathrm{Hom}(i_x,d_y)$$
$$= \mathrm{Hom}(d_x,i_y)\,\mathrm{Hom}(i_x,d_y) + \mathrm{Hom}(i_x,d_y)\,\mathrm{Hom}(d_x,i_y)$$
$$= \mathrm{Hom}(i_xd_x,i_yd_y) - \mathrm{Hom}(d_xi_x,d_yi_y) = 0\ .$$

As in the construction of Tor , we will employ

<u>Lemma 4.1</u>: Suppose X is a left complex and Y is a right complex. Then: i) If X is projective and Y is exact, $\mathrm{Hom}(X,Y)$ is exact.

ii) If X is exact and Y is injective, $\mathrm{Hom}(X,Y)$ is exact.

Proof: Note that $\mathrm{Hom}(X,Y)$ is a right complex. The proof is similar to that of lemma 3.1, using propositions 2.10 and 3.5 of chapter 3 and lemma 2.1 in the second and third steps.

<u>Definition 4.2</u>: Let A and B be left Λ-modules, X a projective resolution of A and Y an injective resolution of B . Define $\mathrm{Ext}^N(A,B) = H^N(\mathrm{Hom}(X,Y))$.

<u>Proposition 4.3</u>: $H^q(\mathrm{Hom}(A,Y))$, $H^q(\mathrm{Hom}(X,Y))$ and $H^q(\mathrm{Hom}(X,B))$ are isomorphic for all q .

Proof: Define Y' by $Y'^q = Y^q$ for $q \neq 0$, $Y'^0 = Y^0/\mathrm{im}(\epsilon)$. Then $0 \longrightarrow B \longrightarrow Y \longrightarrow Y' \longrightarrow 0$ is an exact sequence of translations, $H^q(Y') = 0$ for all q , hence by lemma 4.1 $H^q(\mathrm{Hom}(X,Y')) = 0$ for all q.

Since by proposition 2.10 of chapter 3,

$0 \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Y') \to 0$ is exact, by lemma 2.1

$0 \to H^q(\mathrm{Hom}(X,B)) \to H^q(\mathrm{Hom}(X,Y)) \to 0$ is exact for all q.

Arguing similarly using proposition 3.5 of chapter 3, $H^q(\mathrm{Hom}(X,Y))$ and $H^q(\mathrm{Hom}(A,Y))$ are isomorphic for all q.

Proposition 4.4: $\mathrm{Ext}^q(A,B)$ is independent of the choice of X and Y.

Proof: The proof is similar to that of proposition 3.4. Propositions 1.5 and 1.8 are used to prove independence of X, 1.5' and 1.8' for Y.

Proposition 4.5: i) Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence and B a module. Then there exists a canonical exact sequence

$$0 \to \mathrm{Ext}^0(A'',B) \to \mathrm{Ext}^0(A,B) \to \mathrm{Ext}^0(A',B) \to \cdots$$
$$\to \mathrm{Ext}^{N-1}(A',B) \to \mathrm{Ext}^N(A'',B) \to \mathrm{Ext}^N(A,B) \to \mathrm{Ext}^N(A',B) \to \cdots$$

ii) Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence and A a module. Then there exists a canonical exact sequence

$$0 \to \mathrm{Ext}^0(A,B') \to \mathrm{Ext}^0(A,B) \to \mathrm{Ext}^0(A,B'') \to \cdots$$
$$\to \mathrm{Ext}^{N-1}(A,B'') \to \mathrm{Ext}^N(A,B') \to \mathrm{Ext}^N(A,B) \to \mathrm{Ext}^N(A,B'') \to \cdots .$$

Proof: The proof is similar to that of proposition 3.5:

i) follows using proposition 2.6, corollary 2.7, proposition 2.10 of chapter 3, lemma 2.1, and proposition 2.8.

ii) follows using proposition 2.6', corollary 2.7', proposition 3.5 of chapter 3, lemma 2.1, and proposition 2.8'.

Proposition 4.6: $\mathrm{Ext}^0(A,B)$ is isomorphic to $\mathrm{Hom}(A,B)$.

Proof: Let Y be an injective resolution of B .

$$0 \longrightarrow \mathrm{Hom}(A,B) \xrightarrow{\mathrm{Hom}(i_A,\epsilon)} \mathrm{Hom}(A,Y^0) \xrightarrow{\mathrm{Hom}(i_A,d^0)} \mathrm{Hom}(A,Y^1)$$ is exact,

so $H^0(\mathrm{Hom}(A,Y)) = \ker(\mathrm{Hom}(i_A,d^0))$ is isomorphic to $\mathrm{Hom}(A,B)$.

