Lectures on the Complex Bordism of Finite Complexes
Applications to Stable Homotopy Theory

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Lec. I: Realizing Integral Homology Classes

Let us denote by $\Omega_*^U(\ )$ the singular bordism homology theory associated to the Thom spectrum M. We recall that for a space X the elements of $\Omega_*^U(X)$ are represented by pairs (M,f) where M is a closed manifold whose stable normal bundle has been given a complex structure and f: $M \to X$ is a map. These pairs are subject to a certain equivalence relation called "cobordism," and the resulting equivalence classes form the elements of $\Omega_*^U(X)$, the element corresponding to (M,f) being denoted by [M,f].

In the special case X = point, the resulting equivalence classes, denoted simply by [M], as the map is completely determined, form a ring under cartesian product, called the complex cobordism ring and denoted by Ω_*^U. This ring was computed by Milnor, based on the fundamental work of Thom, who showed: [5].

Thom (Milnor-Thom): $\Omega_*^U \cong \mathbb{Z}[\{M^{2i}\}]$, $i = 1, 2, \ldots$

Later on we may need a more explicit description of Ω_*^U but for the moment this will suffice.

There is a natural transformation of homology theories

$$\Omega_*^U(X) \to H_*(X; \mathbb{Z})$$

called the Thom homomorphism, which is defined by

$$\mu\ [M,f] = f_*[M] \in H_*(X; \mathbb{Z})$$

where $[M,f] \in \Omega_*^U(X)$ and $[M] \in H_*(M; \mathbb{Z})$ is the fundamental class of M.

The classical problem of "representing" an integral homology class

$$u \varepsilon \quad H_*(X; \mathbb{Z})$$

by a weakly complex manifold is easily seen to be equivalent to the requirement that

$$u \varepsilon \quad \mathrm{Im}\{\mu : \ \Omega_*^U (X) \to H_*(X; \mathbb{Z})\}.$$

In the past the study of so representing homology classes has been done "locally," i.e., one class at a time. The problem that we will be concerned with today is the corresponding "globalization." That is we will seek necessary and sufficient conditions for the Thom homomorphism

$$\mu : \ \Omega_*^U (X) \to H_*(X; \mathbb{Z})$$

to be surjective.

The solution to this problem that I will speak of is part of a joint study with P. E. Conner of the complex bordism homology theory and its applications.

Let us note that $\Omega_*^U(X)$ becomes an Ω_*^U-module by setting

$$[N]*[M,f] = [N\mathrm{x}M, \ f \circ p_M] \in \Omega_*^U (X)$$

$$N\mathrm{x}M \xrightarrow{p_M} M \xrightarrow{f} X.$$

The result that we will establish is the following:

Thom: Let X be a finite cw-complex. Then the Thom homomorphism

$$\mu: \Omega_*^U(X) \to H_*(X; \mathbb{Z})$$

is onto iff the projective dimension of $\Omega_*^U(X)$ as a module over Ω_*^U is 0 or 1.

The restriction that X be a finite complex may be relaxed. Just how far is not clear. By working in a suitable stable category such as Boardman's it would seem that one need only require that X be a (-1)-connected cw-spectrum.

For the proof of the theorem we shall require several preliminary results which we turn to now.

Thm 1 (Dold-Serre-Thom): Let X be a finite complex, then the map

$$\mu: \Omega_*^U(X) \to H_*(X; \mathbb{Z})$$

is an epimorphism mod the class of finite groups, and the induced map

$$\tilde{\mu}: \mathbb{Q} \otimes_{\Omega_*^U} \Omega_*^U(X) \to H_*(X; \mathbb{Q})$$

is an isomorphism.

Proof: The first assertion is an easy consequence of Serre's mod $\mathcal{C}$ theory, while the second follows from the first and elementary considerations of the Dold spectral sequence relating $\Omega_*^U(X)$ to $H_*(X; \mathbb{Z})$. $\square$

An important consequence of Thm 1 that we shall need presently is:

Cory 2: Let X be a finite complex and $u \varepsilon H_*(X; \mathbb{Z})$. Then there exists a non-zero integer n and a bordism class

$\alpha \varepsilon \ \Omega^U_* (X) \mid \mu \ (\alpha) = n.u.$

We shall also need the following technical result:

Prop 3: Let A be a finite complex. Then the following conditions are equivalent:

(1) $H_* (A; \mathbb{Z})$ is a free $\mathbb{Z}\cancel{\Omega^U_*}$-module

(2) $\Omega^U_* (A)$ is a free Ω^U_*-module

(3) $\Omega^U_* (A)$ is a projective Ω^U_*-module.

If any of the above three conditions holds then the Thom homomorphism

$$\mu : \ \Omega^U_* (A) \to H_*(A; \mathbb{Z})$$

is surjective.

Proof: There are numerous ways to prove this. The following arrangement of the proof was suggested to me by Nils Andreas Baas of Århus.

Let us introduce the Dold spectral sequence

$$E^r \Longrightarrow \Omega^U_* (A)$$

$$E^2_{p,q} = H_p \ (A; \Omega^U_q).$$

It follows quite easily from Thm 1 that the differentials in this spectral sequence are torsion valued, i.e. tensorized with $\mathbb{Q}$ the spectral sequence is trivial. As Ω_*^U is a free $\mathbb{Z}$-module we find that (1) implies the collapse of this spectral sequence and hence the edge map, which is the Thom homomorphism

$$\mu : \Omega_*^U (A) \to H_*(A; \mathbb{Z})$$

is seen to be onto. Moreover

$$H_*(A; \mathbb{Z}) \otimes_{\mathbb{Z}} \Omega_*^U = E^2_{*,*} = E^\infty_{*,*} = E^\circ \Omega_*^U (A)$$

is a free Ω_*^U-module, and a simple argument shows that $\Omega_*^U (A)$ must therefore also be a free Ω_*^U-module. Thus (1) $\Longrightarrow$(2).

The implication (2) $\Longrightarrow$(3) is trivial and so it will suffice to establish that (3) $\Longrightarrow$(1). So we assume that $\Omega_*^U (A)$ is a projective Ω_*^U-module. Then it is a direct summand in a free Ω_*^U-module and hence

$$\mathbb{Z} \otimes_{\Omega_*^U} \Omega_*^U (A)$$

is seen to be a free $\mathbb{Z}$-module, i.e., a free graded abelian group.

Now suppose contrary to our desire that $H_*(A; \mathbb{Z})$ is <u>not</u> a free $\mathbb{Z}$-module. Then we may choose a torsion element $u \in H_*(A; \mathbb{Z})$ of minimal dimension. Book-keeping considerations in the Dold spectral sequence show that $u \in E^2_{*,0}$ is an infinite cycle, and as it can never be a boundary we find

$$[u] \neq 0 \in E^\infty_{*,0} = \mathbb{Z} \otimes_{\Omega_*^U} \Omega_*^U (A)$$

is a non-zero torsion element which is impossible. Thus $H_*(A; \mathbb{Z})$ is a free $\mathbb{Z}$-module as desired. $\square$

<u>Remark</u>: One may give a completely algebraic proof that (3)$\Longrightarrow$(2). That is a positively graded projective Ω_*^U-module is always a free Ω_*^U-module.

The final preliminary result that we shall need is a representability property for $\Omega_*^U(\ \ )$. This result is of the type that allows one to form "resolutions" of spaces in a suitable sense relative to the homology theory $\Omega_*^U(\ \ )$. The precise result that we need is the following.

<u>Thm 4</u>: Let X be a finite complex. Then there exists a finite complex A and a map

$$\varphi: A \to \Sigma^t X$$

where Σ^t denotes the t-fold suspension functor, such that

(1) $H_*(A; \mathbb{Z})$ is a free $\mathbb{Z}$-module, and

(2) $\varphi_*: \Omega_*^U(A) \to \Omega_*^U(\Sigma^t X)$ is onto.

We will not present the proof of Thm 4 but only remark it follows from the preceeding proposition, the fact that $H_*(\underset{\sim}{MU}; \mathbb{Z})$ is a free $\mathbb{Z}$-module and some Spanier-Whitehead duality arguments. [1,§2] (see also Lecture I of J. F. Adams Vol. 99 of the Springer lecture notes).

<u>Proof of Main Thm</u>: Clearly there are two parts to the proof. First we will show that if

$$\mu : \Omega^U_* (X) \to H_*(X; \mathbb{Z})$$

is onto then*

$$\text{hom. dim}_{\Omega^U_*} \Omega^U_* (X) \leq 1.$$

Let us recall that the Thom homomorphism

$$\mu : \Omega^U_* (\quad) \to H_*(\quad ; \mathbb{Z})$$

is stable. For the Thom homomorphism is induced by the morphism of spectra

$$\mu : \underset{\sim}{MU} \to K(\mathbb{Z})$$

of the Thom-Milnor spectrum to the integral Eilenberg-Mac Lane spectrum that defines the Thom class of MU. Thus for any integer $s \geq o$ we learn that

$$\mu : \Omega^U_* (\Sigma^s X) \to H_* (\Sigma^s X; \mathbb{Z})$$

is onto.

Next let us apply Thm 4 to choose a finite complex A and a map

$$\varphi : A \to \Sigma^t X$$

for some non-negative integer t, such that

(1) $H_* (A; \mathbb{Z})$ is a free $\mathbb{Z}$-module, and

(2) $\varphi_* : \Omega^U_* (A) \to \Omega^U_* (\Sigma^t x)$ is onto.

