Elliptic genera for S^1-manifolds.

1. Elliptic genera.

(1) Let M be a connected closed Spin manifold, and suppose it admits a non-trivial action $S \times M \longrightarrow M$ of the circle group $S = S^1$. Then a well-known result of Atiyah and Hirzebruch (1970) says that $\hat{A}[M] = 0$, where $\hat{A}$ is the $\hat{A}$-genus defined by its characteristic series $\dfrac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)}$. This genus is, in each dimension, a rational combination of Pontrjagin numbers, and any such combination which vanishes on all Spin-manifolds which admit a non-trivial circle action, is a rational multiple of $\hat{A}$.

(2) Connected Spin-manifolds with circle action belong to one of two types: odd manifolds and even manifolds. To understand this, let us remember that a Spin-

structure on M can be defined the following way: Suppose M^{2m} is an oriented Riemannian manifold and $\pi: Q \longrightarrow M^{2m}$ is the principal SO_{2m}-bundle of orthonormal $2m$-frames on M. Then a Spin-structure on M is a diagram:

$$\begin{tikzcd} P \arrow[rr, "s"] \arrow[dr, "\pi'"'] & & Q \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

where $\pi': P \longrightarrow M$ is a principal $Spin_{2m}$-bundle and s is a bundle map which agrees with the standard double covering $Spin_{2m} \longrightarrow SO_{2m}$ on each fiber.

(3) A circle action on M is exactly the same as a periodic action of $\mathbb{R}$ with period 1. Such an action can be canonically lifted to Q and then to P as the latter is a double covering of Q. One easily sees that the action

on E is also periodic but the period is 1 or 2. If the period is 1, we say that M is <u>even</u>. In that case E has a circle action compatible with the action on M. If the period is 2, we say that M is <u>odd</u>

④ Any Spin-manifold admits an even action — the trivial action. If it admits any non-trivial action, it also admits a non-trivial ~~odd~~ even action, for we can double any action using the double covering $S \xrightarrow{2} S$. Thus Spin-manifolds which admit a non-trivial action and those which admit a non-trivial even action are the same, and we don't get new relations between Pontrjagin numbers by considering even actions. The situation is quite different when we consider odd actions. This is where elliptic

genera appear.

⑤ Let k be any commutative $\mathbb{Q}$-algebra, for example a field of characteristic zero. A <u>k-genus</u> is a ring homomorphism $\varphi : \Omega_* \to k$, where Ω_* is the oriented cobordism ring. The ring $\Omega_* \otimes k$ is freely generated by the classes $[\mathbb{C}P_{2i}]$. Thus any k-genus is entirely determined by its logarithm

$$g(u) = \sum_{i \geq 0} \frac{\varphi(\mathbb{C}P_{2i})}{2i+1} u^{2i+1}$$

For example, the signature, which takes any $\mathbb{C}P_{2i}$ to 1, has logarithm $\operatorname{arctanh} u$; and the logarithm of $\hat{A}$ is $2 \operatorname{arcsinh} u/2$.

⑥ The question we are interested in now is: describe all k-genera φ which vanish on all Spin-manifolds with an odd circle action. Here is a possible answer:

Conjecture 1. The following statements are equivalent:

(a) φ vanishes on all Spin-manifolds with an odd circle action;

(b) the logarithm of φ is given by an elliptic integral of the first kind

$$g(u) = \int_0^u \frac{du}{\sqrt{R(u)}} \qquad (\mathrm{I})$$

where $R(u) = 1 - 2\delta u^2 + \varepsilon u^4$, $\delta, \varepsilon \in k$. (This integral has a formal meaning).

We call such genera elliptic genera.

Theorem 1. 1° (a) $\Longrightarrow$ (b) ; 2° (b) $\Longrightarrow$ (a) for semi-free actions (that is actions which are free on the complementary of the fixed point set) and for Spin manifolds which admit an invariant weakly complex structure.

