GENERALIZED COHOMOTOPY GROUPS

by

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ABSTRACT

Let $\pi^n(K)$ denote the n^{th} cohomotopy group of a complex K, and let $\eta^n: \pi^n(K) \longrightarrow H^n(K)$ denote the natural homomorphism defined by Spanier. In this paper we derive results concerning the structure of $\pi^n(K)$ by computing the kernel and cokernel of η^n for various n in terms of the cohomology structure of K. For each prime p, we determine a range of values of n for which η^n gives an isomorphism on the p-primary components of $\pi^n(K)$ and $H^n(K)$; this generalizes the classical Hopf theorem. We also generalize the Steenrod classification theorem by computing the kernel and cokernel of η^n restricted to the p-primary components for a further range of values of n in terms of the Steenrod reduced p^{th} powers. In the second part of this paper, we introduce cohomotopy groups with coefficients in an abelian group G; namely, let $\pi^n(K; G)$ denote the set of homotopy classes of maps of K into a space whose homology is zero except that its n^{th} homology group is G. For a fixed group G, the properties of $\pi^n(K; G)$ are analogous to those of $\pi^n(K)$. Furthermore, if G has no elements of order 2, then a homomorphism $\phi: G \longrightarrow H$ induces a unique homomorphism $\phi_\#: \pi^n(K; G) \longrightarrow \pi^n(K; H)$. Finally, there is a universal coefficient theorem which asserts that

$$0 \longrightarrow \pi^n(K) \otimes G \xrightarrow{\alpha} \pi^n(K; G) \xrightarrow{\beta} \mathrm{Tor}(\pi^{n+1}(K), G) \longrightarrow 0$$

is a split exact sequence. This reduces the problem of computing $\pi^n(K; G)$ to that of computing $\pi^n(K)$.

1. Introduction.

One of the central problems of topology is the computation of the set of homotopy classes of maps of a complex K into the n-sphere S^n. In 1936, Borsuk [6] showed that if the dimension of $K = N \leq 2n - 2$, then this set admits a natural abelian group structure. In this case, this set is called the n^{th} cohomotopy group of K and denoted by $\pi^n(K)$. In 1949, Spanier [22] derived the basic properties of these groups and expressed the existing theorems on the structure of $\pi^n(K)$ by means of an exact sequence [22, p.240]. These theorems are the Hopf theorem [13], which states that the natural homomorphism $\eta^n: \pi^n(K) \longrightarrow H^n(K)$ (= the n^{th} cohomology group of K) is an isomorphism for $n = N$ and is onto in case $n = N - 1$, and the Steenrod theorem [26], which computes the kernel of η^{N-1} and the image of η^{N-2}. Little more is known about the structure of $\pi^n(K)$.

In the first part of this paper, we shall derive further results concerning the structure of $\pi^n(K)$. First, $\pi^n(K)$ is finitely generated when K is finite. Second, $\pi^n(K)$ and $H^n(K)$ have the same rank. Third, the Hopf result is generalized by determining, for each prime p, a range of values of n for which η^n gives an isomorphism on the p-primary components of $\pi^n(K)$ and $H^n(K)$. Finally, the Steenrod result is generalized by giving a computation of the kernel and cokernel of η^n restricted to the p-primary components for a range of values of n where η^n is not an isomorphism. This computation is given in terms of the reduced p^{th} powers of Steenrod [27]. In proving these results, we make use of a cohomotopy exact couple similar to that of Massey [15, part III] and of Serre's technique of "isomorphisms modulo a class of groups" [20].

In the second part of this paper, we shall introduce cohomotopy groups with coefficients in an abelian group G; namely, let $\pi^n(K; G)$ denote the set of homotopy classes of maps of K into an X(G,n)-space, where an X(G,n)-space is a simply connected space whose homology is zero except that its n^{th} homology group is G. $\pi^n(K; G)$ is shown to be independent of the choice of X(G,n)-space. Notice that if $G = Z =$ the additive group of integers, then S^n is an X(Z,n)-space and $\pi^n(K; Z) = \pi^n(K)$. For a fixed group G, the properties of $\pi^n(K; G)$ are analogous to those of $\pi^n(K)$, and the results of part I are generalized so as to apply to $\pi^n(K; G)$. Furthermore, if G has no elements of order 2, then a homomorphism $\phi: G \longrightarrow H$ induces a unique homomorphism $\phi_{\#}: \pi^n(K; G) \longrightarrow \pi^n(K; H)$. Except for this restriction on G, $\phi_{\#}$ enjoys many of the same properties as the induced cohomology homomorphism $\phi_*: H^n(K; G) \longrightarrow H^n(K; H)$. In particular, the sequence corresponding to an exact coefficient sequence is exact (theorem 8.5), and there is a universal coefficient theorem which asserts that

$$0 \longrightarrow \pi^n(K) \otimes G \xrightarrow{\alpha} \pi^n(K; G) \xrightarrow{\beta} \operatorname{Tor}(\pi^{n+1}(K), G) \longrightarrow 0$$

is a split exact sequence. This theorem reduces the problem of computing $\pi^n(K; G)$ to that of computing $\pi^n(K)$. We conclude with a section on cohomotopy operations and a section on homotopy groups with coefficients in G dual to our cohomotopy groups.

In conclusion, I wish to express my warm appreciation to Professor N. E. Steenrod, under whom this paper was written, for his kind advice and encouragement. I also wish to thank Professor J. C. Moore for suggesting the idea of general coefficients.

PART I.

2. Preliminaries.

In this section, we recall the notions and notations which we need in order to state the main results of part I.

We first recall the definition and elementary properties of cohomotopy groups [22]. Let K be a finite dimensional CW-complex [12], and let L be a subcomplex. Let $a: (K,L) \longrightarrow (S^n, pt.)$ be a continuous map, where S^n denotes the n-dimensional sphere and "pt." denotes any fixed point of S^n. Let [a] denote the homotopy class of a. The set of all such homotopy classes has a natural abelian group structure defined on it if dimension $K = N \leq 2n - 2$. We call this the n^{th} cohomotopy group of the CW-pair (K,L) and denote it by $\pi^n(K,L)$. A map $f: (K,L) \longrightarrow (K',L')$ induces a homomorphism $f^{\#}: \pi^n(K',L') \longrightarrow \pi^n(K,L)$ defined by $f^{\#}([a]) = [af]$.

Let $\pi_r(X)$ denote the r^{th} homotopy group of the space X. The process of suspension induces a homomorphism $S_{\#}: \pi_r(S^n) \longrightarrow \pi_{r+1}(S^{n+1})$ which is an isomorphism for $r < 2n - 1$ by the Freudenthal theorem [28]. We identify these groups under this isomorphism and denote the result by $Z_{(r-n)}$.

The homology theory best suited for our investigations is the cellular homology theory as described in [12] or [30]. We denote the n^{th} homology group of (K,L) with coefficients in G by $H_n(K,L; G)$ and the n^{th} cohomology group of (K,L) with coefficients in G by $H^n(K,L; G)$.

We denote the additive group of integers by Z, the group of integers mod n by Z_n, and the p-primary component of a group A by A_p, where the p-primary component of A is the subgroup of all elements of A whose orders are a power of the prime p. (The only exception to this notation is Z_p which denotes the integers mod p.) Let $\phi: A \longrightarrow B$ be a homomorphism. We denote the kernel of ϕ by Ker ϕ, the image of ϕ by Im ϕ, and the cokernel of ϕ (= B/Im ϕ) by Coker ϕ. $A \otimes B$ and Tor(A,B) denote the tensor product and the torsion product respectively [8].

We now recall the notion of a class which was introduced by Serre [20]. A <u>class</u> $\mathcal{C}$ is a non-empty family of abelian groups such that (I) if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence [11], then $A \in \mathcal{C}$ and $C \in \mathcal{C}$ if and only if $B \in \mathcal{C}$. In the applications, one of the following axioms is also assumed: (II_A) if $A \in \mathcal{C}$ and $B \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ and Tor(A,B) $\in \mathcal{C}$, or (II_B) if $A \in \mathcal{C}$ and B is arbitrary, then $A \otimes B \in \mathcal{C}$ and Tor(A,B) $\in \mathcal{C}$.

The important examples of classes are $\mathcal{C}_o$ = the class consisting of the 0 group alone, $\mathcal{C}_T$ = the family of torsion groups, $\mathcal{C}_P$ = the family of torsion groups whose p-primary components are 0, $\mathcal{F}$ = the family of finitely generated groups, $\mathcal{C}_f$ = the family of finite groups, and $\mathcal{C}_{pf}$ = the family of finite groups whose p-primary components are 0. It is easily checked that $\mathcal{C}_o$, $\mathcal{C}_T$, and $\mathcal{C}_P$ satisfy axiom (II_B), while $\mathcal{F}$, $\mathcal{C}_f$, and $\mathcal{C}_{pf}$ satisfy axiom (II_A) but not (II_B).

The notion of class was introduced by Serre to allow us to ignore systematically certain groups. With this in mind, we make the following definitions: a homomorphism $\phi: A \longrightarrow B$ is a <u>$\mathcal{C}$-monomorphism</u> if Ker $\phi \in \mathcal{C}$; it is a <u>$\mathcal{C}$-epimorphism</u> if Coker $\phi \in \mathcal{C}$; it is a <u>$\mathcal{C}$-isomorphism</u> if both Ker ϕ and Coker $\phi \in \mathcal{C}$.

For any class $\mathcal{C}$, let $\alpha_{\mathcal{C}}$ denote the largest integer such that $Z_{(s)} \in \mathcal{C}$ for $0 < s < \alpha_{\mathcal{C}}$.

THEOREM 2.1: (a) $Z_{(r)}$ <u>is finite if</u> $r > 0$, (b) $(Z_{(r)})_p = \begin{cases} Z_p & \text{if } r = 2s(p-1)-1 \text{ for } 0 < s < p \\ 0 & \text{otherwise for } r < 2p(p-1)-2, \end{cases}$
(c) $\alpha_{\mathcal{C}_T} = \infty$, (d) $\alpha_{\mathcal{C}_p} = 2p - 3$, <u>and</u> (e) $\alpha_{\mathcal{C}_0} = 1$.

<u>Proof:</u> (a) is a result of Serre [19]. (b) is an unpublished result of Cartan. (c) follows from (a). (d) follows from (a) and (b). It is well-known that $Z_{(1)} = \pi_{n+1}(S^n) = Z_2$, hence $\alpha_{\mathcal{C}_0} = 1$.

3. <u>The Main Results of Part I.</u>

The purpose of this section is to state our main results on the structure of cohomotopy groups. The following four sections are devoted to the proofs of these results.

There is a natural homomorphism $\eta^r: \pi^r(K,L) \longrightarrow H^r(K,L)$. η^r is defined as follows: let $a \in [a] \in \pi^r(K,L)$, and let u be a chosen generator of $H^r(S^r,pt.)$. Then $\eta^r([a]) = a^*(u) \in H^r(K,L)$ (see section 4 and [22, p.234]). We study the relations between the cohomotopy groups and the cohomology groups using this homomorphism. The classical Hopf theorem [13] states that if K is an N-dimensional complex, then η^N is an isomorphism. Our first result extends this theorem modulo classes.

Let (K,L) be a CW-pair with dimension K = N for this and the following four sections.

THEOREM 3.1: <u>Let</u> $\mathcal{C}$ <u>be a class satisfying condition</u> (II_B) <u>of section 2. Let</u> $n > (N+1)/2$ <u>be such that</u> $H^r(K,L) \in \mathcal{C}$ <u>for every</u> $r > n$. <u>Then</u> η^r <u>is a</u> $\mathcal{C}$<u>-isomorphism if</u> $r > \mathrm{Max}((N+1)/2, n - \alpha_{\mathcal{C}})$, <u>and is a</u> $\mathcal{C}$<u>-epimorphism for</u> $r = n - \alpha_{\mathcal{C}}$ <u>in case</u> $n - \alpha_{\mathcal{C}} > (N+1)/2$.

THEOREM 3.2: <u>Let</u> $\mathcal{C}$ <u>be a class satisfying condition</u> (II_A) <u>of section 2. Let</u> $n > \mathrm{Max}((N+1)/2, N - \alpha_{\mathcal{C}})$ <u>be such that</u> $H^r(K,L) \in \mathcal{C}$ <u>for every</u> $r > n$. <u>Then</u> η^r <u>is a</u> $\mathcal{C}$<u>-isomorphism if</u> $r \geq n$, <u>and is a</u> $\mathcal{C}$<u>-epimorphism for</u> $r = n - 1$.

Theorems 3.1 and 3.2 solve problem 32 of [16].

We have as an immediate corollary

COROLLARY 3.3: Let K and L be two CW-complexes of dimensions M and N respectively, and let $f: L \longrightarrow K$. Let $\mathcal{C}$ be a class satisfying condition (II_B) of section 2, and let $n > \mathrm{Max}((M + 1)/2, (N + 2)/2)$. Then the following two statements are equivalent: (a) $f^*: H^r(K) \longrightarrow H^r(L)$ is a $\mathcal{C}$-isomorphism for $r > n$ and a $\mathcal{C}$-epimorphism for $r = n$, and (b) $f^\#: \pi^r(K) \longrightarrow \pi^r(L)$ is a $\mathcal{C}$-isomorphism for $r > n$ and a $\mathcal{C}$-epimorphism for $r = n$.

Proof: Replace f by a cellular approximation f' [12, p.98]. By the mapping cylinder construction [12, p.108], we may assume f' is an inclusion. Then (a) is true if and only if $H^r(K,L) \in \mathcal{C}$ for $r > n$ by the exact cohomology sequence of a pair. This is true if and only if $\pi^r(K,L) \in \mathcal{C}$ for $r > n$ by 3.1. However, this is true if and only if (b) is true by the exact cohomotopy sequence of a pair.

By specializing $\mathcal{C}$ to particular classes, we have the following four corollaries.

COROLLARY 3.4: Let $n > (N + 1)/2$ be such that $H^r(K,L)$ is finitely generated (finite) for every $r > n$. Then $\pi^r(K,L)$ is finitely generated (finite) for $r > n$.

Proof: This follows immediately from 3.2 by setting $\mathcal{C} = \mathcal{F}$ or $\mathcal{C}_f$ and noting 2.1 (a).

COROLLARY 3.5: Let $n > (N + 1)/2$ be such that $H^r(K,L)$ is finitely generated for every $r > n$. Then $\pi^r(K,L)$ and $H^r(K,L)$

have the same rank for $r > n$. *Furthermore, if* $r > (N + 1)/2$ *and* $u \varepsilon H^r(K,L)$, *then there is an integer* $M \neq 0$ *such that* $Mu \varepsilon \operatorname{Im} \eta^r$.

Proof: By 3.4, $\pi^r(K,L)$ and $H^r(K,L)$ are finitely generated for every $r > n$. Now apply 3.1 with $n = N$, $\mathcal{C} = \mathcal{C}_T$, and note that $\alpha_{\mathcal{C}_T} = \infty$ by 2.1 (c). The conclusion that η^r is a $\mathcal{C}_T$-isomorphism for $r > n$ means that $\pi^r(K,L)$ and $H^r(K,L)$ have the same rank. Furthermore, if for some $u \varepsilon H^r(K,L)$ there did not exist a non-zero integer M such that $Mu \varepsilon \operatorname{Im} \eta^r$, then $\operatorname{Coker} \eta^r \notin \mathcal{C}_T$.

The above is a result of Serre [20; p.288].

COROLLARY 3.6: *Let* $n > (N + 1)/2 + 1$ *be such that* $H^r(K,L) = 0$ *for every* $r > n$. *Then* $\pi^r(K,L) = 0$ *for* $r > n$, η^n *is an isomorphism, and* η^{n-1} *is an epimorphism.*

Proof: Set $\mathcal{C} = \mathcal{C}_0$, and use 3.1. Note that $\mathcal{C}_0$-isomorphism means regular isomorphism.

Let $\phi: A \longrightarrow B$ be a homomorphism. We denote $\phi|A_p$: $A_p \longrightarrow B_p$ by $\phi_{(p)}$.

COROLLARY 3.7: *Let* $n > (N + 1)/2$ *be such that* $H^r(K,L) \varepsilon \mathcal{C}_p$ *for every* $r > n$. *Then* $\eta^r_{(p)}$ *is an isomorphism for* $r > \operatorname{Max}((N + 1)/2, n - 2p + 3)$ *and* $\eta^{n-2p+3}_{(p)}$ *is an epimorphism if* $n - 2p + 3 > (N + 1)/2$.

Proof: Set $\mathcal{C} = \mathcal{C}_p$, and use 3.1. Note that ϕ is a $\mathcal{C}_p$-isomorphism implies that $\phi_{(p)}$ is an isomorphism.

We now extend corollaries 3.6 and 3.7.

Let $Sq^2 : H^{n-2}(K, L) \longrightarrow H^n(K, L; Z_2)$ denote the Steenrod square [26]. $\Lambda : H^n(K, L; Z_2) \longrightarrow \pi^{n-1}(K, L)$ is a homomorphism defined in section 6 (see [22; p.238]). The following theorem is a slight extension of a result of Steenrod [26] and Spanier [22; p.240].

THEOREM 3.8: *Let* $n > (N + 1)/2 + 2$ *be such that* $H^r(K, L) = 0$ *for every* $r > n$. *Then* $\pi^r(K, L) = 0$ *for* $r > n$, η^n *is an isomorphism, and the sequence*

$$\pi^{n-2}(K, L) \xrightarrow{\eta^{n-2}} H^{n-2}(K, L) \xrightarrow{Sq^2} H^n(K, L; Z_2) \xrightarrow{\Lambda} \pi^{n-1}(K, L) \xrightarrow{\eta^{n-1}} H^{n-1}(K, L) \longrightarrow 0$$

is exact.