Clearly we may replace φ by an inclusion, and thus we obtain the following commutative diagram:

*we are of course writting hom-dim. $_{\Omega^U_*} \Omega^U_* (X)$ for the homological dimension, i.e., projective dimension, of $\Omega^U_* (X)$ as a module over Ω^U_*.

$$\begin{tikzcd}
0 & \Omega_*^U(\Sigma^t X) \arrow[l] \arrow[d, "\mu_X"] & \Omega_*^U(A) \arrow[l, "\varphi_*"'] \arrow[d, "\mu_A"] & \tilde{\Omega}_*^U(\Sigma^t X/A \arrow[l, "\Delta_*"'] \arrow[d, "\mu_{X/A}"] & 0 \arrow[l] \\
& H_*(\Sigma^t X;\mathbb{Z}) \arrow[rr, bend right, start anchor=west, end anchor=east] & H_*(A;\mathbb{Z}) \arrow[l, "\varphi_*"'] & \tilde{H}_*(\Sigma^t X/A;\mathbb{Z}) \arrow[l, "\partial_*"'] &
\end{tikzcd}$$

the short exactness of the top row resulting from the long exact sequence of the pair $(\Sigma^t X, A)$ and the fact that

$$\varphi_*: \Omega_*^U(A) \to \Omega_*^U(\Sigma^t X)$$

is onto. Now recall that

$$\mu_X: \Omega_*^U(\Sigma^t X) \to H_*(\Sigma^t X;\mathbb{Z})$$

is onto, and hence the composite

$$\mu_X \varphi_*: \Omega_*^U(A) \to H_*(\Sigma^t X;\mathbb{Z})$$

is clearly onto. Commutativity of the left hand square therefore yields that

$$\varphi_*: H_*(A;\mathbb{Z}) \to H_*(\Sigma^t X;\mathbb{Z})$$

is certainly onto. Therefore the long homology exact sequence of the pair $(\Sigma^t X, A)$ becomes short exact and we may therefore decorate the diagram above to

$$
\begin{array}{ccccccccc}
o & \leftarrow & \Omega_*^U(\Sigma^t X) & \leftarrow & \Omega_*^U(A) & \leftarrow & \tilde{\Omega}_*^U(\Sigma^t X/A) & \leftarrow & o \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
o & \leftarrow & H_*(\Sigma^t X;\mathbb{Z}) & \leftarrow & H_*(A;\mathbb{Z}) & \leftarrow & \tilde{H}_*(\Sigma^t X,A;\mathbb{Z}) & \leftarrow & o
\end{array}
$$

Now recall that A was chosen so that $H_*(A;\mathbb{Z})$ is/was a free $\mathbb{Z}$-module. As $H_*(\Sigma^t X/A;\mathbb{Z})$ is a $\mathbb{Z}$-submodule, we must have that $\tilde{H}_*(\Sigma^t X/A;\mathbb{Z})$ is a free $\mathbb{Z}$-module also. Therefore by Prop 3 $\Omega_*^U(\Sigma^t X/A)$ is a free Ω_*^U-module and hence

$$o \leftarrow \Omega_*^U(\Sigma^t X) \leftarrow \Omega_*^U(A) \leftarrow \Omega_*^U(\Sigma^t X/A) \leftarrow o$$

is a free resolution of $\Omega_*^U(\Sigma^t X)$. Thus

$$\textbf{hom. dim.}_{\Omega_*^U}\ \Omega_*^U(\Omega^t X) \le 1.$$

as desired.

Let us now consider the converse implication. We will suppose that X is a finite complex with

$$\text{hom. dim.}_{\Omega_*^U}\ \Omega_*^U(X) \le 1.$$

We wish to conclude that the Thom homomorphism

$$\mu:\ \Omega_*^U(X) \to H_*(X;\mathbb{Z})$$

is onto. As the Thom homomorphism is stable, clearly it will suffice to show that is onto for some suspension of X. So by applying Thm 4 we may choose a finite complex A, a non-negative integer t , and may be

$$\varphi:\ A \to \Sigma^t X,$$

which as before we may replace, up to homotopy, by an inclusion, such that

(1) $H_*(A;\mathbb{Z})$ is a free $\mathbb{Z}$-module, and

(2) $\varphi_*: \Omega_*^U(A) \to \Omega_*^U(\Sigma^t X)$ is onto.

We may then introduce the commutative diagram

$$\begin{tikzcd}
0 & \Omega_*^U(\Sigma^t X) \arrow[l] \arrow[d, "\mu_X"] & \Omega_*^U(A) \arrow[l, "\varphi_*"'] \arrow[d, "\mu_A"] & \tilde{\Omega}_*^U(\Sigma^t X/A) \arrow[l, "\Delta_*"'] \arrow[d, "\mu_{X/A}"] & 0 \arrow[l] \\
 & H_*(\Sigma^t X;\mathbb{Z}) \arrow[rr, bend right, "J_*"'] & H_*(A;\mathbb{Z}) \arrow[l, "i_*"'] & \tilde{H}_*(\Sigma^t X/A;\mathbb{Z}) \arrow[l, "\partial_*"'] &
\end{tikzcd}$$

Now let us recall the following elementary property of homological dimension: Suppose Λ is a ring with 1 and M is a Λ-module of homological dimension at most 1. Then whenever P is a projective and

$$f: P \to M$$

an epimorphism, ker f is also a projective Λ-module.

Let us consider how this applies to the present situation. According to our choice of A, $H_*(A;\mathbb{Z})$ is a free $\mathbb{Z}$-module and hence by Prop 3 $\Omega_*^U(A)$ is a free Ω_*^U-module. By hypothesis the homological dimension of $\Omega_*^U(X)$ is at most 1 and hence the exactness of the top row in the above diagram shows that $\Omega_*^U(\Sigma^t X/A)$ is a projective Ω_*^U-module and hence by Prop 3 we learn that $H_*(\Sigma^t X/A;\mathbb{Z})$ is free abelian.

Now suppose contrary to our desire that

$$\mu_X : \Omega_*^U(\Sigma^t X) \to H_*(X;\mathbb{Z})$$

is not onto. Choose $u \varepsilon H_*(X;\mathbb{Z})$ with $u \not\varepsilon \operatorname{Im} \mu_X$ we assert that

$$j_* u \neq o \varepsilon \tilde{H}_*(\Sigma^t X/A;\mathbb{Z}).$$

For suppose to the contrary that $j_*(u) = o$. Then

$$u = i_*(v)$$

for some $v \varepsilon H_*(A;\mathbb{Z})$. However by Prop 3

$$v = \mu_A(\alpha)$$

for some $\alpha \varepsilon \Omega_*^U(A)$ and then

$$u = i_* \mu_A(\alpha) = \mu_X \varphi_*(\alpha)$$

and $u \varepsilon \operatorname{Im} \mu_X$ contrary to our choice of u.

Thus

$$j_* u \neq o \varepsilon \tilde{H}_*(\Sigma^t X/A;\mathbb{Z}).$$

Next recall that according to Cory 2 there is a non-zero integer n and a class $\alpha \varepsilon \Omega_*^U(X)$ such that

$$n.u = \mu_X(\alpha).$$

Since φ_* is onto

$$\alpha = \varphi_*(\beta)$$

and hence

$$n.u = \mu_X \varphi_*(\beta) = i_* \mu_A(\beta).$$

Therefore

$$n.j_*(u) = j_*(nu) = j_*i_* \ (\mu_A(\beta)) = o$$

by exactness. Therefore

$$j_*(u) \neq o \ \varepsilon \ \tilde{H}_*(\Sigma^t X/A; \mathbb{Z})$$

is a non-zero torsion class which is impossible. Hence

$$\mu_X: \ \Omega_*^U \ (\Sigma^t X) \to H_*(\Sigma^t X; \mathbb{Z})$$

is onto as desired and the result now follows by stability. $\Box$

In the time remaining let us examine an application of our main result to the study of involutions as closed weakly complex manifolds. (See e.g. the work of Conner-Floyd

Let us denote by $\mathcal{O}_*^U$ the bordism algebra of arbitrary involutions or closed weakly complex manifolds. There is also the bordism algebra $\Omega_*^U(\mathbb{R}\ P(\infty)\ )$ of free involutions on weakly complex manifolds, and as the notation indicates, this may be identified with the complex bordism of $\mathbb{R}\ P(\infty)$). There is also a relati e rodule M_*^U , of involutions on compact U-manifolds with boundary, where the involution is free on the boundary.

Conner and Floyd have established the existance of the fundamental exact sequence

$$o \to \Omega_*^U \to \mathcal{O}_*^U \to M_*^U \to \tilde{\Omega}_*^U \ (\mathbb{R}P(\infty)) \to o$$

and by analysis of the fixed point set have shown

$$M_*^U \cong \ \Omega_*^U \ (BU).$$

Now $H_*(BU;\mathbb{Z})$ is well known to be torsion free and so by Prop 3 we obtain that M^U_* is a free Ω^U_*-module. It is equally well known that

$$\tilde{H}_i(\mathbb{R}P(\infty);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2 & i \text{ odd} \\ & \text{otherwise} \end{cases}$$

and that the natural inclusion

$$\mathbb{R}P(2j+1) \to \mathbb{R}P(\infty)$$

represents the non-zero element of $H_{2j+1}(\mathbb{R}P(\infty);\mathbb{Z})$. As $\mathbb{R}P(2j+1)$ is a closed weakly complex manifold it follows that

$$\mu : \Omega^U_*(\mathbb{R}P(\infty)) \to H_*(\mathbb{R}P(\infty);\mathbb{Z})$$

is onto

$$\text{hom-dim}_{\Omega^U_*} \tilde{\Omega}^U_*(\mathbb{R}P(\infty)) \leq 1.$$

The long exact sequence above yields the exact sequences

$$0 \to \Omega^U_* \to \Theta^U_* \to \tilde{\Theta}^U_* \to 0$$

$$0 \to \tilde{\Theta}^U_* \to M^U_* \to \tilde{\Omega}^U_*(\mathbb{R}P(\infty)) \to 0$$

The last of these shows $\tilde{\Theta}^U_*$ to be a projective Ω^U_*-module. Hence the first show Θ^U_* to be a projective Ω^U_*-module. However at the end of Cory 2 we remarked that projective positively graded Ω^U_* modules are free and hence we have shown:

Thm 5: Θ_*^U is a free Ω_*^U-module. $\square$

Next time we will return to consider some additional consequences of the representability property described in Prop 4.