Proposition 4.7: The following are equivalent:

i) A is projective

ii) $\mathrm{Ext}^1(A,B) = 0$ for all B

iii) $\mathrm{Ext}^q(A,B) = 0$ for all $q \geq 1$ and for all B .

Proof: This follows from proposition 2.10 of chapter 3.

Proposition 4.7': The following are equivalent:

i) B is injective

ii) $\mathrm{Ext}^1(A,B) = 0$ for all A .

iii) $\mathrm{Ext}^q(A,B) = 0$ for all $q \geq 1$ and for all A .

Proof: This follows from proposition 3.5 of chapter 3.

Note that since there is no analog to proposition 5.16 of chapter 3, we do not obtain a complete analog to proposition 3.7.

Remark 4.8: If we had used right modules throughout this section, we would have obtained analogous results. If Λ is commutative, then the values of $\mathrm{Ext}(A,B)$ are Λ-modules.

Proposition 4.9: If A is a finitely generated module over a commutative Noetherian local ring Λ with maximal ideal M , then A is free if and only if $\mathrm{Ext}^1(A,\Lambda/M) = 0$.

Proof: If A is free, then $\mathrm{Ext}^1(A,\Lambda/M)$ is clearly zero. Conversely, we proceed exactly as in the proof of proposition 3.9, obtaining an exact sequence

$0 \longrightarrow B \longrightarrow F \longrightarrow A \longrightarrow 0$ when F is free and F/MF is isomorphic to A/MA . $\mathrm{Hom}(A,\Lambda/M) = \mathrm{Hom}(A/M\,A,\Lambda/M)$ so $\mathrm{Hom}(A,\Lambda/M)$ is isomorphic to Hom(F,Λ/M). $0 \longrightarrow \mathrm{Hom}(B,\Lambda/M) = \mathrm{Hom}(B/MB,\Lambda/M) \longrightarrow \mathrm{Ext}^1(A,\Lambda/M) = 0$ is exact, $\mathrm{Hom}(B/MB,\Lambda/M) = 0$, $B/MB = 0$ and by proposition 4.1 of chapter 5, $B = 0$. Thus A is isomorphic to F.

5. Categories and functors.

In this section, we introduce terminology which greatly simplifies the statements of homological algebra.

<u>Definitions 5.1</u>: Let $\mathcal{F}$ be a set with elements denoted by f, f_1, f_2, etc. such that for certain pairs (f_1,f_2) a product f_1f_2 is defined in $\mathcal{F}$. An element $i \varepsilon \mathcal{F}$ such that $if_1 = f_1$ and $f_2i = f_2$ whenever if_1 and f_2i are defined, is called an identity. $\mathcal{F}$ is called a system of abstract maps provided that

i) If either $f_1(f_2f_3)$ or $(f_1f_2)f_3$ is defined, then so is the other and the two are equal.

ii) If f_1f_2 and f_2f_3 are defined, then so are $(f_1f_2)f_3$ and $f_1(f_2f_3)$.

iii) If $f \varepsilon \mathcal{F}$, then there exist (unique) identities i_1 and i_2 in $\mathcal{F}$ such that fi_1 and i_2f are defined.

<u>Definitions 5.2</u>: A category $\mathcal{C}$ is a system $\mathcal{F}$ of abstract maps together with objects $C, C_1, C_2, \ldots$ which are in 1-1 correspondence with the identities of $\mathcal{F}$. If $f \varepsilon \mathcal{F}$, then the unique objects C_1 and C_2 such that fi_{c_1} and $i_{c_2}f$ are defined, are called the domain and range of f, and we write $f\colon C_1 \longrightarrow C_2$. If for each pair C_1 and C_2 of objects in $\mathcal{C}$, the set of all maps $f\colon C_1 \longrightarrow C_2$ has a natural structure as an Abelian group, then $\mathcal{C}$ is called an additive category. If for each pair C_1 and C_2 the set of maps $f\colon C_1 \longrightarrow C_2$ has a structure as a left (right) Λ-module, then $\mathcal{C}$ is called a left (right) Λ-category.