7 Before going further, (~~I'd like to say some~~) just a few words

on the semi-free case. The Pontryagin numbers do not depend on the Spin-structure. It follows that we are only interested in the ideal I_* in $\Omega_* \otimes \mathbb{Q}$ generated by the classes of Spin-manifolds which admit an odd circle action. Obviously, (a) is equivalent to $\varphi(I_*) = 0$. There is a nice description of I_* due to L. Borsari. She proved in her thesis that I_* is the ideal of all rational combinations of the classes $[\mathbb{C}P(\xi)]$, where ξ is an even-dimensional complex bundle over an oriented manifold. A great number of manifolds of this form are constructed as follows: let $H_{i,j}$ be the Milnor manifold ($i \leq j$), that is a hypersurface of bidegree (1,1) in $\mathbb{C}P_i \times \mathbb{C}P_j$. It is easy to check that $H_{i,j} = \mathbb{C}P(\xi^d)$ for a convenient complex bundle over $\mathbb{C}P_i$. 8 So (a) implies that $\varphi(H_{ij})=0$ for j even. Now, we have an easy expression for $\varphi(H_{ij})$ in terms of the formal group of φ. Let

$$f(u,v)=g^{-1}(g(u)+g(v))$$

For example, Omit it in a hurry

$$f(u,v)=\frac{u+v}{1+uv}$$ for the signature

and

$$f(u,v)=u\sqrt{1+\frac{v^2}{4}}+v\sqrt{1+\frac{u^2}{4}}$$ for the $\hat{A}$-genus.

Then we have

$$\sum \varphi(H_{ij})u^iv^j = g'(u)g'(v)f(u,v).$$

The vanishing of $\varphi(H_{ij})$ is then easily translated in terms of $g(u)$: we see that $1/g'(u)^2$ must be an even polynomial of degree ≤ 4, and this is exactly (b).

(9) I have two comments to make: 1°. Once you know that φ is elliptic, δ and ε are easily found:

$$\delta=\varphi(\mathbb{CP}_2)\ ,\ \varepsilon=\varphi(\mathbb{HP}_2).$$

2°. The formal group of an elliptic genus is

$$f(u,v)=\frac{u\sqrt{R(v)}+v\sqrt{R(u)}}{1-\varepsilon u^2v^2}$$

This is the Euler formal group better known as the addition formula for elliptic integrals. For this group

$$g'(u)g'(v)f(u,v)=\left(\frac{u}{\sqrt{R(u)}}+\frac{v}{\sqrt{R(v)}}\right)(1+\varepsilon u^2v^2+\dots)$$

and we can check that $\varphi(H_{ij})=0$ for even j.

(10) This is how (a) implies (b). The ~~inverse~~ converse, that is the vanishing of $\varphi(\mathbb{CP}(\xi))$ for elliptic φ; can also be proven by using simple properties of the elliptic function obtained by inversion of the integral (I) in the case of $R(u)$ having four distinct roots. I prefer another approach which proves theorem 1 in both cases listed in 2°.

2. Equivariant extensions.

(11) Let Ω_*^S be the cobordism ring of oriented manifolds with a circle action. Any oriented manifold can be equipped with the trivial action and this gives a morphism $\Omega_* \to \Omega_*^S$. Now, suppose given a k-genus φ. Then there are several ways of constructing an equivariant extension of φ. By this I mean a commutative diagram

$$\begin{array}{ccc} \Omega_*^S & \xrightarrow{\varphi_S} & K \\ \uparrow & & \uparrow \\ \Omega_* & \xrightarrow{\varphi} & k \end{array} \qquad (D)$$

where K is an extension of k, or an analogous diagram involving Spin cobordism groups. One way uses S-equivariant cohomology.

(12) Let X be an S-space. Take the Borel construction $\tilde{X} = ES \times_S X$ for X. By definition $H_S^*(X) = H^*(\tilde{X}; k)$. We choose $\mathbb{C}P^\infty$ as a model for BS. Then $H_S^*(pt) = H^*(\mathbb{C}P^\infty) = k[u]$, where u is the canonical generator. The Borel construction comes with a fibration $\pi: \tilde{X} \to \mathbb{C}P^\infty$ and this makes all $H_S^*(X)$ $H_S^*(pt)$-modules. On the other hand, if X is an oriented manifold, there is a Gysin homomorphism $\pi_! : H_S^*(X) \to H_S^*(pt)$.