Lec II. Bordism Resolutions, Generators for Bordism Modules and Related Topics

Last time we discussed when the Thom homomorphism

$$\mu: \Omega_*^U (X) \to H_* (X; \mathbb{Z})$$

was surjective. A cornerstone of our discussion was a certain representability property for the complex bordism homology theory. By iterating this representability property one may form "resolutions" in the sense of the following:

Def: Let X be a finite complex. A partial U-bordism resolution of X consists of

$$\emptyset = A_{-1} \subset A_\circ \subset A_1 \subset \ldots \subset A_k \sim \Sigma^t X$$

such that

(1) $\Omega_*^U (A_i, A_{i-1})$ is a projective Ω_*^U-module for $i = o, 1, \ldots, k-1$, and

(2) $\Omega_*^U (A_i, A_{i-1}) \to \Omega_*^U (A_k, A_{i-1})$ is an epimorphism for $i = o, 1, \ldots, k$.

As each of the triples (A_k, A_i, A_{i-1}) yield a short exact bordism exact sequence

$$o \leftarrow \Omega_*^U(A_k,A_{i-1}) \leftarrow \Omega_*^U(A_i,A_{i-1}) \leftarrow \Omega_*^U(A_k,A_i) \leftarrow o$$

these may be pasted together to yield the long exact sequence

$$\begin{array}{l} \qquad \Omega_*^U(\Sigma^t X) \\ \qquad\quad \updownarrow\cong \\ o \leftarrow \Omega_*^U\ (A_k,A_{-1}) \leftarrow \Omega_*^U\ (A_o,A_{-1}) \leftarrow \ldots \leftarrow \Omega_*^U\ (A_i,A_{i-1}) \leftarrow \ldots \\ \qquad\qquad\qquad\qquad\qquad\qquad \leftarrow \Omega_*^U\ (A_k,\ A_{k-1}) \leftarrow o. \end{array}$$

Note that each of the modules

$$\Omega_*^U\ (A_i,\ A_{i-1})\ :\ \ i = o,\ 1, \ldots,\ k-1$$

are projective Ω_*^U-modules. Only the last module

$$\Omega_*^U\ (A_k,\ A_{k-1})$$

need not be projective. If it is, we call $\emptyset = A_1 \ldots\ A_K \sim \Sigma^t X$ a bordism resolution. However as we may construct such partial resolutions of arbitrary length, i.e., with k as large as we wish, it follows from the following Proposition that by choosing k large enough we may make an actual bordism resolution of X. Note that the above sequence is then an actual projective resolution of $\Omega_*^U(\Sigma^t X)$.

Prop 1: [1; §2], [7]. Let X be a finite complex. Then $\Omega_*^U(X)$ is a finitely generated Ω_*^U-module of finite projective dimension. **

The proof of this result depends on Milnor's theorem that

$$\Omega_*^U \cong \mathbb{Z}[[M^2], [M^4], \ldots]$$

and hence is a coherent ring [7]. Elementary properties of coherence suffice to prove the proposition by induction on the number of cells.

It is perhaps of interest to note that the corresponding result is false for $\Omega_*^{So}(\quad)$ bordism and $\Omega_*^{fr}(\quad)$ bordism. For one knows that $\Omega_*^{So}(\mathbb{R}P(2))$ may be identified with the Wall algebra W_* whose Ω_*^{So}-module structure may be read off from Wall's work and one finds that it is not a finitely generated Ω_*^{So}-module. In the case of framed bordism, i.e., stable homotopy theory one knows that if $\Omega_*^{fr}(X)$ has finite projective dimension then it is actually a free module over $\Omega_*^{fr}(pt) = \pi_*^S$. So for example $\Omega_*^{fr}(\mathbb{R}P(2))$ has infinite projective dimension as an Ω_*^{fr}-module.

Let us suppose now that

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_k \sim \Sigma^t X$$

is an actual bordism resolution of X. Regarding it as a filtered space we may form its homology exact couple. After a suitable reindexing we obtain:

Thm 2: Let X be a finite complex. Then there exists a natural first quadrant homology spectral sequence $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ such that

$$E^r\langle X\rangle \Longrightarrow H_*(X;\mathbb{Z})$$

$$E^2_{p,q}\langle X\rangle = \operatorname{Tor}^{\Omega^U_*}_{p,q}(\mathbb{Z}, \Omega^U_*(X)).$$

Outline of Proof: We note that $E^2\langle X\rangle$ is by definition the homology of the chain complex

$$0 \leftarrow H_*(A_0, A_{-1}) \leftarrow \cdots \leftarrow H_*(A_i, A_{i-1}) \leftarrow \cdots \leftarrow H_*(A_k, A_{k-1}) \leftarrow 0$$

suitably reindexed. As each

$$\Omega^U_*(A_i, A_{i-1}): \quad i = 0, 1, \ldots k$$

is a projective Ω^U_* - module it follows from last time that

$$H_*(A_i, A_{i-1}; \mathbb{Z}): \quad i = 0, 1, \ldots, k$$

is a free $\mathbb{Z}$-module. Thus the Dold spectral sequences for (A_i, A_{i-1}) collapse and the edge map

$$\tilde{\mu}: \mathbb{Z} \otimes_{\Omega^U_*} \Omega^U_*(A_i, A_{i-1}) \to H_*(A_i, A_{i-1}; \mathbb{Z}):$$

$$i = 0, \ldots, k$$

is isomorphic. Thus $E^2 \langle X \rangle$ is the homology of the chain complex

$$\mathbb{Z} \otimes_{\Omega_*} U\Omega_*^U(A_o, A_{-1}) \leftarrow \ldots \mathbb{Z} \otimes_{\Omega_*} U\Omega_*^U(A_i, A_{i-1}) \leftarrow \ldots \mathbb{Z} \otimes_{\Omega_*} U\Omega_*^U(A_k, A_{k-1}) \leftarrow o$$

and as

$$o \leftarrow \Omega_*^U(A_k, A_{-1}) \leftarrow \Omega_*^U(A_o, A_{-1}) \leftarrow \ldots \leftarrow \Omega_*^U(A_i, A_{i-1}) \leftarrow \ldots \leftarrow \Omega_*^U(A_k, A_{k-1}) \leftarrow o$$

is actually a projective resolution of $\Omega_*^U(\Sigma^t X)$ the result follows form the definition of the functor Tor. $\square$

Note that the edge homomorphism

$$\mathbb{Z} \otimes_{\Omega_*} U\Omega_*^U(X) \xrightarrow[\tilde{\mu}]{} H_*(X; \mathbb{Z})$$

is the "reduced" Thom homomorphism. Thus the results of last time may be conveniently rephrased in terms of this spectral sequence. The following technical result is fundamental in this respect.

Thm 3: Let X be a finite complex and $n > o$. Then the following conditions are equivalent:

(1) hom. dim. $_{\Omega_*} U\Omega_*^U(X) \leq n$.

(2) $\mathrm{Tor}_{n,*}^{\Omega_*^U}(\mathbb{Z}, \Omega_*^U(X)) = o$

(3) $\mathrm{Tor}_{j,*}^{\Omega_*^U}(\mathbb{Z}, \Omega_*^U(X)) = o$: $j \geq n$.

The proof of these results is not difficult but due to being rather long and technical arguments is omitted. [1;§4]. Note that this improves by a factor of 1 results to be expected on general homological grounds.

Notice that the results of last time fall neatly under the case n = 1. Namely

Cory 4: Let X be a finite complex. Then the following conditions are equivalent:

(1) $\Omega_*^U(X) \to H_*(X;\mathbb{Z})$ is into;

(2) $\mathbb{Z} \otimes_{\Omega_*^U} \Omega_*^U(X) \to H_*(X;\mathbb{Z})$ is an isomorphism;

(3) hom. dim $_{\Omega_*^U} \Omega_*^U(X) \leq 1$.

(4) $\mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbb{Z}, \Omega_*^U(X)) = 0$

(5) $\mathrm{Tor}_{j,*}^{\Omega_*^U}(\mathbb{Z}, \Omega_*^U(X)) = 0 : j \geq 1$. $\square$

Having constructed the spectral sequence of Thm 2, it is important to inquire into some examples where it is non-trivial. As a start in this direction we note

Thm 5: Let n be a positive integer and let X be either

(1) a large skeleton of $K(\mathbb{Z}_2, n)$

(2) $\mathbb{R}P(2^n) \times \ldots \times \mathbb{R}P(2^n)$.

Then

$$\text{hom. dim}_{\Omega_*^U} \Omega_*^U(X) \geq n$$

and hence

$$E^2_{j,*}\langle X\rangle = \mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbb{Z},\ \Omega^U_*(X)) \neq 0:\quad j \leq n-1.$$

The proof is not hard and uses either

(1) some properties of the Steenrod algebra.

(2) some computations of Conner and Floyd to study the annihilator ideal of

(1) $i \varepsilon\ \Omega^U_n(K(\mathbb{Z}_2,\ n))$

(2) $\gamma \varepsilon\ \Omega^U_n(\mathbb{R}P(2^n) \times \ldots \times \mathbb{R}P(2^n))$

to obtain the desired conclusion. [1;§5], [2], [3].

These examples show that it is possible for $\{E^r\langle\ \ \rangle, d^r\langle\ \ \rangle\}$ to be non-trivial but of themselves are not conclusive. Before going on to produce such an example let us note one more consequence of (2) and (3).

Cory 6: Let X be a finite complex with $\Omega^U_*(X)$ of projective dimension at most 2. Then there is a natural exact sequence

$$0 \to \mathbb{Z} \otimes_{\Omega^U_*} \Omega^U_*(X) \to H_*(X;\mathbb{Z}) \to \mathrm{Tor}^{\Omega^U_*}_{1,*}(\mathbb{Z},\Omega^U_*(X)) \to 0.$$

Hence $\Omega^U_*(X)$ is generated as an Ω^U_* - module by classes of degree at most dim X.

Proof: According to Thm 3

$$E^2_{j,*}\langle X\rangle = \mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbb{Z},\Omega^U_*(X)) = 0:\quad j \geq 2.$$

Thus the exact sequence. The second assertion follows from the injective nature

$$\mathbb{Z} \otimes_{\Omega_*^U} \Omega_*^U (X) \to H_* (X; \mathbb{Z}),$$

the fact that the target madule vanishes in degree > dim X, and that the domain module may be interperted in a standard manner as a set of generators for $\Omega_*^U (X)$. $\square$

Before going further it makes sense to inquire whether indeed there are any examples of finite complexes X with $\Omega_*^U (X)$ of projective dimensions exactly 2 so that we have some non-trivial examples of Cory 6.

