Representability of Functors in Abstract Homotopy

by E. H. Brown, Jr., 30 September 1963

Let C be a category in which $[X, Y]$ denotes maps from X to Y. Let $\mathcal{S}$ be the category of sets and $H : C \to \mathcal{S}$ be a contravariant functor. We consider the problem of imposing conditions on C and H so that $H \approx [\ , Y_H]$ for some $Y_H \in C$.

Let $C_0 \subset C$ be a fixed subcategory. We shall require (C, C_0) to satisfy:

(1) C_0 is a full, small subcategory of C

(2) C_0 has finite sums of objects, C has arbitrary sums, and they agree.

(3) For any $f_i : A \to X_i$, $i = 1, 2$ in C, there exist Z and $g_i : X_i \to Z$ with the following universal property. If $g_i' : X_i \to Z'$ and $g_1' f_1 = g_2' f_2$ then $\exists\, h : Z \to Z'$ for which $g_i' = h g_i$. If f_i are in C_0 we

can choose $g_i \in C_0$.

(4) given $X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow$ there exist X_∞, $g_n : X_n \longrightarrow X_\infty$ with $g_{n+1} f_n = g_n$ and such that: (a) for $Z \in C_0$, the expressions $\longrightarrow [Z, X_n] \xrightarrow{f_{n*}} [Z, X_{n+1}] \longrightarrow \cdots$, $[Z, X_n] \xrightarrow{g_{n*}} [Z, X_\infty]$ satisfy $\varinjlim_{n\to\infty} [Z, X_n] \xrightarrow{\lim g_{n*}} [Z, X_\infty]$ is an isomorphism, and (b) given $Z \in C$ the expressions $\longleftarrow [X_n, Z] \xleftarrow{f_n^*} [X_{n+1}, Z] \longleftarrow \cdots$, $[X_n, Z] \xleftarrow{g_n^*} [X_\infty, Z]$ satisfy $[X_\infty, Z] \xrightarrow{\lim g_n^*} \varprojlim [X_n, Z]$ is onto.

(5) given $f : Y \longrightarrow Y'$ in C, if $f_* : [X, Y] \approx [X, Y']$ for all $X \in C_0$ then f is an equivalence.

<u>Example</u>: Let the objects of C be CW complexes with base points. For $X, Y \in C$ let $[X, Y]$ be the homotopy classes of maps from X to Y. Let C_0 be the finite CW cxs.

Then (C, C_0) satisfies (1) — (5).

For (2) define $X + Y = X \vee Y$

For (3) form Z by taking mapping cylinders Cf_i of $f_i : A \to X_i$ then identifying corresponding points of $A \subset Cf_1$ and $A \subset Cf_2$

For (4) form X_∞ as an identification space of the mapping cylinders Cf_n. Then part (a) follows by a compactness argument and (b) by the homotopy extension theorem.

(5) reduces to a theorem by J.H.C. Whitehead

There are other examples as well.

Let (C, C_0) satisfy the above conditions. Given maps $f_1, f_2 : A \to X$ in C we can form their "equalizer" $h : X \to Z$ as follows: $f_1 + f_2 : A + A \to X$ and $1 + 1 : A + A \to A$, so by (3) there exist $h : X \to Z$ and $g : A \to Z$ such that $h \circ (f_1 + f_2) = g \cdot (1 + 1)$ and the universal property holds. Thus $h.f_1 = h.f_2$. If f_1, f_2 are in C_0, h may be chosen in C_0.

We shall require the functor $H: C \to \mathcal{S}$ to satisfy:

(i) Let $\{X_\alpha\}$ be objects in C and $i_\alpha: X_\alpha \longrightarrow \sum_\beta X_\beta$ be the injections. Then

$$\prod H(i_\alpha) : H(\sum_\beta X_\beta) \longrightarrow \prod_\alpha H(X_\alpha) \text{ is an iso.}$$

(ii) Suppose $f_i: A \longrightarrow X_i$ and $g_i: X_i \longrightarrow Z$ are as in (3). If $u_i \in H(X_i)$ and $H(f_1)u_1 = H(f_2)u_2$ then $\exists\, v \in H(Z)$ such that $H(g_i)v = u_i$

<u>Lemma</u>: Let (C, C_0) and H satisfy the above sets of conditions. Let

$$Y_1 \longrightarrow Y_2 \longrightarrow \cdots \longrightarrow Y_n \xrightarrow{f_n} Y_{n+1} \longrightarrow \cdots, Y_\infty$$

and $g_n: Y_n \longrightarrow Y_\infty$ be as in (4). Then

$$H(Y_\infty) \xrightarrow{\varprojlim H(g_n)} \varprojlim H(Y_n) \text{ is onto.}$$

<u>Proof</u> The maps Σid_n, $\Sigma f_n : \Sigma Y_n \to \Sigma Y_n$ have as an equalizer $\Sigma g_n : \Sigma Y_n \longrightarrow Y_\infty$. For observe that the diagram

$$\begin{tikzcd}
(\Sigma Y_n) + (\Sigma Y_n) \arrow[r, "(\Sigma id_n) + (\Sigma f_n)"] \arrow[d, "(\Sigma id_n) + (\Sigma id_n)"'] & \Sigma Y_n \arrow[d, "\Sigma g_n"] \\
\Sigma Y_n \arrow[r, "\Sigma g_n"'] & Y_\infty
\end{tikzcd}$$

is commutative and satisfies the universal property by (4b). If $u \in H(\Sigma Y_n)$ and

$H(\Sigma id_n)\, u = H(\Sigma f_n)\, u$ then

$$H[(\Sigma id_n) + (\Sigma f_n)](u) = H[(\Sigma id_n) + (\Sigma id_n)](u)$$

Thus by (ii) $\exists\ v \in H(Y_\infty)$ such that $H(\Sigma g_n)(v) = u$. Thus the image of $H(Y_\infty) \xrightarrow{\Sigma H(g_n)} H(\Sigma Y_n) \approx \prod H(Y_n)$ is

$$\{u \in H(\Sigma Y_n) \mid H(\Sigma id_n)(u) = H(\Sigma f_n)(u)\} \approx \overleftarrow{\lim}\, H(Y_n)$$

Hence $H(Y_\infty) \xrightarrow{\overleftarrow{\lim}\, H(g_n)} \overleftarrow{\lim}\, H(Y_n)$ is onto.

Using this lemma one proves:

<u>Theorem</u>: If (C, C_0) satisfies (1) — (5) and H satisfies (i), (ii), then $\exists$ a unique $Y_H \in C$ and a natural equivalence $T : [\ , Y_H] \longrightarrow H$.

<u>Theorem</u>: Let (C, C_0) satisfy (1) — (5), let C_0 have countably many maps, let $H : C_0 \longrightarrow \mathcal{S}$ be a contravariant functor, let H satisfy (i) for finite sums and (ii).

Then there exists a unique $Y_H \in C$ and a natural equivalence $T : [\ , Y_H] \longrightarrow H$ on C_0.

Parallelizability of Sphere Bundles over Spheres

by Wilson Sutherland, 7 October, 1963

Let M be a smooth (C^∞) manifold and $\tau(M)$ be its tangent vector bundle. M is parallelizable, or ∥sible, if $\tau(M)$ is trivial. M is stably parallelizable, or s-∥sible, if $\tau(M) \oplus 1$ is trivial where 1 is the trivial line bundle over M.

Examples: (1) Any compact Lie group is ∥sible.

(2) S^n is ∥sible iff $n = 0, 1, 3, 7$

(3) $S^n \times S^q$

a) If n, q even then $S^n \times S^q$ is not ∥sible since the Euler characteristic $\chi(S^n \times S^q) \neq 0$

b) If n odd, $q \geqq 1$ then $S^n \times S^q$ is ∥sible.

For let τ_n denote the bundle over $S^n \times S^q$ induced from $\tau(S^n)$ by the projection $S^n \times S^q \longrightarrow S^n$, and let τ_q be defined similarly. Then $\tau(S^n \times S^q) = \tau_n \oplus \tau_q = \xi \oplus 1 \oplus \tau_q$ for some ξ since S^n admits a non-zero vector field. Since S^q is s-llsible this $= \xi \oplus (q+1)$ where $(q+1)$ means the trivial $(q+1)$-plane bundle. This $= \tau_n \oplus 1 \oplus (q-1) = (n+q)$ since S^n is s-llsible.

<u>Remark</u>: Let ξ be a sphere bundle with projection $\pi: B \longrightarrow M$ where M is a s-llsible manifold. Let ξ^V be the associated tangent vector bundle. $\{\xi^V\} = \widetilde{KO}(M)$. Then B is s-llsible iff $\{\xi^V\} \varepsilon$ Kernel of $\pi^*: \widetilde{KO}(M) \longrightarrow \widetilde{KO}(B)$.

<u>Proof</u> From the facts

$\tau(B) = \pi^*(\tau(M)) \oplus \hat{\xi}$ and

$\hat{\xi} \oplus 1 = \pi^*(\xi^V)$ (Wu 1952) one deduces $\{\tau(B)\} = \pi^*\{\tau(M)\} + \pi^*\{\xi^\nu\}$ ③

so $\{\tau(B)\} = \pi^*\{\xi^\nu\}$ The remark follows.

We now specialize to the case

$M = S^n \qquad \xi: S^q \longrightarrow B \xrightarrow{\pi} S^n$

Using the homotopy sequence of ξ

$\longrightarrow \pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(S^q) \longrightarrow \cdots$ define

$\Theta = \partial(\iota_n)$. Let $\lambda = S\Theta \in \pi_n(S^{q+1})$.

<u>Theorem 1</u>: $\widetilde{KO}(S^{q+1}) \xrightarrow{\lambda^*} \widetilde{KO}(S^n) \xrightarrow{\pi^*} \widetilde{KO}(B)$

is exact if $n > 2$.