The set of left (or right) Λ-modules and their morphisms is an example of an additive category. If Λ is commutative, the set of Λ-modules and their morphisms is a Λ-category. Diagrams of Λ-modules and their translations give further examples of additive categories (a translation $f: D \longrightarrow D'$ of two similar diagrams D and D' is a family of morphisms $f_j: D_j \longrightarrow D'_j$ such that for each pair (j,k),

$$\begin{array}{ccc} D_j & \xrightarrow{\varphi_{jk}} & D_k \\ \downarrow{\scriptstyle f_j} & & \downarrow{\scriptstyle f_k} \\ D'_j & \xrightarrow{\varphi'_{jk}} & D'_k \end{array}$$

is a commutative diagram where $D_j, D_k, \ldots, D'_j, D'_k, \ldots$ are the modules and $\varphi_{jk}, \ldots, \varphi'_{jk}, \ldots$ are the component morphisms of the diagrams D and D').

<u>Definitions 5.3</u>: Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that for each object $C \varepsilon \mathcal{C}$, an object $T(C) \varepsilon \mathcal{D}$ is given and for each map $f: C \longrightarrow C'$ in $\mathcal{C}$ a map $T(f): T(C) \longrightarrow T(C')$ is given such that

i) If $f = i_C$ then $T(f) = i_{T(C)}$.

ii) If $f'f$ is defined, $T(f'f) = T(f')\,T(f)$.

Then T is said to form a covariant functor from $\mathcal{C}$ to $\mathcal{D}$. If $T(f): T(C') \longrightarrow T(C)$ and $T(f'f) = T(f)T(f')$, T is said to be a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$.

We extend the definition to N variables as follows: Let $\mathcal{C}_1, \ldots, \mathcal{C}_N, \mathcal{D}$ be categories. Let $C_i, C'_i, \ldots, f_i, f'_i \ldots$ be objects and maps in $\mathcal{C}_i$ and let the set $\{1, \ldots, N\}$ be divided into disjoint subsets I and J . Assume that for each set $C_1, \ldots C_N$ of

The set of left (or right) A-modules and their morphisms is an example of an additive category. If A is commutative, the set of A-modules and their morphisms is a A-category. Diagrams of A-modules and their translations give further examples of additive categories (a translation $\varphi: D \longrightarrow D'$ of two similar diagrams D and D' is a family of morphisms $\varphi_j: D_j \longrightarrow D'_j$ such that for each pair (j,k),

$$\begin{array}{ccc} D_j & \xrightarrow{\varphi_{jk}} & D_k \\ \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle \varphi_k} \\ D'_j & \xrightarrow{\varphi'_{jk}} & D'_k \end{array}$$

is a commutative diagram where $D_j, D_k, \ldots, D'_j, D'_k, \ldots$ are the modules and $\varphi_{jk}, \ldots, \varphi'_{jk}, \ldots$ are the component morphisms of the diagrams D and D').

Definition 7.3: Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that for each object $C \in \mathcal{C}$, an object $T(C) \in \mathcal{D}$ is given and for each map $f: C \longrightarrow C'$ in $\mathcal{C}$ a map $T(f): T(C) \longrightarrow T(C')$ is given such that

i) If $f = 1_C$ then $T(f) = 1_{T(C)}$.

ii) If $f'f$ is defined, $T(f'f) = T(f')\,T(f)$.

Then T is said to be a covariant functor from $\mathcal{C}$ to $\mathcal{D}$. If $T(f): T(C') \longrightarrow T(C)$ and $T(f'f) = T(f)T(f')$, T is said to be a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$.

We extend the definition of T variables as follows:

Let $\mathcal{C}_1, \ldots, \mathcal{C}_N, \mathcal{D}$ be categories. Let $C_1, C'_1, \ldots, f_1, f'_1, \ldots$ be objects and maps in $\mathcal{C}_k$ and let the set $\{1, \ldots, N\}$ be divided into disjoint subsets I and J. Assume that for each set $C_1, \ldots, C_N$ of objects there is given an object $T(C_1,\ldots,C_N) \varepsilon \mathcal{D}$, and for each set $f_1,\ldots f_N$ of maps, $f_i\colon C_i \longrightarrow C'_i$ for $i\varepsilon I$, $f_j\colon C'_j \longrightarrow C_j$ for $j\varepsilon J$, there is given a map in $\mathcal{D}$ $T(f_1,\ldots,f_N)\colon T(C_1,\ldots,C_N) \longrightarrow T(C'_1,\ldots,C'_N)$. Then T is a functor, covariant in the variables in I , contravariant in those in J , provided that

i) If $f_1,\ldots,f_N$ are identities, then so is $T(f_1,\ldots,f_N)$.