The answer is yes and the method of constructing a flock of such examples leads to a whole cavalcade of new and in many cases unanswered questions.

To construct these examples let $\eta \epsilon \pi_1^S \cong \mathbb{Z}_2$ denote the non-zero element in the stable 1-stem. Then for k - large the composite

$$S^{k+1} \xrightarrow{\eta} S^k \xrightarrow{2} S^k$$

is null homotopic. Hence we may form a coextension

$$\tilde{\eta}: \quad S^{k+2} \to S^k \cup_2 e^{k+1}$$

It may be shown by explicit computation that

$$[S^{k+2}, \tilde{\eta}] = [\mathbb{C}P(1)] \cdot \sigma_k$$

where

$$\sigma_k = [S^k, i] \in \Omega_k^U (S^k \cup_2 e^{k+1}).$$

Let W be the mapping cone of $\tilde{\eta}$. Then one finds

$$\tilde{\Omega}_*^U (W) \cong \Omega_*^U/(2, [\mathbb{C}P(1)]) \quad \sigma_k \oplus \Omega_*^U e_{2\,k+3}$$

and

$$\tilde{H}_i (W; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 : & i = k \\ \mathbb{Z} : & i = k+3 \\ 0 : & \text{otherwise} \end{cases}$$

The projective dimension of $\tilde{\Omega}_*^U (W)$ is exactly 2 and the generator is $H_{k+3} (W; \mathbb{Z})$ is not representable while twice it is.

Applying the reasoning of Cory 6 we find that regardless of the homological dimension of $\Omega_*^U (X)$ the collapse of the spectral sequence (2) implies that

$$\mathbb{Z} \otimes_{\Omega_*^U} \Omega_*^U (X) \to H_* (X; \mathbb{Z})$$

is monic and hence that as an Ω_*^U - module, $\Omega_*^U (X)$ is generated by classes of degree at most dim X. Thus if we find a finite complex V for which it is <u>not</u> possible to choose generators for $\Omega_*^U (V)$ of dimension at most dim V, we will also have found a finite complex V for which the spectral sequence (2) is non-trivial. It is now time to reveal how to construct such a complex.

Let $\alpha_1 \in \pi_3^S \cong \mathbb{Z}_{24}$ be the element of order 3. As before we may obtain a coextension

$$\bar{\alpha}_1 : \; S^{k+4} \to S^k \cup_3 e^{k+1}$$

of ${}_1$. This in turn may be extended to a map

$$\tilde{\alpha}_1 : S^{k+4} \cup_3 e^{k+5} \to S^k \cup_3 e^{k+1}$$

(for $\{p, \alpha_1, p\} = o$ according to Toda [11]), (which should be familiar to readers of Adams J(X) IV paper). Let V(1) denote the mapping cone of $\tilde{\alpha}_1$. (The rationale for the notation will be explained next time.) Now it may be shown that [9].

$$\tilde{\Omega}_*^U (V(1)) \cong \Omega_*^U/(3, [\mathbb{C}P(2)]) . \sigma_k .$$

Let $\beta_1 \in \pi_{10}^S \cong \mathbb{Z}_6$ denote the element of order 3. By explicit computation we construct a map [9] [12]

$$\bar{\beta}_1 : \; S^{k+16} \to S^k \cup_3 e^k \cup_{\tilde{\alpha}_1} e^{k+5} \cup_3 e^{k+6}$$

such that

$$\begin{tikzcd}
S^{k+16} \arrow[r, "\bar{\beta}_1"] \arrow[dr, "\beta_1"'] & S^k \cup_3 e^k \cup_{\tilde{\alpha}_1} e^{k+5} \cup_3 e^{k+6} \arrow[d, "q"] \\
& S^{k+6}
\end{tikzcd}$$

commutes. We thus receive the bordism class

$$[S^{k+16}, \bar{\beta}_1] = [M^{16}]\sigma_k \in \tilde{\Omega}^U_{16}(V(1)).$$

The characteristic numbers of the manifold M^{16} may be computed by use of the RR theorem and one finds that $[M^{16}] \in \Omega^U_{16}$ is an acceptable generator for the ring $\Omega^U_* \cong \mathbb{Z}[[M^2], [M^4], \ldots]$.

Now let $V(1\frac{1}{4})$ denote the mapping cone of $\bar{\beta}_1$. Thus we have the cofibration

$$S^{k+16} \xrightarrow[\beta_1]{} V(1) \xrightarrow[j]{} V(1\tfrac{1}{4})$$

from which we obtain the exact triangle

$$\begin{tikzcd}
\tilde{\Omega}^U_*(S^{k+16}) \arrow[rr, "\bar{\beta}_{1*}"'] & & \tilde{\Omega}^U_*(V(1)) \arrow[dl, "j_*"] \\
& \tilde{\Omega}^U_*(V(1\frac{1}{4})) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

We note that

$$\mathrm{Im}\,\partial_* = \ker \bar{\beta}_{1*} = \{[M]\, i_{k+6} \mid [M][M^{16}]\,\sigma_k = o\}$$

i.e. $\mathrm{Im}\,\partial_* \cong \Sigma^{k+16} A([M^{16}]\ \sigma_k)$, the k+16 - fold suspension of the anihilator ideal of the class

$$[S^{k+16}, \bar{\beta}_1] = [M^{16}]\ \sigma_k \in \Omega^U_{16+k}(V(1)).$$

As the structure of the module $\tilde{\Omega}^U_*(V(1))$ is known (it was written down above) we find

$$A([M^{16}]\sigma_k) = (3, [\mathbb{C}P(2)]).$$

From the epimorphism

$$\Omega_*^U(V(1\tfrac{1}{4})) \to \Sigma^{16} A([M^{16}]\sigma_k) \to o$$

we therefore find that $\Omega_*^U(V(1\frac{1}{4}))$ contains an indecomposable class of dimension 18 corresponding to $\Sigma^{16}[\mathbb{C}P(2)]$. But the cell structure of $V(1\frac{1}{4})$ is

$$V(1\tfrac{1}{4}) = S^k \cup_3 e^k \cup_{\tilde{\alpha}_1} e^{k+5} \cup_3 e^{k+6} \cup_{\bar{\beta}_1} e^{k+17}$$

which shows that dim $V(1\frac{1}{4}) = 17<18$. Thus indeed $\Omega_*^U(V(1\frac{1}{4}))$ must contain a generator of dimension in excess of dim $V(1\frac{1}{4}))$ and hence the spectral sequence $\{E^r\langle V(1\frac{1}{4})\rangle, d^r\langle V(1\frac{1}{4})\rangle\}$ is non-trivial. Note that as

$$\text{hom. dim}_{\Omega_*^U} \Omega_*^U(V(1\tfrac{1}{4})) = 3$$

this is in a sense the simplest example of this phenomenon.

The previous results and examples suggest that the following might make an interesting problem: Give an a priori estimate of the highest dimension an indecomposable class occurs in $\Omega_*^U(X)$ in terms of "other" invariants of X.

In this connection let me mention in closing a recent result proved jointly with P.E. Conner.

Thm 7: Let X be a finite complex of dimension d with $\Omega_*^U(X)$ of projective dimension at most 3. Then $\Omega_*^U(X)$ is generated as an Ω_*^U-module by classes of dimension at most 2d+1.

The proof is not difficult but makes use of the relation between $\Omega_*^U(\quad)$ and connective $k_*(\quad)$ theory which is analogous to that discussed for homology. The details are not hard but will be deferred to another occasion. (See the appendix to this lecture).

Appendix to Lecture II: The Relation of Cobordism to K-Theories

There are two varieties of K-theory that will enter into our discussion. First of all there is the classical $\mathbb{Z}_2$ graded K-theory denoted by $K_*(\quad)$ which in its $\mathbb{Z}$-graded form is the homology theory associated to the spectrum

$$\underset{\sim}{BU} = \{\ldots, BU, \ U, \ BU, \ U \quad \ldots\}.$$

There is second of all the homology theory associated to the connective bu - spectrum

$$\underset{\sim}{bu} = \{BU, U \quad \ldots, BU(2n \ldots \infty), U(2n+1 \ldots \infty), \ldots\}$$

whose associated homology theory is denoted by $k_*(\quad)$, and called little K-theory. There are morphisms of ring spectra

$$\begin{tikzcd} & bu \arrow[d, "\lambda"] \\ MU \arrow[ur, "\eta"] \arrow[r, "\hat{\zeta}"'] & BU \end{tikzcd}$$

where ζ is the standard K-theory orientation of MU, η its connective analog, and λ is a sort of localization map. We recall

$$\left.\begin{aligned} k_*(pt) &= \mathbb{Z}[t] \\ K_*(pt) &= \mathbb{Z}[t, t^{-1}] \end{aligned}\right\} \quad \deg t = 2$$

$$\lambda : \mathbb{Z}[t] \to \mathbb{Z}[t, t^{-1}]$$

is the usual inclusion and

$$\eta: \ \Omega_*^U \to \mathbb{Z}[t] \ : \quad \eta[M^{2n}] = Td[M]t^n$$

where Td denotes the Todd genus

The relation between complex bordism and K-theory discovered by Conner and Floyd may be put as follows.

<u>Thm 1</u>: Let X be a finite complex. Then the natural map

$$\tilde{\zeta}: \mathbb{Z}[t, t^{-1}] \otimes_{\Omega_*^U} \Omega_*^U(X) \to K_*(X)$$

is an isomorphism.

<u>Outline of Proof</u>: We shall need a few preliminary steps.

<u>Prop A</u>: If A is a finite complex with torsion-free homology then the natural map

$$\tilde{\xi}: \mathbb{Z}[t, t^{-1}] \otimes_{\Omega_*^U} \Omega_*^U (A) \to K_* (A)$$

is an isomorphism.