Let $\mathcal{P}^1 : H^s(K, L) \longrightarrow H^{s+2p-2}(K, L; Z_p)$ denote the first Steenrod reduced p^{th} power [27]. $\mu : H^{r+2p-3}(K, L; Z_p) \longrightarrow \pi^r(K, L)_p$ is a homomorphism defined in section 7. In section 7 we prove

THEOREM 3.9: *Let* $n > (N + 1)/2$ *be such that* $H^r(K, L) \in \mathcal{C}_p$ *for every* $r > n$. *Then* $\eta^r_{(p)}$ *is an isomorphism for* $r > \text{Max}((N + 1)/2, n - 2p + 3)$ *and the sequences*

$$H^{r-1}(K, L) \xrightarrow{\mathcal{P}^1} H^{r+2p-3}(K, L; Z_p) \xrightarrow{\mu} \pi^r(K, L)_p \xrightarrow{\eta^r_{(p)}} H^r(K, L)_p \xrightarrow{\mathcal{P}^1_{(p)}} H^{r+2p-2}(K, L; Z_p)$$

for $r > \text{Max}((N + 1)/2, n - 4p + 5)$ *and*

$$\pi^{n-4p+5}(K, L)_p \xrightarrow{\eta^{n-4p+5}_{(p)}} H^{n-4p+5}(K, L)_p \xrightarrow{\mathcal{P}^1_{(p)}} H^{n-2p+3}(K, L; Z_p)$$

if $n - 4p + 5 > (N + 1)/2 + 1$ *are exact.*

For a range of values of r, note that 3.9 computes the kernel and the cokernel of $\eta^r_{(p)}$ in terms of the cohomology groups of (K,L) and the first reduced p^{th} power. Theorems 3.8 and 3.9 partially solve problem 33 of [16].

4. The Cohomotopy Exact Couple.

The proofs of our main theorems as well as subsequent results in part II are based on a cohomotopy exact couple of the pair (K, L) similar to the one studied by Massey [15; part III]. Since it differs from Massey's cohomotopy exact couple, we describe it in detail.

Let (K, L) be a CW-pair with dimension $K = N$. Let z be the least integer $> (N + 1)/2$; i. e. z is the least integer n for which $\pi^n(K, L)$ has a natural group structure. Let K^s denote the union of L with the s-dimensional skeleton of K. Our exact couple is based on the exact cohomotopy sequence of the triple (K, K^s, K^{s-1}):

$$\pi^z(K, K^s) \xrightarrow{j} \pi^z(K, K^{s-1}) \longrightarrow \cdots \longrightarrow \pi^r(K, K^s) \xrightarrow{j} \pi^r(K, K^{s-1})$$
$$\xrightarrow{i} \pi^r(K^s, K^{s-1}) \xrightarrow{\Delta} \pi^{r+1}(K, K^s) \longrightarrow \cdots,$$ where i and j are the homomorphisms induced by the inclusions $(K^s, K^{s-1}) \longrightarrow (K, K^{s-1})$ and $(K, K^{s-1}) \longrightarrow (K, K^s)$ respectively and Δ is the coboundary operator of the triple (K, K^s, K^{s-1}) (see [22; p. 229] or section 9 for the definition of Δ and the proof of exactness).

For notational convenience, we set $A^{r,s} = \pi^r(K, K^s)$ and $C^{r,s} = \pi^r(K^s, K^{s-1})$ for $r \geq z$. Also let $j^{r,s}: A^{r,s} \longrightarrow A^{r,s-1}$, $i^{r,s}: A^{r,s-1} \longrightarrow C^{r,s}$, and $\Delta^{r,s}: C^{r,s} \longrightarrow A^{r+1,s}$ be the appropriate j, i, or Δ for $r \geq z$. In order to extend the above sequence to an exact sequence extending indefinitely in both directions, we set $C^{z-1,s} = \operatorname{Ker} j^{z,s}$, $C^{r,s} = 0$ for

$r < z-1$, $A^{r,s} = 0$ for $r < z$, $\Delta^{z-1,s}: C^{z-1,s} \longrightarrow A^{z,s}$ to be the inclusion, and the remaining homomorphisms i, j, and Δ to be zero. The indices on the homomorphisms i, j, and Δ are determined by their domains, and thus we omit them whenever possible.

The groups and homomorphisms defined above fit together in a lattice as in figure 1. Any path in figure 1 which moves downward and to the right in a zig-zag pattern traces out an exact sequence. This follows immediately from the exact sequence of the triple (K, K^s, K^{s-1}) and our definitions.

We now compute some of the groups of the cohomotopy exact couple. The following groups are obviously 0 for $m \geq 1$: $C^{r,r-m}$, $C^{N+m,s}$, $A^{N+m,s}$, $A^{r,N+m-1}$, and $C^{r,N+m}$. Also $j: A^{r,r-m} \longrightarrow A^{r,r-m-1}$ is an isomorphism for $m \geq 2$ and is onto for $m = 1$. This follows because $C^{r-1,r-m} \longrightarrow A^{r,r-m} \longrightarrow A^{r,r-m-1} \longrightarrow C^{r,r-m}$ is exact and for $m \geq 1$, $C^{r,r-m} = 0$. Hence $A^{r,r-2} \approx A^{r,-1} = \pi^r(K,L)$.

Define a homomorphism $\psi: \pi^r(K^s, K^{s-1}) \longrightarrow C^s(K, L; \pi_s(S^r, pt.))$ as follows. The cellular homology and cohomology theory is based on $\pi_s(K^s, K^{s-1})$ as the group of chains in dimension s [12]. Hence $C^s(K, L; \pi_s(S^r, pt.)) = \mathrm{Hom}(\pi_s(K^s, K^{s-1}), \pi_s(S^r, pt.))$, where Hom(A, B) denotes the groups of all homomorphisms from A to B. Let $[b] \in \pi_s(K^s, K^{s-1})$, $[a] \in \pi^r(K^s, K^{s-1})$, then define $\psi([a])([b]) = [ab] \in \pi_s(S^r, pt.)$. It is shown in [22, p.222] and in [24] that ψ has the following properties for $r \geq z$ and $s \leq N$:

$$
\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
 & \cdots \longrightarrow & A^{r,s} & \xrightarrow{i} & C^{r,s+1} & \xrightarrow{\Delta} & A^{r+1,s+1} & \longrightarrow \cdots & \\
 & & \downarrow j & & & & \downarrow j & & \\
\cdots \longrightarrow C^{r-1,s-1} & \xrightarrow{\Delta} & A^{r,s-1} & \xrightarrow{i} & C^{r,s} & \xrightarrow{\Delta} & A^{r+1,s} & \longrightarrow \cdots & \\
 & & \downarrow j & & & & \downarrow j & & \\
 & \cdots \longrightarrow & A^{r,s-2} & \xrightarrow{i} & C^{r,s-1} & \xrightarrow{\Delta} & A^{r+1,s-1} & \longrightarrow \cdots & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & & & \vdots & &
\end{array}
$$

Figure 1.

$$
\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
 & \cdots \longrightarrow & {}'A^{r,s} & \xrightarrow{'i} & {}'C^{r,s+1} & \xrightarrow{'\Delta} & {}'A^{r+1,s+2} & \longrightarrow \cdots & \\
 & & \downarrow {}'j & & & & \downarrow {}'j & & \\
\cdots \longrightarrow {}'C^{r-1,s-2} & \xrightarrow{'\Delta} & {}'A^{r,s-1} & \xrightarrow{'i} & {}'C^{r,s} & \xrightarrow{'\Delta} & {}'A^{r+1,s+1} & \longrightarrow \cdots & \\
 & & \downarrow {}'j & & & & \downarrow {}'j & & \\
 & \cdots \longrightarrow & {}'A^{r,s-2} & \xrightarrow{'i} & {}'C^{r,s-1} & \xrightarrow{'\Delta} & {}'A^{r+1,s} & \longrightarrow \cdots & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & & & \vdots & &
\end{array}
$$

Figure 2.

1) ψ is an isomorphism,

2) ψ is natural with respect to cellular maps $f: (K,L) \longrightarrow (K',L')$, and

3) the following diagram is commutative:

$$\begin{array}{ccc} \pi^r(K^s,K^{s-1}) & \xrightarrow{i\Delta} & \pi^{r+1}(K^{s+1},K^s) \\ \downarrow \psi & & \downarrow \psi \\ C^s(K,L; \pi_s(S^r,pt.)) & \xrightarrow{S_{\#}\delta} & C^{s+1}(K,L; \pi_{s+1}(S^{r+1},pt.)), \end{array}$$

where $\delta: C^s(K,L; \pi_s(S^r,pt.)) \longrightarrow C^{s+1}(K,L; \pi_s(S^r,pt.))$ is the coboundary homomorphism, and $S_{\#}: C^{s+1}(K,L; \pi_s(S^r,pt.)) \longrightarrow C^{s+1}(K,L; \pi_{s+1}(S^{r+1},pt.))$ is the homomorphism induced by suspending the coefficients. Under our assumptions on r and s, $S_{\#}$ is an isomorphism on the coefficients, and we may write the commutative diagram as

$$\begin{array}{ccc} \pi^r(K^s,K^{s-1}) & \xrightarrow{i\Delta} & \pi^{r+1}(K^{s+1},K^s) \\ \downarrow \psi & & \downarrow \psi \\ C^s(K,L; Z_{(s-r)}) & \xrightarrow{\delta} & C^{s+1}(K,L; Z_{(s-r)}) \end{array}$$

(see section 2 for the definiton of $Z_{(s-r)}$).

We now describe the first derived cohomotopy exact couple (see [15; part I] for the precise definitions and the proof that this is an exact couple). Define $'C^{r,s} = H(C^{r,s}) = \mathrm{Ker}(i^{r+1,s+1}\Delta^{r,s}) / \mathrm{Im}(i^{r,s}\Delta^{r-1,s-1})$, $'A^{r,s} = \mathrm{Im}\, j^{r,s}$, $'j^{r,s}: {'A^{r,s}} \longrightarrow {'A^{r,s-1}}$ is defined by $j^{r,s-1}$, $'i^{r,s}: {'A^{r,s-1}} \longrightarrow {'C^{r,s}}$ is defined by $i^{r,s}(j^{r,s-1})^{-1}$, and $'\Delta^{r,s}: {'C^{r,s}} \longrightarrow {'A^{r+1,s+1}}$ is defined by $\Delta^{r,s}$. These groups and homomorphism fit together in a lattice as in figure 2.

From the remarks above, the following groups are obviously 0 for $m \geq 1$: ${}'C^{r,r-m}$, ${}'C^{N+m,s}$, ${}'A^{N+m,s}$, ${}'A^{r,N+m-1}$, and ${}'C^{r,N+m}$. Also ${}'A^{r,r-1} \approx {}'A^{r,r-2} \approx \dots \approx \pi^r(K,L)$ for $r \geq z$. Furthermore, ${}'C^{r,s} \approx H^s(K,L; Z_{(s-r)})$ for $r \geq z + 1$ by the above identifications and definition of ${}'C^{r,s}$.

Under these identifications, ${}'i^{r,r}$: $\pi^r(K,L) \longrightarrow H^r(K,L)$ is defined geometrically as follows. Let $[a] \in \pi^r(K,L)$. We may assume that a: $(K,L) \longrightarrow (S^r, pt.)$ is such that $a(K^{r-1}) = pt.$ Restrict a to a map a': $(K^r, K^{r-1}) \longrightarrow (S^r, pt.)$, then $[a']$ is an r-cochain of (K,L) which is a cocycle. Its cohomology class is ${}'i^{r,r}([a])$. We denote the homomorphism ${}'i^{r,r}$ by η^r. It is easily checked that this gives the same definition as given in section 3.

We recall for reference later the definition of a cohomology operation θ of type $(n,q; A,B)$. θ is a function θ: $H^n(K,L; A) \longrightarrow H^q(K,L; B)$, defined for every CW-pair (K,L), such that if f: $(K,L) \longrightarrow (K',L')$, then $f^*\theta = \theta f^*$: $H^n(K',L'; A) \longrightarrow H^q(K,L; B)$. A theorem of Serre [21; p.220] states that the cohomology operations of a given type $(n,q; A,B)$ are in 1-1 correspondence with the elements of $H^q(A,n; B) = H^q(K(A,n); B)$, where $K(A,n)$ is an Eilenberg-MacLane complex (see [21] for details). This theorem holds if θ is only defined for every CW-pair (K,L) with dimension $K \leq N$ and $N \geq q + 1$.

Let f: $(K,L) \longrightarrow (K',L')$ be a cellular map; i. e. $f(K^n) \subset K'^n$. Then f induces homomorphisms $f^\#$: $\pi^r(K'^s, K'^{s-1}) \longrightarrow \pi^r(K^s, K^{s-1})$ and $f^\#$: $\pi^r(K', K'^s) \longrightarrow \pi^r(K, K^s)$ which commute with i, j, and Δ when $r \geq z$ and z', where z' = least integer $>$ (dimension $K' + 1)/2$. Hence f induces a homomorphism of the cohomotopy exact couple of (K', L') into the cohomotopy exact couple of (K, L), and hence a homomorphism f^* of the first derived cohomotopy exact couples. Thus f^* commutes with ${}'i'\Delta : {}'C^{r,s} \longrightarrow {}'C^{r+1,s+2}$. Hence ${}'i'\Delta$ is a cohomology operation because any two cellular approximations f_1 and f_2 to an arbitrary map $g: (K, L) \longrightarrow (K', L')$ induce the same homomorphism g^* on ${}'C^{r,s}$ and ${}'C^{r+1,s+2}$ (see [12, p.98] for the definition and properties of cellular approximations).

5. Proof of the Hopf Theorem Mod $\mathcal{C}$.

In this section we give the proofs of theorems 3.1 and 3.2.

Proof of 3.1: The proof is based on figure 3, a portion of the first derived cohomotopy exact couple. In order to prove that η^r is a $\mathcal{C}$-isomorphism, it suffices to show that $'A^{r,r} \in \mathcal{C}$ for $r > \operatorname{Max}((N+1)/2, n - \alpha_{\mathcal{C}})$ because $'A^{r,r} \xrightarrow{'j} \pi^r(K,L) \xrightarrow{\eta^r} H^r(K,L) \xrightarrow{'\Delta} {'A^{r+1,r+1}}$ is exact. Again by exactness (and the fact that $'A^{r,N} = 0 \in \mathcal{C}$), it suffices to prove that $H^{r+1}(K,L; Z_{(1)}) \in \mathcal{C}, \ldots, H^N(K,L; Z_{(N-r)}) \in \mathcal{C}$ for $r > \operatorname{Max}((N+1)/2, n - \alpha_{\mathcal{C}})$. Now $n - r < n - (n - \alpha_{\mathcal{C}}) = \alpha_{\mathcal{C}}$, hence $Z_{(1)} \in \mathcal{C}, \ldots, Z_{(n-r)} \in \mathcal{C}$ by definition of $\alpha_{\mathcal{C}}$. Since $Z_{(s)}$ is finitely generated by 2.1 (a), we may use the universal coefficient theorem for cohomology (2.3 of the appendix); i. e. $0 \longrightarrow H^{r+s}(K,L) \otimes Z_{(s)} \xrightarrow{\alpha} H^{r+s}(K,L; Z_{(s)}) \xrightarrow{\beta} \operatorname{Tor}(H^{r+s+1}(K,L), Z_{(s)}) \longrightarrow 0$ is exact. $Z_{(s)} \in \mathcal{C}$ for $s \leq n - r$, hence $H^{r+s}(K,L; Z_{(s)}) \in \mathcal{C}$ for $s \leq n - r$ by properties (I) and (II_B) of classes. For $s > n - r$, $H^{r+s}(K,L) \in \mathcal{C}$ and $H^{r+s+1}(K,L) \in \mathcal{C}$ by hypothesis, and again by the universal coefficient theorem, $H^{r+s}(K,L; Z_{(s)}) \in \mathcal{C}$ for $s > n - r$. This completes the proof.

Proof of 3.2: This proof is very similar to that of 3.1. Again we need to show that $H^{r+1}(K,L; Z_{(1)}) \in \mathcal{C}, \ldots, H^N(K,L; Z_{(N-r)}) \in \mathcal{C}$ for $r \geq n$. We use the above universal coefficient theorem. Since $H^{r+s}(K,L) \in \mathcal{C}$ for $s \geq 1$ and $Z_{(s)} \in \mathcal{C}$ for $s \leq N - n < N - (N - \alpha_{\mathcal{C}}) = \alpha_{\mathcal{C}}$, by conditions ($II_A$) and (I) on classes the result is proven.

$$
\begin{array}{ccccccccccc}
 & & \vdots & & & & \vdots & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
\cdots & \longrightarrow & {}'A^{r,N-1} & \xrightarrow{'i} & H^{N}(K,L;\ Z_{(N-r)}) & \longrightarrow & 0 & \longrightarrow \cdots & & & \\
 & & \downarrow {'j} & & & & \downarrow & & & & \\
\cdots & \longrightarrow & {}'A^{r,N-2} & \xrightarrow{'i} & H^{N-1}(K,L;\ Z_{(N-r-1)}) & \longrightarrow & 0 & \longrightarrow \cdots & & & \\
 & & \downarrow {'j} & & & & \downarrow & & & & \\
\cdots & \longrightarrow & {}'A^{r,N-3} & \xrightarrow{'i} & H^{N-2}(K,L;\ Z_{(N-r-2)}) & \xrightarrow{'\Delta} & {}'A^{r+1,N-1} & \longrightarrow \cdots & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & \vdots & & & & \vdots & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
\cdots & \longrightarrow & {}'A^{r,r+2} & \xrightarrow{'i} & H^{r+3}(K,L;\ Z_{(3)}) & \xrightarrow{'\Delta} & {}'A^{r+1,r+4} & \longrightarrow \cdots & & & \\
 & & \downarrow {'j} & & & & \downarrow {'j} & & & & \\
\cdots & \longrightarrow & {}'A^{r,r+1} & \xrightarrow{'i} & H^{r+2}(K,L;\ Z_{(2)}) & \xrightarrow{'\Delta} & {}'A^{r+1,r+3} & \longrightarrow \cdots & & & \\
 & & \downarrow {'j} & & & & \downarrow {'j} & & & & \\
\cdots & \longrightarrow & {}'A^{r,r} & \xrightarrow{'i} & H^{r+1}(K,L;\ Z_{(1)}) & \xrightarrow{'\Delta} & {}'A^{r+1,r+2} & \longrightarrow \cdots & & & \\
 & & \downarrow {'j} & & & & \downarrow {'j} & & & & \\
\cdots & \longrightarrow & \pi^{r}(K,L) & \xrightarrow{\eta^{r}} & H^{r}(K,L) & \xrightarrow{'\Delta} & {}'A^{r+1,r+1} & \xrightarrow{'i} & H^{r+2}(K,L;\ Z_{(1)}) & \longrightarrow \cdots & \\
 & & \downarrow & & & & \downarrow {'j} & & & & \\
 & & \vdots & & & \cdots \longrightarrow & \pi^{r+1}(K,L) & \xrightarrow{\eta^{r+1}} & H^{r+1}(K,L) & \longrightarrow \cdots & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & \vdots & & & &
\end{array}
$$

Figure 3.