ii) If $f_1,\ldots,f_N,f'_1,\ldots f'_N$ are such that $f'_i f_i$, $i\varepsilon I$, and $f_j f'_j$, $j\varepsilon J$, are defined, then

$$T(\ldots,f'_i f_i,\ldots,f_j f'_j,\ldots) = T(f'_1,\ldots f'_N)T(f_1,\ldots,f_N) .$$

If $\mathcal{C}_1,\ldots,\mathcal{C}_N,\mathcal{D}$ are all additive categories, and $f_1,\ldots,f_N$, $g_1,\ldots,g_N$ are maps in $\mathcal{C}_1,\ldots,\mathcal{C}_N$ such that f_i and g_i have the same domain and range for each i , then if

$T(f_1,\ldots,f_r + g_r,\ldots,f_N) = T(f_1,\ldots,f_r,\ldots,f_N) + T(f_1,\ldots,g_r,\ldots f_N)$, $1 \leq r \leq N$, T is said to be an additive functor. We will only be concerned with additive functors.

<u>Definitions 5.4</u>: Let T and U be functors of N variables from $(\mathcal{C}_1,\ldots,\mathcal{C}_N)$ to $\mathcal{D}$. Let $\{1,\ldots,N\}$ be partitioned into I and J with both T and U covariant in the variables C_i, $i\varepsilon I$, contravariant in C_j , $j\varepsilon J$. Denote $(C_1,\ldots,C_N)$ by (C) . If for each set (C) , there is a map $\mu_{(c)}\colon T(C) \longrightarrow U(C)$ such that whenever $f_1,\ldots f_N$ are maps, $f_i\colon C_i \longrightarrow C'_i$, $i\varepsilon I$, $f_j\colon C'_j \longrightarrow C_j$, $j\varepsilon J$, then the diagram

$$\begin{array}{ccc} T(C) & \xrightarrow{T(f_1,\ldots,f_N)} & T(C') \\ \downarrow{\scriptstyle \mu_{(c)}} & & \downarrow{\scriptstyle \mu_{(c')}} \\ U(C) & \xrightarrow{U(f_1,\ldots,f_N)} & U(C') \end{array}$$

is commutative, we say that μ is a natural transformation of T into U . If $\mu_{(c)}$ is an equivalence for all (C) , we say that $\mu_{(c)}$ is a natural equivalence objects there is given an object $T(C_1,\ldots,C_N) \varepsilon \mathcal{C}$, and for each set $f_1,\ldots,f_N$ of maps, $f_i: C_i \to C'_i$ for $i \varepsilon I$, $f_j: C'_j \to C_j$ for $j \varepsilon J$, there is given a map in $\mathcal{C}$ $T(f_1,\ldots,f_N): T(C_1,\ldots,C_N) \to T(C'_1,\ldots,C'_N)$. Then T is a functor, covariant in the variables in I, contravariant in those in J, provided that

a) If $f_1,\ldots,f_N$ are identities, then so is $T(f_1,\ldots,f_N)$.

b) If $f_1,\ldots,f_N, f'_1,\ldots,f'_N$ are such that $f'_i f_i$, $i \varepsilon I$, and $f_j f'_j$, $j \varepsilon J$, are defined, then $T(\ldots,f'_i f_i,\ldots,f_j f'_j,\ldots)$
$= T(f'_1,\ldots,f'_N)T(f_1,\ldots,f_N)$.

If $\mathcal{C}_1,\ldots,\mathcal{C}_N, \mathcal{C}$ are all additive categories, and $f_1,\ldots,f_N$, $g_1,\ldots,g_N$ are maps in $\mathcal{C}_1,\ldots,\mathcal{C}_N$ such that f_i and g_i have the same domain and range for each i, then if

$T(f_1,\ldots,f_r + g_r,\ldots,f_N) = T(f_1,\ldots,f_r,\ldots,f_N) + T(f_1,\ldots,g_r,\ldots,f_N)$,

$1 \leq r \leq N$, T is said to be an additive functor. We will only be concerned with additive functors.