<u>Corollary 1</u>: The q-sphere bundles ξ

over S^n $(n>2)$ which have s-||sible

total space are those for which

$\{\xi^\nu\} \in$ image λ^*

<u>Proof of Thm 1</u>: B admits a cell

structure $B = S^q \cup_\Theta e^n \cup e^{n+q}$. Let

$\pi' = \pi / S^q \cup_\Theta e^n$ Let $T =$ mapping

cone of π = Thom complex of ξ
We then have a diagram

$$\begin{array}{ccccc}
\widetilde{KO}(T) & \xrightarrow{j^*} & \widetilde{KO}(S^n) & \xrightarrow{\pi^*} & \widetilde{KO}(B) \\
\downarrow i^* & & \downarrow 1 & & \downarrow k^* \\
\widetilde{KO}(S^{q+1}) & \xrightarrow{\lambda^*} & \widetilde{KO}(S^n) & \xrightarrow{\pi'^*} & \widetilde{KO}(S^q \cup_\theta S^n)
\end{array}$$

i^* is onto since the bundle admits a spin structure. Theorem 1 then follows from the known exactness in the diagram.

<u>Defn</u> For M of dimension $2r-1$ define the <u>mod 2</u> <u>semi-characteristic</u>

$$\chi_*(M) = \sum_{i=0}^{r-1} b_i \text{ where } b_i = \text{rank } H_i(M, Z_2)$$

<u>Theorem 2</u> (Kervaire + Adams) Let M be a smooth S-$\parallel$sible n manifold without boundary. Then

<u>K.1</u> When n is even M is $\parallel$sible $\Leftrightarrow$ $\chi(M) = 0$

<u>K.2</u> When n is odd, M is $\parallel$sible $\Leftrightarrow$ either $n = 1, 3, 7$ or $\chi_*(M)$ even

<u>Corollary 2</u>: Let B be the total space of a q-sphere bundle over S^n and suppose B is S-‖sible. Then one + only one of the following is true:

(1) B is ‖sible

(2) $n+q$ is even

(3) $n = 8$, $q = 7$, and $H_7(B, Z_2) = 0$,

e.g. the Hopf bundle $S^{15} \longrightarrow S^8$.

<u>Proof</u> (Cor. 2) If $n+q$ is even our assertion in Cor 2 is a consequence of K.1. If $n+q$ is odd we distinguish two cases:

(a) $H_*(B, Z_2)$ is same as $H_*(S^n \times S^q, Z_2)$ Then B is ‖sible by K.2 since $\chi_*(B)$ is even.

(b) Otherwise $q = n-1$ and $B = S^{n-1} \cup_\theta e^n \cup e^{2n-1}$; θ is of odd degree. But the class of θ is in the image of $p_* : \pi_{n-1}(SO(n)) \longrightarrow \pi_{n-1}(S^{n-1})$ whose $p : SO(n) \longrightarrow S^{n-1}$ is the projection of the principal tangent bundle to S^{n-1}. So p_* is onto, hence this principal bundle admits a cross-section, i.e. S^{n-1} is ‖sible and so $n = 1, 2, 4$ or 8. If $n = 1, 2,$ or 4, B is $1, 3,$ or 7 dimensional and hence B is ‖sible by K.2. If $n = 8$, B is not ‖sible again by K.2.

Given a q-sphere bundle

$\xi : S^q \longrightarrow B \xrightarrow{\pi} S^n$ we apply Cor 1 to test for s-‖sibility. If so apply Cor 2 to test for ‖-sibility.

Remarks a) If $q \geq n$ then B ‖sible $\Rightarrow$ ξ is trivial

b) There exist bundles ξ which are not stably trivial and yet B is ‖sible.

Proposition: Let $O_{n,r}$ = Stiefel manifold of r-frames in n-space over the reals, complex nos, or quaternions. $O_{n,r}$ is ‖sible whenever $r > 1$

<u>Proof</u>: By Theorem 2, it suffices to show $O_{n,r}$ is S-‖sible. For convenience we use real Stiefel manifolds $V_{n,r}$.

$$
\begin{tikzcd}
V_{n,r+1} \arrow[d, "p_{r+1}"] & & \\
V_{n,r} \arrow[r, "\tau_r"] \arrow[d, dotted] & BSO(n-r) \arrow[d, dotted] & \\
V_{n,2} \arrow[r] \arrow[d, "p_2"] & BSO(n-2) \arrow[d, "q_2"] & \\
S^{n-1} \arrow[r, "\tau_1"] & BSO(n-1) \arrow[r, "q_1"] & BSO(n)
\end{tikzcd}
$$

Let ξ^V_τ be the vector bundle associated with the sphere bundle whose projection is p_{r+1}. Claim that $\{\xi^V_\tau\} = 0$ since the diagram commutes and $q_1 \tau_1$ is trivial. We induct on r. If $r = 1$, S^{n-1} is S-‖sible. Now suppose $V_{n,r}$ is S-‖sible.

$$\{\tau(V_{n,r+1})\} = p^*_{r+1}\{\tau(V_{n-r})\} + p^*_{r+1}\{\xi^V_\tau\} = 0$$

The proposition then follows.

<u>Remark</u> Let $Y_{m,r}$ be a Cayley Stiefel manifold. ξ: $S^{8m-9} \longrightarrow Y_{m,2} \longrightarrow S^{8m-1}$

This is infact a sphere bundle and $Y_{m,2}$ is $\parallel$sible manifold.

Sectional Curvature

by John Thorpe   14 October 1963

Let X be a Riemannian manifold and $G_2(X)$ be the Grassmann bundle of 2-planes tangent to X.

We define the <u>Riemannian</u> <u>sectional</u> curvature $\gamma : G_2(X) \longrightarrow \mathbb{R}$. For $(x, \underline{P}) \in G_2(X)$ $\gamma(x, \underline{P})$ = Gaussian curvature of the geodesic submanifold of X tangent to $\underline{P}$ at x.

<u>Remark</u>: If u_1, u_2 is an orthonormal basis for $\underline{P}$ then $\gamma(x, \underline{P}) = -\langle R(u_1, u_2)u_1, u_2\rangle$.

γ determines the curvature tensor R and thus all curvature properties of X.

We wish to generalize these classical notions of curvature. By the <u>Gauss-Bonnet theorem</u>, if X compact, orientable, of dim 2, then $\frac{1}{2\pi}\int_X \kappa \, dV = \chi(X)$ where

κ = Gaussian curvature, χ = Euler char.

<u>Allendoerfer</u> - <u>Weil</u> - <u>Chern</u> generalize this result. If X compact, orientable, of dim n (even) then $\frac{2}{c_n}\int_X K \, dV = \chi(X)$ where

c_n = volume of unit Euclidean n-sphere, K = <u>Lipschitz</u> - <u>Killing</u> <u>curvature</u>. K is a candidate to replace Gaussian curvature.

<u>Remark</u> If n odd then $K \equiv 0$

Otherwise K is expressable in terms of the curvature tensor R of X. For the case $X^n \subseteq E^{n+m}$ the second fundamental form $A_\nu : X(x) \longrightarrow X(x)$ linearly for each normal vector ν to x The Lipschitz - Killing curvature $K(x)$ is then given by

$K(x) = c(n,m)\int_S \det A_\nu \, dV$ where S = normal sphere at x.

Let $G_p(X)$ = Grassmann bundle of p-planes tangent to X. We define the p^{th} *sectional curvature* of X $\gamma_p : G_p(X) \longrightarrow R$ for p even, $2 \le p \le n$. For $(x, P) \in G_p(X)$, $\gamma_p(x, P)$ = Lipschitz–Killing curvature of the geodesic p-submanifold of X tangent to P at x.

Remark: If $u_1, \ldots, u_p$ is an orthonormal basis for P then

$$\gamma_p(x, P) = \frac{(-1)^{p/2}}{2^{p/2}\, p!} \sum \varepsilon_{i_1 \ldots i_p} \varepsilon_{j_1 \ldots j_p} \langle R(u_{i_1}, u_{i_2}) u_{j_1}, u_{j_2} \rangle \cdots \langle R(u_{i_{p-1}}, u_{i_p}) u_{j_{p-1}}, u_{j_p} \rangle$$

Thus γ_p is smooth on $G_p(X)$ and

$\gamma_2(x, P) = -\langle R(u_1, u_2) u_1, u_2 \rangle = \gamma(x, P)$

γ_n for n even is of course the Lipschitz–Killing curvature.

Examples (1) If $\gamma_2 = K =$ const then $\gamma_p = K^{p/2} =$ const (p even and $\leq n$) In particular if $X = S^n_r$ (Euclidean n-sphere of radius r) then $\gamma_p = \frac{1}{r^p}$.

(2) If Y is flat then $\gamma_p(X \times Y) = 0$ for all $p > \dim X$

(3) If the metric of complex projective space $P_n(\mathbb{C})$ is normalized so $\frac{1}{4} \leq \gamma_2(P_n(\mathbb{C})) \leq 1$ then $\frac{1}{16} \leq \gamma_4(P_n(\mathbb{C})) \leq \frac{1}{2}$

(4) If either $\gamma_2 \geq 0$ everywhere or $\gamma_2 \leq 0$ everywhere then $\gamma_4 \geq 0$

Theorem 1: For $p < n$ (p even, $n = \dim X$) if γ_p is const on each fiber of $G_p(X)$ then γ_p is const.