6. Proof of Theorem 3.8.

The first part of 3.8 is just corollary 3.6. The first open problems are then to determine the kernel of η^{n-1} and the image of η^{n-2} from the cohomology structure of the pair (K, L). This is achieved by the exact sequence of 3.8. In order to prove 3.8, we first prove an unpublished result of Adem (see [15; p.263]).

These results lead us to the next problem, namely, to determine the kernel of η^{n-2} from the cohomology structure of the pair (K, L). This requires a study of secondary operations and partial results have been obtained by Adem[1].

Define a homomorphism $\lambda: Z_{(s)} \longrightarrow Z_{(s+1)}$ by $\lambda(\alpha) = \alpha\gamma$, where $\alpha \in Z_{(s)}$ and γ is the non-zero element of $Z_{(1)}$. Since $2\gamma = 0$ and composition is bilinear in the stable range, $\lambda(2Z_{(s)}) = 0$. Hence λ induces a homomorphism $\lambda': Z_{(s)}/2Z_{(s)} \longrightarrow Z_{(s+1)}$.

THEOREM 6.1: (Adem). <u>In the first derived cohomotopy exact couple,</u> $d = {}'1'\Delta: H^r(K, L; Z_{(s)}) \longrightarrow H^{r+2}(K, L; Z_{(s+1)})$ <u>is the Steenrod square,</u> Sq^2, <u>defined with respect to the homomorphism</u> λ'.

Proof: d is a cohomology operation (see section 4), and hence d corresponds to an element of $H^{r+2}(Z_{(s)}, r; Z_{(s+1)}) \approx \mathrm{Hom}(Z_{(s)}/2Z_{(s)}, Z_{(s+1)})$ (see [10; part II] for the calculation of $H^{r+2}(Z_{(s)}, r; Z_{(s+1)})$). We now show in a special case that d corresponds to the above homomorphism λ' under these isomorphisms, and then the theorem follows from theorem 17.4 of [10; part III].

Let $(K,L) = (M,x_0)$, where $M = S^r \cup e^{r+2}$, the cell e^{r+2} being attached to S^r by the non-zero element γ of $\pi_{r+1}(S^r)$. This is the space considered by Steenrod in [26]. Then obviously, $H^r(M,x_0; Z_{(s)}) \approx Z_{(s)}$ and $H^{r+2}(M,x_0; Z_{(s+1)}) \approx Z_{(s+1)}$, and hence this special case contains all possible cohomology operations of type $(r, r+2; Z_{(s)}, Z_{(s+1)})$. We are to show that the operation d above corresponds to the homomorphism λ'. Let $u \in H^r(M,x_0; Z_{(s)})$, and compute d(u) (see section 4 for the definition of $d = {}'i^{r+1,r+s+1}, \Delta^{r,r+s})$. Let $u' = [a] \in \pi^{r-s}(S^r,x_0) \approx C^r(M,x_0; Z_{(s)}) \approx Z_{(s)}$ be a representative cocycle of the cohomology class u. $a: (S^r,x_0) \longrightarrow (S^{r-s}, pt.)$. Extend a to a map $a': (M,S^r) \longrightarrow (E_+^{r-s+1}, S^{r-s})$, where E_+^{r-s+1} is the upper hemisphere of S^{r-s+1} and S^{r-s} is the equator of S^{r-s+1}. Let $h: (E_+^{r-s+1}, S^{r-s}) \longrightarrow (S^{r-s+1}, pt.)$ be the canonical map of degree 1 pulling S^{r-s} to the base point. Then $\Delta^{r,r+s}([a]) = [ha'] \in \pi^{r-s+1}(M,S^r) \approx C^{r+2}(M,x_0; Z_{(s+1)}) \approx Z_{(s+1)}$ by definition. The cohomology class of $[ha']$ is d(u) by definition ($[ha']$ is obviously a cocycle and there are no coboundaries). However, $[ha'] = S_\#([a]\gamma) \in Z_{(s+1)}$ (see [28]), and d corresponds to λ' as stated.

Theorem 3.8 now follows as a corollary to theorem 6.1.

<u>Proof of 3.8</u>: The proof is based on figure 4, a portion of the first derived cohomotopy exact couple.

$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \vdots \\
 & & & & \downarrow & & & & \\
 & & & \cdots \to & 0 & \to \cdots & & & \vdots \\
 & & & & \downarrow & & & & \downarrow \\
\pi^{n-2}(K,L) & \xrightarrow{\eta^{n-2}} & H^{n-2}(K,L) & \xrightarrow{{}'\triangle^{n-2,n-2}} & {}'A^{n-1,n-1} & \xrightarrow{{}'i^{n-1,n}} & H^{n}(K,L;\, Z_2) & \to & 0 \to \\
 & & & & \downarrow {}'j^{n-1,n-1} & & & & \downarrow \\
 & & & \cdots \to & \pi^{n-1}(K,L) & \xrightarrow{\eta^{n-1}} & H^{n-1}(K,L) & \to & 0 \to \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & \vdots & & & & \vdots
\end{array}
$$

Figure 4.

${}'i^{n-1,n}$: ${}'A^{n-1,n-1} \longrightarrow H^{n}(K,L;\, Z_2)$ is an isomorphism by exactness. Furthermore, ${}'i^{n-1,n}\,{}'\triangle^{n-2,n-2} = Sq^2$ by 6.1 because the Sq^2 defined with respect to the homomorphism $\lambda: Z_{(0)} \longrightarrow Z_{(1)}$ is the usual Sq^2: $H^{n-2}(K,L) \longrightarrow H^{n}(K,L;\, Z_2)$. The exact sequence of 3.8 follows immediately from figure 4 with Λ defined by $\Lambda = {}'j^{n-1,n-1}({}'i^{n-1,n})^{-1}$.

7. Proof of Theorem 3.9.

The first part of 3.9 is just corollary 3.7. The first open problems are than to determine the kernel of $\eta_{(p)}^{n-2p+3}$ and the image of $\eta_{(p)}^{n-2p+2}$ from the cohomology structure of the pair (K, L). This and more is achieved by the exact sequences of 3.9. In order to prove 3.9, we first prove a result, analogous to theorem 6.1, which introduces the reduced p^{th} powers into the first derived cohomotopy exact couple.

These results lead us to the next problem, namely, to determine the kernel of $\eta_{(p)}^{n-4p+5}$ from the cohomology structure of the pair (K, L). This requires a further study of secondary operations. Some results on secondary (and higher order) operations have been obtained by the author and will appear at a later date.

LEMMA 7.1: *In the first derived cohomotopy exact couple, ${}'A^{r+1,r+s}$ is a torsion group for $s \geq 1$. Furthermore, $('j)_{(p)}: ({}'A^{r+1,r+s})_p \longrightarrow ({}'A^{r+1,r+s-1})_p$ is an isomorphism for $2 \leq s < 2p - 2$.*

Proof: The following exact sequence is part of the first derived cohomotopy exact couple: $H^{r+s-1}(K, L; Z_{(s-1)}) \xrightarrow{'\Delta} {}'A^{r+1,r+s} \xrightarrow{'j} {}'A^{r+1,r+s-1} \xrightarrow{'i} H^{r+s}(K, L; Z_{(s-1)})$. By 2.1 (a), $Z_{(s-1)} \in \mathcal{C}_T$ for $s \geq 2$, hence $H^{r+s-1}(K, L; Z_{(s-1)}) \in \mathcal{C}_T$ for $s \geq 2$. Also, ${}'A^{r+1,N} = 0 \in \mathcal{C}_T$, therefore by induction, ${}'A^{r+1,r+s-1} \in \mathcal{C}_T$ for $s \geq 2$. Similarily, $Z_{(s-1)} \in \mathcal{C}_p$ for $2 \leq s < 2p - 2$ by 2.1 (b), hence

$H^{r+s-1}(K,L;\ Z_{(s-1)}) \in \mathcal{C}_p$ and $H^{r+s}(K,L;\ Z_{(s-1)}) \in \mathcal{C}_p$ for $2 \leq s < 2p - 2$. Therefore $'j_{(p)}: ({'A}^{r+1,r+s})_p \longrightarrow ({'A}^{r+1,r+s-1})_p$ is an isomorphism for $2 \leq s < 2p - 2$. This completes the proof.

Let $r > (N + 1)/2$. Refering to figure 5, we define $d: H^r(K,L) \longrightarrow H^{r+2p-2}(K,L;\ Z_{(2p-3)})_p = H^{r+2p-2}(K,L;\ Z_p)$ by $d(u) = {'i}_{(p)}^{r+1,r+2p-2}({'j}_{(p)}^{r+1,r+2p-3})^{-1} \cdots ({'j}_{(p)}^{r+1,r+2})^{-1} P\, \Delta^{r,r}(u)$, where $P: {'A}^{r+1,r+1} \longrightarrow ({'A}^{r+1,r+1})_p$ is the natural projection onto the p-primary component (P is naturally defined because ${'A}^{r+1,r+1}$ is a torsion group [14; p.5]).

THEOREM 7.2: <u>In the first derived cohomotopy exact couple, the homomorphism</u> $d: H^r(K,L) \longrightarrow H^{r+2p-2}(K,L;\ Z_p)$ <u>is a multiple</u> β <u>of</u> $\mathcal{P}'$, <u>where</u> $\beta \not\equiv 0 \pmod p$.

<u>Proof</u>: Refer to figure 5. Since all the homomorphisms in the definition of d are natural with respect to cellular maps $f: (K,L) \longrightarrow (K',L')$, d is also natural with respect to such maps and hence with respect to all maps (as in section 4). Thus d is a cohomology operation and d corresponds to an element of $H^{r+2p-2}(Z,r;\ Z_p) \approx Z_p$ (see the calculations of Cartan [7]). Therefore $d = \beta\mathcal{P}'$, and it suffices to exhibit a complex K for which $d \neq 0$ for then $\beta \not\equiv 0 \pmod p$.

Let $(K,L) = (M,x_0)$, where $M = S^r \cup e^{r+2p-2}$, the cell e^{r+2p-2} being attached to S^r by a non-zero element of $\pi_{r+2p-3}(S^r)_p = Z_p$. M has the property that $\mathcal{P}'(u) \neq 0$, where u is a generator of $H^r(M,x_0)$ (this is a result of

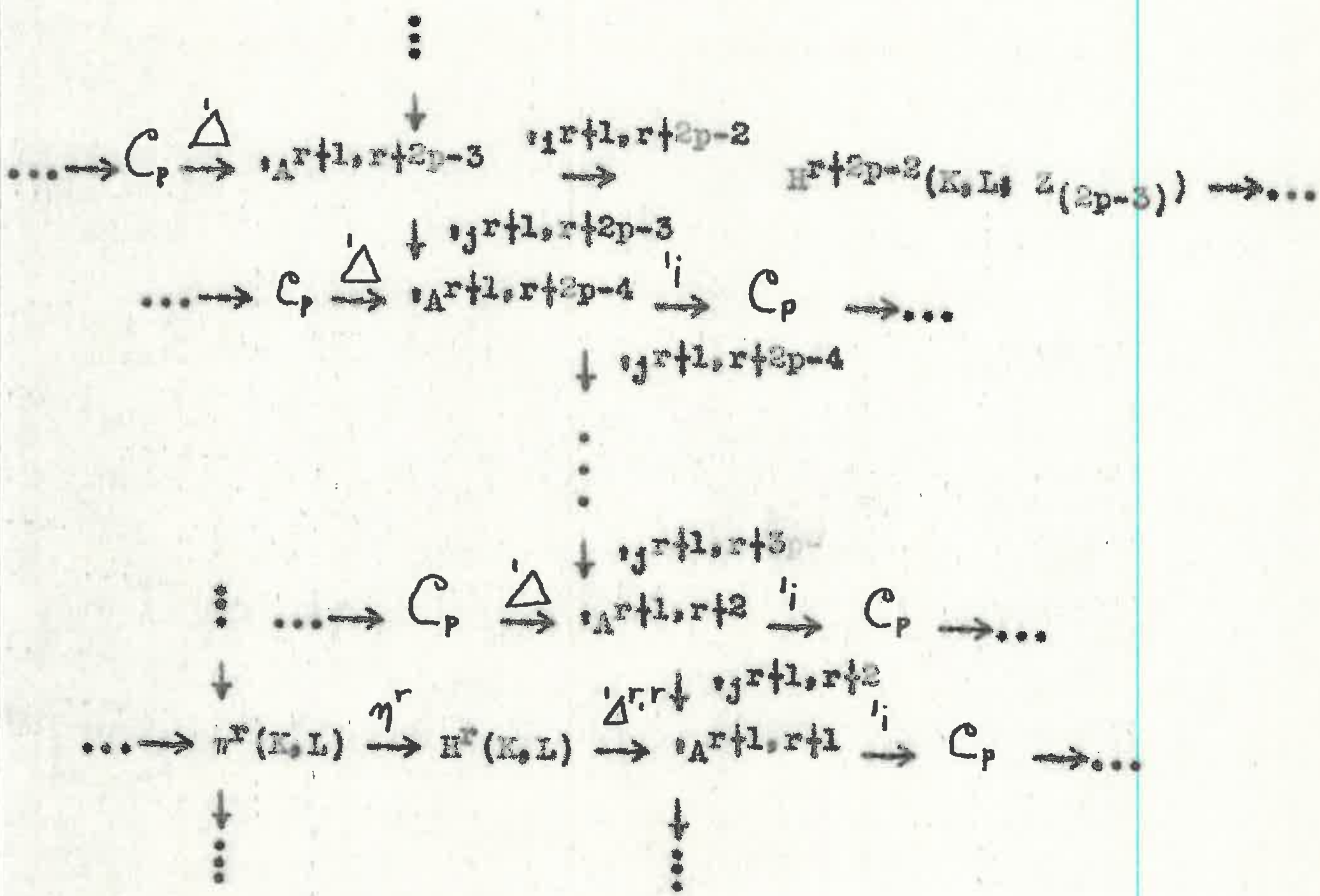
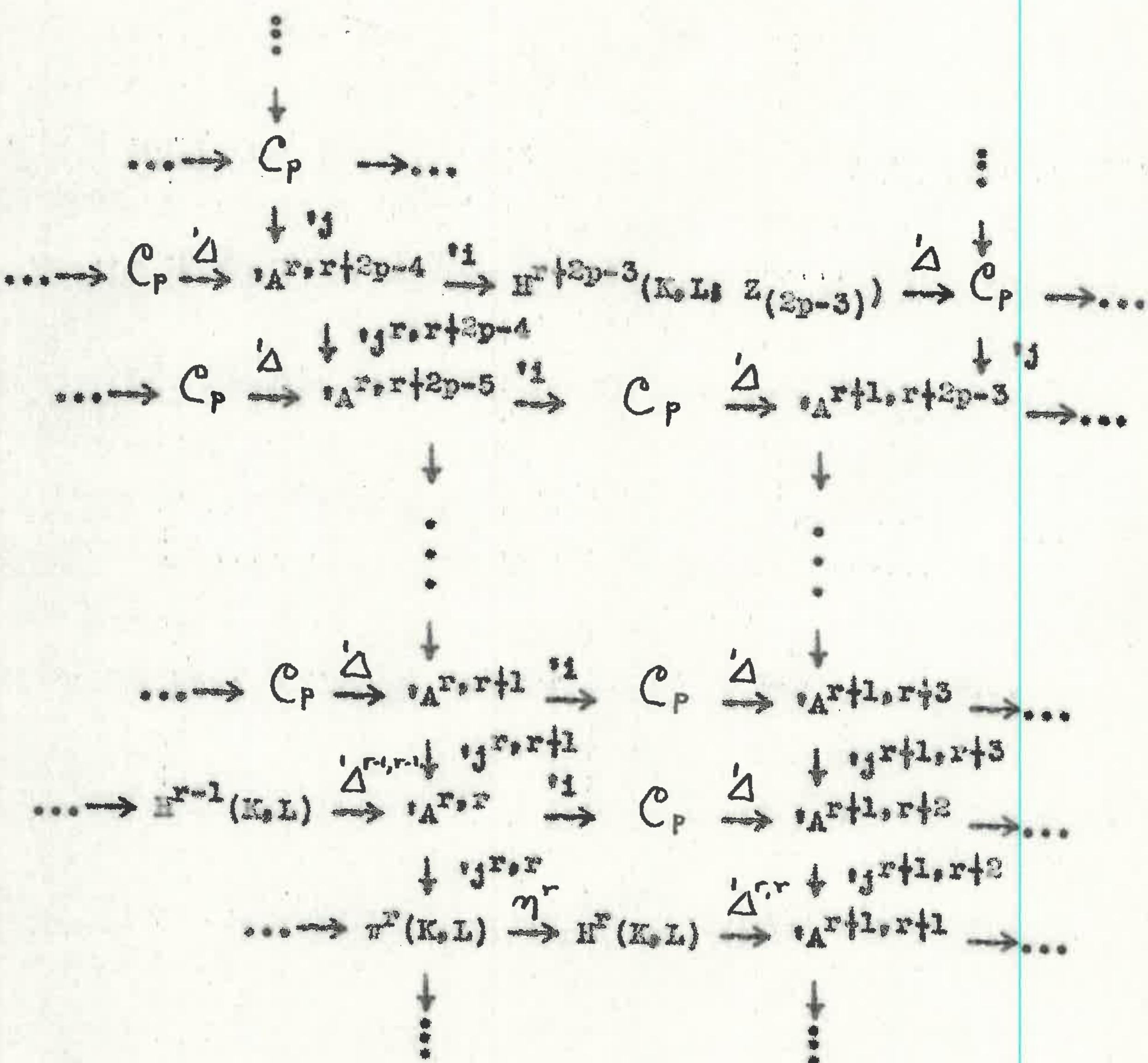


Figure 5.