Definition 5.4: Let T and U be functors of N variables from $(\mathcal{C}_1,\ldots,\mathcal{C}_N)$ to $\mathcal{C}$. Let $\{1,\ldots,N\}$ be partitioned into I and J with both T and U covariant in the variables C_i, $i \varepsilon I$, contravariant in C_j, $j \varepsilon J$. Denote $(C_1,\ldots,C_N)$ by (C). If for each set (C), there is a map $\mu_{(C)}: T(C) \to U(C)$ such that whenever $f_1,\ldots,f_N$ are maps, $f_i: C_i \to C'_i$, $i \varepsilon I$, $f_j: C'_j \to C_j$, $j \varepsilon J$, then the diagram

$$\begin{array}{ccc} T(C) & \xrightarrow{T(f_1,\ldots,f_N)} & T(C') \\ \downarrow{\scriptstyle \mu(C)} & & \downarrow{\scriptstyle \mu(C')} \\ U(C) & \xrightarrow{U(f_1,\ldots,f_N)} & U(C') \end{array}$$

is commutative,

we say that μ is a natural transformation of T into U. If $\mu_{(C)}$ is an equivalence for all (C), we say that $\mu_{(C)}$ is a natural equivalence or a natural isomorphism. (A map f is an equivalence if there exists a map g such that fg and gf are both identities.)

Let Λ_1, Λ_2, Λ be rings. We restrict ourselves (for notational convenience) for the remainder of this section to an additive functor $T(A,C)$ define for Λ_1-modules A and Λ_2-modules C with values as Λ-modules. We assume that T is covariant in A, contravariant in C.

Proposition 5.5: If $A_\alpha \xrightarrow{i_\alpha} A \xrightarrow{j_\alpha} A_\alpha$ and $C_\beta \xrightarrow{k_\beta} C \xrightarrow{\ell_\beta} C_\beta$ are finite direct sum representations of A and C, then $T(A_\alpha,C_\beta) \xrightarrow{T(i_\alpha,\ell_\beta)} T(A,C) \xrightarrow{T(j_k,k_\beta)} T(A_\alpha,C_\beta)$ is a direct sum representation of $T(A,C)$.

Proof: $T(j_{\alpha'},k_{\beta'})T(i_\alpha,\ell_\beta) = T(j_{\alpha'}i_\alpha,\ell_\beta k_{\beta'})$, is the identity if $(\alpha,\beta) = (\alpha',\beta')$ and zero otherwise. Also

$$\Sigma_{\alpha,\beta}T(i_\alpha,\ell_\beta)T(j_\alpha,k_\beta) = \Sigma_{\alpha,\beta}T(i_\alpha j_\alpha,k_\beta\ell_\beta) = T(\Sigma_{\alpha,\beta}i_\alpha j_\alpha,\Sigma_{\alpha,\beta}k_\beta\ell_\beta)$$
$$= \text{identity}.$$

Corollary 5.6: If $0 \to A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C'' \to 0$ are split exact sequences, then so are $0 \to T(A',C) \to T(A,C) \to T(A'',C) \to 0$ and $0 \to T(A,C'') \to T(A,C) \to T(A,C') \to 0$.

Definitions 5.7: Let $A' \to A \to A''$, $C' \to C \to C''$ be exact sequences. If $T(A',C) \to T(A,C) \to T(A'',C)$ and $T(A,C'') \to T(A,C) \to T(A,C')$ are also exact, then T is said to be an exact functor. Let $0 \to A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C'' \to 0$ be exact sequences.

or a natural isomorphism. (A map f is an equivalence if there exists a map g such that fg and gf are both identities.)

Let A_1, A_2, A be rings. We restrict ourselves (for notational convenience) for the remainder of this section to an additive functor $T(A,C)$ define for A_1-modules A and A_2-modules C with values as A-modules. We assume that T is covariant in A, contravariant in C.

Proposition 5.5: If $A_\alpha \xrightarrow{i_\alpha} A \xrightarrow{j_\alpha} A_\alpha$, and $C_\beta \xrightarrow{i_\beta} C \xrightarrow{j_\beta} C_\beta$ are finite direct sum representations of A and C, then $T(A_\alpha, C_\beta) \xrightarrow{T(i_\alpha, j_\beta)} T(A,C) \xrightarrow{T(j_\alpha, i_\beta)} T(A_\alpha, C_\beta)$ is a direct sum representation of $T(A,C)$.

Proof: $T(j_{\alpha'}, i_{\beta'})T(i_\alpha, j_\beta) = T(j_{\alpha'} i_\alpha, j_\beta i_{\beta'})$ is the identity if $(\alpha,\beta) = (\alpha',\beta')$ and zero otherwise. Also

$$\Sigma_{\alpha,\beta} T(i_\alpha, j_\beta)T(j_\alpha, i_\beta) = \Sigma_{\alpha,\beta} T(i_\alpha j_\alpha, i_\beta j_\beta) = T(\Sigma_{\alpha,\beta} i_\alpha j_\alpha, \Sigma_{\alpha,\beta} i_\beta j_\beta)$$

$= \text{identity}.$

Corollary 5.6: If $0 \to A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C'' \to 0$ are split exact sequences, then so are $0 \to T(A',C) \to T(A,C) \to T(A'',C) \to 0$ and $0 \to T(A,C'') \to T(A,C) \to T(A,C') \to 0$.