(13) Unlike the ordinary cohomology, H_S^* has an infinite-dimensional coefficient ring. We shall always take the completed cohomology $\hat{H}^* = \prod_i H^i$, that is

$$\hat{H}_S^*(X) = \hat{H}^*(\tilde{X}) = \prod_i H^i(\tilde{X}).$$

For example, $\hat{H}_S^*(pt) = k[[u]]$, and more generally, $\hat{H}_S^*(X)$ is the u-adic completion of $H_S^*(X)$. Note that $\pi_!$ is continuous and extends on $\hat{H}_S^*(X)$.

(14) The Borel construction applies also to S-bundles over X. If ξ is an oriented S-bundle, one constructs a bundle $\tilde{\xi}$ over $\tilde{X}$. Then any usual characteristic

class $\chi \in \hat{H}^*(BSO)$ defines an equivariant class $\chi_S(\xi) = \chi(\tilde{\xi}) \in \hat{H}^*_S(X)$.

Now take any k-genus φ with logarithm $g(u)$. The series $u/g^{-1}(u)$ being even defines a multiplicative sequence $\Phi_0 = 1, \Phi_1(p_1), \dots, \Phi_n(p_1 \dots p_n), \dots$, and a class $\Phi \in \hat{H}^*(BSO)$. Define $\Phi_S(M) = \Phi_S(TM) \in \hat{H}^*_S(M)$.

Definition. $\varphi_S(M) = \pi_! \, \Phi_S(M) \in \hat{H}^*_S(pt)$.

(15) This is easily seen to be an extension of φ in the above sense with $K = \hat{H}^*_S(pt)$. I'll call it the formal extension of φ. This definition was suggested to me by Peter Landweber. It is quite general. But for particular genera, better extensions can be constructed, which I'll call, for the needs of this talk, substantial extensions.

(16) To understand this, let us look at the Atiyah-

Hirzebruch proof of the vanishing of $\hat{A}[M]$. The main point is that this number is the index of an elliptic operator, the Dirac operator. Remember that $Spin_{2m}$ has a canonical complex representation S which splits into a sum $S^+ \oplus S^-$ of two irreducible representations. Once a Spin structure $s : P \to Q$ is given, one defines a vector bundle $V = P \times_{Spin_{2m}} S$ which also splits as $V^+ \oplus V^-$. The Dirac operator, defined by means of the Riemannian connexion on M, operates in $\Gamma(V)$ and sends $\Gamma(V^\pm)$ into $\Gamma(V^\mp)$. Its kernel $H = H^+ \oplus H^-$ is finite dimensional, and the index theorem says that

$$\operatorname{ind} D = \dim H^+ - \dim H^- = \hat{A}[M].$$

(17) Now suppose the circle acts on M and P in a compatible way (we have seen that this implies that the action is even). Then D commutes with the action,

$H^{\pm}$ are canonically complex representations of S and one defines

$$\mathrm{ind}_S\, D = [H^+] - [H^-] \in R(S)$$

It is well-known that the characters of circle representations are restrictions on $S \subset \mathbb{C}$ of Laurent polynomials. We obtain an extension diagram:

$$\begin{tikzcd} \Omega_*^{\mathrm{Spin},ev,S} \arrow[r, "\mathrm{ind}_S D"] & R(S)\otimes\mathbb{Q} = \mathbb{Q}[z,z^{-1}] = K \\ \Omega_*^{\mathrm{Spin}} \arrow[u] \arrow[r, "\hat{A}"] & \mathbb{Q} = k \arrow[u] \end{tikzcd}$$

On the other hand It can be shown, using the index theorem for families of elliptic operators, that $\mathrm{ind}_S\, D$ is related to the formal extension $\hat{A}_S$ by

$$\hat{A}_S(u) = ch(\mathrm{ind}_S D) = (\mathrm{ind}_S D)(e^u)$$

where $ch: R(S) \longrightarrow \hat{H}^*(BS;\mathbb{Q}) = \mathbb{Q}[[u]]$ sends z to e^u. It follows that $\hat{A}_S(u)$ is the power-series expansion of a trigonometric function regular at 0.