Borel and Serre [5; p.425]). Assume $d = 0$ for this complex. Since $H^s(M,x_0; G) = 0$ unless $s = r$ or $s = r + 2p - 2$, ${}'A^{r+1,r+1} \approx H^{r+2p-2}(M,x_0; Z_{(2p-3)})$, and $d(u) = p\,{}'\Delta^{r,r}(u)$ for $u \in H^r(M,x_0)$. Let u generate $H^r(M,x_0) \approx Z$. ${}'\Delta^{r,r}(u)$ is an element of finite order, hence there is an integer $D \equiv 1 \pmod p$ such that $p\,{}'\Delta^{r,r}(u) = D\,{}'\Delta^{r,r}(u) = {}'\Delta^{r,r}(Du)$. Since $d = 0$, ${}'\Delta^{r,r}(Du) = 0$, and by exactness, $Du = \eta^r([a])$, where $a: (M,x_0) \longrightarrow (S^r,pt.)$. $a|S^r: (S^r,x_0) \longrightarrow (S^r,pt.)$ is a map of degree D, and hence $a^*(u') = Du$, where u' is a generator of $H^r(S^r,pt.)$. Thus $0 \neq D\mathcal{P}'(u) = \mathcal{P}'(Du) = \mathcal{P}'(a^*(u')) = a^*(\mathcal{P}'(u')) = 0$ because $D \equiv 1 \pmod p$ and $\mathcal{P}'(u') = 0$ in S^r. This is a contradiction, and hence $d \neq 0$. This completes the proof.

Using a more computational proof similar to the proof of theorem 6.1, it can be shown that $\beta \equiv 1 \pmod p$. However, to prove theorem 3.9, it is not necessary to know $\beta \equiv 1 \pmod p$ because $\mathrm{Im}\,(\beta\mathcal{P}') = \mathrm{Im}\,\mathcal{P}'$ and $\mathrm{Ker}\,(\beta\mathcal{P}') = \mathrm{Ker}\,\mathcal{P}'$ as long as $\beta \not\equiv 0 \pmod p$. Using this remark, we now prove 3.9.

Proof of 3.9: The proof is based on figure 6, a portion of the first derived cohomotopy exact couple. The groups marked $\mathcal{C}_p$ are in that class by arguments given in 7.1. By hypothesis, $r + 4p - 5 > n$, and hence $H^{r+4p-5}(K,L; Z_{(4p-5)}) \in \mathcal{C}_p$. It follows that ${}'A^{r,r+2p-3} \in \mathcal{C}_p$ and ${}'A^{r+1,r+2p-2} \in \mathcal{C}_p$ as in figure 6. Hence $({}'A^{r,r})_p \approx H^{r+2p-3}(K,L; Z_{(2p-3)})_p = H^{r+2p-3}(K,L; Z_p)$ under the isomorphism



... → C_p → ...
'j
... → C_p 'Δ → 'A^{r,r+2p-4} 'i → H^{r+2p-3}(K,L; Z_(2p-3)) 'Δ → C_p → ...
'j^{r,r+2p-4} 'j
... → C_p 'Δ → 'A^{r,r+2p-5} 'i → C_p 'Δ → 'A^{r+1,r+2p-3} → ...
... → C_p 'Δ → 'A^{r,r+1} 'i → C_p 'Δ → 'A^{r+1,r+3} → ...
'j^{r,r+1} 'j^{r+1,r+3}
... → H^{r-1}(K,L) 'Δ^{r-1,r-1} → 'A^{r,r} 'i → C_p 'Δ → 'A^{r+1,r+2} → ...
'j^{r,r} 'j^{r+1,r+2}
... → π^r(K,L) η^r → H^r(K,L) 'Δ^{r,r} → 'A^{r+1,r+1} → ...

Figure 6.

$'i^{r,r+2p-3}_{(p)}('j^{r,r+2p-4}_{(p)})^{-1}\dots('j^{r,r}_{(p)})^{-1}$. Furthermore,
$\text{Im}\,(p'\Delta^{r-1,r-1}) = (\text{Im}\,'\Delta^{r-1,r-1}) \cap ('A^{r,r})_p$. Similar remarks for $'\Delta^{r,r}$ and $'A^{r+1,r+1}$ hold. From figure 6 we obtain the exact sequence

$$H^{r-1}(K,L) \xrightarrow{p'\Delta^{r-1,r-1}} ('A^{r,r})_p \xrightarrow{'j^{r,r}_{(p)}} \pi^r(K,L)_p \xrightarrow{\eta^r_{(p)}} H^r(K,L)_p \xrightarrow{'\Delta^{r,r}_{(p)}} ('A^{r+1,r+1})_p.$$

Using theorem 7.2 and the above isomorphisms, we obtain the exact sequence of 3.9 with μ defined by $\mu = 'j^{r,r}_{(p)} \dots 'j^{r,r+2p-4}_{(p)}('i^{r,r+2p-3}_{(p)})^{-1}$:
$H^{r+2p-3}(K,L; Z_p) \longrightarrow \pi^r(K,L)_p$. For $r = n - 4p + 5$, we have only the statement on the cokernel of $\eta^r_{(p)}$.

PART II.

8. Cohomotopy Groups with Coefficients in G.

In this section we define our generalized cohomotopy groups and state their elementary properties. In making the definition certain arbitrary choices are necessary, and we prove independence of these choices under certain restrictions on G. We also state our main results on generalized cohomotopy groups; the proofs are given in the following five sections.

Let (K,L) be a CW-pair with dimension K = N (we shall assume this for the remainder of this paper). Let X be an (n - 1)-connected space [3; p.183], and let $x_0 \varepsilon X$. Let $\pi(K,L; X,x_0)$ denote the set of homotopy classes of maps a: $(K,L) \longrightarrow (X,x_0)$. By virtue of the results of Spanier and Whitehead [24], if $n > (N + 1)/2$, then $\pi(K,L; X,x_0)$ has a natural abelian group structure (see section 9). If $n > (N' + 1)/2$ where N' = dimension K', then a map f: $(K,L) \longrightarrow (K',L')$ induces a homomorphism $f^{\#}: \pi(K',L'; X,x_0) \longrightarrow \pi(K,L; X,x_0)$ defined by $f^{\#}([a]) = [af]$. Also if $n' > (N + 1)/2$ where X' is an (n' - 1)-connected space, then a map $\phi: (X,x_0) \longrightarrow (X',x_0')$ induces a homomorphism $\phi_{\#}: \pi(K,L; X,x_0) \longrightarrow \pi(K,L; X',x_0')$ defined by $\phi_{\#}([a]) = [\phi a]$. In the sequel, when we say a structure is natural with respect to maps f: $(K,L) \longrightarrow (K',L')$ and maps $\phi: (X,x_0) \longrightarrow (X',x_0')$, we assume (K',L') and (X',x_0') satisfy the same dimensional or connectedness assumptions that (K, L) and (X, x_0) satisfy.

If $X \neq \emptyset$, let $'SX$ denote the reduced suspension of X [23; p.656]; namely, $'SX$ is the space obtained from $X \times I$ by identifying $X \times \{0\} \cup X \times \{1\} \cup x_0 \times I$ to a point x_0. (x_0 is used to denote the base point of both X and $'SX$.) Also define $'S^r X = 'S('S^{r-1}X)$ and $'S^0 X = X$. If $X = \emptyset$, define $'S\emptyset$ to be a pair of points.

Let G be an abelian group and let $n > 1$. An $(n + 1)$-dimensional CW-complex X is said to be an <u>$X(G,n)$-space</u> if $\pi_1(X) = 0$, $H_i(X) = 0$ for $i \neq n$, and $H_n(X) = G$. (This concept was introduced by Moore [18; p.550].) Note that if X is an $X(G,n)$-space, then $'SX$ is an $X(G,n+1)$-space.

In section 10 we prove

LEMMA 8.1: <u>For given</u> G <u>and</u> n, <u>there exists an</u> $X(G,n)$-<u>space</u>.

Let X be an $X(G,n)$-space. We define the <u>n^{th} cohomotopy group of</u> (K,L) <u>with coefficients in</u> G to be $\pi(K,L; X,x_0)$, and denote it by $\pi^n(K,L; G)$. When we use the notation $\pi^n(K,L; G)$, we assume $n > (N + 1)/2$ and thus $\pi^n(K,L; G)$ has a natural group structure. As defined, $\pi^n(K,L; G)$ depends on the choice of $X(G,n)$-space. We show below that $\pi^n(K,L; G)$ is naturally independent of this choice when G has no elements of order 2. However, if we do not change coefficients during a discussion, it suffices to choose a fixed $X(G,t)$-space Y and use $'S^{n-t}Y$ as the $X(G,n)$-space for each $n \geq t \geq 2$. We do this in the following paragraph.

Define $\Delta: \pi^n(L; G) \longrightarrow \pi^{n+1}(K,L; G)$ as follows: let $b \in [b] \in \pi^n(L; G)$. Extend b to a map $b': (K,L) \longrightarrow (CX,X)$, where CX denotes the cone on X [23; p.656] and $X = {}'S^{n-t}Y$. Let $h: (CX,X) \longrightarrow ({}'SX,x_0)$ be the canonical map [23; p. 657]. The composition $hb': (K,L) \longrightarrow ({}'SX,x_0) = ({}'S^{n-t+1}Y,x_0)$ represents $\Delta([b])$. (This homomorphism is strictly analogous to the homomorphism Δ of section 2 for ordinary cohomotopy; see [22; p.216].) Let $i: L \longrightarrow K$ and $j: K \longrightarrow (K,L)$ be inclusions. Then the cohomotopy sequence of the pair (K,L) is defined to be the following sequence of groups and homomorphisms:

$$\cdots \longrightarrow \pi^n(K,L; G) \xrightarrow{j^\#} \pi^n(K; G) \xrightarrow{i^\#} \pi^n(L; G) \xrightarrow{\Delta} \pi^{n+1}(K,L; G)$$

$\longrightarrow \cdots$. In section 9 we prove that this sequence is exact for $n > (N + 1)/2$. In section 10 we prove

THEOREM 8.2: $\pi^n(K,L; G)$ *satisfies all the axioms for cohomology of Eilenberg and Steenrod* [11; p.13] *in those dimensions where a natural group structure is defined.*

We now return to the question of independence of the choice of the $X(G,n)$-space. Let X be an $X(G,n)$-space, Y an $X(H,n)$-space. There is a natural homomorphism η: $\pi(X,x_0; Y,y_0) \longrightarrow \mathrm{Hom}(G,H)$ defined by $\eta([a]) = a_*: H_n(X,x_0) = G \longrightarrow H_n(Y,y_0) = H$. In section 10 we prove

THEOREM 8.3: *If* $n \geq 3$, *then* $\eta: \pi(X,x_0; Y,y_0) \longrightarrow \mathrm{Hom}(G,H)$ *is an epimorphism and has a kernel isomorphic to* $\mathrm{Ext}(G, H \otimes Z_2)$ (see [8] or [11; p.161] for the definition and properties of Ext).

Let $\mathcal{D}$ be the family of abelian groups having no elements of order 2. In section 4 of the appendix we prove

LEMMA 8.4: <u>If</u> $G \varepsilon \mathcal{D}$, <u>then</u> $Ext(G, H \otimes Z_2) = 0$.

Let ϕ: $G \longrightarrow H$ be a homomorphism, and let X be an $X(G,n)$-space, Y an $X(H,n)$-space. There exists a map $\underline{\phi}$: $(X, x_0) \longrightarrow (Y, y_0)$ such that $\underline{\phi}_* = \phi$: $H_n(X, x_0) \longrightarrow H_n(Y, y_0)$ by 8.3. In fact, if $G \varepsilon \mathcal{D}$, then $\underline{\phi}$ is unique up to homotopy by 8.3 and 8.4. Furthermore, if ψ: $H \longrightarrow J$, Z is an $X(J,n)$-space, and $H \varepsilon \mathcal{D}$ also, then $\underline{\psi}\,\underline{\phi} \simeq (\underline{\psi\phi})$: $(X, x_0) \longrightarrow (Z, z_0)$.

Now let X and X' be two different $X(G,n)$-spaces, let $G \varepsilon \mathcal{D}$, and let ϕ: $G \longrightarrow G$ be the identity homomorphism. Then there exist maps $\underline{\phi}$: $(X, x_0) \longrightarrow (X', x_0')$ and $\underline{\phi}'$: $(X', x_0') \longrightarrow (X, x_0)$ inducing ϕ such that $\underline{\phi}'\underline{\phi}$ and $\underline{\phi}\underline{\phi}'$ are homotopic to the identity maps. Furthermore, $\underline{\phi}$ and $\underline{\phi}'$ are unique up to homotopy. Hence $\underline{\phi}$ and $\underline{\phi}'$ induce unique isomorphisms $\underline{\phi}_{\#}$: $\pi(K, L; X, x_0) \longrightarrow \pi(K, L; X', x_0')$ and $\underline{\phi}'_{\#}$: $\pi(K, L; X', x_0') \longrightarrow \pi(K, L; X, x_0)$ which are inverses of each other. Hence the set of groups $\{\pi(K, L; X, x_0)\}$ for all $X(G,n)$-spaces X form a transitive system of groups [11; p.17], and we have shown that $\pi^n(K, L; G)$ is independent of the choice of $X(G,n)$-space. In case $G \notin \mathcal{D}$, $\underline{\phi}_{\#}$ is an isomorphism (see lemma 9.4), but it is not a unique isomorphism. In this case, we assume a fixed $X(G,2)$-space during any given discussion.

As a further corollary of the above discussion, if $G \varepsilon \mathcal{D}$, then a homomorphism ϕ: $G \longrightarrow H$ induces a unique homomorphism $\phi_{\#}$: $\pi^n(K, L; G) \longrightarrow \pi^n(K, L; H)$. This is natural in the sense that if $\phi: G \longrightarrow H$, $\psi: H \longrightarrow J$ and G and $H \in \mathcal{D}$, then $(\psi\phi)_{\#} = \psi_{\#}\phi_{\#}$.

We now state our main results on cohomotopy groups with coefficients in G. Let G' and G $\in \mathcal{D}$, and let

$0 \longrightarrow G' \xrightarrow{\phi} G \xrightarrow{\psi} G'' \longrightarrow 0$ be an exact coefficient sequence. In section 11 we define a homomorphism $\delta_{\#}: \pi^r(K,L; G'') \longrightarrow \pi^{r+1}(K,L; G')$ for $r \geq z$, where z is the least integer > $(N+1)/2$ as in section 4. Define the sequence corresponding to this exact coefficient sequence to be the following sequence of groups and homomorphisms:

$\ldots \longrightarrow \pi^r(K,L; G') \xrightarrow{\phi_{\#}} \pi^r(K,L; G) \xrightarrow{\psi_{\#}} \pi^r(K,L; G'') \xrightarrow{\delta_{\#}} \pi^{r+1}(K,L; G') \longrightarrow \ldots$. In section 11 we prove

THEOREM 8.5: For $r \geq z$, the sequence corresponding to an exact coefficient sequence is exact. This sequence is natural with respect to maps $f: (K,L) \longrightarrow (K',L')$, and if G', G, G'', H', and H $\in \mathcal{D}$, then it is natural with respect to a homomorphism of one exact coefficient sequence into another:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G' & \xrightarrow{\phi} & G & \xrightarrow{\psi} & G'' & \longrightarrow & 0 \\ & & \downarrow \xi' & & \downarrow \xi & & \downarrow \xi'' & & \\ 0 & \longrightarrow & H' & \xrightarrow{\phi'} & H & \xrightarrow{\psi'} & H'' & \longrightarrow & 0. \end{array}$$

The above is a generalization of an exact sequence of Moore [18; p.552].

Define a function $\pi^r(K,L) \times G \longrightarrow \pi^r(K,L; G)$ by $([a], [g]) \longrightarrow [ga]$, where $[g] \in \pi_r(X,x_0) = G$, $[a] \in \pi^r(K,L)$, and X is an $X(G,r)$-space. By 9.3, this is a bilinear function for $r \geq z$, and hence it induces a homomorphism $\alpha: \pi^r(K,L) \otimes G \longrightarrow \pi^r(K,L; G)$. In section 12 we define a homomorphism $\beta: \pi^r(K,L; G) \longrightarrow \mathrm{Tor}(\pi^{r+1}(K,L),G)$ and prove the following universal coefficient theorems:

THEOREM 8.6: *Let G be a finitely generated abelian group. Then the sequence*

$$(*)\quad 0 \longrightarrow \pi^r(K,L)\otimes G \xrightarrow{\alpha} \pi^r(K,L; G) \xrightarrow{\beta} \mathrm{Tor}(\pi^{r+1}(K,L),G) \longrightarrow 0$$

is exact for $r \geq z$. This sequence is natural with respect to maps $f: (K,L) \longrightarrow (K',L')$, and if $G \in \mathcal{D}$, then it is natural with respect to homomorphisms $\phi: G \longrightarrow H$. Furthermore, if (K,L) is a finite CW-pair and $G \in \mathcal{D}$, then the exact sequence $(*)$ splits (i. e. $\mathrm{Im}\ \alpha$ is a direct summand of $\pi^r(K,L; G)$).

THEOREM 8.7: *Let $G \in \mathcal{D}$, and let (K,L) be a finite CW-pair. Then the sequence $(*)$ is exact for $r \geq z$. This sequence is is natural with respect to maps $f: (K,L) \longrightarrow (K',L')$ and homomorphisms $\phi: G \longrightarrow H$.*

In section 12 we give an example to show that if G is not finitely generated and (K,L) is not finite, then the sequence $(*)$ is not necessarily exact.

Theorem 8.6 generalizes an exact sequence of Serre [20; p.284]; Serre's sequence is 8.6 for the case $(K,L) = (S^n, pt.)$

and $G = $ a cyclic group. He notes that his sequence does not split when $G = Z_2$.

Theorems 8.6 and 8.7, beside giving a further analogy between cohomotopy groups with arbitrary coefficients and cohomology groups with arbitrary coefficients, reduce the problem of calculating $\pi^r(K,L; G)$ to the standard problem of calculating $\pi^r(K,L)$. Because of this, theorems 8.6 and 8.7 are two of our main results.