Definition 5.7: Let $A' \to A \to A''$, $C' \to C \to C''$ be exact sequences. If $T(A',C) \to T(A,C) \to T(A'',C)$ and $T(A,C'') \to T(A,C) \to T(A,C')$ are also exact, then T is said to be an exact functor. Let $0 \to A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C'' \to 0$ be exact sequences.

If $0 \to T(A',C) \to T(A,C) \to T(A'',C)$ and $0 \to T(A,C'') \to T(A,C) \to T(A,C')$ are exact, then T is said to be left exact. If $T(A',C) \to T(A,C) \to T(A'',C) \to 0$ and $T(A,C'') \to T(A,C) \to T(A,C') \to 0$ are exact, then T is said to be right exact.

The proofs of the following propositions are straightforward and will be omitted.

Proposition 5.8: T is exact if and only if T is both right and left exact.

Proposition 5.9: The following are equivalent:

i) T is left exact

ii) If $0 \to A' \to A \to A''$ and $C' \to C \to C'' \to 0$ are exact, then so are $0 \to T(A',C) \to T(A,C) \to T(A'',C)$ and $0 \to T(A,C'') \to T(A,C) \to T(A,C')$.

iii) If $0 \to A' \to A \to A''$ and $C' \to C \to C'' \to 0$ are exact, then so is $0 \to T(A',C'') \to T(A,C) \xrightarrow{\psi} T(A'',C) \oplus T(A,C')$ where ψ has coordinates $T(A,C) \to T(A'',C)$; $T(A,C) \to T(A,C')$

Proposition 5.10: The following are equivalent:

i) T is right exact

ii) If $A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C''$ are exact, then so are $T(A',C) \to T(A,C) \to T(A'',C) \to 0$ and $T(A,C'') \to T(A,C) \to T(A,C') \to 0$.

iii) If $A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C''$ are exact, then so is $T(A',C) \oplus T(A,C'') \xrightarrow{\phi} T(A,C) \to T(A'',C') \to 0$, where ϕ has coordinates $T(A',C) \to T(A,C)$ and $T(A,C'') \to T(A,C)$.

We observe that by propositions 1.10, 2.7 and 2.8 of chapter 3, the functor $\otimes_\Lambda$ is right exact and the functor $\hom_\Lambda$ is left exact.

If $0 \to T(A',C) \to T(A,C) \to T(A'',C)$ and $0 \to T(A,C'') \to T(A,C) \to T(A,C')$ are exact, then T is said to be left exact. If $T(A',C) \to T(A,C) \to T(A'',C) \to 0$ and $T(A,C'') \to T(A,C) \to T(A,C') \to 0$ are exact, then T is said to be right exact.

The proofs of the following propositions are straightforward and will be omitted.

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(iii) If $0 \to A' \to A \to A''$ and $C' \to C \to C'' \to 0$ are exact, then so is $0 \to T(A',C'') \to T(A,C) \xrightarrow{\psi} T(A'',C) \oplus T(A,C')$ where ψ has coordinates $T(A,C) \to T(A'',C)$; $T(A,C) \to T(A,C')$

Proposition 5.10: The following are equivalent:

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(iii) If $A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C''$ are exact, then so is $T(A',C) \oplus T(A,C'') \xrightarrow{\phi} T(A,C) \to T(A'',C') \to 0$, where ϕ has coordinates $T(A',C) \to T(A,C)$ and $T(A,C'') \to T(A,C)$.

We observe that by propositions 1.10, 2.7 and 2.8 of chapter 5, the functor $\otimes_\Lambda$ is right exact and the functor $\hom_\Lambda$ is left exact.

Definitions 5.11: A connected sequence of covariant functors is a family $T = \{T^N\}$ of covariant functors (of one variable) together with connecting morphisms $T^N(A'') \to T^{N+1}(A')$ defined for each exact sequence $0 \to A' \to A \to A'' \to 0$ subject to the conditions:

i) $\cdots \to T^{N-1}(A'') \to T^N(A') \to T^N(A) \to T^N(A'') \to T^{N+1}(A') \to \cdots$

is a zero sequence.

ii) If

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B' & \to & B & \to & B'' & \to & 0 \end{array}$$

is a commutative diagram with exact rows then

$$\begin{array}{ccc} T^N(A'') & \to & T^{N+1}(A') \\ \downarrow & & \downarrow \\ T^N(B'') & \to & T^{N+1}(B') \end{array}$$

is a commutative diagram.