But what is more important is that $\mathrm{ind}_S\, D$, as a rational function can only have poles at 0 and infinity. Atiyah and Hirzebruch use a Lefschetz type localization theorem for the index in order to eliminate these possibilities and show that $\mathrm{ind}_S\, D$ is the zero constant. As a consequence, $\hat{A}_S(u)$ is the zero-series.

(18) Edward Witten has considered the case of the Rarita-Schwinger operator D_T which operates in $\Gamma(V \otimes_{\mathbb{R}} T(M))$; its index is given by

$$\mathrm{ind}\, D_T = ch(T(M)\otimes\mathbb{C})\,\hat{A}(M)\,[M]. \qquad (RS)$$

and there is also an equivariant extension

$$\mathrm{ind}_S\, D_T \in R(S).$$

Witten conjectured that this index is also constant and proved it for homogeneous spaces. For general Spin-manifolds, despite the similarities with the

$\hat{A}$-genus, it does not seem that the Atiyah-Hirzebruch method works. This stimulated the attempts to prove the conjecture by cobordism methods and motivated Borsari's thesis and Landweber-Stong's research on characteristic numbers of Spin-manifolds with a circle action.

(19) The Rarita-Schwinger index is quite simply related to elliptic genera. First note that this characteristic number is neither multiplicative nor stable. But it can be included in a multiplicative genus by the following device: put $k = \mathbb{Q}[t]/t^2=0$ and define a k-genus φ by

$$\varphi(M) = \hat{A}[M] + ch(T\otimes\mathbb{C} - \dim T)\,\hat{A}(M)[M]\cdot t .$$

It is easy to check that φ is an elliptic k-genus with $R(u) = 1 + (\frac{1}{4} - 6t)u^2 - tu^4$. If we prove that $\varphi_S(u)$ is a constant series, then Witten's

conjecture will easily follow. Now we can state:

(20) <u>Conjecture 2.</u> Let φ be a k-genus. The following statements are equivalent:

(a) $\varphi_S(u) = \text{const}$ for all Spin manifolds with a circle action.

(b) φ is elliptic.

and

<u>Theorem 2.</u> 1° (a) $\Rightarrow$ (b) (R. Stong)

2° (b) $\Rightarrow$ (a) for semi-free actions or for manifolds with an invariant weakly complex structure.

To prove (b) $\Rightarrow$ (a) we need a substantial extension of elliptic genera.

3. <u>A substantial extension of elliptic genera.</u>

(21) In order to include Witten's conjecture we need to prove conjecture 2 for any elliptic genus over a

$\mathbb{Q}$-algebra. But it is easy to reduce this general case to the case of a non-degenerate elliptic genus over $\mathbb{Q}$. Then again we can take any field extension of $\mathbb{Q}$. We shall assume that k is an algebraically closed field of characteristic zero. Then, take an elliptic genus over k with polynomial $R(u) = 1 - 2\delta u^2 + \varepsilon u^4$. Let $\alpha \in k^*$ be a fourth root of ε. The elliptic genus corresponding to $R^*(u) = 1 - \frac{2\delta}{\alpha^2} u^2 + u^4$ is related to φ by $\alpha^{2n} \varphi^*(M^{4n}) = \varphi(M^{4n})$, and there is a similar relation between formal extensions:

$$\alpha^{2n} \varphi_S^*(M^{4n})(u) = \varphi_S(M^{4n})(\alpha^{-1} u).$$

This reduces the problem to the case of $R(u) = 1 - 2\rho u^2 + u^4$.