<u>Proof</u>: One has that the Dold-Atiyah-Hirzebruch spectral sequences for $\Omega_*^U (A)$ and $K_* (A)$ both collapse while $\tilde{\xi}$ induces an isomorphism of the terms E^2. The result easily follows. **

<u>Prop B</u>: If X is a finite complex then there exists a finite complex A, an integer t, and a map

$$f: A \to \Sigma^t X$$

such that

(1) $H_* (A; \mathbb{Z})$ is free abelian

(2) $K_* (A) \xrightarrow[f_*]{} K_*(\Sigma^t X)$ is onto

<u>Proof</u>: The proof is similar to the analagous representability property for $\Omega_*^U (\quad)$ and is skipped. **

To prove the theorem we proceed in two parts. We first show that $\tilde{\xi}$ is always onto. To this end we let X be a finite complex and we use Prop B to choose

$$f: A \to \Sigma^t X$$

so that

(1) $H_*(A;\mathbb{Z})$ is free abelian

(2) $K_*(A) \xrightarrow[f_*]{} K_*(\Sigma^t X)$ is onto.

From the commutative diagram

$$\begin{array}{ccc} \Omega_*^U(A) & \to & \Omega_*^U(\Sigma^t X) \\ \downarrow & & \downarrow \\ K_*(A) & \to & K_*(\Sigma^t X) \end{array}$$

we obtain in view of (2) and Prop A the diagram

$$\begin{tikzcd} \mathbb{Z}[t,t^{-1}] \otimes_{\Omega_*^U} \Omega_*^U(A) \arrow[r] \arrow[d, "\cong"', "\tilde{\zeta}_A"] & \mathbb{Z}[t,t^{-1}] \otimes_{\Omega_*^U} \Omega_*^U(\Sigma^t X) \arrow[d, "\tilde{\zeta}"] & \\ K_*(A) \arrow[r] & K_*(\Sigma^t X) \arrow[r] & 0 \end{tikzcd}$$

and hence

$$\tilde{\zeta}: \mathbb{Z}[t,t^{-1}] \otimes_{\Omega_*^U} \Omega_*^U(\Sigma^t X) \to K_*(\Sigma^t X)$$

is onto and the result follows from stability.

Finally we must show that $\tilde{\zeta}$ is always monic. This we do by induction on hom. dim. $_{\Omega_*^U} \Omega_*^U(X)$, the case where the projective dimension is 0 being taken care of by Prop A. We thus suppose that hom. dim $_{\Omega_*^U} \Omega_*^U(X) = n>o$ and that the result is already established for n-1. Using the representability property of $\Omega_*^U(\quad)$ we choose

$$g\colon\ B \to \Sigma^s X$$

such that

(1) $H_*(B;\mathbb{Z})$ is free abelian

(2) $\Omega^U_*(B) \xrightarrow[g_*]{} \Omega^U_*(\Sigma^s X)$ is onto.

we thus obtain a short exact sequence

$$o \leftarrow \Omega^U_*(\Sigma^s X) \leftarrow \Omega^U_*(B) \leftarrow \tilde{\Omega}^U_*(\Sigma^s X/B) \leftarrow o$$

that shows hom. dim $_{\Omega^U_*}\tilde{\Omega}^U_*(\Sigma^s X/B) = n-1$. Consider the diagram

$$\begin{tikzcd}
o & \mathbb{Z}[b,t^{-1}]\otimes_{\Omega^U_*}\Omega^U_*(\Sigma^s X) \arrow[l] \arrow[d, "\tilde{\zeta}_1"', "\text{onto}"] & \mathbb{Z}[t,t^{-1}]\otimes_{\Omega^U_*}\Omega^U_*(B) \arrow[l, "\tilde{g}_*"'] \arrow[d, "\tilde{\zeta}_2", "\cong"'] & \mathbb{Z}[t,t^{-1}]\otimes_{\Omega^U_*}\tilde{\Omega}^U_*(\Sigma^s X/B) \arrow[l] \arrow[d, "\tilde{\zeta}_3\ (\text{inductive})", "\cong"'] \\
 & K_*(\Sigma^s X) \arrow[rr, bend right, "\partial_*"'] & K_*(B) \arrow[l, "g_*"'] & \tilde{K}_*(\Sigma^s X/B) \arrow[l, "h_*"']
\end{tikzcd}$$

Now note:

$$\tilde{g}_* \quad \text{and} \quad \tilde{\zeta}_1 \quad \text{are onto.}$$

Therefore by commutativity of the left hand square g_* is onto. Hence $\partial_* = o$ and the diagram has become

$$\begin{array}{ccccccccc}
0 \leftarrow \mathbb{Z}[t,t^{-1}] & \otimes_{\Omega_*^U\Omega_*^U} \Omega_*^U(\Sigma^s X) & \leftarrow & \mathbb{Z}[t,t^{-1}] \otimes_{\Omega_*^U\Omega_*^U} \Omega_*^U(B) & \leftarrow & \mathbb{Z}[t,t^{-1}] \otimes_{\Omega_*^U\Omega_*^U} \Omega_*^U(\Sigma^s X/B) & \leftarrow 0 \\
& \text{onto} \downarrow \tilde{\zeta}_1 & & \cong \downarrow \tilde{\zeta}_2 & & \cong \downarrow \tilde{\zeta}_3 & \\
0 \longleftarrow & K_*(\Sigma^s X) & \longleftarrow & K_*(B) & \longleftarrow & K_*(\Sigma^s X)B) & \longleftarrow 0
\end{array}$$

and thus the five lemma and stability shows that

$$\tilde{\zeta} : \mathbb{Z}[b,t^{-1}] \otimes_{\Omega_*^U\Omega_*^U} (X) \to K_*(X)$$

is iso. This completes the inductive step and as hom. dim $_{\Omega_*^U\Omega_*^U}(X) < \infty$ for a finite complex [1; §2] the result follows. $\Box$

The relation between $\Omega_*(\ )$ and little K-theory $k_*(\ )$ follows more the pattern of the relation between bordism and homology. First by forming the $k_*(\ )$ exact couple of a bordism resolution and suitably reindexing we obtain:

<u>Thm 2</u>: Let X be a finite complex. Then there exists a natural first quadrant homology spectral sequence

$\{E^r[X], d^r[X]\}$ with

$$E^r[X] \Longrightarrow k_*(X)$$

$$E^2_{p,q}[X] = \mathrm{Tor}^{\Omega_*^U}_{p,q}(\mathbb{Z}[t], \Omega_*^U(X)).$$

The edge map

$$E^2_{o,*}[X] \to k_*(X)$$

coinciding with $\tilde{\zeta}$. **

By analogy with the situation for hom. dim 1 and homology we obtain [1; §10]:

Thm 3: Let X be a finite complex. Then

$$\tilde{\zeta}: \mathbb{Z}[t]\otimes_{\Omega_*^U}\Omega_*^U(X) \longrightarrow k_*(X)$$

is an isomorphism iff

$$\text{hom. dim}_{\Omega_*^U}\ \Omega_*^U(X) \leq 2. \qquad **$$

The proof of Thm 3 is similar to its homology analog. The theorem of Conner-Floyd (Thm 1) enters to establish certain key technical results, viz. [1; §10]:

Thm 4: Let X be a finite complex and $n > o$. Then the following conditions are equivalent

(1) $\text{hom. dim.}_{\Omega_*^U}\Omega_*^U(X) \leq n+1.$

(2) $\text{Tor}_{n,*}^{\Omega_*^U}(\mathbb{Z}[t], \Omega_*^U(X)) = o$

(3) $\text{Tor}_{j\,*}^{\Omega_*^U}(\mathbb{Z}[t], \Omega_*^U(X)) = o: \ j \geq n.$ **

Thus we obtain:

Cory 5: Suppose that X is a finite complex with hom. dim. $_{\Omega_*^U}\Omega_*^U(X) \leq 3$. Then there is a natural exact sequence

$$0 \to \mathbb{Z}[t] \otimes_{\Omega^U_*} \Omega^U_* (X) \to k_*(X) \to \mathrm{Tor}^{\Omega^U_*}_{1,*}(\mathbb{Z}[t], \ \Omega^U_* (X)) \to 0.$$

Proof: This results from the spectral sequence of Thm 2 upon noting that

$$E^2_{j\ *}[X] = 0 \qquad j > 1$$

according to Thm 4. $\square$

We are now prepared to outline a proof of the theorem stated at the end of lecture II dealing with the dimension of generator of complex bordism modules. We recall the precise result is:

Thm Let X be a finite complex of dimension d with $\Omega^U_*(X)$ of projective dimension at most 3 over Ω^U_*. Then $\Omega^U_*(X)$ is generated by classes of degree at most 2d+1.

Outline of Proof: Let us first observe that according to the co bordism version of Thm 1 the map

$$\Omega^*_U(Y) \to K^*(Y)$$

is seen to be surjective for all finite complexes Y whenever $* < 2$. From the commutative diagram

$$\begin{tikzcd} & k^*(Y) \arrow[d, "\lambda"] \\ \Omega^*_U(Y) \arrow[ur, "\eta"] \arrow[r, "\zeta"'] & K^*(Y) \end{tikzcd}$$

and the fact that λ is an isomorphism for $* < 2$ we therefore learn that

$$\eta: \quad \Omega_U^*(Y) \to k^*(Y)$$

is onto for $* < 2$.