Theorem 2: If $\gamma_p = K_p$ (const) and $\gamma_q = K_q$ (const) then $\gamma_{p+q} = K_p \cdot K_q$ ($p+q \leq n$)

Cor: If $\gamma_{p_i} = K_{p_i}$ $(i=1,\ldots,k)$ and $q = \sum_{i=1}^{k} m_i p_i \leq n$ where $m_1, \ldots, m_k$ are integers ≥ 0, then $\gamma_q = (K_{p_1})^{m_1} \cdots (K_{p_k})^{m_k}$

Theorem 3: If $\gamma_p \equiv 0$ for some $p \leq n$ then $\gamma_q \equiv 0$ for all $q \geq p$

Let X now be compact, orientable, of even dim n. Thus $\chi(X) = \frac{2}{c_n} \int_X \gamma_n \, dV$

By Theorem 2, if $\gamma_p = K_p = \text{const}$ for some $p \mid n$, then $\gamma_n = (K_p)^{n/p}$ so $\operatorname{sign}(\chi(X)) = \operatorname{sign}(K_p)^{n/p}$

By Theorem 3, if $\gamma_p \equiv 0$ for some $p \leq n$ then $\gamma_n \equiv 0$ so $\chi(X) = 0$.

Theorem 4: If $\gamma_p = \text{const}$ for some (even) p, then the k^{th} Pontryagin class $P_k(X) = 0$ for k any multiple of $p/2$.

If $\gamma_p \equiv 0$ then $P_k(X) = 0$ for all $k \geq p/2$.

Cor: If $\gamma_2 = \text{const}$, then $P_k(X) = 0$ for all k.

We now indicate a little of the method used in proving these results.

$$\begin{tikzcd} F(X) \arrow[d, "\pi"] \\ X \end{tikzcd}$$

Let $F(X)$ be the principal $O(n)$ bundle of orthogonal frames on X.

Define 1-forms $\omega_1, \dots, \omega_n$ on $F(X)$ by $\omega_i(z)(v) = \langle \pi_* v, f_i \rangle$ where $z = (x, f_1, \dots, f_n) \in F(X)$.

Define 2-forms Ω_{ij} on $F(X)$ by

$$\Omega_{ij}(z)(u, v) = -\langle R(\pi_* u, \pi_* v) f_i, f_j \rangle$$

Theorem 5: γ_2 is const $\Longleftrightarrow$ $\exists$ K (real) such that $\Omega_{ij} = K\, \omega_i \wedge \omega_j$

Define p-forms $\Theta^{(p)}_{i_1 \cdots i_p}$ on $F(X)$ for $i_1, \dots, i_p \in \{1, \dots, n\}$ by $\Theta^{(p)}_{i_1 \cdots i_p} =$

$$\frac{1}{p!} \sum_{(j)} \delta(i_1, \dots, i_p, j_1, \dots, j_p)\, \Omega_{j_1 j_2} \wedge \dots \wedge \Omega_{j_{p-1} j_p}$$

where $\delta(i_1, \ldots, i_p; j_1, \ldots, j_p)$ is $+1$ (resp. -1) if $(j_1, \ldots, j_p)$ is an even (resp. odd) permutation of $(i_1, \ldots, i_p)$ and zero otherwise.

<u>Define</u> a p-form $\Theta^{(p)}$ on X with values in the bundle of p-vectors on X as follows. For $u_i \in X(x)$, $\Theta^{(p)}_{(x)}(u_1, \ldots, u_p) =$

$$\sum_{i_1 < \cdots < i_p} \Theta^{(p)}_{i_1 \cdots i_p}(z)\,(u_1', \ldots, u_p')\, f_{i_1} \vee \cdots \vee f_{i_p}$$ where

$u_1', \ldots, u_p'$ are lifts of $u_1, \ldots, u_p$.

<u>Theorem 6</u>: $\gamma_p(x, P) = \langle \Theta^{(p)}_{(x)}(\tilde{P}), \tilde{P} \rangle$ where $\tilde{P}$ is P with an orientation.

<u>Theorem 7</u>. $\gamma_p = \text{const} \Leftrightarrow \exists\, K_p$ such that $\Theta^{(p)}_{i_1 \cdots i_p} = K_p\, w_{i_1} \vee \cdots \vee w_{i_p}$

Let $\gamma_p = K_p = \text{const}$

$P_K(X) \in H^{4k}(X, \mathbb{R})$ is represented, according to De Rham's theorem, by a closed differential $4k$-form.

<u>Theorem 8</u> (Chern)

$$P_k(X) \sim \text{const} \sum_{(i)} \Theta^{(2k)}_{i_1 \cdots i_{2k}} \vee \Theta^{(2k)}_{i_1 \cdots i_{2k}}$$

Theorem 4 is a consequence of this.

A Universal Coefficient Theorem for Generalized Homology Theories

by Daniel Kahn 21 October 1963

A generalized homology theory (G.H.T.) is one which satisfies all the Eilenberg-Steenrod axioms except the dimension axiom.
A *spectrum* E is a sequence $\{E_n \mid n \in Z\}$ of spaces together with a sequence of maps $\varepsilon_n: SE_n \longrightarrow E_{n+1}$, where SE_n denotes the suspension $S^1 \wedge E_n$ of E_n. By "space" we always mean a countable C.W. complex with base point. Every G.H.T. arises from a spectrum as follows. Given a space X, form the reduced join $E_n \wedge X$, i.e. the identification space of $E_n \times X$ which collapses $E_n \vee X$ to a point. Using the natural map $S(E_n \wedge X) \longrightarrow E_{n+1} \wedge X$ we obtain a sequence

$$\cdots \longrightarrow \pi_{n+k}(E_k \wedge X) \longrightarrow \pi_{n+k+1}(S(E_k \wedge X))$$

$$\longrightarrow \pi_{n+k+1}(E_{k+1} \wedge X) \longrightarrow \cdots$$

Let $\pi_n(\underline{E} \wedge X)$ be the direct limit. Then define $\tilde{H}_n(X, \underline{E}) = \pi_n(\underline{E} \wedge X)$ (Reference: G. W. Whitehead, _Generalized Homology Theories_, Trans. A.M.S. 102 (1962), pp. 227–283)

If M is a module over a principal ideal domain, the universal coefficient theorem (UCT) gives a splittable exact sequence:

$$0 \longrightarrow \tilde{H}_n(X) \otimes M \longrightarrow \tilde{H}_n(X; M) \longrightarrow \mathrm{Tor}(\tilde{H}_{n-1}(X), M) \longrightarrow 0$$

as a step toward a U.C.T. for a G.H.T. we examine $\tilde{H}_*(X; M)$ where M is a left A module, A not being a principal ideal domain. Let

$$0 \longleftarrow M \xleftarrow{\varepsilon} M_0 \xleftarrow{d^1} M_1 \xleftarrow{d^2} M_2 \longleftarrow \cdots$$

be a free (acyclic) resolution of M.

Let $\tilde{M}_i = \begin{cases} M_i & i \geq 0 \\ M & i = -1 \\ 0 & i < -1 \end{cases}$

Let $C_*(X)$ denote the singular chain complex of X with coeffs. in A.

Since $\tilde{M}$ is acyclic, $H_*(C_*(X) \otimes_A \tilde{M}) = 0$

$\therefore \; H_n(C_*(X) \otimes_A M) \approx H_{n+1}(C_*(X) \otimes_A \tilde{M}/M)$

$C_*(X) \otimes_A \tilde{M}/M$ admits a filtration

$G_p(C_*(X) \otimes_A \tilde{M}/M) = \sum_{k=0}^{p} C_*(X) \otimes_A M_k$ This gives a spectral sequence with $E^0_{p,q} = C_q(X) \otimes_A M_p$ and $d^0_{p,q} : C_q(X) \otimes_A M_p \longrightarrow C_{q-1}(X) \otimes_A M_p$

$$x \otimes y \longmapsto \partial x \otimes y$$

Since M_p is free over A we have

$$\begin{tikzcd} \downarrow & & \downarrow \\ E^1_{p,q} \arrow[d, "d^1"] & = & H_q(X;A) \otimes_A M_p \arrow[d, "1 \otimes d"] \\ E^1_{p-1,q} & = & H_q(X;A) \otimes_A M_{p-1} \\ \downarrow & & \downarrow \end{tikzcd}$$

where $(1 \otimes d)(x^q \otimes y) = (-1)^q x \otimes dy$

Hence $E^2_{p,q} \approx \mathrm{Tor}^A_p(H_q(X;A), M) = \mathrm{Tor}^A_{p,q}(H_*(X;A), M)$

and $E^r_{p,q} \Longrightarrow \tilde{H}_*(X;M)$, that is

$E^\infty_{p,q} = \dfrac{G_p \tilde{H}_{p+q}(X;M)}{G_{p-1} \tilde{H}_{p+q}(X;M)}$ and $E^r_{p,q} = E^\infty_{p,q}$ for $r > \max(p, q+1)$

where $G_p \tilde{H}_{p+q}(X;M) = \mathrm{Im}\; \tilde{H}_{p+q}(G_p(C_* \otimes \frac{\tilde{M}}{M})) \longrightarrow H_{p+q}(C_* \otimes \frac{\tilde{M}}{M})$

To obtain a UCT for a G.H.T. we mimic the above procedure. For the spectrum $\underline{M}$ we shall find $E^r_{p,q} \Longrightarrow \tilde{H}_*(X;\underline{M})$ with $E^2_{p,q} = \mathrm{Tor}^{-}_{p,q}(\tilde{H}_*(X;-), \pi_*(\underline{M}))$

The role of the ring is played by a spectrum $\underline{A}$ with maps $A_j \wedge M_k \longrightarrow M_{j+k}$ and $A_i \wedge A_j \longrightarrow A_{i+j}$ We require strict associativity in the latter together with other properties. $\pi_*(\underline{A})$ becomes a graded ring with unit and we have a pairing $\pi_*(\underline{A}) \otimes \pi_*(\underline{M}) \longrightarrow \pi_*(\underline{M})$ which makes $\pi_*(M)$ into a $\pi_*(\underline{A})$ - module.