If the roles of A' and A'' are reversed, $\{T^N\}$ is a connected sequence of contravariant functors.

Generalizing, a multiply connected sequence of functors is a sequence $\{T^N\}$ of functors of the same variables and variance such that there are connecting morphisms with respect to each variable for which

i) $\{T^N\}$ is a connected sequence of functors with respect to each variable separately.

ii) If

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_1' & \to & A_1 & \to & A_1'' & \to & 0 \end{array} \quad , \quad \begin{array}{ccccccccc} 0 & \to & C_1' & \to & C_1 & \to & C_1'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C' & \to & C & \to & C'' & \to & 0 \end{array}$$

are commutative diagrams with exact rows, then

$$\begin{array}{ccc} T^N(A'',C) & \to & T^{N+1}(A',C) \\ \downarrow & & \downarrow \\ T^N(A'',C_1) & \to & T^{N+1}(A',C_1) \end{array}$$

Definition 5.11: A connected sequence of covariant functors is a family $T = \{T^n\}$ of covariant functors (of one variable) together with connecting morphisms $T^n(A'') \longrightarrow T^{n+1}(A')$ defined for each exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ subject to the conditions:

i) $\ldots \longrightarrow T^{n-1}(A'') \longrightarrow T^n(A') \longrightarrow T^n(A) \longrightarrow T^n(A'') \longrightarrow T^{n+1}(A') \longrightarrow \ldots$

is a zero sequence.

ii) If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

with exact rows then $T^n(A'') \longrightarrow T^{n+1}(A')$ is a commutative diagram

$$\begin{array}{ccc} T^n(A'') & \longrightarrow & T^{n+1}(A') \\ \downarrow & & \downarrow \\ T^n(B'') & \longrightarrow & T^{n+1}(B'). \end{array}$$

If the roles of A' and A'' are reversed, $\{T^n\}$ is a connected sequence of contravariant functors.

Generalizing, a multiply connected sequence of functors is a sequence $\{T^n\}$ of functors of the same variables and variance such that there are connecting morphisms with respect to each variable for which

i) $\{T^n\}$ is a connected sequence of functors with respect to each variable separately.

ii) If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1' & \longrightarrow & A_1 & \longrightarrow & A_1'' & \longrightarrow & 0 \end{array} \quad , \quad \begin{array}{ccccccccc} 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_1' & \longrightarrow & C_1 & \longrightarrow & C_1'' & \longrightarrow & 0 \end{array}$$

are commutative diagrams with exact rows, then

$$\begin{array}{ccc} T^n(A'',C) & \longrightarrow & T^{n+1}(A',C) \\ \downarrow & & \downarrow \\ T^n(A'',C_1) & \longrightarrow & T^{n+1}(A',C_1) \end{array}$$

and
$$\begin{array}{ccc} T^N(A,C') & \longrightarrow & T^{N+1}(A,C'') \\ \downarrow & & \downarrow \\ T^N(A_1,C') & \longrightarrow & T^{N+1}(A_1,C'') \end{array}$$
are commutative diagrams (where we have chosen $\{T^N\}$ all covariant in A, contravariant in C, as a typical example). If $\{T^N\}$ and $\{U^N\}$ are multiply connected sequences of functors, a morphism $\Phi: \{T^N\} \longrightarrow \{U^N\}$ is a sequence of natural transformations $\Phi^N: T^N \longrightarrow U^N$ which commute with the connecting morphisms.

$\{\mathrm{Tor}^{\Lambda}_N(A,C)\}$ and $\{\mathrm{Ext}^N_{\Lambda}(A,C)\}$ are examples of exact multiply connected sequences of functors (that condition ii) is satisfied follows from the constructions, and propositions 1.8, 1.8', 2.8 and 2.8'.). The method by which Tor^{Λ} and Ext_{Λ} were constructed is actually quite general. Let $T(A_1,\ldots,A_r)$ be any functor of r variables. Assume that $A_1,\ldots,A_r$ are all graded. Define $T^{N_1,\ldots,N_r}(A_1,\ldots,A_r) = T(A_1^{\epsilon_1 N_1},\ldots,A_r^{\epsilon_r N_r})$ where $\epsilon_r = +1$ if A_i is a covariant variable, -1 if contravariant, and define $T^N(A_1,\ldots,A_r) = \prod_{\sum N_i = N} T^{N_1,\ldots,N_r}(A_1,\ldots,A_r)$ [for $\otimes_{\Lambda}$, if all complexes are left complexes as in the construction of Tor^{Λ}, this is the same as $\bigoplus_{\sum N_i = N} T^{N_1,\ldots,N_r}(A_1,\ldots,A_r)$]. If $A_1',\ldots,A_r'$ is another set of graded modules and $f_i: A_i \longrightarrow A_i'$ are given for A_i a covariant variable, $f_i: A_i' \longrightarrow A_i$ for A_i contravariant, where f_i is of degree p_i, define $T^{N_1,\ldots,N_r}(f_1,\ldots f_r)$ on $T^{N_1,\ldots,N_r}(A_1,\ldots,A_r)$ as