(22) The equation
$$y^2 = 1 - 2\rho x^2 + x^4 \qquad (Q)$$
defines a Jacobi quartic of modulus $\rho \neq \pm 1$. Such quartics canonically correspond to elliptic curves over k.

One can proceed for instance as in one of the Igusa papers on transformation theory of elliptic functions: Let C be any elliptic curve over k, and $o \in C$ the neutral element of the group law. The sixteen fourth division points on C form a subgroup $C[4] \cong \mathbb{Z}/4 \times \mathbb{Z}/4$. Choose one of the primitive points in $C[4]$, that is a point $r \in C$ of order exactly 4, and consider the divisor
$$\mathfrak{a} = \sum_{w \in C[2]} \{(w) - (w+r)\}.$$
It is easy to check that $\mathfrak{a}$ is a principal divisor, that is the divisor of a function x. One sees easily that x is odd, and that $x(u+r) = c\, x(u)^{-1}$ for some $c \in k^*$. Normalize x in such a way as to have $c = 1$.

(23) Igusa shows that there is a unique function y such that $y(o) = 1$ and $y^2 = 1 - 2\rho x^2 + x^4$ where $\rho = \frac{1}{8} \sum x(s)^2$, the sum being over primitive $s \in C[4]$

such that $2s \neq 2r$. The curve C is mapped by (x,y) on the quartic Q and appears as the non-singular model for Q.

(24) The functions x and y have important symmetry properties. Let $\sigma : C[2] \longrightarrow \{\pm 1\}$ be the unique homomorphism such that $\sigma(r) = 1$. Then we have

$$x(u+\omega) = \sigma(\omega)\, x(u) \quad \text{for all } \omega \in C[2].$$
$$y(u+\omega) = \sigma(\omega)\, y(u).$$

(25) Let now $\mathcal{O}$ be the local ring at o. This is a DVR and its completion $\hat{\mathcal{O}}$ is also a DVR. It is well-known that if $u \in \hat{\mathcal{O}}$ is a local parameter, $\hat{\mathcal{O}}$ is canonically isomorphic to $k[[u]]$. The differential dx/y is the invariant differential on C. As the characteristic is zero, there is a unique local parameter u such that $du = dx/y$.

Then define $\hat{\mathcal{O}} \xrightarrow{\cong} \hat{H}^*_S(pt)$ by sending u to

the canonical generator of this ring. We shall always identify $\mathcal{O}$ with its image in $\hat{H}^*_S(pt)$. This embedding extends to an embedding of the whole function field $K = k(C)$ into $\hat{H}^*_S(pt)[u^{-1}] = k[[u]][u^{-1}]$ — the field of formal Laurent series in u.

Theorem 3. For any oriented S-manifold M,

$$\varphi_S(M) = \pi_!\, \Phi_S(M) \in \mathcal{O}.$$

I'll only give a sketch of the proof which is based on a localization formula.

(26) First of all, consider the product $\prod_{i=1}^{m} x(u_i)$ which is an element in $\hat{H}^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty)$. Obviously, the Weyl group $W(SO_{2m})$ leaves it unchanged. Thus we have a characteristic class $\mathcal{E} \in \hat{H}^*(BSO_{2m})$ — the "elliptic" Euler class. Let M be an oriented S-manifold. Then if F is a component of the fixed

point set, its normal bundle ν_P is an oriented S-bundle. Then we have

$$(L) \qquad \pi_! \, \varphi_S(M) = \sum_P \pi_!^P \left(\mathcal{E}_S(\nu_P)^{-1} \, \varphi_S(P) \right)$$

The right hand term is taken in a convenient localization of $\hat{H}^*_S(P)$, and the sum is at first sight in $k[[u]][u^{-1}]$ but actually belongs to $k[[u]]$. We only have to prove that each term of the sum is in K, for $\mathcal{O} = K \cap k[[u]]$.