Now it is possible to embed X in S^{2d+1}. Letting $Y \subset S^{2d+1}$ be a finite complex that is a deformation retract of $S^{2d+1}-X$ we obtain the commutative diagram

$$\begin{array}{ccc} \Omega_*^U(X) & \xleftrightarrow[\cong]{D} & \Omega_U^{2d+1-*}(Y) \\ \downarrow\zeta & & \downarrow\zeta \\ k_*(X) & \xleftrightarrow[\cong]{D} & k^{2d+1-*}(Y) \end{array}$$

where D is the Spanier-Whitehead duality isomorphism. Thus we learn that

$$(*) \qquad \Omega_*^U(X) \to k_*(X) \text{ is onto: } * > 2d-1.$$

Suppose now to the contrary that $u \varepsilon \Omega_j^U(X)$ is an indecomposable class with $j > 2d+1$. Then

$$u \neq o \; \varepsilon [\mathbb{Z} \otimes_{\Omega_*^U} \Omega_*^U(X)]_j$$

and as $H_j(X; \mathbb{Z}) = o$ it follows that u must be killed in the spectral sequence

$$E^r\langle X\rangle \Longrightarrow H_*(X;\mathbb{Z})$$

$$E^2_{p-q}\langle X\rangle = \mathrm{Tor}^{\Omega^U_*}_{p-q}(\mathbb{Z}, \Omega^U_*(X)).$$

Now according to Thm 3 of Lec. 2

$$\mathrm{Tor}^{\Omega^U_*}_{p,q}(\mathbb{Z}, \Omega^U_*(X)) = o: \quad p>2$$

and hence the only possible way that

$$u \in E^2_{o,j}\langle X\rangle$$

can die in the spectral sequence is for there to be a

$$v \in E^2_{2,\ j-1}\langle X\rangle$$

with

$$d^2 v = u.$$

Consider now the exact sequence of Ω^U_* - modules

$$o \to \mathbb{Z}[t] \to \mathbb{Z}[t] \to \mathbb{Z} \to o$$

Applying the functer $-\otimes_{\Omega^U_*}\Omega^U_*(X)$ and Thm 4 above we obtain the exact sequence

$$o \to \mathrm{Tor}_{2,*}^{\Omega^U_*}(\mathbb{Z}, \Omega^U_*(X)) \to \mathrm{Tor}_{1\ *}^{\Omega^U_*}(\mathbb{Z}[t], \Omega^U_*(X)) \to \ldots.$$

$$\ldots \to \mathbb{Z}\otimes_{\Omega^U_*}\Omega^U_*(X) \to o.$$

The zero on the left results from the case n = 2 of Thm 4.

Hence the element

$$v \varepsilon \quad \mathrm{Tor}_{2,j-1}^{\Omega_*^U} (\mathbb{Z}, \Omega_*^U(X))$$

determines a non-zero element

$$w \varepsilon \quad \mathrm{Tor}_{1,j-3}^{\Omega_*^U} (\mathbb{Z}[t], \Omega_*^U(X)).$$

Note that m_t has degree 2 and hence the connecting morphism is of bi degree (-1, -2).

The exact sequence of Cory 5

$$o \to \mathbb{Z}[t] \otimes_{\Omega_*^U} \Omega_*^U(X) \xrightarrow{\tilde{\zeta}} k_*(X) \to \mathrm{Tor}_{1\ *}^{\Omega_*^U} (\mathbb{Z}[t], \Omega_*^U(X)) \to o$$

then shows that w determines a non-zero element

$$w \varepsilon \quad k_{j,2}(X)$$

that is not in the image of $\tilde{\zeta}$. As j>2d+1, j-2>2d-1 and this is contrary to (*) and hence u cannot exist yielding the result. $\square$

Lec. III Detecting Homotopy Elements by Realizing Bordism Modules

Our objective in this and the next lecture will be to explain how our investigations into realizing bordism modules by spaces has led to the rediscovery of an old infinite family in the stable homotopy of spheres and the discovery of a new family. The new family that we have found consists of classes of order p

$$[\Psi_t] \in \pi^S_{2p(tp-1)-2t}: \quad t = 1, 2, \ldots$$

where p is a prime strictly greater than 3. As of the present moment I am unable to determine what, if any, are the corresponding elements for $p = 3$. The elements $[\Psi_t]$ all lie in the cokernel of the J-homomorphism. Upon observing that

$$2p(tp-1)-2t = 2(p-1)(tp+t-1)-2$$

and consulting Todas tables, it is natural to conjecture that appropriately constructed, the elements $[\Psi_t]$ generalize the elements

$$\beta_1, \ldots, \quad \beta_{p-1} : \quad \beta_t \in \pi^S_{2(p-1)(tp+t-1)-2}$$

constructed by Toda. This is certainly true for β_1 as the p-component of $\pi^S_{2(p-1)p-2}$ is cyclic of order p with generator either β_1 or $[\Psi_1]$.

I believe that my method of construction shows that

$$[\Psi_{t+1}] \in \{\Psi_t, p, \alpha_1, p, \Psi_1\}$$

where $\alpha_1 \in \pi^S_{2p-3}$ is the element of Hopf invariant one and the five fold Toda bracket has been defined in some suitable way. Now Gershenson has shown

$$\beta_{s+1} \in \{\beta_s\ p, \beta_1, p, \alpha_s\} \quad : s+1<p$$

so unless I have misinterperted my construction a suitable permutation formula for 5-fold brackets will be needed to show $\{[\Psi_t]\}$ generalizes $\{\beta_t\}$ in the precise sense.

I leave these questions to the experts whose aid will be greatly appreciated.

Having described our main result in terms of its application to the stable stems let me now back track and describe the bordism problem that led up to all this. Rather than try to explain the motivation of this problem, which lies deeply in the study of numerous examples, let me simply pose it in its own right. We will need some preliminary facts and notation concerning the cobordism ring Ω^U_* .

Recollections on Ω_*^U : As we have remarked before, Milnor has shown that [5]

$$\Omega_*^U \cong \mathbb{Z}[[M^2], [M^4], \ldots, [M^{2n}], \ldots]$$

for suitable manifolds $\{M\}$. What constitutes a suitable manifold? Milnor tells us. We must examine the characteristic number

$$s_n(c)\ [M^{2n}]$$

where $s_n(c)$ is a certain polynomial in the Chern classes which yields the primitive element of $H^{2n}(BU; \mathbb{Z})$. The criterion of Milnor is that $[M^{2n}]$ is a possible choice for a generator (is an indecomposable element) of dimension 2n iff

$$s_n(c)\ [M^{2n}] = \begin{cases} \pm 1 & \text{if } n \neq p^i-1 \text{ any prime } p \\ \pm p & \text{if } n = p^i-1 : p \text{ a prime.} \end{cases}$$

There is a special choice of a generator in the dimensions $2p^i-2$, p a prime, called "Milnor manifolds" by Conner-Floyd and denoted by $[V^{2p^i-2}]$. The manifold V^{2p-2} may be, and will be chosen to be $\mathbb{C}P(p-1)$. The manifold $p \in \Omega_o^U$ will for technical reasons often be denoted by V^o.

Problem: Let p be a prime and n a non-negative integer. Does there exist a finite complex V(n) such that

$$\tilde{\Omega}^U_* (V(n)) \cong \Omega^U_*/(p, [V^{2p-2}], \ldots, [V^{2p^n-2}])\nu$$

as Ω^U_* - modules, ie does there exist a class

$$\nu \in \tilde{\Omega}^U_* (V(n))$$

such that $\tilde{\Omega}^U_*$ (V(n)) is the cyclic Ω^U_* - module generated by ν subject to the relations

$$A(\nu) = (p, [V^{2p-2}], \quad , [V^{2p^n-2}]).$$

($A(\nu)$ = annihilator ideal of ν.

An additional problem that has the ulterior motive of constructing the spaces V(n) by induction may be phrased as follows:

What is the image of

$$\tilde{\Omega}^{fr}_* (V(n)) \to \Omega^U_*(V(n)).$$

ie. which classes $\sigma \varepsilon \tilde{\Omega}^U_*$ (V(n)) may be represented in the form $\Sigma^s\sigma = [S^m, f]$ for some map $f: S^m \to \Sigma^s V(n)$ of a sphere into a high suspension of V(n)?

As we will be concerned almost exclusively with stable phenomena we will introduce such simple minded notations as

$$S^o \cup_p e^1$$

to denote a complex

$$S^N \cup_p e^{N+1} \quad : \quad N \ \text{large}$$

The notation extends to more than one cell.

Let us begin with what is clearly obvious. Namely we may choose

$$V(o) = S^o \cup_p e^1 .$$

Let us therefore proceed to the second of our questions which concerns which of the classes in $\Omega_*^U(V(o))$ are stably spherical i.e., in the image of framed.

It is at this point that it is convenient to divide the case $p = 2$ from the case $p > 2$. The former is filled with special problems, and while the complete answer is known it will be convenient to concentrate our attention on the case $p>2$.

Henceforth $p>2$ even if we forget to so qualify our statements.

It will be of great use to us to have available a "direct" description of a homotopy class

$$\nu : S^{2p-2} \to \underset{\sim}{MU}$$

that represents V^{2p-2}, that is, satisfies

$$[\nu] = [V^{2p-2}] \ \varepsilon \ \pi_*^s(\underset{\sim}{MU}).$$

To this end (ignoring the motivation for all this) let

$$\alpha_1 \in \pi^S_{2p-3}$$

denote the element of Hopf invariant 1 mod p.

Consider the mappings

$$\underset{\sim}{M}U \xleftarrow{i} S^o \xleftarrow{\alpha_1} S^{2p-3} \xleftarrow{p} S^{2p-3}$$

where

$$i : S^o = \underset{\sim}{S} \rightarrow \underset{\sim}{M}U$$

is the unit of the ring spectrum $\underset{\sim}{M}U$. Recall that a framed manifold of positive dimension always bounds a U-manifold and hence the composite

$$i . \alpha_1 \sim o.$$

Since α_1 is of order p

$$\alpha_1 \circ p \sim o.$$

Thus there is defined a Toda bracket

$$\{i, \alpha_1, p\} : S^{2p-2} \rightarrow \underset{\sim}{M}U.$$

The result that we require is:

Thm 1: With the above notations

$$\{i, \alpha_1, p\} : S^{2p-2} \rightarrow \underset{\sim}{M}U$$

represents $[V^{2p-2}] = [\mathbb{C}P(p-1)] \in \pi^S_*(\underset{\sim}{M}U) = \Omega^U_*$.

Outline of Proof: Let us recall that we may introduce a spectrum MU/S by constructing the cofibration*

$$\underset{\sim}{S} \to \underset{\sim}{MU} \xrightarrow{F} \underset{\sim}{MU}/S.$$

The elements of π_* ($\underset{\sim}{MU}$/S) may be viewed as equivalence classes of compact U-manifolds with a computable framing on their boundaries. The resulting bordism group $\Omega_*^{U,fr}$ has been studied by Conner and Floyd.