Note The principal example is for $\underline{A} = \underline{S}$, the sphere spectrum with $\varepsilon_k : SS^k \longrightarrow S^{k+1}$ the identity. Here $S^1 \wedge M_k \longrightarrow M_{k+1}$ is the map in $\underline{M}$ and $S^j \wedge M_k \longrightarrow M_{j+k}$ is defined inductively using the homeo $S^j \approx \overbrace{S^1 \wedge \cdots \wedge S^1}^{j \text{ times}}$.
$S^j \wedge S^k \longrightarrow S^{j+k}$ is the identity.

The role of the free resolution is played by a sequence $\underline{M} = \underline{M}^{-1} \subset \underline{M}^0 \subset \underline{M}^1 \subset \cdots$ (where "$\subset$" denotes "subspectrum") such that $i_* : \pi_*(\underline{M}^k) \longrightarrow \pi_*(\underline{M}^{k+1})$ is the zero homomorphism and $\underline{M}^{k+1}/\underline{M}^k$ is a wedge of spectra of the form $\underline{A} \wedge S^r$ with indices shifted. It follows that the sequence: $0 \leftarrow \pi_*(\underline{M}) \leftarrow \pi_*(\underline{M}^0, \underline{M}) \leftarrow \pi_*(\underline{M}^1, \underline{M}^0) \leftarrow \pi_*(\underline{M}^2, \underline{M}^1) \leftarrow$ is a free $\pi_*(A)$-resolution of $\pi_*(\underline{M})$. Form the homotopy exact couple of $(\tilde{\underline{M}}/\underline{M}) \wedge X \supset \cdots \supset (\underline{M}^k/\underline{M}) \wedge X \supset \cdots$ Then one can show that $E^2_{p,q}$ (with indices shifted) satisfies $E^2_{p,q} \approx \mathrm{Tor}^{\pi_*(A)}_{p,q}(\tilde{H}_*(X, A), \pi_*(M))$ and that the associated filtration of $\pi_*((\tilde{\underline{M}}/\underline{M}) \wedge X) \approx \tilde{H}_{*-1}(X; \underline{M})$ is complete (see Eilenberg and Moore, Limits and Spectral Sequences, Topology 1 (1962), 1–23) Hence the spectral sequence converges.

Note that if $\underline{M} = A \wedge Y$, one has a sort of Kunneth Theorem.

Cobordism Exact Sequences

by R. Holzsager 28 October 1963

Source: C.T.C. Wall, "Cobordism Exact Sequences for Differential and Combinatorial Manifolds," Annals of Mathematics, Jan. 1963.

We denote compact manifolds (combinatorial or differential) by V, W, M, N. There is a 1:1 correspondence between elements of $H^1(V; Z_2)$ and double coverings of V. If $W \subset_1 V$ (i.e. W is a submanifold of codimension 1) then W determines a dual cohomology class in $H^1(V; Z_2)$. Thus W determines a double covering of V. This covering is trivial on V−W and the sheets are cross-joined over W.

If $W \subset_1 V$ then locally W separates V into two sides, thus determining the normal covering of W in V.

Lemma 1 If $W \subset_1 V$ and p is the covering determined by W, then $p|W$ is iso to the normal covering of W.

We compose double coverings of manifold W by addition in $H^1(W, Z_2)$.

Lemma 2: Let $W \subset_1 V$. The composition of the orientation covering of W with the normal covering is the restriction to W of the orientation cov. of V.

In the differential case let $N \subset M$, V be manifolds and $f: V \longrightarrow M$ smooth.

Def f is t-regular on V if for each $x \in V$ with $f(x) \in N$ we have $df(V_x) + N_{f(x)} = M_{f(x)}$

In the combinatorial case, let L be a subset of a simplicial cx K.

Def L is in general position for r in K if for any (closed) simplex σ in K which meets L, $\sigma \cap L$ is the intersection of σ with a hyperplane of codim. r.

Lemma 3c: If L is in general position for r in K, $g: V \longrightarrow K$ simplicial, then $g^{-1}(L)$ is in general position for r in V.

Lemma 3d: If $M \supset N$ and $f: V \longrightarrow M$ a map, then f can be approximated by a smooth map g, t-regular on N.

Lemma 4c: $W \subset_1 V \Longleftrightarrow \exists$ a Δ-ation of V with W in general position for 1.

<u>Lemma 4d</u>: If $N \subset M$, $g: V \longrightarrow M$ smooth, t-regular on N, then $g^{-1}(N)$ is a submanifold of V.

<u>Lemma 5</u>: Any double cov. $p: \tilde{V} \longrightarrow V$ can <u>be</u> defined by a submanifold $W \subset_1 V$.

To prove this one uses that for sufficiently large n, $P^n(\mathbb{R})$ is a universal space for Z_2. Thus for some $f: V \longrightarrow P^n(\mathbb{R})$, p is induced as shown.

$$\begin{tikzcd}
\tilde{V} \arrow[r] \arrow[d, "p"] & S^n \arrow[d] \\
V \arrow[r, "f"] & P^n(\mathbb{R}) \supset P^{n-1}(\mathbb{R})
\end{tikzcd}$$

Since $S^n \longrightarrow P^n(\mathbb{R})$ consists of two sheets over $P^n(R) - P^{n-1}(\mathbb{R})$ cross-joined along $P^{n-1}(\mathbb{R})$, Lemma 5 follows from Lemmas 3c, 4c, 3d, 4d.

<u>Lemma 6</u>: Let $W \subset_1 V$ Then $\exists$ $g: V \longrightarrow P^n(\mathbb{R})$ such that $W = g^{-1}(P^{n-1})$ where g is smooth and either t-regular on N or simplicial for some Δ-ation of M with N in general position.

<u>Lemma 7</u>: Let $p: \tilde{V} \longrightarrow V$ be a double cover and X a submanifold of ∂V defining $p|\partial V$. Then $\exists$ $W \subset_1 V$ defining p with $\partial W = X$.

<u>Lemma 8</u>: If the normal cov. of W in V is trivial and W defines ρ, then ρ can be induced by a map of V to S^1. Conversely if ρ can be so induced, we may define ρ by a submanifold W with trivial normal covering in V.

Let $\mathfrak{N}$ be the unoriented cobordism ring

Let Ω be the oriented cobordism ring

Let $\mathfrak{W}$ be the subset of $\mathfrak{N}$ consisting of classes containing a manifold M satisfying

(A) The orientation covering of M is induced by a map of M into S^1.

$\mathfrak{W}$ is a subring of $\mathfrak{N}$ since if M, M' satisfy (A) so do $M \cup M'$ and $M \times M'$ (the maps into S^1 can be multiplied since S^1 a group)

We have ring homomorphisms where i is inclusion and r, s drop orientation.

$$\begin{tikzcd} \Omega \arrow[r, "r"] \arrow[dr, "s"'] & \mathfrak{N} \\ & \mathfrak{W} \arrow[u, "i"'] \end{tikzcd} \qquad r = is$$

<u>Lemma 9</u>: Let $\tilde{M} \longrightarrow M$ be a double covering of the closed orientable manifold M. Then $\tilde{M}$ with the induced orientation is orientably cobordant to $2M$.

Lemma 10: There exists an additive homo $\partial : \mathfrak{N} \longrightarrow \Omega$ of degree -1 such that $\partial\{M\} = [V]$ where $V \subset_1 M$ and V defines the orientation covering of M.

Lemma 11: There exists an additive homo $d : \mathfrak{N} \longrightarrow \mathfrak{N}$ of degree -2 $\ni$ $d\{M\} = \{B\}$ where $B \subset_2 V$ defines the normal cover of V in M (V as in Lemma 10)

Theorem: The following sequences are exact:

$$\Omega \xrightarrow{2} \Omega \xrightarrow{s} \mathfrak{W} \xrightarrow{\partial i} \Omega \xrightarrow{2} \Omega$$

$$0 \longrightarrow \mathfrak{W} \xrightarrow{i} \mathfrak{N} \xrightarrow{d} \mathfrak{N} \longrightarrow 0$$

$$\Omega + \mathfrak{N} \xrightarrow{(2,0)} \Omega \xrightarrow{r} \mathfrak{N} \xrightarrow{(\partial, d)} \Omega + \mathfrak{N} \xrightarrow{(2,0)} \Omega$$

Handlebody Decompositions I

by Dr. Derwent 4 November 1963

All manifolds will be compact, C^∞, and of dimension $n = p+q$; all maps C^∞. Let W be a manifold with boundary and $f: S^{p-1} \times D^q \longrightarrow \text{bd}\,W$ be an embedding.
sphere disk

Form the manifold $W + h_p = W \cup_f D^p \times D^q$ with corners rounded. To round, observe that $D^p \times D^q = (S^{p-1} \times I) \times (S^{q-1} \times I) / \text{Identification}$. Using the coordinates (x_0, s, y_0, t), it suffices to round the corner of a square. This gives a unique diff structure on $W + h_p$ (W with p handle attached) and the following are embedded in $W + h_p$: $S^{p-1} \times D^q$ (the attaching manifold), $D^p \times S^{q-1}$ (the transverse manif), $S^{p-1} \times 0$ (the attaching sphere), $0 \times S^{q-1}$ (the transverse sphere), as well as W.

Remarks: 1) Let f_t be an isotopy of $f \mid S^{p-1} \times 0$. Then $\exists$ an isotopy F_t of f which extends f_t.

2) Let f_t be an isotopy of $f_0 : S^{p-1} \times D^q \longrightarrow \text{bd}\,W$. Then $W + h_p^1 \underset{\text{diffeomorphic}}{=} W + h_p^0$

<u>Prop.</u> given $W + h_p + h_r$ [strictly $(W+h_p)+h_r$] with $p \geq r$, we can assume the handles disjoint.

From now on, let W be an n-manifold with $\mathrm{bd}\, W = M_1 \cup M_2$ where $M_1 \cap M_2 = \emptyset$. Let $f : W \longrightarrow R$. Call x a critical point of f if $(df)_x = 0$. Call x non-degenerate if $\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right) \neq 0$ Say x has index p if $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)$ has p negative and q positive eigenvalues.

