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NUMERICAL INVARIANTS OF HOMOTOPY TYPE

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1. Introduction.

This report is the author's interpretation of joint work with I. Berstein, T. Ganea, and P.J.Hilton. There are many questions one may ask about a space X . For example, is X an H-space ?, or what is the Lusternik-Schnirelmann category of X ? There are other similar notions such as weak category, nilpotency, conilpotency, cocategory, etc. With all of these numerical invariants, the problem of computation is not easy.

We define here an abstract situation which includes all but one of the above notions and many more. Under very general hypotheses, we show how to attack the problem of computation, assuming X is given with cells and attaching maps or via its Postnikov system. We also give a few methods of defining new notions from old and give a few examples. We give no proofs (those of propositions in section 4 being trivial); nor do we delve deeply into the examples. One is referred to the papers of the aforementioned authors for deeper properties of the examples.

2. Definition of Structure.

We will work in the category C of countable CW-complexes with base point and based homotopy classes of maps.

Definition 2.1r. A right structure is three covariant functors, T, U, V, from C to itself[2] and two natural transformations of functors, $\Delta : V \longrightarrow T$ and $r: U \longrightarrow T$. (In practice, V will usually be the identity functor, I.) Let $X \in C$. Then X will be called structured with respect to the given structure map t if $t\colon V(X) \longrightarrow U(X)$ and $r(X)t = \Delta(X)$. Let X and Y be structured with structure maps t_X and t_Y. Then $f\colon X \longrightarrow Y$ will be called a homomorphism with respect to the given structure and structure maps if $U(f)t_X = t_Y V(f)$.

Definition 2.1.l. A left structure is three covariant functors P, Q, R, from C to itself and two natural transformations of functors, $\nabla : P \longrightarrow R$ and $l\colon P \longrightarrow Q$. (In practice, R will usually be the identity functor.) Let $X \in C$. Then X will be called structured with respect to the given structure with structure map t if $t\colon Q(X) \longrightarrow R(X)$ and $t\, l(X) = \nabla(X)$. In a similar way, homomorphisms are defined as in definition 2.1r.

Examples. 1. cat $\leq n$ is the right structure $(\prod_1^{n+1}, F\bigvee_1^{n+1}, I)$, where $\prod_1^{n+1}$ is the $(n+1)$-fold self cartesian product, $F\bigvee_1^{n+1}$ is the fat wedge, i.e. the subset of $\prod_1^{n+1}$ consisting of points with at least one coordinate at the base point, Δ is the diagonal map, and r is the inclusion.

2. H-space is the left structure $(F\bigvee_1^2, \prod_1^2, I)$, where ∇ is the folding map and l is the inclusion. It is also the left structure (I, ∞, I) where $\infty(X) = X_\infty = \Omega S(X)$, l is the inclusion, and ∇ is the identity.

Assumption 2.2. For most of the applications, structures will be assumed to satisfy the following assumption:

(2.2r) $\quad \pi(SV(Y); U(X)) \xrightarrow{r(X)_*} \pi(SV(Y); T(X)) \longrightarrow 0$

is exact for all $X, Y \in C$.

(2.2l) $\quad \pi(Q(X); \Omega R(Y)) \xrightarrow{l(X)^*} \pi(P(X); \Omega R(Y)) \longrightarrow 0$

is exact for all $X, Y \in C$.

We note that the above examples satisfy this assumption.

3. The Main Theorem.

Definition 3.1r. Let (T, U, V) be a right structure. Let Y be structured by t_Y. Let $f\colon X \longrightarrow Y$. Then f is called primitive if there exists a map $t'\colon V(X) \longrightarrow U(X)$ (not necessarily a structure map) such that $U(f)t' = t_Y V(f)$.

Definition 3.1l. Let (P, Q, R) be a left structure. Let X be structured by t_X. Let $f\colon X \longrightarrow Y$. Then f is called primitive if there exists a map $t': Q(Y) \longrightarrow R(Y)$ (not necessarily a structure map) such that $R(f)t_X = t'Q(f)$.

Theorem 3.2r. Let $f: X \longrightarrow Y$. Let Y be structured by t_Y. Let $p: Y \longrightarrow Z = Y \cup_f CX$. Assume (T,U,V) satisfies (2.2r) and that V preserves cofibrations. Then, if f is primitive, then Z is structured and p is a homomorphism. Conversely, if $\pi(V(X); U(X)) \longrightarrow \pi(V(X); U(Y)) \longrightarrow \pi(V(X); U(Z))$ is exact and Z is structured with p a homomorphism, then f is primitive.[3]

Theorem 3.2l. Let $f: X \longrightarrow Y$. Let X be structured by t_X . Let $p: W \longrightarrow X$, where W is the induced fibre space from the path space on Y . Assume (P,Q,R) satisfies (2.2l) and that R preserves fibrations. Then, if f is primitive, then W is structured and p is a homomorphism. Conversely, if $\pi(Q(Y); R(Y)) \longrightarrow \pi(Q(X); R(Y)) \longrightarrow \pi(Q(W);R(Y))$ is exact and W is structured with p a homomorphism, then f is primitive.