(27) Now notice that P is a trivial S-space, that is $\tilde{P} = \mathbb{C}P^\infty \times P$, and $\varphi_S(P)$ is simply the image of $\varphi(P)$ under the projection on P. Then we have to calculate $\mathcal{E}_S(\nu_P)$. The representation of S in the fiber of ν_P has no trivial factors. It follows that ν_P has a complex structure commuting with the circle action, and it splits as $\nu_P = \sum_{i \neq 0} \xi_i$, where ξ_i

is a complex bundle on which $z \in S$ acts as z^i. Only a finite number of ξ_i is non-zero. Those i for which it happens are called the rotation numbers. One sees that the extension $\tilde{\nu}_P$ of ν_P is $\sum \eta^i \otimes \xi_i$.

Remember now that the formal group of φ, which can be defined by $f(x(u), x(v)) = x(u+v)$, is given by

$$f(u,v) = \frac{u\sqrt{R(v)} + v\sqrt{R(u)}}{1 - u^2 v^2},$$

that is

$$x(u+v) = \frac{x(u)\,y(v) + x(v)\,y(u)}{1 - x(u)^2 x(v)^2}.$$

With the help of this formula and the splitting principle for the ξ_i's, one obtains

$$\mathcal{E}(\tilde{\nu}_P) = \sum_{\text{finite}} P_\alpha \otimes h_\alpha$$

where $h_\alpha \in H^*(P)$, and P_α is a polynomial in $x(iu)$ and $y(iu)$, where i runs through all rotation numbers. The theorem follows easily.

(28) A closer examination of the formulae gives the following additions to the theorem.

1°. One has a certain control over the poles of $\varphi_S(M)$. Actually, a point $q \in C$ can be a pole only if iq is a zero of x for a rotation number i.

Corollary 1. The poles of $\varphi_S(M)$ are division points on C.

(29) 2°. The function

$$f_P = \pi_!^P\left(\mathcal{E}_S^{-1}(\vartheta_P)\,\varphi_S(P)\right)$$

has a symmetry expressed by the formula

$$f_P(u+\omega) = \sigma(\omega)^{d_P} f_P(u)$$

where $\omega \in C[2]$ and $d_P = \sum_i i \dim_{\mathbb{C}} \xi_i$. It is well-known that in the case of a Spin manifold, all d_P's are even or odd according to the parity of the action. Thus:

Corollary 2. If M is a connected Spin-manifold, then

$$\varphi_S(u+\omega) = \sigma(\omega)^d \varphi_S(u) \qquad \omega \in C[2]$$

where $d = 0$ or 1 according to the parity of the action.

Corollary 3. Conjecture 2 $\Rightarrow$ Conjecture 1.

Indeed, for $d=1$, $\varphi_S(u+\omega) = -\varphi_S(u)$ for certain $\omega \in C[2]$. The only constant which does this is zero.

(30) We also can easily prove now theorem 2 for semi-free actions. Indeed, all the rotation numbers are ± 1 in this case. This implies that all the poles of $\varphi_S(M)$ are zeros of x, that is belong to $C[2]$. But these points are eliminated by corollary 2.

(31) When the action is not semi-free, other poles are possible and we need further argumentation to eliminate them. One way of doing so is suggested by a series of papers of Krichever, where he studies, for weakly complex manifolds, the case of the $T_{x,y}$

genus defined by $T_{x,y}(\mathbb{C}P^n) = \sum_i x^i y^{n-i}$. One notices that the localization formula (L) is still valid if ℓ runs through all components of the fixed point set of a cyclic subgroup $\mathbb{Z}/n \subset S$. Then a calculation similar to the above but more complicated enables us to eliminate poles of order n. However, there is a technical difficulty here: the normal bundles ν_ℓ no longer need to be complex, and the proof fails to work if they are not. The proof does work for weakly complex manifolds, and this completes the proof of theorem 2.

(32) For general Spin-manifolds, we need further argumentation, perhaps the construction of a more "substantial" extension for elliptic genera. A geometric understanding of elliptic genera could be of help here.

I've been told that other participants of this Conference have ideas on this question. As I don't have any, I'd better stop now.