<u>Theorem</u> There exists $f : W \longrightarrow [0,1]$ such that f has no degenerate critical points, $f^{-1}(0) = M_1$, $f^{-1}(1) = M_2$, and all critical points lie in the interior of W.

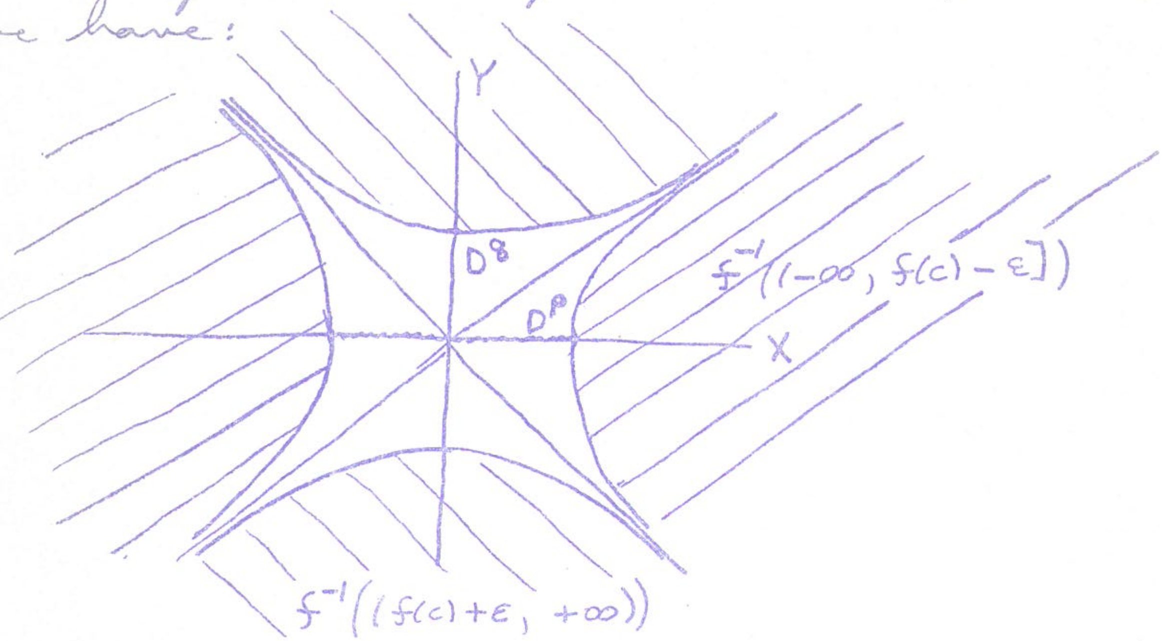
<u>Morse Lemma</u>: If c is a non-degenerate critical point of f of index p, then there exist coords $(x_1, \ldots, x_n)$ s.t.

$$f(x) = f(c) - x_1^2 - x_2^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_n^2$$

$$\stackrel{\text{def}}{=} f(c) - x^2 + y^2$$

<u>Corollary</u> (to Lemma) The non-degenerate critical points are isolated.

Assuming there is only one critical point c we have:



Result: $f^{-1}((-\infty, f(c)+\varepsilon]) = f^{-1}(-\infty, f(c)-\varepsilon]) + h_p$

From the theorem, $W = M_1 \times I$ + handles, one handle for each critical point. By interchanging the order of critical pts. with different indices we obtain

$W = M_1 \times I$ + 0-handles + 1-handles + --- + n-handles

Fix f with c_p critical point of index p. There is induced an ordered handle decomposition of W: $W = K_0 \cup \ldots \cup K_n$

$K_0 = M_1 \times I$ + 0-handles

$K_1 = M_1 \times I$ + 0-handles + 1-handles

- - - -

<u>Theorem</u>: given $W = UK_p$ there exists a relative CW complex (X, M_1) such that:

1) $X_p = p$-skeleton $\subset K_p$ as strong deformation retract.
2) $(K_p, K_{p-1}) \sim (X_p, X_{p-1})$
3) <u>If</u> $M_2 = \emptyset$ then $X = W$

We form dually a decomposition of W by applying $M_2 \times I$ to get

$$W = \underbrace{M_1 \times I + 0\text{-handles} + \cdots + p\text{-handles}}_{K_p} \,\|\, \underbrace{+ \cdots + n\text{-handles} + M_2 \times I}_{L_{q-1}}$$

and viewing the composition in the opposite order.

$$S^{p-1} \times D^q \subset \partial K_{p-1}, \quad D^p \times S^{q-1} \subset \partial L_{q-2}$$

We have free abelian chain complexes

$$K = \{ H_q(K_q, K_{q-1}), \partial_q \}$$

$$L = \{ H_q(L_q, L_{q-1}), \partial_q' \}$$

$$H(K; G) = H(W, M_1; G)$$

$$H(L; G) = H(W, M_2; G)$$

Strong Morse Inequalities

$$c_p = R_p(W, M_1) + \underbrace{B_p(W, M_1)}_{\geq T_p(W, M_1)} + \underbrace{B_{p-1}(W, M_1)}_{= T_{p-1}(W, M_1)}$$

Thus $\quad c_p \geq R_p(W, M_1) + T_p(W, M_1) + T_{p-1}(W, M_1)$

We shall eventually show that simple connectedness and dimension ≥ 6 $\Rightarrow$ there exists f such that the above are equal.

Handlebody Decompositions II

by Dr. John Derwent 18 November 1963

as before we are given a manifold W with $\dim W = n$, $p+q=n$. Also bdry $W = M_1 \cup M_2$ where $M_1 \cap M_2 = \phi$ and either or both of M_1, M_2 may be empty. W has a handlebody decomp:

$$W = \underbrace{M_1 \times I + h_0^1 + \cdots + h_p^1 + \cdots + h_p^{c_p}}_{K_p} + \underbrace{\cdots + h_n^{c_n} + M_2 \times I}_{L_{q-1}}$$

$K_p = K_{p-1} + p$-handles, $K_p \cap L_{q-1} = \partial K_p = \partial L_{q-1}$

a p-handle is, strictly speaking, a map $h_p^i : D^p \times D^q \to W$

We obtain free abelian chain complexes

$K = \{ H_p(K_p, K_{p-1}), \partial_p \}$

$L = \{ H_q(L_q, L_{q-1}), \partial_q' \}$

with basis elements corresponding to oriented cells

We now investigate the duality implicit in the decomposition of W. Let W be oriented. Orient the handles $h_p^i(D^p \times D^q)$ to agree with the orientation of W (assume infact that we orient each factor)

This determines homology and cohomology classes:

$D_i^p \times 0 \in H_p(K_p, K_{p-1})$, $0 \times D_i^q \in H_q(L_q, L_{q-1})$

$S_i^{p-1} \times 0 \in H_{p-1}(\partial K_{p-1})$, $0 \times S_i^{q-1} \in H_{q-1}(\partial L_{q-1})$

$\bar{D}_i^p \times 0 \in H^p(K_p, K_{p-1})$, $0 \times \bar{D}_i^q \in H^q(L_q, L_{q-1})$

Incidence matrices are determined by

$$\partial_{p+1}(D_i^{p+1} \times 0) = \sum_j x_{ij}^{p+1} (D_j^p \times 0)$$

$$\partial'_q(0 \times D_j^q) = \sum_i y_{ji}^q (0 \times D_i^{q-1})$$

Let $r : \partial K_p \longrightarrow (K_p, K_{p-1})$

$s : \partial L_q \longrightarrow (L_q, L_{q-1})$

Results: ① $\partial_{p+1}(D_i^{p+1} \times 0) = r_*(S_i^p \times 0)$

$\partial'_q(0 \times D_i^q) = s_*(0 \times S_i^{q-1})$

② $r^*(\bar{D}_i^p \times 0) \cap [\partial K_p] = (-1)^{pq}\, 0 \times S_i^{q-1}$

$s^*(0 \times \bar{D}_i^q) \cap [\partial L_q] = (-1)^{pq}\, S_i^{p-1} \times 0$

This is proved by applying the "permanence" relation for relative cap products several times using the following diagram (for first equality).

$$\begin{tikzcd}
 & (D^p, S^{p-1}) & \\
(D^p \times S^{q-1}, S^{p-1} \times S^{q-1}) \arrow[ur, "\text{projection}"] \arrow[rr] \arrow[d, "\text{excision}"', "h'_p \text{ restricted}"] & & (D^p \times D^q, S^{p-1} \times D^q) \arrow[ul, "\text{projection}"'] \arrow[d, "\text{excision}"] \\
(\partial K_p, \partial K_p - 0 \times S^{q-1}) \arrow[rr] & & (K_p, K_p - 0 \times D^q) \\
\partial K_p \arrow[u] \arrow[rr, "r"] & & (K_p, K_{p-1}) \arrow[u]
\end{tikzcd}$$

③ $x_{ij}^{p+1} = (-1)^{p+1} y_{ji}^{q}$ In view of

$$\partial_{p+1}(D_i^{p+1} \times 0) = \sum_j x_{ij}^{p+1} (D_j^p \times 0)$$

$$\partial'_q (0 \times D_j^q) = \sum_i y_{ji}^q (0 \times D_i^{q-1})$$

this gives the desired duality. We prove ③ by expressing $x_{ij}^{p+1} = \langle \bar{D}_j^p \times 0, r_*(S_i^p \times 0) \rangle$

$= \langle r^*(\bar{D}_j^p \times 0), S_i^p \times 0 \rangle = \langle s^*(0 \times \bar{D}_i^{q-1}) \cup r^*(\bar{D}_j^p \times 0), [\,] \rangle$

and comparing this with the expression for y_{ji}^q

④ $x_{ij}^{p+1} = (S_i^p \times 0) \cdot (0 \times S_j^{q-1})$ the intersection no. in ∂K_p

Given a free f.g. chain cx, we can put it in standard form. This can be done geometrically!