Let X be an $K(G,n)$-space. As in the case of homotopy groups of spheres, $\pi_{n+s}(X)$ is independent of n when $s < n - 1$ [18; p.551]. We denote this group by $G_{(s)}$ (see section 2). For any class $\mathcal{C}$, let $\alpha_{\mathcal{C}}(G)$ denote the largest integer such that $G_{(s)} \varepsilon \mathcal{C}$ for $0 < s < \alpha_{\mathcal{C}}(G)$. In section 9 we define a natural homomorphism $\eta^r: \pi^r(K,L; G) \longrightarrow H^r(K,L; G)$ analogous to η^r for ordinary cohomotopy. In section 13 we prove

THEOREM 8.8: <u>Let $\mathcal{C}$ be a class satisfying condition</u> (II_B) <u>of section 2, and let G be finitely generated. Let</u> $n >$ $(N + 1)/2$ <u>be such that</u> $H^r(K,L; G) \varepsilon \mathcal{C}$ <u>for every</u> $r > n$. <u>Then</u> $\eta^r: \pi^r(K,L; G) \longrightarrow H^r(K,L; G)$ <u>is a</u> $\mathcal{C}$<u>-isomorphism if</u> $r > \text{Max}((N + 1)/2, n - \alpha_{\mathcal{C}}(G))$, <u>and is a</u> $\mathcal{C}$<u>-epimorphism for</u> $r = n - \alpha_{\mathcal{C}}(G)$ <u>in case</u> $n - \alpha_{\mathcal{C}}(G) > (N + 1)/2$.

Theorems 2.1 and 8.6 give information on $\alpha_{\mathcal{C}}(G)$ for various $\mathcal{C}$ and G, and we may draw consequences of 8.8 similar to 3.2, 3.3, 3.4, and 3.5. Furthermore, the result of Adem (theorem 6.1) carries over to general coefficients and there is a theorem analogous to 3.8. Also, for the case $G = Z_p$, there is a theorem analogous to 3.9. Rather than considering these in detail, let us note that any result on the structure of $\pi^r(K,L)$ gives a result on the structure of $\pi^r(K,L; G)$ by 8.6 or 8.7.

9. Group Structure and the Exact Sequence of a Pair.

In this section, we review some known results on the existence of a group structure on $\pi(K,L; X,x_0)$. We then prove the exactness of the cohomotopy sequence of a pair and thus show that the cohomotopy exact couple of part I can be generalized; some of our main results are based on this generalized cohomotopy exact couple.

If $X \neq \emptyset$, let SX denote the suspension of X [24]; namely, SX is the space obtained from $X \times I$ by identifying $X \times \{0\}$ and $X \times \{1\}$ to points. Also define $S^r X = S(S^{r-1}X)$ and $S^0 X = X$. If $X = \emptyset$, define $S\emptyset$ to be a pair of points. (Note the difference between SX and 'SX defined in section 8.) As in [24], suspension induces a function $S_{\#}: \pi(K,L; X,x_0) \longrightarrow \pi(SK,SL; SX,Sx_0)$. (Since Sx_0 is contractible to its midpoint, $x_0 \times \{\frac{1}{2}\}$, which we again denote by x_0, $\pi(SK,SL; SX,Sx_0) \approx \pi(SK,SL; SX,x_0)$. We shall do this throughout in order to preserve the base point.) $S_{\#}$ is natural with respect to maps $f: (K,L) \longrightarrow (K',L')$ and maps $\phi: (X,x_0) \longrightarrow (X',x_0')$.

THEOREM 9.1: *If X is an (n - 1)-connected space and (K,L) is a CW-pair with dimension $K \leq 2n - 2$, then $S_{\#}$ is a 1-1 correspondence.*

Proof: This is an immediate consequence of corollary 7.2 of [24].

Let $(X,x_0)^{K,L}$ denote the function space of maps a: $(K,L) \longrightarrow (X,x_0)$ with the compact-open topology. The constant map at x_0 serves as the base point for this function space. In [2], a function $\lambda: \pi_r((X,x_0)^{K,L}) \longrightarrow \pi(S^rK, S^rL; X,x_0)$ is defined and the following theorem is proven [2; p.81]:

THEOREM 9.2: *λ is a 1-1 correspondence. This correspondence is natural with respect to maps $f: (K,L) \longrightarrow (K',L')$ and maps $\phi: (X,x_0) \longrightarrow (X',x_0')$.*

As in [24], we now prove the main result on the existence of a group structure on $\pi(K,L; X,x_0)$.

THEOREM 9.3: *If X is an (n - 1)-connected space and (K,L) is a CW-pair with dimension $K \leq 2n - 2$, then $\pi(K,L; X,x_0)$ is an abelian group. This group structure is natural with respect to maps $f: (K,L) \longrightarrow (K',L')$ and maps $\phi: (X,x_0) \longrightarrow (X',x_0')$.*

Proof: By 9.1, $S_\#^2: \pi(K,L; X,x_0) \longrightarrow \pi(S^2K, S^2L; S^2X,x_0)$ is a natural 1-1 correspondence. Also, by 9.2, $\lambda: \pi_2((S^2X,x_0)^{K,L}) \longrightarrow \pi(S^2K,S^2L; S^2X,x_0)$ is a natural 1-1 correspondence, and $\pi_2((S^2X,x_0)^{K,L})$ is an abelian group. We define the group structure on $\pi(K,L; X,x_0)$ using the 1-1 correspondence $\lambda^{-1}S_\#^2$. (This addition of homotopy classes is analogous to that defined in [22].) This group structure is natural with respect to maps $f: (K,L) \longrightarrow (K',L')$ and maps $\phi: (X,x_0) \longrightarrow (X',x_0')$ because the 1-1 correspondences λ and $S_\#$ are; i. e. these maps induce homomorphisms $f^\#$ and $\phi_\#$.

For use later, we now prove

LEMMA 9.4: *Let* (K,L) *be a CW-pair with dimension* $K \leq r - 2$. *Let* $k: (X,x_0) \longrightarrow (X',x_0')$ *be such that* $k_{\#}: \pi_s(X,x_0) \longrightarrow \pi_s(X',x_0')$ *is an isomorphism for* $s < r - 1$ *and an epimorphism for* $s = r - 1$. *Also assume* $\pi_1(X) = 0$. *Then* $k_{\#}: \pi(K,L; X,x_0) \longrightarrow \pi(K,L; X',x_0')$ *is a 1-1 correspondence.*

Proof: We may assume that $k: (X,x_0) \longrightarrow (X',x_0')$ is an inclusion by the mapping cylinder construction. The lemma is an easy consequence of the deformation obstruction theory as described in [3; p.186] because the pair (X',X) is $(r - 1)$-connected [3; p.183]. The details of the proof are left to the reader.

Let $\mu: (SX,x_0) \longrightarrow ('SX,x_0)$ be the canonical map identifying $x_0 \times I$ to the point x_0. If X is connected, then $\pi_1(SX) = 0$ and μ satisfies the hypotheses of lemma 9.4 with $r = \infty$. Hence $\mu_{\#}: \pi(K,L; S^tX,x_0) \longrightarrow \pi(K,L; 'S^tX,x_0)$ is a 1-1 correspondence for $t \geq 0$.

Let X be an $(n - 1)$-connected space. Let dimension $K = N \leq 2(s - r + n) - 2$ and let $s \geq r$. We define θ: $\pi_r(('S^sX,x_0)^{K,L}) \longrightarrow \pi(K,L; 'S^{s-r}X,x_0)$ to be the following composition:

$$\pi_r(('S^sX,x_0)^{K,L}) \xrightarrow{\lambda} \pi(S^rK,S^rL; 'S^sX,x_0) \xrightarrow{\mu_{\#}^{-1}} \pi(S^rK,S^rL; S^sX,x_0) \xrightarrow{(S_{\#}^{-1})^r} \pi(K,L; S^{s-r}X,x_0) \xrightarrow{\mu_{\#}} \pi(K,L; 'S^{s-r}X,x_0).$$

λ is the

isomorphism of 9.2, $S_{\#}^{-1}$ is the isomorphism of 9.1 ($S_{\#}$ is an isomorphism under the above restrictions on s, r, n, and N), and $\mu_{\#}$ is the above isomorphism. Furthermore, λ, $\mu_{\#}$, and $S_{\#}$ are natural with respect to maps f: $(K,L) \longrightarrow (K',L')$ and maps ϕ: $(X,x_0) \longrightarrow (X',x_0')$, and thus the isomorphism θ is also natural with respect to such maps.

Before considering the cohomotopy sequence of a pair, we prove the following lemma. Let (K,L) be a CW-pair with $L \neq \emptyset$. Let p: $X^K \longrightarrow X^L$ be defined by restriction; i. e. $p(a) = a|L$ for $a \in X^K$.

LEMMA 9.5: p: $X^K \longrightarrow X^L$ <u>is a fibre space in the sense of Serre</u> [19; p.443], <u>and the fibre over the constant map at</u> x_0 <u>is</u> $(X,x_0)^{K,L}$. <u>A map</u> f: $(K,L) \longrightarrow (K',L')$ <u>induces a fibre preserving map</u> $\tilde{f}$: $(X^{K'},p',X^{L'}) \longrightarrow (X^K,p,X^L)$, <u>and a map</u> ϕ: $(X,x_0) \longrightarrow (X',x_0')$ <u>induces a fibre preserving map</u> $\tilde{\phi}$: $(X^K,p,X^L) \longrightarrow (X'^K,p',X'^L)$.

<u>Proof</u>: The fact that p: $X^K \longrightarrow X^L$ is a fibre space is essentially a restatement of the homotopy extension theorem for CW-complexes [12; p.97]. The details of the proof are straightforward and are left to the reader.

Using lemma 9.5 we now give a proof of the exactness of the cohomotopy sequence of the pair (K,L) (see section 8). This exact sequence is important because it allows us to obtain the generalized cohomotopy exact couple below. Let X be an X(G,n)-space, then $'S^sX$ is an X(G,n+s)-space.

By 9.5, $p\colon {}'S^sX^K \longrightarrow {}'S^sX^L$ is a fibre space with fibre $({}'S^sX, x_0)^{K,L}$. Consider the exact homotopy sequence of this fibre space (the base point is the constant map at x_0) and use the isomorphism θ defined above:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_r(({}'S^sX,x_0)^{K,L}) & \xrightarrow{{}'j_\#} & \pi_r({}'S^sX^K) & \xrightarrow{p_\#} & \pi_r({}'S^sX^L) & \xrightarrow{\partial} & \pi_{r-1}(({}'S^sX,x_0)^{K,L}) & \longrightarrow \cdots \\
 & \downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \downarrow & \\
\cdots \longrightarrow & \pi(K,L\,;\ {}'S^{s-r}X,x_0) & \xrightarrow{j^\#} & \pi(K;\ {}'S^{s-r}X) & \xrightarrow{i^\#} & \pi(L;\ {}'S^{s-r}X) & \xrightarrow{\Delta} & \pi(K,L;\ {}'S^{s-r+1}X,x_0) & \longrightarrow \cdots .
\end{array}$$

∂ is the boundary operator in the homotopy sequence of the fibre space [19; p.444], ${}'j_\#$ is induced by the inclusion ${}'j\colon ({}'S^sX,x_0)^{K,L} \longrightarrow {}'S^sX^K$, and $p_\#$ is induced by p. θ is defined and is an isomorphism when $N \leq 2(s - r + n) - 2$. Furthermore, $j^\#\theta = \theta\,{}'j_\#$ and $i^\#\theta = \theta p_\#$ by the naturality of θ, and it can be checked that $\Delta\theta = \theta\partial$. Since ${}'S^{s-r}X$ is an $X(G,n+s-r)$-space, this proves that the cohomotopy sequence of the pair (K,L) is exact in dimensions $> (N + 1)/2$. Naturality of this exact sequence with respect to maps $f\colon (K,L) \longrightarrow (K',L')$ and maps $\phi\colon (X,x_0) \longrightarrow (X',x_0')$ follows immediately from the naturality statements of lemma 9.5.

The results of section 4 on the cohomotopy exact couple now carry over to our more general situation. Let X be an $(n - 1)$-connected space. We replace S^{n+t} in section 4 by ${}'S^tX$ for $t \geq 0$. The exact cohomotopy sequence of a pair

(K, L) gives immediately the exact cohomotopy sequence of a triple (K, L, M) [11; p.25]. Hence with the above substitution, all of the results of section 4 are true; the main identifications now being that $'C^{r,s} \approx H^s(K,L; \pi_{s-r+n}(X))$ for $r \geq \mathrm{Max}(n+1, z)$ and $'A^{r,r-1} \approx \pi(K,L; 'S^{r-n}X, x_0)$ for $r \geq \mathrm{Max}(n, z)$. Again we denote $'i^{r,r}: \pi(K,L; 'S^{r-n}X, x_0) \rightarrow H^r(K,L; \pi_n(X))$ by η^r (see section 4 for explicit details).

10. Independence of the Choice of X(G,n)-space.

In this section we give the proofs of 8.1, 8.2, and 8.3 as well as giving some further properties of $\pi^n(X, L; G)$. These further properties will be used in section 12.

Proof of 8.1: Let $G = F/R$, where F is a free abelian group on generators $\{x_\alpha\}_{\alpha \in A}$ and R is the group of relations. R is free abelian [11; p.134] with basis $\{y_\beta\}_{\beta \in B}$. Let T be the CW-complex $\bigvee_{\alpha \in A} S^n_\alpha$, a union of n-spheres S^n_α with a single point in common. By the Hurewicz theorem, $\pi_n(T) \overset{\eta}{\approx} H_n(T) = F$. For each $\beta \in B$, attach an $(n+1)$-cell e_β to T by a map representing $\eta^{-1}(y_\beta) \in \pi_n(T)$. Call the resulting space X. By construction, $\pi_1(X) = 0$, $H_i(X) = 0$ for $i < n$, $\pi_n(X) \approx H_n(X) = G$, and $H_i(X) = 0$ for $i > n+1$. Moreover, since any non-zero $(n+1)$-cycle in X would imply a non-trivial relation among the $\{y_\beta\}_{\beta \in B}$, $H_{n+1}(X) = 0$.

In order to prove 8.3, we first prove

LEMMA 10.1: Let X be an X(G,n)-space. Then $\pi_{n+1}(X) \approx G \otimes Z_2$.

Proof: The proof is based on figure 7, a portion of the first derived homotopy exact couple of Massey [15; part II]. $H_{n+1}(X) = 0$ and hence by exactness $\pi_{n+1}(X, x_0) \approx \Gamma \approx H_n(X; Z_2)$. By the universal coefficient theorem (1.1 of the appendix), $H_n(X; Z_2) \approx G \otimes Z_2$.

$$\begin{array}{ccccccccc}
 & & \vdots & & & & & & \\
 & & \downarrow & & & & & & \\
 & \cdots\rightarrow & 0 & \rightarrow\cdots & & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
\cdots\rightarrow 0 & \rightarrow & \Gamma & \rightarrow & H_n(X; Z_2) & \rightarrow & 0 & \rightarrow\cdots & \\
 & & \downarrow & & & & \downarrow & & \\
 & \cdots\rightarrow & \pi_{n+1}(X,x_0) & \rightarrow & H_{n+1}(X) & \rightarrow & 0 & \rightarrow\cdots & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & & & \vdots & &
\end{array}$$

Figure 7.

<u>Proof of 8.3</u>: The proof is based on figure 8, a portion of the first derived generalized cohomotopy exact couple. Let $(K,L) = (X,x_0)$, $N = n + 1$, and let (Y,y_0) be the coefficient space.

$$\begin{array}{ccccccccc}
 & & \vdots & & & & & & \\
 & & \downarrow & & & & & & \\
 & \cdots\rightarrow & 0 & \rightarrow\cdots & & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
\cdots\rightarrow 0 & \rightarrow & {}'A^{n,n} & \xrightarrow{'i} & H^{n+1}(X,x_0; \pi_{n+1}(Y,y_0)) & \rightarrow & 0 & \rightarrow\cdots & \\
 & & \downarrow {}'j & & & & \downarrow & & \\
 & \cdots\rightarrow & \pi(X,x_0; Y,y_0) & \xrightarrow{\eta} & H^n(X,x_0; \pi_n(Y,y_0)) & \rightarrow & 0 & \rightarrow\cdots & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & & & \vdots & &
\end{array}$$

Figure 8.

Now $H^n(X,x_0; \pi_n(Y,y_0)) \approx H^n(X,x_0; H) \approx \mathrm{Hom}(G,H)$ by the universal coefficient theorem (1.1 of the appendix). By 10.1 and the universal coefficient theorem, $H^{n+1}(X,x_0; \pi_{n+1}(Y,y_0)) \approx H^{n+1}(X,x_0; H\otimes Z_2) \approx \mathrm{Ext}(G,H\otimes Z_2)$. By exactness η is an epimorphism with kernel isomorphic to ${}^*A^{n,n} \approx \mathrm{Ext}(G,H\otimes Z_2)$.

Before we prove 8.2, let us draw some further corollaries of 8.3.

COROLLARY 10.2: <u>If</u> $G \varepsilon \mathcal{D}$, <u>then</u> $\pi^n(K,L; G)$ <u>is a unitary left module over any subring of the ring of endomorphisms of</u> G.

<u>Proof:</u> By the naturality statements of 9.3, $\pi^n(K,L; G)$ is a unitary left module over $\pi(X,x_0; X,x_0)$ where X is an X(G,n)-space (the multiplication is composition). By 8.3 and 8.4, $\eta : \pi(X,x_0; X,x_0) \longrightarrow \mathrm{Hom}(G,G)$ is an isomorphism if $G \varepsilon \mathcal{D}$, and the corollary follows.

COROLLARY 10.3: <u>If</u> G <u>is a field and</u> $G \varepsilon \mathcal{D}$, <u>then</u> $\pi^n(K,L; G)$ <u>is a vector space over</u> G.

<u>Proof:</u> Any field is a subring of its ring of additive endomorphisms; namely, $g \varepsilon G$ acts on G by left multiplication. The corollary now follows from corollary 10.2.

We now complete the proof of 8.2. The results of section 9 show that $\pi^n(K,L; G)$ satisfy axioms 1 through 4 of Eilenberg and Steenrod.

Let (K,L) be a CW-pair with $L \neq \emptyset$. Let (K_L,p_L) be the CW-pair obtained by identifying L to a point p_L. Let coefficient theorem (1.1 of the appendix). By 10.1 and the universal coefficient theorem, $H^{n+1}(X,x_0; \pi_{n+1}(Y,y_0)) \approx H^{n+1}(X,x_0; H\otimes Z_2) \approx \mathrm{Ext}(G,H\otimes Z_2)$. By exactness η is an epimorphism with kernel isomorphic to $'A^{n,n} \approx \mathrm{Ext}(G,H\otimes Z_2)$.

Before we prove 8.2, let us draw some further corollaries of 8.3.