Now we claim that the following constitutes a correct description of an element in $\{i, \alpha_1, p\}$. (Details to the reader.) Choose a (U,fr) manifold [W] with

$$\partial_*[W] = [\alpha_1].$$

This is possible because the sequence

$$o \to \Omega_n^U \to \Omega_n^{U,fr} \xrightarrow{\partial_*} \Omega_{n-1}^{fr} \to o.$$

is exact when n>o. (Of course the classical Thom construction is used to identify π_*^S with Ω_*^{fr}.)

Then

$$\partial_*(p[W]) = p[\alpha_1] = o$$

and hence

$$p[W]=F_*[C]$$

for some closed U-manifold C. Then

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*See: Conner and Floyd, The Relation of Cobordism Theories to K-Theories Springer Lecture Notes No. (1966).

$$[C] \in \{i, \alpha_1, p\} .$$

Now a simple characteristic number check shows that a class $[C] \in \Omega^U_{2p-2}$ has $F_*[C] = p[W]$ iff $[C] = p[A] + [\mathbb{C}P(p-1)]$ as $p\,\Omega^{U,fr}_{2p-2}$ is part of the indeterminacy of $\{i,\alpha_1,p\}$ we may safely conclude that

$$[C] = [\mathbb{C}P\,(p-1)]$$

yielding the desired conclusion. $\square$

We may now obtain:

<u>Cory 2:</u> Suppose that X is a finite complex and $\nu \in \mathfrak{N}^{fr}_*(X)$ is a class such that $p\nu = o \in \mathfrak{N}^{fr}_*(X)$. Then $\{\alpha_1, p, \gamma\}$ $\mathfrak{N}^{fr}_*(X)$ is defined and*

$$\Phi_*\{\alpha_1, p, \gamma\} = [\mathbb{C}P(p-1)]\Phi_*(\nu) \in \mathfrak{N}^U_*(X).$$

<u>Proof:</u> That $F_*\{\alpha_1, p, \gamma\}$ consists of a single class in $\mathfrak{N}^U_*(X)$ is easily checked by observing that F_* maps the indeterminacy of $\{\alpha_1, p, \gamma\}$ to zero.

The morphism of homology theories

$$\Phi_*: \mathfrak{N}^{fr}_*(\ ) \to \mathfrak{N}^U_*(\ )$$

is induced by the natural map

$$i: \underset{\sim}{S} \to \underset{\sim}{M}U.$$

Thus we have

$$\Phi_*\{\alpha_1, p, \gamma\} = i\circ\{\alpha_1, p, \gamma\}$$
$$= \{i, \alpha_1, p\}\,\Phi_*(\gamma)$$
$$= [\mathbb{C}P(p-1)]\,\Phi_*(\gamma)$$

By a standard Toda bracket identity

as required. $\square$

In particular, if we let

$$\gamma \in \tilde{\Omega}_o(V(o))$$

be the canonical class we may conclude that there exists a map

$$S^{2p-2} \to V(o)$$

such that

$$[S^{2p-2}, \varphi] = [\mathbb{C}P(p-1)]\gamma \in \tilde{\Omega}^U_{2p-2}(V(o)).$$

Now recall that as p is odd Toda has shown that

$$\varphi = \{\alpha_1, p \;\; \gamma\}: S^{2p.2} \to V(o)$$

one may also just compute $\pi_{2p-2}(V(1))$ and see its p-component

is of order p.

Therefore we may obtain an extension

$$\bar{\varphi}: S^{2p-2} \cup_p e^{2p-3} \to V(o)$$

of φ . Upon observing that

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*Here $\Phi_*\ \tilde{\Omega}^{fr}_*(\ ) \to \tilde{\Omega}^U_*(\ )$ is the natural forgetful functor that views a framed manifold as a U-manifold.

$$S^{2p-2} \cup_p e^{2p-3} = S^{2p-2}V(o)$$

we may form the iteration

$$\bar{\varphi}_t : S^{2t(p-1)}V(o) \xrightarrow{\Sigma^{t-1}\bar{\varphi}} S^{2(t-1)(p-1)}V(o) \to \cdots \xrightarrow{\bar{\varphi}} V(o).$$

Let φ_t denote the composite

$$\varphi_t : \begin{array}{l} S^{2t(p-1)} \quad \text{inclusion of bottom sphere} \\ \quad \downarrow \\ S^{2t(p-1)}\,V(o) \xrightarrow{\bar{\varphi}_t} V(o). \end{array}$$

The following is then clear:

<u>Thm 3</u>: $\quad [S^{2t(p-1)}, \varphi_t] = [\mathbb{C}P(p-1)]^t \gamma \varepsilon\ \Omega^U_{2+(p-1)}(V(o))$

and hence the classes $[\mathbb{C}P(p-1)]^t\ \gamma$: $t = o,1,\ldots$, all lie in the image of Φ_*: $\Omega^{fr}_*(V(o)) \to \Omega^U_*\ (V(o))$. $\square$

<u>Remark</u>: It may be shown by a more delicate argument that Thm 3 exhausts the list of framed bordism classes on V(o).

We now arrive at our first application to the stable homotopy groups of spheres. Let

$$S^o \xrightarrow[i]{} V(o) \xrightarrow[q]{} S^1$$

be the natural cofibration.

<u>Thm 4</u>: The composites $q\varphi_t$: $t = 1, 2, \ldots$ represent non-zero elements of order p in $\pi^s_{2t(p-1)-1}$ i.e.

$$[q \cdot \varphi_t] = [\Phi_t] \neq o \varepsilon\, \pi^s_{2t(p-1)-1}$$

<u>Proof</u>: Suppose to the contrary that

$$[q \circ \varphi_t] = o \varepsilon\, \pi^s_* .$$

From the exact sequence

$$\Omega^{fr}_{2t(p-1)}(S^o) \xrightarrow[i_*]{} \Omega^{fr}_{2t(p-1)}(V(o)) \xrightarrow[q_*]{} \Omega^{fr}_{2t(p-1)}(S^1)$$

we therefore find

$$[\varphi_t] = i_*[\theta_t]$$

for some $[\theta_t] \varepsilon\, \pi^s_{2t(p-1)}$.

But from the commutative diagram

Because a framed manifold of positive dimension bounds a U-manifold

$$o = \begin{array}{ccc} \Omega^{fr}_{2t(p-1)}(S^o) & \xrightarrow{i_*} & \Omega^{fr}_{2t(p-1)}(V(o)) \\ \downarrow \Phi_* & & \downarrow \Phi_* \\ \Omega^{U}_{2t(p-1)}(S^o) & \xrightarrow[i_*]{} & \Omega^{U}_{2t(p-1)}(V(o)) \end{array}$$

we learn

$$\Phi_*[q\varphi_t] = o$$

contrary to Thm 3 yielding the results. $\square$

Remarks: (1) $[\Phi_t] = \alpha_t$ of Toda

(2) $\bar{\Phi}_t = A_t$ of Adams.

The next lecture will show how these results yield V(1) and lead naturally to the classes $[\Psi_t]$ discussed earlier.

Lec. 4 Detecting Homotopy Elements by Realizing Bordism Modules

Let us briefly recall the results of last time. We were concerned with the following problem:

Let p be a prime and n a non-negative integer. Does there exist a finite complex V(n) such that

$$\tilde{\Omega}^U_*(V(n)) \cong \Omega^U_*/(p,[V^{2p-2}],\ldots,[V^{2p^n-2}])\gamma \quad \text{as} \quad \Omega^U_*\text{-modules?}$$

Last time we found a solution for $p>2$ and $n = o, 1$. You recall that

$$V(o) = S^o \cup_p e^1.$$

and that there was a map

$$\varphi : \quad S^{2p-2} \to V(o)$$

of order p in stable homotopy such that

$$[S^{2p-2},\varphi] = [V^{2p-2}]\gamma \ .$$

We went on to give applications to the stable homology of spheres. Our objective today is to return to the original problem.

<u>Thm 1</u>: Let

$$\bar{\varphi} : S^{2p-2}\, V(o) \to V(o)$$

be the extension of φ . Let W be the mapping cone of $\bar{\varphi}$.
Then

$$\tilde{\Omega}^U_* \,(W) \cong \Omega^U_*/(p, [V^{2p-2}],\ [V^{2p^2-2}])\gamma \text{ as } \Omega^U_*\text{-modules}$$

<u>Proof</u>: Consider the exact Ω^U_* ( ) triangle of the cofibration

$$S^{2p-2}V(o) \xrightarrow{\bar{\varphi}} V(o) \longrightarrow W.$$

We have

$$\begin{tikzcd} \Omega^U_*(S^{2p-2}V(o)) \arrow[rr, "\bar{\varphi}_*"] & & \Omega^U_*(V(o)) \arrow[dl] \\ & \tilde{\Omega}^U_* \,(W) \arrow[ul] & \end{tikzcd}$$

Now note

$$\Omega^U_* \,(S^{2p-2}V(o)) \cong \Omega^U_*/(p)\ s^{2p-2}\ \gamma$$

and

$$\bar{\varphi}_*\ s^{2p-2}\gamma = [V^{2p-2}]\gamma$$

Since $\Omega^U_*/(p)$ has no zero divisors, $\bar{\varphi}_*$ is monic so the above exact triangle becomes $o \to \Omega^U_*/(p)\ .\ s^{2p-2}\gamma \xrightarrow[\bar{\varphi}_*]{} \Omega^U_*/(p) \to \Omega^U_*(W) \to o$

and thus we find

$$\Omega_*^U(W) \cong \Omega_*^U/(p, [V^{2p-2}]) \quad \text{as required.} \ \square$$

Thus W is a space satisfying the conditions of V(1) and hence we will write V(1) for W in the sequel. That is

Cory 2: The mapping cone of

$$\overline{\varphi} : S^{2p-2}V(o) \to V(o)$$

may be taken as V(1) for p>2. $\square$

Remarks: For the prime 2 the analog of the space V(1) does not exist. However instead there are some rather strange 2-primary spaces connected with the Hopf maps.