<u>Theorem</u>: If $n \geq 4$ and $H_1(W, M_1) = H_1(W, M_2) = 0$ then $\exists$ an oriented handlebody decomposition with the ∂_p's in the usual standard form for free abelian chain complexes.

Lemma: Let $f_1, f_2 : S^{p-1} \times D^q \longrightarrow N$ be embeddings, $\dim N = p-1+q$, $q \geq 2$. Let $\varepsilon = \pm 1$. Then $\exists$ an embedding $f_3 : S^{p-1} \times D^q \longrightarrow N$ such that:

1) $f_{3*} = f_{1*} + \varepsilon f_{2*}$
2) $N \times I + h_p^1 + h_p^2 = N \times I + h_p^2 + h_p^3$

Using this lemma we can perform the necessary operations on the incidence matrices to put them in standard form.

Theorem (Whitney) If $f_1 : V^p \longrightarrow M^n$ and $f_2 : W^q \longrightarrow M^n$ are embeddings, $p+q=n$, $p \geq 3$, $q \geq 3$, and $\pi_1(M^n) = 0$, then we can assume after an isotopy that $V^p \cdot W^q$ = actual no. of intersections.

Theorem (Smale) Let $V = W + h_p + h_{p+1}$. If the attaching sphere of h_{p+1} intersects the transverse sphere of h_p exactly once and transversely, then $V = W$

Theorem If W is connected and simply connected and each component of M_1 and M_2 is simply connected + $n \geq 6$, then

all excess handles can be removed,
i.e. $c_p = R_p + T_p + T_{p-1}$, $R_p = p^{th}$ Betti no. of (W, M_1)
T_p = number of torsion coefficients in dimensions p of (W, M_1)

<u>Consequences</u>: (All due originally to Smale)

1. With the same hypotheses, there is a nondegenerate function on W with $R_p + T_p + T_{p-1}$ critical pts of index p.

2. h-cobordism theorem: If W is simply connected and M_1 and M_2 are both deformation retracts of W, and $n \geqq 6$, then $W = M_1 \times I = M_2 \times I$

3. If M is a differentiable homotopy sphere of dimension $n \geqq 5$, M is homeomorphic to S^n via a homeomorphism which is a diffeomorphism except at one point.

4. If W is contractible, has simply connected boundary and $\dim W = n \geqq 6$ then $W = D^n$

<u>Remark</u>: 3 follows from 1 for $n \geqq 6$ and from 2 for $n = 5$ via an argument of Milnor and Mazur.

Piecewise Linear Microbundles and Cobordism

by Dr. Robert Williamson   2 December 1963

Reference: Milnor's notes on microbundles

We shall be concerned with piecewise linear manifolds and maps.

Def A (PL) microbundle $\underline{x}$ over B consists of $\underline{x} : B \xrightarrow{i} E \xrightarrow{p} B$ where $p \circ i = \text{ident}$, p is a bundle map, and neighborhoods U, V always exist such that

$$\begin{tikzcd}
B \arrow[r, "i"] \arrow[d, phantom, "\cup"] & E \arrow[r, "p"] \arrow[d, phantom, "\cup"] & B \arrow[d, phantom, "\cup"] \\
U \arrow[r] \arrow[dr, "1\times 0"'] & V \arrow[r] & U \\
& U \times R^n \arrow[ur, "proj"'] &
\end{tikzcd}$$

commutes.

given microbundles $\underline{y}$, $\underline{z}$ over X one can define their Whitney sum $\underline{y} \oplus \underline{z}$. For any microbundle $\underline{y}$ there exists $\underline{z}$ such that $\underline{y} \oplus \underline{z} = \underline{e}^X$ trivial.

Def Let $(\underline{y})_{PL}$ = the stable class of $\underline{y}$ and let $K_{PL}(X)$ = group of stable classes over X.

2

Theorem: If X is a PL manifold with compatible differentiable structure α, then there is a natural homo $K_0(X) \longrightarrow K_{PL}(X)$ such that $(\tau_\alpha)_0 \longrightarrow (\underline{t}_X)_{PL}$ where τ_α is the tangent bundle and $\underline{t}_X : X \longrightarrow X \times X \longrightarrow X$ is the natural microbundle.

Def: A normal microbundle for PL embedded $M \subset N$ is a nbhd V of M and a map $j: V \longrightarrow M$ such that $M \subset V \xrightarrow{j} M$ is a microbundle.

Theorem: $M \times 0$ has a normal microbundle $\underline{n}$ in $N \times R^q$ for q sufficiently large. Furthermore $\underline{t}_M \oplus \underline{n} = \underline{t}_N | M$

Corollary: M has a nbhd PL homeo to $M \times R^q$ in $N \times R^s$ for some s
$\Longleftrightarrow (\underline{t}_M)_{PL} = (\underline{t}_N | M)_{PL}$

Theorem (Milnor): Let M be a PL manifold. If $\exists$ a stable class $(\xi)_0 \in K_0(M)$ such that $(\xi)_0 \longrightarrow (\underline{t}_M)_{PL}$ then M has a differentiable structure with tangent fibre bundle τ_α such that $(\tau_\alpha)_0 = (\xi)_0$.

Proof: Put $M \subset V$ open set in R^n such that $V \simeq M$. Take ξ' over V, $\xi'|M = \xi$, $M \subset E(\xi')$

$(T_{E(\xi)}|M)_0 = (\xi)_0 \Rightarrow (\underline{t}_{E(\xi)}|M)_{PL} = (\underline{t}_M)$

By a theorem of Hirsch [$M \times R^8$ has a differentiable structure $\Rightarrow$ M has a differentiable structure] we find that M has a diff structure.

Note: There exists a universal space B_{PL} for microbundles s.t. $k_{PL}(X) = [X, B_{PL}]$.
The natural homo is induced by $B_0 \longrightarrow B_{PL}$

$$\begin{tikzcd} & B_0 \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "f"'] & B_{PL} \end{tikzcd}$$

f classifying map for $\underline{t}_M$

Obstructions lie in $H^i(M, \pi_i(B_{PL}, B_0))$
Munkres' obstruction lies in $H^i(M, \Gamma_{i-1})$
where Γ_i = group of diff structures on S^i

Theorem (Hirsch) $\pi_i(B_{PL}, B_0) \approx \Gamma_{i-1}$

$$0 \longrightarrow \pi_i(B_0) \longrightarrow \pi_i(B_{PL}) \longrightarrow \pi_i(B_{PL}, B_0) \longrightarrow 0$$

$$0 \longrightarrow k_0(S^i) \longrightarrow k_{PL}(S^i) \longrightarrow \Gamma_{i-1} \longrightarrow 0$$

Theorem (Milnor) $k_0(S^7 \cup_7 E^8) \longrightarrow k_{PL}(S^7 \cup_7 E^8)$

map of degree 7

has kernel $\neq 0$.

Thus there is an open neighborhood of $S^7 \cup_7 E^8$ in Euclidean space which can be given a differentiable structure which is not parallelizable

Consider the cobordism groups:

Ω_p^n = cobordism group of PL manifolds

Ω^n = cobordism group of C^∞ manifolds

Theorem (Thom) There exist isos and monos such that

$$\begin{array}{ccc} \Omega^n & \approx & \lim \pi_{n+k}(MO(k)) \\ {\scriptstyle \text{mono}}\downarrow & & \downarrow{\scriptstyle \text{mono}} \\ \Omega^n_{PL} & \approx & \lim \pi_{n+k}(MPL(k)) \end{array}$$

commutes.

The mono $\Omega^n \longrightarrow \Omega^n_{PL}$ comes from the work of J.H.C. Whitehead.

Note: If $\underline{x}$ is a microbundle over X then the Stiefel-Whitney classes $w_i(\underline{x}) \in H^i(X, Z_2)$ and the Pontrjagin classes $p_i(x) \in H^{4i}(X, Q)$ are defined.

From $\Omega^n_{PL}/\Omega^n = \lim \pi_{n+k}(MPL(k), MO(k)) \xrightarrow{Hur} H_{n+k}(MP\ldots)$

Thom SS

$$\Gamma_{n-1} = \pi_n(B_{PL}, B_0) \xrightarrow{Hur} H_n(B_{PL}, B_0)$$

we conclude:

<u>Theorem</u>: If $\Gamma_{n-1} = 0$ for $n-1 < N$ then

$\Gamma_{N-1} \approx \Omega^N_{PL}/\Omega^N$

One can show:

$\Omega^8_{PL}/\Omega^8 = Z_{28}$

$\Omega^9_{PL}/\Omega^9 =$ 2 tors

$\Omega^{10}_{PL}/\Omega^{10} =$ 2 tors

$\Omega^{11}_{PL}/\Omega^{11} = Z_3 \oplus$ 2 tors

Higher Order Generalized Whitehead Products ①

by Dr. Gerald Porter 9 December 1963

Let $f \in \pi_n(X)$, $g \in \pi_m(X)$ Then the Whitehead product $[f, g] \in \pi_{m+n-1}(X)$ is represented by the composition

$$(*) \quad S^{m+n-1} = \partial(I^m \times I^n) = I^m \times S^{n-1} \cup S^{m-1} \times I^n$$

$$\longrightarrow S^m \vee S^n \xrightarrow{g|f} X$$

We shall generalize this.

<u>Definitions</u>

Let $\mathcal{C}$ = category of countable CW cxs with base point

$\mathcal{C}^n = \mathcal{C} \times \cdots \times \mathcal{C}$ n-tuples

$T_i : \mathcal{C}^n \longrightarrow \mathcal{C}$ is the functor with

$T_i(X_1, \ldots, X_n) \subset X_1 \times \cdots \times X_n$ consisting of those pts having at least i coords at a base pt $*$.