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COROLLARY 10.3: If G is a field and $G \varepsilon \mathcal{D}$, then $\pi^n(K,L; G)$ is a vector space over G.

Proof: Any field is a subring of its ring of additive endomorphisms; namely, $g \varepsilon G$ acts on G by left multiplication. The corollary now follows from corollary 10.2.

We now complete the proof of 8.2. The results of section 9 show that $\pi^n(K,L; G)$ satisfy axioms 1 through 4 of Eilenberg and Steenrod.

Let (K,L) be a CW-pair with $L \neq \emptyset$. Let (K_L, p_L) be the CW-pair obtained by identifying L to a point p_L. Let

$f: (K,L) \longrightarrow (K_L, p_L)$ be the canonical map.

LEMMA 10.4: $f^{\#}: \pi(K_L, p_L; X, x_0) \longrightarrow \pi(K, L; X, x_0)$ is a 1-1 correspondence.

Proof: Same as in [22; p.215].

Let (K,L) be a CW-pair. Let $M \subset L$ be such that $K - M$ is a subcomplex of K. Let $i: (K - M, L - M) \longrightarrow (K,L)$ be the inclusion.

THEOREM 10.5: (Excision Axiom). $i^{\#}: \pi^r(K,L; G) \longrightarrow \pi^r(K - M, L - M; G)$ is an isomorphism for $r \geq z$.

Proof: This follows immediately from 10.4 as in [22; p.215].

Let $f, g: (K,L) \longrightarrow (K',L')$ be homotopic.

THEOREM 10.6: (Homotopy Axiom). $f^{\#}$ and $g^{\#}: \pi^r(K',L'; G) \longrightarrow \pi^r(K,L; G)$ are equal.

Proof: Obvious.

Theorem 8.2 has now been proven. A corollary to this theorem is the fact that any theorem derivable from the axioms for cohomology of Eilenberg and Steenrod which does not make use of the lower dimensional groups holds for cohomotopy groups with coefficients in G. An example of this is the Mayer-Vietoris sequence of a triad [11; p.39].

11. Proof of Theorem 8.5.

This section is devoted to the proof that the sequence corresponding to an exact coefficient sequence is exact. The preliminary results are only steps in this proof.

LEMMA 11.1: *Let $G \varepsilon \mathcal{F}$, and let $n > (N + 1)/2$. Let X be an $X(G,n)$-space. Let Y be an $(n - 1)$-connected space such that $H_n(Y,y_0) = G$ and $H_r(Y,y_0) = 0$ for $n < r < 2n$. Then there is a map k: $(X,x_0) \longrightarrow (Y,y_0)$, unique up to homotopy, such that k_*: $H_n(X,x_0) = G \longrightarrow H_n(Y,y_0) = G$ is the identity and $k_\#$: $\pi^n(K,L; G) \longrightarrow \pi(K,L; Y,y_0)$ is a natural isomorphism.*

Proof: Similarly to the argument of 8.3, there is a map k: $(X,x_0) \longrightarrow (Y,y_0)$ such that k_*: $H_n(X,x_0) \longrightarrow H_n(Y,y_0)$ is the identity. Hence k_*: $H_r(X,x_0) \longrightarrow H_r(Y,y_0)$ is an isomorphism for $r < 2n$; it follows by a theorem of J. H. C. Whitehead [30] that $k_\#$: $\pi_r(X,x_0) \longrightarrow \pi_r(Y,y_0)$ is an isomorphism for $r < 2n - 1$ and an epimorphism for $r = 2n - 1$. Now apply lemma 9.4.

Let p: $E \longrightarrow B$ be a fibre space in the sense of Serre, with fibre F over $b_0 \varepsilon B$. Let K be a CW-complex. Define a map p_1: $E^K \longrightarrow B^K$ by $p_1(a) = pa$.

LEMMA 11.2: *p_1: $E^K \longrightarrow B^K$ is a fibre space with fibre F^K. A map f: $K \longrightarrow K'$ induces a fibre preserving map $\tilde{f}$: $(E^{K'},p_1',B^{K'}) \longrightarrow (E^K,p_1,B^K)$, and a fibre preserving map ϕ: $(E,p,B) \longrightarrow (E',p',B')$ induces a fibre preserving map $\tilde{\phi}$: $(E^K,p_1,B^K) \longrightarrow (E'^K,p_1',B'^K)$.*

Proof: The proof is a straight forward application of the covering homotopy theorem for (E,p,B), and the details are omitted.

Let G' and $G \in \mathcal{D}$, and let $0 \longrightarrow G' \xrightarrow{\phi} G \xrightarrow{\psi} G'' \longrightarrow 0$ be an exact coefficient sequence. Let X be an $X(G,N+1)$-space, and let X'' be an $X(G'',N+1)$-space. Let $\underline{\psi}: (X,x_0) \longrightarrow (X'',x_0'')$ be a map inducing ψ. We replace $\underline{\psi}$ by a fibre mapping as follows: assume $\underline{\psi}$ is an inclusion by the mapping cylinder construction. Let Y be the space of paths in X'' which end in X. X is contained in Y as a deformation retract [11; p.30] by $x \longrightarrow$ constant path at x. Define $p: Y \longrightarrow X''$ by $p(f) = f(0)$, p is a fibre map [19; p.479] with fibre F' = the space of paths starting at $x_0 \in X''$ and ending in X. We may assume $x_0 \in X \subset X''$. p is our replacement of $\underline{\psi}$.

Using the technique of spectral sequences, we now prove the following lemma. We may use this technique because the ordinary singular homology theory and the cubical singular homology theory of Serre [19] are isomorphic [9]. The proof of this lemma is the only place we use spectral sequences.

LEMMA 11.3: $H_{N+1}(F') = G'$, $H_r(F') = 0$ <u>otherwise for</u> $r < 2(N+1) - 1$.

Proof: Let E be the space of paths in X'' starting at x_0; E is a contractible space. Let $p_1: (E,F') \longrightarrow (X'',X)$ be defined by $p_1(f) = f(1)$. This is a relative fibre space with fibre F = the space of loops in X'' at x_0. F is $(N-1)$-connected because X'' is N-connected. As in [17; p.330], there is a spectral sequence of this relative fibre space with $E_2^{p,q} \approx H_p(X'', X; H_q(F))$ and E_∞ is the graded group associated with $H(E, F')$. From the exact homology sequence of the pair (X'', X):

$\cdots \longrightarrow H_{N+2}(X'') \longrightarrow H_{N+2}(X'', X) \longrightarrow H_{N+1}(X) \longrightarrow H_{N+1}(X'') \longrightarrow$ $H_{N+1}(X'', X) \longrightarrow H_N(X) \longrightarrow \cdots$ we see that $H_{N+2}(X'', X) = G'$ and $H_r(X'', X) = 0$ otherwise. Hence $E_2^{p,q} = 0$ for $q < N$ except that $E_2^{N+2,0} = G'$ and $E_2^{p,q} = 0$ for $p < N + 2$. Thus $H_r(E, F') = 0$ for $r < 2N + 2$ except that $H_{N+2}(E, F') = G'$. However E is contractible, and hence by the exact homology sequence of the pair (E, F'), $H_{N+1}(F') = G'$ and $H_r(F') = 0$ otherwise for $r < 2N + 1$.

In order to prove the naturality of theorem 8.5, we first prove the following lemma.

LEMMA 11.4: *Given a homomorphism of one exact coefficient sequence into another:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G' & \xrightarrow{\phi} & G & \xrightarrow{\psi} & G'' & \longrightarrow & 0 \\ & & \downarrow \xi' & & \downarrow \xi & & \downarrow \xi'' & & \\ 0 & \longrightarrow & H' & \xrightarrow{\phi'} & H & \xrightarrow{\psi'} & H'' & \longrightarrow & 0. \end{array}$$

Let G', G, G'', H', *and* $H \in \mathcal{D}$. *Refer to the above construction. Then there is a fibre preserving map* $\bar{\xi}: (Y, p, X'') \longrightarrow (Y_H, p_H, X''_H)$ *which is homotopic to* ξ *on* Y *and to* ξ'' *on* X''.

Proof:

$$\begin{array}{ccc} G & \xrightarrow{\psi} & G'' \\ \downarrow \xi & & \downarrow \xi'' \\ H & \xrightarrow{\psi'} & H'' \end{array}$$

is commutative and hence

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X'' \\ \downarrow \xi & & \downarrow \xi'' \\ X_H & \xrightarrow{\psi'} & X''_H \end{array}$$

is commutative up to homotopy by the results of section 8. Assume ψ and ψ' are inclusions. $\underline{\xi}: X \longrightarrow X_H$ is such that $\psi'\underline{\xi} \simeq \underline{\xi}''\underline{\psi}$, i. e. $\underline{\xi}$ is homotopic in X''_H to a map which can be extended to all of X'', namely $\underline{\xi}''|X$. Use the homotopy extension theorem to define a map $\xi_1: (X'',X) \longrightarrow (X''_H, X_H)$ such that $\xi_1|X'' \simeq \underline{\xi}''$ and $\xi_1|X \simeq \underline{\xi}$. ξ_1 induces a fibre preserving map $\bar{\bar{\xi}}: (Y,p,X'') \longrightarrow (Y_H,p_H,X''_H)$ having the correct properties. This completes the proof.

We now prove theorem 8.5, the aim of this section.

<u>Proof of 8.5</u>: By 10.4 and the exact cohomotopy sequence of the pair (K,x_0), it suffices to prove this theorem for the case when $L = \emptyset$. Use the above construction and 11.2 to obtain the fibre space (Y^K, p_1, X''^K) with fibre F'^K. Consider the homotopy sequence of this fibre space and use the isomorphism θ of section 9:

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & \pi_{N+1-r}(F'^K) & \xrightarrow{'j_\#} & \pi_{N+1-r}(Y^K) & \xrightarrow{p_{1\#}} & \pi_{N+1-r}(X''^K) & \xrightarrow{\partial} & \pi_{N-r}(F'^K) & \longrightarrow\cdots \\
& \downarrow k_\#^{-1}\theta & & \downarrow \theta & & \downarrow \theta & & \downarrow k_\#^{-1}\theta & \\
\cdots\longrightarrow & \pi^r(K;G') & \xrightarrow{\phi_\#} & \pi^r(K;G) & \xrightarrow{\psi_\#} & \pi^r(K;G'') & \xrightarrow{\delta_\#} & \pi^{r+1}(K;G') & \longrightarrow\cdots
\end{array}$$

∂ is the boundary operator in the homotopy sequence of the fibre space, $'j_\#$ is induced by the inclusion $'j: F'^K \longrightarrow Y^K$, and $p_{1\#}$ is induced by p_1. θ is defined and is an isomorphism for $r \geq z$ because $2(N+1-N-1+r)-2 = 2r-2 \geq 2z-2 \geq N$.

$k_{\#}: \pi^r(K; G') \longrightarrow \pi(K; F')$ is the isomorphism of 11.1 and 11.3, and $\delta_{\#}: \pi^r(K; G'') \longrightarrow \pi^{r+1}(K; G')$ is defined by $\delta_{\#} = k_{\#}^{-1}\theta\partial\theta^{-1}$. Furthermore, $\theta' j_{\#} = \phi_{\#} k_{\#}^{-1}\theta$ and $\psi_{\#}\theta = \theta p_{1\#}$ by the naturality of θ. Also, $\delta_{\#}\theta = k_{\#}^{-1}\theta\partial$ by definition. Hence the sequence corresponding to the exact coefficient sequence is exact.

A map $f: K \longrightarrow K'$ induces a fibre preserving map $\bar{\bar{f}}: (Y^{K'}, p_1', X''^{K'}) \longrightarrow (Y^K, p_1, X''^K)$ and hence $f^{\#}$ commutes with $\delta_{\#}$ because ∂ and θ are natural. $f^{\#}$ commutes with $\phi_{\#}$ and $\psi_{\#}$ also. Thus f induces a homomorphism of the sequence of K' into that of K. Also, a homomorphism of one exact coefficient sequence into another

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G' & \xrightarrow{\phi} & G & \xrightarrow{\psi} & G'' & \longrightarrow & 0 \\ & & \downarrow \xi' & & \downarrow \xi & & \downarrow \xi'' & & \\ 0 & \longrightarrow & H' & \xrightarrow{\phi'} & H & \xrightarrow{\psi'} & H'' & \longrightarrow & 0 \end{array}$$

induces a fibre preserving map $\bar{\bar{\xi}}: (Y^K, p_1, X''^K) \longrightarrow (Y_H{}^K, p_H, X''_H{}^K)$ by lemma 11.4 when G', G, G'', H', and $H \in \mathcal{B}$. Hence $\xi_{\#}$ commutes with $\delta_{\#}$ as well as with $\phi_{\#}$ and $\psi_{\#}$. Thus a homomorphism of the exact coefficient sequence induces a homomorphism of the sequence corresponding to that exact coefficient sequence. This completes the proof.

12. *Proof of the Universal Coefficient Theorem.*

This section is devoted to proving that the sequence

$$(*)\ 0 \longrightarrow \pi^r(K,L)\otimes G \xrightarrow{\alpha} \pi^r(K,L;\, G) \xrightarrow{\beta} \mathrm{Tor}(\pi^{r+1}(K,L),G) \longrightarrow 0$$

is exact under various hypotheses.

LEMMA 12.1: *$\alpha : \pi^r(K,L)\otimes G \longrightarrow \pi^r(K,L;\, G)$ is natural with respect to maps $f: (K,L) \longrightarrow (K',L')$, and if $G \in \mathcal{D}$, then it is natural with respect to homomorphisms $\phi: G \longrightarrow H$. Moreover, if G is finitely generated and free, then α is an isomorphism.*

Proof: The naturality statements are obvious. If $G = Z$, α is obviously an isomorphism because $\pi^r(K,L;\, Z) = \pi^r(K,L)$. Moreover, the functors $\pi^r(K,L)\otimes G$ and $\pi^r(K,L;\, G)$ are additive [8] on $\mathcal{D}$ by the results of section 9. Hence they commute with finite direct sums, and α is an isomorphism if G is finitely generated and free.

Proof of the first part of 8.6: Let $0 \longrightarrow R \xrightarrow{i} F \xrightarrow{j} G \longrightarrow 0$ be exact, where F is a finitely generated free abelian group. R is finitely generated and free also. By 8.5, the sequence corresponding to this coefficient sequence is exact (R and $F \in \mathcal{D}$):

$$\cdots \longrightarrow \pi^r(K,L;\, R) \xrightarrow{i_\#} \pi^r(K,L;\, F) \xrightarrow{j_\#} \pi^r(K,L;\, G) \xrightarrow{\delta_\#} \pi^{r+1}(K,L;\, R)$$

$$\xrightarrow{i_\#} \pi^{r+1}(K,L;\, F) \longrightarrow \cdots.$$

Hence the following sequence is exact: $0 \longrightarrow \mathrm{Ker}\,\delta_\# \longrightarrow \pi^r(K,L;\, G) \longrightarrow \mathrm{Im}\,\delta_\# \longrightarrow 0.$ However, $\mathrm{Ker}\,\delta_\# = \mathrm{Im}\, j_\# \approx \mathrm{Coker}\, i_\#$ and $\mathrm{Im}\,\delta_\# = \mathrm{Ker}\, i_\#$ by exactness. By 12.1, we see that $\mathrm{Coker}\, i_\# \approx$

Coker $(\pi^r(K,L)\otimes R \longrightarrow \pi^r(K,L)\otimes F) \approx \pi^r(K,L)\otimes G$ and Ker $i_\# \approx$ Ker$(\pi^{r+1}(K,L)\otimes R \longrightarrow \pi^{r+1}(K,L)\otimes F) = \mathrm{Tor}(\pi^{r+1}(K,L),G)$ (see [2] for the elementary properties of Tor needed in this proof). Under these isomorphisms, the inclusion Ker $\delta_\# \longrightarrow \pi^r(K,L;G)$ goes over to $\alpha : \pi^r(K,L)\otimes G \longrightarrow \pi^r(K,L;G)$, and $\delta_\#$ defines $\beta : \pi^r(K,L;G) \longrightarrow \mathrm{Tor}(\pi^{r+1}(K,L),G)$. Hence the sequence (*) is exact. (*) is natural with respect to maps $f: (K,L) \longrightarrow (K',L')$ because of 12.1 and the exact sequence of 8.5 is natural. A homomorphism $\phi: G \longrightarrow H$ gives rise to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{i} & F & \xrightarrow{j} & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R' & \xrightarrow{i'} & F' & \xrightarrow{j'} & H & \longrightarrow & 0,
\end{array}$$

where R, R', F, and F' are finitely generated free abelian groups. Since R, R', F, F', and $G \in \mathcal{D}$, by 8.5 we obtain a commutative diagram:

$$\begin{array}{ccccccc}
\cdots\longrightarrow & \pi^r(K,L;G) & \xrightarrow{\delta_\#} & \pi^{r+1}(K,L;R) & \xrightarrow{i_\#} & \pi^{r+1}(K,L;F) & \longrightarrow\cdots \\
 & \downarrow \phi_\# & & \downarrow & & \downarrow & \\
\cdots\longrightarrow & \pi^r(K,L;H) & \xrightarrow{\delta_\#'} & \pi^{r+1}(K,L;R') & \xrightarrow{i_\#'} & \pi^{r+1}(K,L;F') & \longrightarrow\cdots.
\end{array}$$

Hence the induced map $\mathrm{Tor}(\pi^{r+1}(K,L),G) \longrightarrow \mathrm{Tor}(\pi^{r+1}(K,L),H)$ commutes with β. By 12.1, $\phi_\#$ commutes with α. This completes the proof of the first part of 8.6.

An exact sequence $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ splits if there exists a homomorphism $k: C \longrightarrow B$ such that $jk =$ the identity on C. For abelian groups, this is equivalent to the statement that $B = A + C$, the direct sum of A and C.

LEMMA 12.2: <u>The exact sequence</u> (*) <u>splits for</u> $G = Z_p$, p <u>an odd prime</u>.

<u>Proof</u>: By 10.3, $\pi^r(K, L; Z_p)$ is a vector space over Z_p and hence, $\pi^r(K, L; Z_p)$, as a group, is a direct sum of copies of Z_p. This implies that (*) splits when $G = Z_p$.

LEMMA 12.3: <u>If</u> (K, L) <u>is a finite CW-pair, then the exact sequence</u> (*) <u>splits for</u> $G = Z_{p^s}$, p <u>an odd prime</u>.