We give the diagram used in the proof of theorem 3.2r.

$$\begin{array}{ccccccc}
 & & & & \pi(V(Y);T(Y)) & & \pi(V(X);U(X)) \\
 & & & & \uparrow & & \downarrow \\
 & & & & \pi(V(Y);U(Y)) & \rightarrow & \pi(V(X);U(Y)) \\
 & & & & \downarrow & & \downarrow \\
\pi(V(S(X));U(Z)) & \rightarrow & \pi(V(Z);U(Z)) & \rightarrow & \pi(V(Y);U(Z)) & \rightarrow & \pi(V(X);U(Z)) \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi(V(S(X));T(Z)) & \rightarrow & \pi(V(Z);T(Z)) & \rightarrow & \pi(V(Y);T(Z)) & \leftarrow &
\end{array}$$

The cases of $X = S^n$ or a Moore space in theorem 3.2r or $Y = K(\Pi,n)$ in theorem 3.2l give an inductive technique for deciding whether or not a given space is right or left structured respectively.

4. Further Notions.

We give some methods for constructing new structures from old. For brevity we restrict ourselves to right structures.

Let (T,U,V) be a right structure. Let (I,Q,I) be a left structure. Define a new right structure (T',U',V') by $T' = TQ$, $U' = UQ$, $V' = V$, $\Delta' = T(1)\Delta$, and $r' = rQ$.

Proposition 4.1r. Let X be right structured with respect to (T,U,V). Then X is right structured with respect to (T', U', V'). Conversely, let X be right structures with respect to (T', U', V') and left structured with respect to (I,Q,I). Then X is right structured with respect to (T,U,V). Furthermore, if (T,U,V) satisfies (2.2r), so does (T',U',V').

Example. Let (T,U,V) be $\text{cat} \leq n$. Let $Q = \infty$. Then (T',U',V') is called $\text{cat}_\infty \leq n$. Proposition 4.1r shows that if X is an H-space, then $\text{cat}_\infty X = \text{cat}\, X$.[4]

Definition 4.2r. Let (T,U,V) be a right structure. Define the associated right weak structure, (wT, wU, wV), by $wT = T$, wU = fibre of $p: T \longrightarrow T/U$, $wV = V$, $w\Delta = \Delta$, and wr is the inclusion of the fibre in T. Note that X is structured with respect to the weak structure if and only if $p(X)\,\Delta(X) = 0$.

Proposition 4.3r. If X is structured with respect to (T,U,V), then X is structured with respect to (wT, wU, wV). If (T,U,V) satisfies (2.2r), then so does (wT, wU, wV).

Example. If (T,U,V) is $\text{cat} \leq n$, then (wT, wU, wV) is $\text{wcat} \leq n$ and $\text{wcat}\, X \leq n$ if and only if the composition $X \longrightarrow \prod_1^{n+1} X \longrightarrow X \# \cdots \# X$ is trivial.

Clearly one can iterate the above construction and define (w^mT, w^mU, w^mV). The author doubts that $w^m\text{cat} \leq n$ is interesting if $m > 1$.

Definition 4.4r. Let (T,U,V) be a right structure. Define the suspension of this structure, (ST, SU, SV), by $ST = S(T)$, $SU = S(U)$, $SV = S(V)$, $S\Delta = S(\Delta)$, and $Sr = S(r)$.

Proposition 4.5r. If X is structured with respect to (T,U,V), then X is structured with respect to (ST, SU, SV).

Example. In [4] it is proved that $\text{Swcat}\, X = \text{conil}\, X$ (see [1]).

All of the numerical invariants of homotopy type considered by the aforementioned authors fit into our general description except cocat $\leqq n$ (see [3] for an inductive definition). One of the main properties cocat $\leqq n$ should have is that cocat $X \leqq 1$ if and only if X is an H-space. As a possible alternative definition of cocat $\leqq n$ we suggest (P_n, Q_n, R_n), where $P_n = I$, $Q_n(X) = (K_n \# X)^{K_n}$, $R_n = I, \nabla_n$ is the identity, l_n is the obvious inclusion, and K_n is a universal space with cat $K_n \leqq n$ and $K_1 = S^1$. Ganea has shown that (P_n, Q_n, R_n) satisfies (2.21) if K_n dominates S^1.

References.

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2. I. Berstein and P.J.Hilton, Category and generalized Hopf invariants, Illinois J.Math. 4(1960), 437-451.
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4. T. Ganea, P.J.Hilton, and F.P.Peterson, On the homotopy-commutativity of loop spaces and suspensions, Topology, to appear.

Footnotes.

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[2] This restriction on the category is not essential. In some applications T, U, and V may be functors from C' to C".

[3] The case cat $\leqq n$ of this theorem is proved in [2]. The general proof is essentially the same.

[4] In general, it is not true that $\text{cat}_\infty X = \text{cat}\, X$. One only has $\text{cat}_\infty X \leqq \text{cat}\, X$.