$$(-1)^{\gamma} T(f_1^{\ell_1},\ldots,f_r^{\ell_r}): T(A_1^{\epsilon_1 N_1},\ldots,A_r^{\epsilon_r N_r}) \longrightarrow T(A_1'^{\epsilon_1(N_1+P_1)},\ldots,A_r'^{\epsilon_r(N_r+P_r)}),$$

where $\gamma = \sum_{i<j} N_i p_j$, $\ell_i = N_i$ if T is covariant in A_i, $\ell_i = -(N_i + p_i)$ if T is contravariant in A_i. If $g_i: A_i' \longrightarrow A_i''$ (resp. $g_i: A_i'' \longrightarrow A_i'$) are morphisms of degree q_i, we then verify that

$T(g_1f_1,\ldots,g_rf_r) = (-1)^{\eta_i} T(g_1,\ldots,g_r)T(f_1,\ldots,f_r)$, where $\eta = \sum_{i<j} p_iq_j$. Now suppose each A_i is a complex with differentiation d_i and let $\delta_i = T(i_{A_1},\ldots,d_i,\ldots,i_{A_r})$. The δ_i anticommute, hence define $T(A_1,\ldots,A_r)$ as a complex with differentiation $\Sigma\,\delta_i$. If $f_1,\ldots,f_r$ and $f_1',\ldots,f_r'$ are respectively homotopic translations of complexes and s_i, $1 \leq i \leq r$ are homotopies, then for $\sigma_i = T(i_{A_1},\ldots,s_i,\ldots,i_{A_r})$, $\Sigma\,\sigma_i$ defines a homotopy between $T(f_1,\ldots,f_r)$ and $T(f_1',\ldots,f_r')$.

Now if we are given a functor T of modules and we replace all covariant variables A_i by projective resolutions X_i, all contravariant variables A_i by injective resolutions X_i we obtain a left complex. We define $LT(A_1,\ldots,A_r)$ as $HT(X_1,\ldots,X_r)$. By propositions 1.8, 1.8' and the previous paragraph, LT is independent of the choices of the X_i. LT is graded, and the component of degree N gives a functor L_NT called the Nth left derived functor of T. $L_NT = 0$ for $n < 0$. By corollary 5.6, lemma 2.1, and propositions 1.8, 1.8', 2.8 and 2.8', $\{L_NT\}$ is an exact multiply connected sequence of functors. The augmentation morphisms $\epsilon_i\colon X_i \longrightarrow A_i$ (resp. $\epsilon_i\colon A_i \longrightarrow X_i$) induce a natural transformation $\gamma_0\colon L_0T(A_1,\ldots,A_r) \longrightarrow T(A_1,\ldots,A_r)$. γ_0 is a natural equivalence if and only if T is right exact: the condition is necessary since L_0T is right exact; if T is right exact, considering our typical case $T(A,C)$, and letting X be a projective resolution of A, Y an injective resolution of C, the sequence $T(X_1,Y^0) \oplus T(X_0,Y^1) \xrightarrow{\phi} T(X_0,Y^0) \longrightarrow T(A,C) \longrightarrow 0$ is exact by proposition 5.10; since $H_0(T(X,Y)) = \text{coker}\ (\phi)$, this gives the result. For this reason, left derived functors are of interest for the study of right exact functors. Similarly, if we replace all covariant variables by injective resolutions and all contravariant variables by projective resolutions in a functor T of modules, we obtain a multiply connected sequence of functors $\{R^N T\}$, called the right derived functors of T. $R^O T$ is naturally equivalent to T if and only if T is left exact.

It is clear that $\{\mathrm{Tor}_N^\Lambda\}$ are the left derived functors of $\otimes_\Lambda$ and that $\{\mathrm{Ext}_N^\Lambda\}$ are the right derived functors of Hom_Λ.