<u>Proof</u>: Corresponding to the coefficient homomorphism $\phi: Z_{p^s} \longrightarrow Z_p$ sending a generator into a generator, we have the commutative diagram (by 12.1)

$$\begin{array}{ccc} \pi^r(K, L) \otimes Z_{p^s} & \xrightarrow{1 \otimes \phi} & \pi^r(K, L) \otimes Z_p \\ \downarrow \alpha_1 & & \downarrow \alpha \\ \pi^r(K, L; Z_{p^s}) & \xrightarrow{\phi_\#} & \pi^r(K, L; Z_p). \end{array}$$

$1 \otimes \phi$ is onto, and α and α_1 are monomorphisms. We can write $\pi^r(K, L) = Z + \dots + Z + Z_{p_1^{m_1}} + \dots + Z_{p_k^{m_k}}$ because by 3.4, $\pi^r(K, L)$ is a finitely generated abelian group. We obtain generators $\{x_j\}$ of $\pi^r(K, L) \otimes Z_{p^s}$ from generators of Z and $Z_{p_i^{m_i}}$ for $p_i = p$. It is obvious that $(1 \otimes \phi)(x_j) \neq 0$ for all j, and hence $\alpha(1 \otimes \phi)(x_j) \neq 0$ for all j. However, if (*) does not split for $G = Z_{p^s}$, then some x_j is such that $\alpha_1(x_j)$ is divisible by p in $\pi^r(K, L; Z_{p^s})$. Hence $\phi_\# \alpha_1(x_j)$ is divisible by p in $\pi^r(K, L; Z_p)$, but each element of $\pi^r(K, L; Z_p)$ has order p. Thus $0 = \phi_\# \alpha_1(x_j) = \alpha(1 \otimes \phi)(x_j) \neq 0$. This is a contradiction.

LEMMA 12.4: (H-lemma). *Given a commutative diagram where each row and each column is exact. If the first and and third columns and the middle row split, then every row and every column splits.*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow c_1 & & \downarrow d_1 & & \downarrow e_1 & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \longrightarrow & 0 \\
 & & \downarrow c_2 & & \downarrow d_2 & & \downarrow e_2 & & \\
0 & \longrightarrow & A_3 & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Proof: By hypothesis, there are homomorphisms $c_2^{-1}: A_3 \longrightarrow A_2$, $e_2^{-1}: C_3 \longrightarrow C_2$, and $g_2^{-1}: C_2 \longrightarrow B_2$ such that $c_2c_2^{-1} = 1$, $e_2e_2^{-1} = 1$, and $g_2g_2^{-1} = 1$ on A_3, C_3, and C_2 respectively. Define $g_3^{-1}: C_3 \longrightarrow B_3$ by $g_3^{-1} = d_2g_2^{-1}e_2^{-1}$, then $g_3g_3^{-1} = g_3d_2g_2^{-1}e_2^{-1} = e_2g_2g_2^{-1}e_2^{-1} = e_2e_2^{-1} = 1$, and hence the bottom row splits. Let f_3^{-1} be the other component of the direct sum decomposition, i. e. $f_3^{-1}: B_3 \longrightarrow A_3$ is such that $1 = f_3f_3^{-1} + g_3^{-1}g_3$ on B_3. Define $d_2^{-1} = g_2^{-1}e_2^{-1}g_3 + f_2c_2^{-1}f_3^{-1}: B_3 \longrightarrow B_2$. Then $d_2d_2^{-1} = d_2g_2^{-1}e_2^{-1}g_3 + d_2f_2c_2^{-1}f_3^{-1} = g_3^{-1}g_3 + f_3c_2c_2^{-1}f_3^{-1} = g_3^{-1}g_3 + f_3f_3^{-1} = 1$, and hence the middle column splits. Let f_2^{-1} and c_1^{-1} be the other components of the direct sum decompositions of the second row and the first column, i. e. $f_2^{-1}f_2 = 1$ and $c_1^{-1}c_1 = 1$ on A_2 and A_1 respectively. Define $f_1^{-1}: B_1 \longrightarrow A_1$ by $f_1^{-1} = c_1^{-1}f_2^{-1}d_1$. Then $f_1^{-1}f_1 = c_1^{-1}f_2^{-1}d_1f_1 = c_1^{-1}f_2^{-1}f_2c_1 = c_1^{-1}c_1 = 1$, and hence every row and every column splits.

LEMMA 12.5: <u>If the exact sequence</u> (*) <u>splits for</u> G' <u>and</u> G'', <u>then it splits for</u> $G' + G'' = G$, <u>where</u> G' <u>and</u> $G'' \in \mathcal{D}$.

<u>Proof:</u> $0 \to G' \to G \to G'' \to 0$ is a split exact sequence. The following is a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \pi^r(K,L)\otimes G' & \to & \pi^r(K,L)\otimes G & \to & \pi^r(K,L)\otimes G'' & \to & 0\\
 & & \downarrow\alpha_1 & & \downarrow\alpha & & \downarrow\alpha_2 & & \\
0 & \to & \pi^r(K,L;G') & \to & \pi^r(K,L;G) & \to & \pi^r(K,L;G'') & \to & 0 \\
 & & \downarrow\beta_1 & & \downarrow\beta & & \downarrow\beta_2 & & \\
0 & \to & \mathrm{Tor}(\pi^{r+1}(K,L),G') & \to & \mathrm{Tor}(\pi^{r+1}(K,L),G) & \to & \mathrm{Tor}(\pi^{r+1}(K,L),G'') & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The rows are exact and split because $\otimes$, Tor, and $\pi^r(K,L;G)$ are additive functors when $G \in \mathcal{D}$. The columns are exact by above, and the first and third columns split by hypothesis. Hence by 12.4, the middle column splits.

The last part of 8.6 now follows immediately from 12.3 and 12.5.

In order to prove 8.7, we first establish a direct limit theorem. Let $G \in \mathcal{D}$, and let $\{G^\alpha ; \pi_\alpha^\beta\}$ be the direct system of finitely generated subgroups of G [11, Chapt. VIII]. Then $G = \text{dir}_\alpha \lim G^\alpha$. Note that each $G^\alpha \in \mathcal{D}$. $\{\pi^r(K,L;G^\alpha); (\pi_\alpha^\beta)_\#\}$ is obviously a direct system of groups for $r \geq 2$. We define $\xi : \text{dir}_\alpha \lim \pi^r(K,L;G^\alpha) \to \pi^r(K,L;G)$

by $\xi(\{[a]\}) = (\pi_\alpha)_\#([a])$, where $[a] \in \pi^r(K,L; G^\alpha)$, $\{[a]\}$ is the element in $\operatorname{dir}_\alpha \lim \pi^r(K,L; G^\alpha)$ represented by $[a]$, and $\pi_\alpha : G^\alpha \longrightarrow G$ is the canonical homomorphism. Since $\pi_\alpha = \pi_\beta \pi_\alpha^\beta$ and all $G^\alpha \in \mathcal{D}$, $(\pi_\alpha)_\# = (\pi_\beta)_\#(\pi_\alpha^\beta)_\#$ and ξ is well defined (note that $G \in \mathcal{D}$ is necessary here).

THEOREM 12.6: <u>If</u> (K,L) <u>is a finite CW-pair, then</u> ξ <u>is an isomorphism for</u> $r \geq z$. <u>This isomorphism is natural with respect to maps</u> $f: (K,L) \longrightarrow (K',L')$ <u>and homomorphisms</u> $\phi: G \longrightarrow H$.

<u>Proof:</u> Let X be an $X(G,r)$-space, and let $[a] \in \pi^r(K,L; G)$. Since K is compact, $a(K) \subset X$ is also compact. Thus $a(K)$ is contained in a finite subcomplex X' of X [12, p.96]. Assume X is the standard $X(G,r)$-space constructed in the proof of 8.1. Then a finite subcomplex of X must an $X(G^\beta,r)$-space for some β. Hence a defines a map $a': (K,L) \longrightarrow (X',x_0)$ such that $(\pi_\beta)_\#([a']) = [a]$. Hence ξ is an epimorphism. A similar argument shows ξ is a monomorphism because $K \times I$ is also a compact CW-complex. The naturality statements are obvious.

Theorem 8.7 now follows immediately from the exact sequence (*), 12.6, and the fact that direct limits preserve exactness, $\otimes$, and Tor [8].

The following example shows that the universal coefficient theorem of 8.6 and 8.7 does not hold in general, even if $G \in \mathcal{D}$. Let $G = Q$ = rationals and let K be an $X(Q,n)$-space.

Then by 8.3, $\pi^n(K; Q) \approx \mathrm{Hom}(Q,Q) = Q$ and $\pi^n(K; Z) \approx \mathrm{Hom}(Q,Z) = 0$. The sequence (*), if exact, would give here that $0 \to 0 \otimes Q \to Q \to \mathrm{Tor}(\pi^{n+1}(K), Q) \to 0$ is exact. However $\mathrm{Tor}(\pi^{n+1}(K), Q) = 0$, and hence $Q = 0$. This is a contradiction.

13. Proof of Theorem 8.8.

Proof of 8.8: Refer to figure 3 in the proof of 3.1, but where we use the generalized cohomotopy exact couple; i. e. Z is replaced everywhere by G. Again it suffices to prove that $H^{r+1}(K,L; G_{(1)}) \in \mathcal{C}, \ldots, H^N(K,L; G_{(N-r)}) \in \mathcal{C}$ for $r > \mathrm{Max}((N+1)/2, n - \alpha_{\mathcal{C}}(G))$. Again $n - r < n - (n - \alpha_{\mathcal{C}}(G)) = \alpha_{\mathcal{C}}(G)$, hence $G_{(1)} \in \mathcal{C}, \ldots, G_{(n-r)} \in \mathcal{C}$ by definition of $\alpha_{\mathcal{C}}(G)$. Since $G_{(s)}$ is finitely generated by 8.6, we may use the universal coefficient theorem for cohomology; i. e.

$$0 \longrightarrow H^{r+s}(K,L) \otimes G_{(s)} \xrightarrow{\alpha} H^{r+s}(K,L; G_{(s)}) \xrightarrow{\beta} \mathrm{Tor}(H^{r+s+1}(K,L), G_{(s)})$$

$\longrightarrow 0$ is exact. $G_{(s)} \in \mathcal{C}$ for $s \leq n - r$, therefore $H^{r+s}(K,L; G_{(s)}) \in \mathcal{C}$ for $s \leq n - r$. The proof now differs from that of 3.1. We have yet to show that $H^{r+s}(K,L; G_{(s)}) \in \mathcal{C}$ for $s > n - r$ from the assumption that $H^{r+s}(K,L; G) \in \mathcal{C}$ for $s > n - r$. Since $(G + G')_{(s)} \approx G_{(s)} + G'_{(s)}$, $H^r(K,L; G + G') \approx H^r(K,L; G) + H^r(K,L; G')$, and G is finitely generated, it suffices to show this for $G = Z$ or $G = Z_p t$. The case $G = Z$ is obvious as in the proof of 3.1. We now consider the case $G = Z_p t$. By 8.6, $(Z_p t)_{(s)}$ can have only a p-primary component. Hence it suffices to show that if $H^r(K,L; Z_p t) \in \mathcal{C}$ for $r > n$, then $H^r(K,L; Z_p s) \in \mathcal{C}$ for $r > n$ and all $s \geq 1$ (we have changed the notation to simplify the rest of the proof).

If $t = 1$, then it is obvious by induction on s using the exact sequence

$H^r(K, L; Z_p s-1) \longrightarrow H^r(K, L; Z_p s) \longrightarrow H^r(K, L; Z_p)$ corresponding to the exact coefficient sequence $0 \longrightarrow Z_p s-1 \longrightarrow Z_p s \longrightarrow Z_p \longrightarrow 0$ (see 1.3 of the appendix). If $t > 1$, then let r_0 be the largest integer r such that $H^r(K, L; Z_p q) \notin \mathcal{C}$ for some q with $1 \leq q < t$, and assume $r_0 > n$. Then corresponding to the exact coefficient sequence $0 \longrightarrow Z_p t-q \longrightarrow Z_p t \longrightarrow Z_p q \longrightarrow 0$, we have an exact sequence $H^{r_0}(K, L; Z_p t) \longrightarrow H^{r_0}(K, L; Z_p q) \longrightarrow H^{r_0+1}(K, L; Z_p t-q)$ by 1.3 of the appendix. However, $H^{r_0}(K, L; Z_p t) \in \mathcal{C}$ and $H^{r_0+1}(K, L; Z_p t-q) \in \mathcal{C}$, hence $H^{r_0}(K, L; Z_p q) \in \mathcal{C}$ which is a contradiction. Thus $r_0 \leq n$ and $H^r(K, L; Z_p) \in \mathcal{C}$ for $r > n$ and as remarked above, this implies that $H^r(K, L; Z_p s) \in \mathcal{C}$ for $r > n$ and all $s \geq 1$. This completes the proof of 8.8.

14. Cohomotopy Operations.

We conclude our discussion of generalized cohomotopy groups by defining the concept of a universally defined cohomotopy operation analogous to universally defined homotopy operations [4]. We classify these operations and compute the classifying groups.

Let A and B ε $\mathcal{D}$ throughout this section. A cohomotopy operation of type $(n, q; A, B)$ is a function θ: $\pi^n(K, L; A) \longrightarrow \pi^q(K, L; B)$, defined for every CW-pair (K, L) with $N \leq \mathrm{Min}(2n - 2, 2q - 2)$, such that if $f: (K, L) \longrightarrow (K', L')$, then $\theta f^{\#} = f^{\#}\theta: \pi^n(K', L'; A) \longrightarrow \pi^q(K, L; B)$.

Let $n + 1 \leq 2q - 2$, and let $[b] \varepsilon \pi^q(X, x_0; B)$, where X is an $X(A, n)$-space. If $[a] \varepsilon \pi^n(K, L; A)$ and $N \leq \mathrm{Min}(2n - 2, 2q - 2)$, then define $\theta_b([a]) = [ba] \varepsilon \pi^q(K, L; B)$. Clearly θ_b is a cohomotopy operation of type $(n, q; A, B)$. Thus we have a function $\chi: \pi^q(X, x_0; B) \longrightarrow$ the set of cohomotopy operations of type $(n, q; A, B)$, defined by $\chi([b]) = \theta_b$.

THEOREM 14.1: If $n + 1 \leq 2q - 2$, then χ is a 1-1 correspondence.

Proof: Let ι ε $\pi^n(X, x_0; A)$ denote the class of the identity map of (X, x_0) into (X, x_0). Let $a \varepsilon [a] \varepsilon \pi^n(K, L; A)$, $a: (K, L) \longrightarrow (X, x_0)$. Then $a^{\#}(\iota) = [a]$. Hence if θ is a given cohomotopy operation of type $(n, q; A, B)$, then $\theta([a]) = \theta(a^{\#}(\iota)) = a^{\#}\theta(\iota) = [\theta(\iota)a] = \theta_b([a])$ where $[b] = \theta(\iota)$ ε $\pi^q(X, x_0; B)$. Thus χ is onto. If $\theta_b = \theta_{b'}$, then

$[b] = [b\iota] = \theta_b(\iota) = \theta_{b'}(\iota) = [b'\iota] = [b']$. Thus χ is 1-1.

CORROLLARY 14.2: *If* $n + 1 \leq 2q - 2$, *then each* θ *is a homomorphism.*

Proof: By 14.1, $\theta = \theta_b$ for some $[b] \in \pi^q(X, x_0; B)$. Let $[a]$ and $[a'] \in \pi^n(K, L; A)$, then $\theta_b([a] + [a']) = [b]([a] + [a']) = [ba] + [ba'] = \theta_b([a]) + \theta_b([a'])$ by 9.3.

We now compute $\pi^q(X, x_0; B)$, where X is an $X(A, n)$-space.