Motivated by our construction of V(1) from V(o) and the applications to stable homotopy obtained in the process we will now investigate the image of the natural map

$$\Omega_*^{fr}(V(1)) \to \Omega_*^U(V(1)).$$

Unlike the situation for V(o) it does not seem possible to proceed via simple minded Toda bracket constructions. Higher order brackets seem to be needed. Rather than proceed in this way we will investigate this problem by means of the Adams spectral sequence. To motivate the reason for doing this, if such is needed, let me explain that the construction of our spaces $V(n)$ has an analogous formulation strictly in terms of cohomology. So let me turn to this now.

A Cohomology Problem: Let us denote by p an odd prime and by $A^*(p)$ the mod p steenrod algebra. According to Milnor the dual algebra $A_*(p)$ has a particularly simple description that runs as follows.

$$\deg \lambda_i = 2p^i - 1$$
$$\deg \mu_j = 2p^j - 2$$

$A_*(p) \cong E[\lambda_o, \lambda_1, \ldots] \otimes \mathbb{Z}_p[\mu_1, \mu_2, \ldots]$ with diagonal

$$\nabla^* \mu_k = \Sigma_{i=o}^{k} \mu_{k-1}^{p^i} \otimes \mu_i$$

and

$$\nabla^* \lambda_k = \lambda_k \otimes 1 + \Sigma_{i=o}^{k} \mu_{k-i}^{p^i} \otimes \lambda_i$$

These formulas show that

$$E[\lambda_o, \ldots, \lambda_n] \subset A_*(p)$$

is both a subalgebra and a left sub comodule. Thus its dual will be a left quotient module of $A^*(p)$ that is in addition a coalgebra over $A^*(p)$.

Problem: Does there exist a finite complex $W(n)$ such that

$$\tilde{H}_*(W(n); \mathbb{Z}_p) \cong E[\lambda_o, \ldots, \lambda_n]$$

as a left A^*-comodule?

Remark: It is possible to give a direct description of $H^*(W(n);\mathbb{Z}_p)$ as an $A^*(p)$-module. Namely let $Q_i \varepsilon A^{2p^i-1}(p)$ be the primitive element (i.e. the dual of λ_i). Then

$$\tilde{H}^*(W(n);\mathbb{Z}_p) \cong A^*/A^*(Q_{n+1},\ldots) + A^*(P^1,\ldots P^i,\ldots)$$

The following result is easily proved by means of the Adams spectral sequence.

Thm 3: Let p be a prime and n a non-negative integer. Then if one of the spaces $V(n)$, $W(n)$ exists so does the other and they may be assumed to be Spanier-Whitehead duals of each other. **

Remarks: (1°) This cohomology problem seems no more tractable than the corresponding bordism problem. But it did suggest the use of the Adams spectral sequence.

(2°) Note that the proposed homology for $W(n)$ is a ring. Thus it makes good sense to ask if we can choose $W(n)$ and/or $V(n)$ to be a ring spectrum. In this connection note

(a) $W(o)$, $V(o)$ are ring spectra: Toda $\left\{\begin{array}{l}\text{remember}\\ p>2\end{array}\right.$

(b) $W(1)$, $V(1)$, are ring spectra iff $p>3$: Smith, Toda. The proof of (b) is "hard" homotopy theory. It was suggested by the existence of $V(2)$ iff $p>3$ and one possible method of its construction.

After this semi-digression let us return to the main stream of the study. Suffice it to say that by use of the Adams spectral sequence we can prove [9].

Thm 4: Let p be an odd prime. Then there exists

$$\psi: \quad S^{2p^2-2} \to V(1)$$

such that

$$[S^{2p^2-2}, \psi] = [V^{2p^2-2}]\gamma \in \tilde{\Omega}^U_* \,(V(1)).$$

The element $[\psi] \in \pi^S_{2p^2-2}(V(1)$ has order p. **

Again reasoning by analogy with the previous lectures program we inquire into the possibility of extending

$$\psi: \quad S^{2p^2-2} \to V(1)$$

to a map

$$\bar{\psi}: \quad S^{2p^2-2}\, V(1) \to V(1).$$

In this connection we have:

Thm 5: Let p be a prime, p>3. Let

$$\psi: \quad S^{2p^2-2} \to V(1)$$

be the map with

$$[S^{2p^2-2}, \psi] = [V^{2p^2-2}]\gamma \in \tilde{\Omega}^U_* \,(V(1)).$$

Then there exists an extension

$$\bar{\psi}: S^{2p^2-2}\, V(1) \to V(1)$$

of ψ.

<u>Outline of Proof</u>: According to Thm 4 we may extend ψ to a map

$$\tilde{\psi} : S^{2p^2-2} V(o) \to V(1)$$

and so what we must do is show that the composite

$$S^{2p^2+2p-1} V(o) \xrightarrow[S^{2p^2-2}\bar{\varphi}]{} S^{2p^2-2} V(o) \xrightarrow[\tilde{\psi}]{} V(1)$$

is null homotopic. This is accomplished by using the Adams spectral sequence to show

$$\pi^s_{2p^2+2p-4} (V(o) , V(1)) = o: \quad p>3.$$

For p=3 this group

$$\pi^s_{2o} (V(o), V(1)) \neq o$$

and so the argument fails. $\square$

<u>Alternate Proof</u>: As remarked in our digression on the cohomology analog the space V(1) regarded as a spectrum is a ring spectrum for p>3. Let

$$\mu : V(1) \wedge V(1) \to V(1)$$

be the ring structure. Then

$$S^{2p^2-2} \wedge V(1) \xrightarrow[\psi \wedge i]{} V(1) \wedge V(1) \xrightarrow[\mu]{} V(1)$$

is easily seen to be the required extension. $\square$

Thus as with $\bar{\varphi}$: $S^{2p-2}V(o) \dashrightarrow V(o)$ we may form the iteration

$$\bar{\psi}_t:\ S^{2t(p^2-1)}\ V(1) \xrightarrow{\Sigma^{t-1}\bar{\psi}} S^{2(t-1)(p^2-1)}\ V(1) \to \cdots \xrightarrow{\bar{\psi}} V(1),$$

and we let ψ_t be the composite

$$\psi_t:\ S^{2t(p^2-1)} \underset{\substack{\text{inclusion}\\ \text{of bottom}\\ \text{sphere}}}{\to} S^{2t(p^2-1)}\ V(1) \underset{\bar{\psi}_t}{\to} V(1).$$

The following is now clear.

<u>Thm 6</u>: Let p be a prime strictly greater than 3 and t a non-negative integer. Then

$$[S^{2t(p^2-1)}, \psi_t] = [v^{2p^2-2}]^t\ \gamma\varepsilon\ \Omega^U_*\ (V(1)).$$

Hence all the classes $[v^{2p^2-2}]^t\ \gamma$: $t = o,1,\ldots$ lie in the image of $\Omega^{fr}_*(V(1)) \to \Omega^U_*\ (V(1))$. And are non-zero.

Now notice that

$$V(1) = S^o \cup_p e^1 \cup_{\bar{\varphi}} e^{2p-1} \cup_p e^{2p}\ .$$

Let

$$q:\ V(1) \to S^{2p}$$

be the collapse onto the top cell. Introduce the composite

$$\psi_t:\ S^{2t(p^2-1)} \underset{\psi_t}{\to} V(1) \underset{q}{\to} S^{2p}$$

Our main application to stable homotopy theory is then

Thm 7: The elements

$$[\Psi_t] \varepsilon\ \pi^S_{2t(p^2-1)-2p} \ : \ t = 1, 2,$$

are all non-zero and of order p for all primes p>3.

Outline of Proof. Let us denote by V(1/2) the subcomplex of V(1) whose cell structure is

$$V(1/2) = S^o \cup_p e^1 \cup_{\overline{\varphi}} e^{2p-1} .$$

Note that

$$V(1/2) \xrightarrow{i} V(1) \xrightarrow{q} S^{2p}$$

is a cofibration. Note that

$$\tilde{\Omega}^U_* (V(1/2)) \cong \Omega^U_*/(p)\ \gamma \oplus \Omega^U_*\ \sigma_{\ 2p-1}$$

and that

$$i_* \gamma = \gamma \ , \quad i_* \sigma_{2p-1} = o.$$

Suppose to the contrary that

$$[\Psi_t] = o.$$

Then

$$\psi_t: \ S^{2+(p^2-1)} \to V(1)$$

may be compressed to a map

$$\hat{\psi}_t: \ S^{2t(p^2-1)} \to \ V(1/2).$$

A moment's study of Thm 6 and the bordism of V(1/2) shows that then we must have

$$[S^{2t(p^2-1)}, \hat{\psi}_t] = [V^{2p^2-2}]^t\gamma \varepsilon \ \tilde{\Omega}^U_* (V(1/2)).$$

That is, the bordism class $[V^{2p^2-2}]^t$ is spherical on the space V(1/2). Thus we are led to a study of the possible spherical bordism classes on V(1/2). Now it turns out that

$$[M]\gamma \ \varepsilon \ \tilde{\Omega}^U_* \ (V(1/2))$$

is spherical iff

$$[M] = \{\theta, \alpha_1, p\}$$

for some element

$$\theta \ \varepsilon \ \Omega^{U,fr}_* \ .$$

With the aid of Adams results on the $e_{\mathbb{C}}$ invariant and a detailed study of K-theory characteristic numbers for (U,fr)-manifolds we succeed in showing that

$$\gamma, \quad \sigma_{2p-1} \quad \varepsilon \quad \tilde{\Omega}^U_* \ (V(1/2))$$

are the only spherical classes or V(1/2) which shows that

$$[\psi_t] \neq o \ \varepsilon \ \pi^S_{2t(p^2-1)-2p}$$

as desired. $\square$

Closing Remark: It is perhaps worth noting that our investigation of the characteristic numbers of (U, fr.) manifolds contains several results that are perhaps of interest in their own right. Among them is the following Hattori - Stong type theorem.

Thm 8: All odd primary relations among the integral cohomology Chern numbers of compact almost complex manifolds with framed boundaries come from K-theory. That is, the natural mapping

$$\frac{\Omega_*^{U,fr}}{\text{torsion}} \otimes \mathbb{Z}[1/2] \to K_*(MU/S) \otimes \mathbb{Z}[1/2]$$

is a monomorphism onto a $\mathbb{Z}'[1/2]$ direct summand. **

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