Thus T_0 = cartesian product

T_1 = "fat" wedge

T_{n-1} = "thin" wedge $\vee$

The "smash" $\wedge : \mathcal{C}^n \longrightarrow \mathcal{C}$ is the identification

$\wedge = T_0 / T_1$

Let the suspension $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ be $\wedge(S^1, \cdot)$ and the cone $c : \mathcal{C} \longrightarrow \mathcal{C}$ be $\wedge(I^1, \cdot)$

Let $\Sigma : \mathcal{C}^n \longrightarrow \mathcal{C}^n$ and $c : \mathcal{C}^n \longrightarrow \mathcal{C}^n$ be defined coordinatewise.

Let the n-fold suspension $\Sigma^n : C \longrightarrow C$
be $\Sigma^n = \Sigma(\Sigma^{n-1})$

Let $p : I' \longrightarrow S'$ induce the transformation $p : C \longrightarrow \Sigma$

Let $\rho = T_0(P) : T_0 C \longrightarrow T_0 \Sigma$

$Q : C^n \longrightarrow C$ be $Q = \rho^{-1}(T_1 \Sigma)$

$\bar{\rho} = \rho | Q$

Note $Q(X_1, \ldots, X_n) = \bigcup_{i=1}^{n} T_0(CX_1, \ldots, X_i, \ldots, CX_n)$

$(\rho, \bar{\rho}) : (T_0 C, Q) \longrightarrow (T_0 \Sigma, T_1 \Sigma)$ is a relative homeomorphism,

Theorem: There exists a homot. equiv. trans.
$\bar{h} : \Sigma^{n-1} \wedge \longrightarrow Q$ (each with domain C^n)

We obtain

$\Sigma^{n-1} \wedge \xrightarrow{\bar{h}} Q \xrightarrow{\bar{\rho}} T_1 \Sigma \longrightarrow X$ in place of (*)

Def Let $\varphi : T_1 \Sigma(A_1, \ldots, A_n) \longrightarrow X$
Then the n^{th} order Generalized Whitehead product
$W(\varphi) \in \pi(\Sigma^{n-1} \wedge (A_1, \ldots, A_n), X)$ is $\varphi_* \{\bar{\rho}\bar{h}\}$

Let $f_k : \Sigma A_k \longrightarrow X$ These induce
$f_1 | f_2 | \cdots | f_n : T_{n-1} \Sigma(A_1, \ldots, A_n) \longrightarrow X$
If $\varphi : T_1 \Sigma(A_1, \ldots, A_n) \longrightarrow X$ say $\varphi \in (f_1, \ldots, f_n)$
if φ extends $f_1 | f_2 | \cdots | f_n$
Let $[f_1, \ldots, f_n] = \{W(\varphi) : \varphi \in (f_1, \ldots, f_n)\}$

Naturality: Let $T_1\Sigma(B_1,\dots,B_n) \xrightarrow{\varphi} X \xrightarrow{g} Y$

$f_i : A_i \longrightarrow B_i \qquad \varphi \in (h_1,\dots,h_n)$

Then $W(\varphi T_1\Sigma(f_1,\dots,f_n)) = \Sigma^{n-1}\Lambda(f_1,\dots,f_n)^* W(\varphi)$

$\Sigma^{n-1}\Lambda(f_1,\dots,f_n)^*[h_1,\dots,h_n] \subset [h_1(\Sigma f_1),\dots,h_n(\Sigma f_n)]$

$W(g\circ\varphi) = g_* W(\varphi)$

$g_*[h_1,\dots,h_n] \subset [gh_1,\dots,gh_n]$

Remark: Let $j : T_1\Sigma \longrightarrow T_0\Sigma$ be inclusion

Then $W(j) = 0$

This is proved with the aid of

$$\begin{tikzcd}
c\,\Sigma^{n-1}\Lambda \arrow[r, "c\bar{h}"] & CQ \arrow[r] & CT_0C \arrow[r] & T_0C \arrow[r] & T_0\Sigma \\
\Sigma^{n-1}\Lambda \arrow[u] \arrow[rrr] & & & Q \arrow[u] \arrow[r] & T_1\Sigma \arrow[u]
\end{tikzcd}$$

which commutes.

Prop Let $\varphi : T_1\Sigma \longrightarrow X$

$$\begin{tikzcd}
T_1\Sigma \arrow[r, "\varphi"] \arrow[d, "j"'] & X \\
T_0\Sigma \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

Then φ is extendable to ψ $\Longleftrightarrow$ $W(\varphi) = 0$

Fix n Let $S_i = \{\sigma : [1,\dots,n-i+1] \longrightarrow [1,\dots,n] \mid \sigma$ is order preserving $\}$

Define $\sigma : \mathcal{C}^n \longrightarrow \mathcal{C}^{n-i+1}$ by

$\sigma(A_1,\dots,A_n) = (A_{\sigma(1)},\dots,A_{\sigma(n-i+1)})$

$j_\sigma : T_i \Sigma\sigma \longrightarrow T_i\Sigma$

$\Theta : \bigvee_{\sigma\in S_i} \Sigma^{n-1}\wedge\sigma \longrightarrow T_i\Sigma$

$\Theta \mid \Sigma^{n-1}\wedge\sigma = w(j\sigma)$

Theorem The following functors are homot equiv:

$T_{i-1}\Sigma$

$T_i\Sigma \cup_\Theta c\bigvee_{\sigma\in S_i}\Sigma^{n-1}\wedge\sigma$

$T_i\Sigma \cup_\Theta \bigvee_{\sigma\in S_i} c\Sigma^{n-1}\wedge\sigma$

Theorem: Let $\varphi : T_i\Sigma \longrightarrow X$

$$\begin{tikzcd} T_i\Sigma \arrow[r, "\varphi"] \arrow[dr] & X \\ & T_{i-1}\Sigma \arrow[u, dashed, "\psi"'] \end{tikzcd}$$

Then φ is extendable to ψ $\Longleftrightarrow$ $w(\varphi\cdot j_\sigma) = 0$ for all $\sigma\in S_i$

Theorem: Let $\varphi : T_1\Sigma \longrightarrow X$ If X is an H-space then $w(\varphi) = 0$

To prove this use Hilton's result that $\Sigma T_1\Sigma(\quad)$ is a retract of $\Sigma T_0\Sigma(\ )$

$$\begin{tikzcd} \Sigma T_1\Sigma \arrow[r, "\Sigma j"] \arrow[d, "\Sigma\varphi"] & \Sigma T_0\Sigma \arrow[dl, "\psi"] \\ \Sigma X & \end{tikzcd}$$

Obtain the extension ψ

$$\begin{tikzcd} T_1\Sigma \arrow[r, "\varphi"] \arrow[d, "j"] & X \arrow[d, "k"] \\ T_0\Sigma \arrow[r, "\psi'"] & \Omega\Sigma X \arrow[d, "r"] \\ & {} \end{tikzcd}$$

$k(x)(t) = (t, x)$

$\therefore W(\varphi) = 0$

Cor $[f_1, \ldots, f_n] = 0$

$$W(\varphi)^* : \Pi(X, H) \longrightarrow \Pi(\Sigma^{n-1}\Lambda(\ ), H)$$

$$W(\varphi)^*[f] = W(f\varphi)$$

Cor If H is a $K(\pi, n)$ then $W(\varphi)^*$ is the 0-map

$$W(\varphi)^* : H^n(X, \pi) \longrightarrow H^n(\Sigma^{n-1}\Lambda(\ ), \pi)$$

Cor $\Sigma_* W(\varphi) = 0 \iff \underbrace{k\, W(\varphi)}_{= W(k\varphi)} = 0$

Note There are examples of spaces with all Whitehead products 0 which are not H-spaces

A Combinatorial Transversality Theorem

by Dr. V. Poenaru   6 January 1964

We use locally finite simpl. cxs and p.w. linear maps.

Let X be a set. A <u>complex in X</u> is an injection $f : K \longrightarrow X$ where K a complex.

Two complexes in X $\quad f_1 : K_1 \longrightarrow X$, $f_2 : K_2 \longrightarrow X$ are <u>coherent</u> if $\exists$ $f_3 : K_3 \longrightarrow X$ with $\operatorname{Im} f_3 = \operatorname{Im} f_1 \cap \operatorname{Im} f_2$ and $f_1^{-1} f_3$, $f_2^{-1} f_3$ pw linear.

<u>Def</u> A <u>polystructure</u> on X is a family $\mathcal{Z}$ of complexes in X s.t. 1) $X = \bigcup_{f \in \mathcal{Z}} \operatorname{Im} f$

2) If $f_1, f_2 \in \mathcal{Z}$ then f_1 and f_2 are coherent.

3) The sets $\operatorname{Im} f$, $f \in \mathcal{Z}$ form a lattice

4) If $f : K \longrightarrow X$ is in $\mathcal{Z}$ and $g : K' \longrightarrow K$ is a pw linear injection then $f \circ g \in \mathcal{Z}$.

The pair $X, \mathcal{Z}$ is a <u>polyspace</u>

The <u>polytopology</u> of a polyspace is given by $\quad U \subset X$ open $\Longleftrightarrow$ $f^{-1}(U)$ open $\forall f \in \mathcal{Z}$

X is a <u>polyhedron</u> if it admits a covering by open sets which are images of $f \in \mathcal{Z}$

Let X be a polyspace and $Y \subset X$ be such that Y is a union of images of $f \in \mathcal{F}$. Then we get a polystructure on Y and $T(\mathcal{F}|Y) = T(\mathcal{F})|Y$ where $T(\mathcal{F})$ is the polytopology.

A polysubspace of a polysubspace is a polysubspace.