THEOREM 14.3: (a) $\pi^n(X, x_0; B) \approx \mathrm{Hom}(A, B)$,
(b) $\pi^{n+1}(X, x_0; B) \approx \mathrm{Ext}(A, B)$,
(c) $\pi^q(X, x_0; B) = 0$ *for* $q > n + 1$, *and*
(d) *if* B *is finitely generated and* $q < n$, *then*

$$0 \longrightarrow \mathrm{Ext}(A, Z_{(n+1-q)} \otimes B + \mathrm{Tor}(Z_{(n-q)}, B)) \longrightarrow \pi^q(X, x_0; B) \longrightarrow$$

$$\mathrm{Hom}(A, Z_{(n-q)} \otimes B + \mathrm{Tor}(Z_{(n-q-1)}, B)) \longrightarrow 0$$ *is an exact sequence.*

Proof: (a) follows from 8.3 and 8.4. (b) follows from 8.8 and the universal coefficient theorem for cohomology because X has dimension $n + 1$. (c) is trivial because X has dimension $n + 1$. If B is finitely generated, then $B_{(s)} \approx Z_{(s)} \otimes B + \mathrm{Tor}(Z_{(s-1)}, B)$ by theorem 8.6. Consider figure 9, a portion of the generalized cohomotopy exact couple with $G = B$, $(K, L) = (X, x_0)$, and $N = n + 1$. It is clear from figure 9 that the sequence $0 \longrightarrow {}'A^{q,n} \longrightarrow {}'A^{q,n-1} \longrightarrow H^n(X, x_0; B_{(n-q)}) \longrightarrow 0$ is exact. However, ${}'A^{q,n} \approx H^{n+1}(X, x_0; B_{(n+1-q)}) \approx \mathrm{Ext}(A, B_{(n+1-q)})$, ${}'A^{q,n-1} \approx {}'A^{q,q} \approx \pi^q(X, x_0; B)$, and

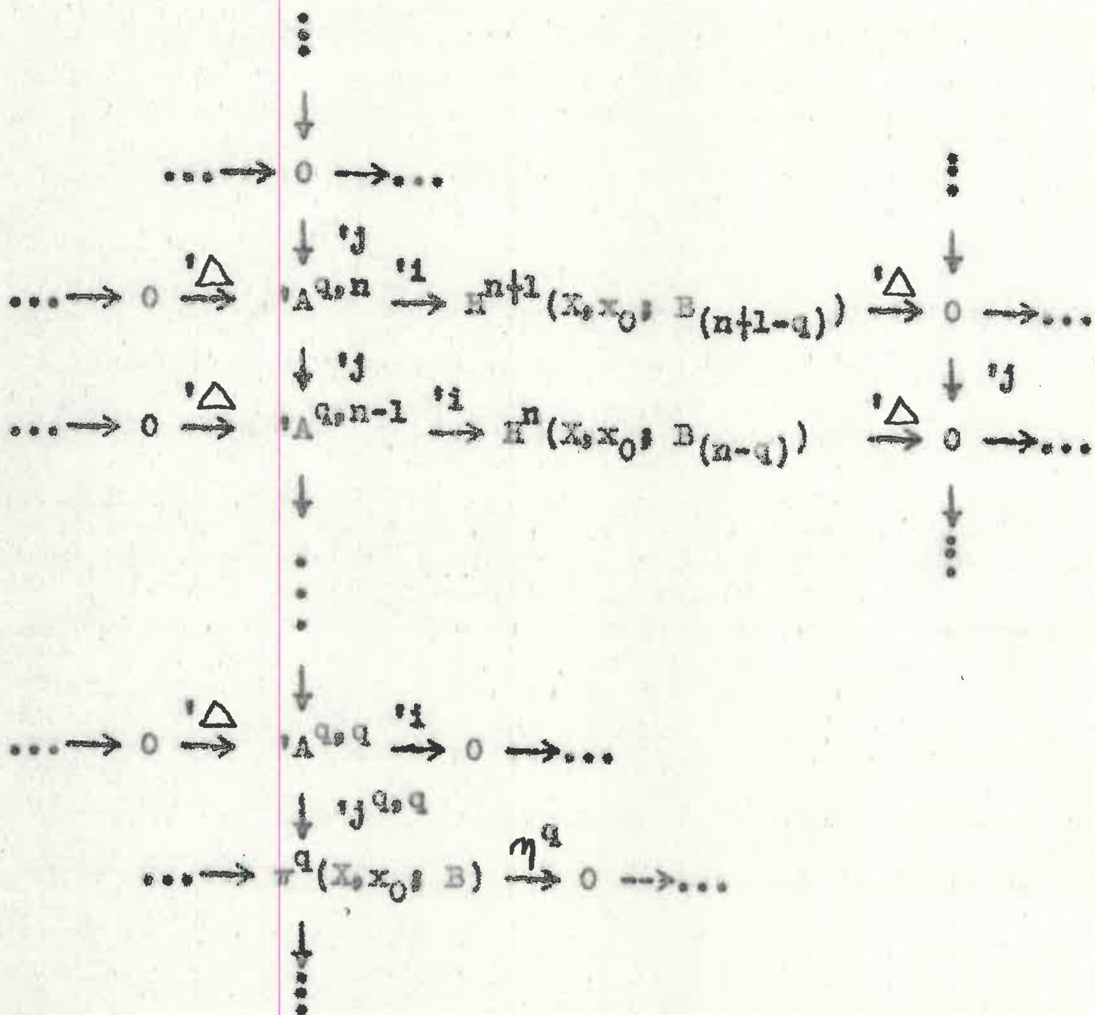

Figure 9.

$H^n(X,x_0; B_{(n-q)}) \approx \mathrm{Hom}(A, B_{(n-q)})$. Combined with the above, this completes the proof.

15. Generalized Homotopy Groups.

We conclude this paper with a brief discussion of generalized homotopy groups.

As one might expect, there is a theory of homotopy groups with coefficients in G which is dual in an intuitive sense to the theory developed above. The results of Spanier and Whitehead [25] make this duality precise. By these results, we are led to consider spaces having only one non-vanishing cohomology group; in particular, we consider homotopy classes of maps of such a space into arbitrary spaces. The duality of [25] gives theorems dual to our theorems on cohomotopy groups with coefficients in G. These theorems are only valid in the stable range. However, as in ordinary homotopy theory, there is a natural group structure defined outside of the stable range, and many of the theorems extend beyond the stable range. In this section we outline these results without giving proofs.

Let G be a finitely generated abelian group, and let $n > 2$. An n-dimensional CW-complex X is said to be an <u>X(n,G)-space</u> if $\pi_1(X) = 0$, $H^i(X) = 0$ for $i \neq n$, and $H^n(X) = G$. Note that if X is an X(n,G)-space, then 'SX is an X(n+1,G)-space.

LEMMA 15.1: *For given G and n, there exists an X(n,G)-space.*

Let $y_0 \in B \subset Y$. Let X be an X(n-1,G)-space ($n > 3$), and let CX denote the cone on X. $x_0 \in X \subset CX$ as the base. We define the $\underline{n^{th}}$ homotopy group of (Y,B) with coefficients in G to be $\pi(CX,X,x_0; Y,B,y_0)$, and we denote it by $\pi_n(Y,B; G)$. This is a group by 9.2. In case $B = \{y_0\}$, $\pi(CX,X,x_0; Y,y_0,y_0) = \pi('SX,x_0; Y,y_0) = \pi_n(Y; G)$ is defined for $n > 2$. As defined, $\pi_n(Y,B; G)$ depends on the choice of X(n-1,G)-space. We show below that $\pi_n(Y,B; G)$ is naturally independent of this choice when G has no elements of order 2. However, if we do not change coefficients during a discussion, it suffices to choose a fixed X(t,G)-space and use suspensions of it.

A map $f: (Y,B) \longrightarrow (Y',B')$ induces a homomorphism $f_{\#}: \pi_n(Y,B; G) \longrightarrow \pi_n(Y',B'; G)$. Let $i: B \longrightarrow Y$ and $j: Y \longrightarrow (Y,B)$ be inclusions. There is a natural homomorphism $\partial: \pi_n(Y,B; G) \longrightarrow \pi_{n-1}(B; G)$ analogous to the usual homotopy boundary homomorphism.

THEOREM 15.2: The homotopy sequence of a pair (Y,B): $\dots \longrightarrow \pi_n(B; G) \xrightarrow{i_{\#}} \pi_n(Y; G) \xrightarrow{j_{\#}} \pi_n(Y,B; G) \xrightarrow{\partial} \pi_{n-1}(B; G) \longrightarrow \dots$ is exact. This exact sequence is natural with respect to maps $f: (Y,B) \longrightarrow (Y',B')$.

If (Y,B) is a CW-pair, then we can obtain a generalized homotopy exact couple analogous to the generalized cohomotopy exact couple.

Let X be an X(n,G)-space, Y an X(n,H)-space. There is a natural homomorphism $\eta: \pi(Y,y_0; X,x_0) \longrightarrow \mathrm{Hom}(G,H)$ defined by $\eta([a]) = a^*: H^n(X,x_0) = G \longrightarrow H^n(Y,y_0) = H$. Using the generalized homotopy exact couple, we can prove

THEOREM 15.3: *If* $n > 3$, *then* $\eta : \pi(Y, y_0; X, x_0) \longrightarrow \mathrm{Hom}(G, H)$ *is an epimorphism and has a kernel isomorphic to* $\mathrm{Ext}(G, H \otimes Z_2)$.

As in section 8, if G is finitely generated and $G \varepsilon \mathcal{D}$, then $\pi_n(Y, B; G)$ is naturally independent of the choice of $X(n-1, G)$-space and a homomorphism $\phi: G \longrightarrow H$ induces a unique homomorphism $\phi_{\#}: \pi_n(Y, B; G) \longrightarrow \pi_n(Y, B; H)$.

Let G' and $G \varepsilon \mathcal{D}$, and let $0 \longrightarrow G' \xrightarrow{\phi} G \xrightarrow{\Psi} G'' \longrightarrow 0$ be an exact sequence of coefficients. There is a natural homomorphism $\partial_{\#}: \pi_r(Y, B; G'') \longrightarrow \pi_{r-1}(Y, B; G')$ analogous to $\delta_{\#}$ of 8.5.

THEOREM 15.4: *The homotopy sequence corresponding to this exact coefficient sequence:*

$$\cdots \longrightarrow \pi_r(Y, B; G') \xrightarrow{\phi_{\#}} \pi_r(Y, B; G) \xrightarrow{\Psi_{\#}} \pi_r(Y, B; G'') \xrightarrow{\partial_{\#}} \pi_{r-1}(Y, B; G') \longrightarrow \cdots$$

is exact. This sequence is natural with respect to maps $f: (Y, B) \longrightarrow (Y', B')$, *and if* G', G, G'', H', *and* $H \varepsilon \mathcal{D}$, *then it is natural with respect to homomorphisms of one exact coefficient sequence into another.*

There are homomorphims $\alpha: \pi_r(Y, B) \otimes G \longrightarrow \pi_r(Y, B; G)$ and $\beta : \pi_r(Y, B; G) \longrightarrow \mathrm{Tor}(\pi_{r-1}(Y, B), G)$ analogous to those in 8.6.

THEOREM 15.5: *The sequence* $0 \longrightarrow \pi_r(Y, B) \otimes G \xrightarrow{\alpha} \pi_r(Y, B; G) \xrightarrow{\beta} \mathrm{Tor}(\pi_{r-1}(Y, B), G) \longrightarrow 0$ *is exact. If* $\pi_r(Y, B)$ *is finitely generated and* $G \varepsilon \mathcal{D}$, *then this exact sequence splits.*

<u>This exact sequence is natural with respect to maps</u> $f: (Y, B) \to (Y', B')$, <u>and if</u> $G \in \mathcal{D}$, <u>then it is natural with respect to homomorphisms</u> $\phi: G \to H$.

If (Y, B) is a CW-pair, then there is a natural homomorphism $\eta_r: \pi_r(Y, B; G) \to H_r(Y, B; G)$. Using this and the generalized homotopy exact couple, we can state and prove a generalized Hurewicz theorem mod $\mathcal{C}$. The details of this are left to the reader. It is conjectured that the conditions of (Y, B) being a CW-pair and the connectedness condition on (Y, B) needed for the homotopy exact couple proof can be relaxed, and a proof similar to that in [20] can be given. Many of the other results of section 3 also dualize to give information on homotopy groups. In general however, it seems to be easier to work with cohomotopy groups because the theory of cohomology operations is so well developed. However, the homotopy exact couple contains much valuable information on homotopy groups (see the work of J. H. C. Whitehead and the English school).

APPENDIX

1. Results from Eilenberg and Steenrod [11].

The purpose of the first three sections of the appendix is to discuss the universal coefficient theorem for cohomology.

THEOREM 1.1: *Let K be a chain complex composed of free groups. For an arbitrary abelian group G, the sequences*

$$0 \longrightarrow H_r(K) \otimes G \xrightarrow{\alpha} H_r(K; G) \xrightarrow{\beta} \mathrm{Tor}(H_{r-1}(K), G) \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow \mathrm{Ext}(H_{r-1}(K), G) \xrightarrow{\bar{\beta}} H^r(K; G) \xrightarrow{\bar{\alpha}} \mathrm{Hom}(H_r(K), G) \longrightarrow 0$$

are exact and split. These exact sequences are natural with respect to chain maps $f: K \longrightarrow K'$ *and homomorphisms* $\phi: G \longrightarrow H$.

Proof: See exercise G-3 in [11; Chapt. V].

THEOREM 1.2: *Let K be a chain complex composed of finitely generated free groups. For an arbitrary abelian group G, the sequences*

$$(*)\ 0 \longrightarrow H^r(K) \otimes G \xrightarrow{\alpha} H^r(K; G) \xrightarrow{\beta} \mathrm{Tor}(H^{r+1}(K), G) \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow \mathrm{Ext}(H^{r+1}(K), G) \xrightarrow{\bar{\beta}} H_r(K; G) \xrightarrow{\bar{\alpha}} \mathrm{Hom}(H^r(K), G) \longrightarrow 0$$

are exact and split. These exact sequences are natural with respect to chain maps $f: K \longrightarrow K'$ *and homomorphisms* $\phi: G \longrightarrow H$.

Proof: See exercises F-3, F-4, and G-3 in [11; Chapt. V].

Let $0 \longrightarrow G' \xrightarrow{\phi} G \xrightarrow{\psi} G'' \longrightarrow 0$ be an exact coefficient sequence. $\delta_*: H^r(K; G'') \longrightarrow H^{r+1}(K; G')$ is defined in [11; p.158]. Define the sequence corresponding to this exact coefficient sequence to be the following sequence of groups and homomorphisms:

$$\cdots \longrightarrow H^r(K; G') \xrightarrow{\phi_*} H^r(K; G) \xrightarrow{\psi_*} H^r(K; G'') \xrightarrow{\delta_*} H^{r+1}(K; G') \longrightarrow \cdots.$$

THEOREM 1.3: *Let K be a chain complex composed of free groups. Then the sequence corresponding to the above exact coefficient sequence is exact. This exact sequence is natural with respect to chain maps $f: K \longrightarrow K'$ and homomorphisms of one exact coefficient sequence into another:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G' & \xrightarrow{\phi} & G & \xrightarrow{\psi} & G'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H' & \xrightarrow{\phi'} & H & \xrightarrow{\psi'} & H'' & \longrightarrow & 0. \end{array}$$

Proof: See exercise C-3 in [11; Chapt. V].

2. A New Universal Coefficient Theorem.

The sequence (*) of 1.2 does not hold if K is not finitely generated. We now prove a similar theorem by assuming G is finitely generated with K arbitrary.

LEMMA 2.1: <u>If G is finitely generated and free, then α : $H^r(K) \otimes G \longrightarrow H^r(K; G)$ is an isomorphism.</u>

<u>Proof:</u> α is obviously an isomorphism in case $G = Z$. Furthermore, the functors $H^r(K) \otimes G$ and $H^r(K; G)$ are additive with respect to G and thus commute with finite direct sums [8]. Hence α is an isomorphism if G is finitely generated and free.

THEOREM 2.2: <u>Let K be a chain complex composed of free groups. Let G be finitely generated. Then the sequence (*) of 2.1 is exact. This exact sequence is natural with respect to chain maps f: $K \longrightarrow K'$ and homomorphisms ϕ: $G \longrightarrow H$.</u>

<u>Proof:</u> Using 2.1 and 1.3 of the appendix, the proof is formally the same as that of theorem 8.6 and is thus omitted.

COROLLARY 2.3: <u>The universal coefficient theorem for cohomology</u>

$$0 \longrightarrow H^r(X,A) \otimes G \xrightarrow{\alpha} H^r(X,A; G) \xrightarrow{\beta} \mathrm{Tor}(H^{r+1}(X,A), G) \longrightarrow 0$$

<u>holds in the following cases:</u>

1) <u>simplicial (or cellular) theory for finite complexes and G arbitrary, or not necessarily finite complexes and G finitely generated;</u>

2) <u>singular theory for G finitely generated; and</u>

3) <u>Cech theory for (X,A) compact and G arbitrary, or (X,A) paracompact and G finitely generated.</u>

Proof: This follows immediately from 1.2 and 2.2 of the appendix and the fact that direct limits preserve exactness, $\otimes$, and Tor [8].

3. A Counter-example.

This example shows that the exact sequence of 2.3 of the appendix does not hold in general for singular theory or cellular theory. Let X be an (n - 1)-connected CW-complex such that $H_n(X) = Q$ = the additive group of rationals. Let G = Q. By 1.1 of the appendix, $H^n(X; Z) = Hom(Q, Z) = 0$, and $H^n(X; Q) = Hom(Q, Q) = Q$. However, 2.3 of the appendix would give that $0 \longrightarrow 0 \otimes Q \longrightarrow Q \longrightarrow Tor(H^{n+1}(X), Q) \longrightarrow 0$ is exact; but $Tor(H^{n+1}(X), Q) = 0$, and hence $Q = 0$. This is a contradiction.

4. Proof of Lemma 8.4.

Proof of 8.4:* For the properties of Ext needed in this proof, see [8]. There is a natural homomorphism χ: $\mathrm{Hom}(A,B)\otimes Z_2 \longrightarrow \mathrm{Hom}(A,B\otimes Z_2)$ defined by $[\chi(f\otimes 1)](a) = f(a)\otimes 1$, where $f \varepsilon \mathrm{Hom}(A,B)$ and 1 is the non-zero element of Z_2. We first show that χ is an isomorphism if A is free. Let $A = \sum_i Z$, $i \varepsilon I$. Then $\mathrm{Hom}(A,B) = \mathrm{Hom}(\sum_i Z_i,B) \approx \prod_i \mathrm{Hom}(Z_i,B) = \prod_i B_i$ and $\mathrm{Hom}(A,B\otimes Z_2) = \mathrm{Hom}(\sum_i Z_i,B\otimes Z_2) \approx \prod_i \mathrm{Hom}(Z_i,B\otimes Z_2) = \prod_i (B\otimes Z_2)_i$. Moreover, the natural homomorphism $(\prod_i B_i)\otimes Z_2 \longrightarrow \prod_i (B\otimes Z_2)_i$ is an isomorphism (see exercise E-6 in [11; Chapt. V]). Hence χ is an isomorphism if A is free.

Let $0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0$ be exact, where F and R are free abelian groups. By definition of Ext, $\mathrm{Hom}(F,H) \longrightarrow \mathrm{Hom}(R,H) \longrightarrow \mathrm{Ext}(G,H) \longrightarrow 0$ is exact. Thus the following is a commutative diagram with the rows exact:

$$\begin{array}{ccccccccc}
\mathrm{Hom}(F,H)\otimes Z_2 & \longrightarrow & \mathrm{Hom}(R,H)\otimes Z_2 & \longrightarrow & \mathrm{Ext}(G,H)\otimes Z_2 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(F,H\otimes Z_2) & \longrightarrow & \mathrm{Hom}(R,H\otimes Z_2) & \longrightarrow & \mathrm{Ext}(G,H\otimes Z_2) & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

The first two and the last two vertical homomorphisms are isomorphisms, and hence by the five-lemma [11; p.16], $\mathrm{Ext}(G,H)\otimes Z_2 \approx \mathrm{Ext}(G,H\otimes Z_2)$. Now by hypothesis, $0 \longrightarrow G \xrightarrow{\xi} G$

*This proof was worked out with the help of D. A. Buchsbaum.

is exact, where $\xi(g) = 2g$. Thus $\mathrm{Ext}(G,H) \xrightarrow{\xi^*} \mathrm{Ext}(G,H) \rightarrow 0$ is exact (see [8]), where ξ^* is multiplication by 2. Hence every element of $\mathrm{Ext}(G,H)$ is divisible by 2 and $\mathrm{Ext}(G,H \otimes Z_2) \approx \mathrm{Ext}(G,H) \otimes Z_2 = 0$. This completes the proof.

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