Let $X, \mathcal{F}$ and $Y, \mathcal{F}'$ be polyspaces. Then $f: X \longrightarrow Y$ is a <u>polymap</u> if for each $\varphi \in \mathcal{F}$ there exist $g: K \longrightarrow L$ p.w. linear and $\psi \in \mathcal{F}'$ as shown

$$\begin{array}{ccc} K & \xrightarrow{g} & L \\ \varphi\downarrow & & \downarrow\psi \\ X & \xrightarrow{f} & Y \end{array} \qquad \text{comm.}$$

<u>Remark</u>: If X and Y are polyhedra then f is a polymap $\Longleftrightarrow$ f is p.w. linear

<u>Remark</u>: If $X, \mathcal{F}$, and $Y, \mathcal{F}_2$ are polyspaces then $Hom(X, Y)$ admits a polystructure $\mathcal{F}$. An injection $f: K \longrightarrow Hom(X, Y)$ is in $\mathcal{F}$ $\Longleftrightarrow$ $\forall\ g: L \longrightarrow X$, $g \in \mathcal{F}$; the induced map $K \times L \longrightarrow Y$ factors through some injection of $\mathcal{F}_2$ $\quad K \times L \longrightarrow E \xrightarrow{\in \mathcal{F}_2} Y$

<u>Remark</u>: The polyspace $Hom(X, Y)$ is in general not a polyhedron.

A *combinatorial manifold* $M, \mathcal{T}$ is a polyhedron which is a manifold and where M is covered by images of coordinate maps $f : E_m \longrightarrow M$, $f \in \mathcal{T}$.

A submanifold $M_n \subset M_m$ is *locally flat* if about each $x \in M_n$ $\exists$ a coord nbhd $f : E_m = E_n \times E_{m-n} \longrightarrow M_m$ such that $f^{-1}(M_n) = E_n$

Remark: If $m-n \geq 3$ then $M_n \subset M_m$ is always locally flat.

By manifold we mean combinatorial manifold, and submanifolds are assumed locally flat.

Def: Let P be a polyhedron and $M_m \supset M_n$ be a manifold and submanifold. Then $f : P \longrightarrow M_m \supset M_n$ is *transversal* if for each $x \in P$ with $f(x) \in M_n$ there exists a Δation K of P with $x \in \sigma$ and a coord nbhd $\varphi : E_m = E_n \times E_{m-n} \longrightarrow M_m$ with the properties:

$\varphi(0) = f(x)$, $\varphi^{-1}(M_n) = E_n$

$f(\text{star}(\sigma, K)) \subset \varphi(E_m)$

$\varphi^{-1} f : \text{star}(\sigma, K) \longrightarrow E_m$ linear

$\varphi^{-1} f|\sigma : \sigma \longrightarrow E_{m-n}$ non-degenerate

$\varphi^{-1} f(\text{link}(\sigma, K)) \subset E_n$

Remarks 1) Let $P \xrightarrow{f} M_m \supset M_n$, $x \in P$

If f is transversal at x then it is transversal in a nbhd of x.

2) If $P \xrightarrow{f} M_m \supset M_n$ is transversal, P a manifold and f an injection then $f(P)$, M_n are in general position in M_m

3) Weak combinatorial implicit function theorem

Let $P \xrightarrow{f} M_m \supset M_n$ as in 2. Then $f^{-1}(M_n)$ is a locally flat submodule of P with codim $m-n$

Def N, M be manifolds. Two maps $f, g : N \longrightarrow M$ are isotopic if $\exists$ a map h as shown

$$\begin{tikzcd} N \times I \arrow[rr, "h"] \arrow[dr, "proj"'] & & M \times I \arrow[dl, "proj"] \\ & I & \end{tikzcd}$$

such that

$h(t) : N \longrightarrow M$ satisfies $h(0) = f$, $h(1) = g$

The maps f, g are ambiently isotopic if in addition $\exists$ an iso F as shown

$$\begin{tikzcd} M \times I \arrow[rr, "F"] \arrow[dr] & & M \times I \arrow[dl] \\ & I & \end{tikzcd}$$

with $F(0) = \text{ident}$ and $h(t) = F(t) \circ f$

Remark: In the differentiable case we have isotopy = ambient isotopy

Absolute Transversality Theorem:

Let $f : P \longrightarrow M_m \supset M_n$. Then $\exists\ \varphi : P \to M_m \supset M_n$ such that $\varphi \sim f$ ambiently and φ is transversal.

Relative Transversality Theorem:

Let P, $M_m \supset M_n$ be manifolds with boundary. We consider only proper maps, i.e. maps which carry boundary to boundary and interior to interior. Assume $(\partial M_m) \cap M_n = \partial M_n$. Let $f : P \longrightarrow M_m \supset M_n$ with $f|\partial P$, $\partial P \longrightarrow \partial M_m \supset \partial M_n$ transversal. Then $\exists\ \varphi : P \to M_m \supset M_n$ with $\varphi \sim f$ ambiently and φ transversal.

Def Let $M_m \supset M_n$ with M_n locally flat. We say M_n admits a normal microbundle if $\exists$ a nbhd N_m with $M_m \supset \underset{\xrightarrow{\text{proj } p}}{N_m} \supset M_n$

such that $M_n \underset{i}{\subsetneq} N_m \xrightarrow[p]{} M_n$ is a microbundle.

Let $M_m \supset M_n$ with M_n admitting a normal microbundle ν. Let $f : P \to M_m \supset M_n$ be transversal. Then $f^{-1}M_n$ admits a normal microbundle μ such that if $\varphi = f|f^{-1}(M_n)$ then $\varphi^*\nu = \mu$.

Higher Order Whitehead Products Distinguished by Cohomology

Dr. Gerald Porter 20 January 1964

We shall use mod p coefficients.

Def 1: Let $f: S^n \longrightarrow X$, $x \in H^n(X)$ Then f signifies x if $f^*(x) \neq 0$

Let $x_{j_i} \in H^{m_{j_i}}(X)$ where $m_{j_i} > 0$ Let Φ be a natural universally defined cohom. op. which vanishes on the cohom of cartesian products of spheres [e.g. Φ is a Steenrod operation] Let $y \in H^q(X)$.

Def 2: We say (Φ, y) distinguishes the product $x_{k_1} \cdots x_{k_n}$ if

$$\Phi(y) = \sum \alpha(j_1, \ldots, j_n)\, x_{j_1} \cdots x_{j_n} \mod (H^*(X))^{n+1}$$

where 1) $\alpha(k_1, \ldots, k_n) \neq 0$

2) If $m_{k_i} = m_{k_j}$ then $k_i = k_j$

3) Given $(j_1, \ldots, j_n) \neq (k_1, \ldots, k_n)$ and $\alpha(j_1, \ldots, j_n) \neq 0$ $\exists\, i \ni m_{j_i} \neq m_{k_r}$ $1 \leq r \leq n$

4) No p of the k_i's are the same.

Theorem: Suppose $f_i: S^{m_{k_i}} \longrightarrow X$ signifies x_{k_i} for $1 \leq i \leq n$, and $f_i = f_j$ if $k_i = k_j$ Suppose further that (Φ, y) distinguishes $x_{k_1} \cdots x_{k_n}$ then $0 \notin [f_1, \ldots, f_n]$

Proof [See 9 December 1963 talk by Porter for definition of $[f_1, \dots, f_n]$ etc.] ②

We are done if $[f_1, \dots, f_n]$ empty so assume $w(\phi) \in [f_1, \dots, f_n]$ Let $T_0 = S^{m_{k_1}} \times \dots \times S^{m_{k_n}}$

If $w(\phi) = 0$ $\exists\ \psi : T_0 \longrightarrow X$, $\psi \in (f_1, \dots, f_n)$

Since $\Phi = 0$ in $H^*(T_0)$, $0 = \Phi\psi^*(y) = \psi^*\Phi(y)$ We shall show that infact $\psi^*\Phi(y) \neq 0$.

Observe that $(H^*(T_0))^{n+1} = 0$

$$\psi^*(x_{j_i}) = \sum_{\{r \mid m_{j_i} = m_{k_r}\}} 1 \otimes \dots \otimes f_r^*(x_{j_i}) \otimes \dots \otimes 1 \quad \mathrm{mod}\,(H^*(T_0))^2$$

Suppose $(j_1, \dots, j_n) \neq (k_1, \dots, k_n)$ and $\alpha(j_1, \dots, j_n) \neq 0$

Then by cond 3 of Def 2 $\exists$ some i such that $m_{j_i} \neq m_{k_r}$ for $1 \leq r \leq n$ Hence $\psi^*(x_{j_i}) \in (H^*(T_0))^2$

Therefore $\psi^*(x_{j_1} \cdots x_{j_n}) \in (H^*(T_0))^{n+1} = 0$

Thus $\psi^*\Phi(y) = \alpha(k_1, \dots, k_n)\, \psi^*(x_{k_1} \cdots x_{k_n})$

By naturality of cup product

$$\psi^*(x_{k_1} \cdots x_{k_n}) = \prod_{i=1}^{n} \sum 1 \otimes \dots \otimes f_n^*(x_{k_i}) \otimes \dots \otimes 1$$

Write $x_{k_1} \cdots x_{k_n} = \pm x_{l_1}^{\nu_1} \cdots x_{l_t}^{\nu_t}$ where l_i are distinct. Let $M = \prod_{i=1}^{t} \nu_i$ By property 4 of Def 2 $M \neq 0 \ (\mathrm{mod}\ p)$. Using prop 2 of Def 2 it is easily seen that

$$\psi^*(x_{k_1} \cdots x_{k_n}) = M f_1^*(x_{k_1}) \otimes \dots \otimes f_n^*(x_{k_n})$$

Since f_i signifies x_{k_i}, $f_i^*(x_{k_i}) = \gamma_i s_i$, $\gamma_i \neq 0$

Therefore $\psi^*\Phi(y) = \alpha(k_1, \dots, k_n) M \gamma_1 \cdots \gamma_n\, s_1 \otimes \dots \otimes s_n$

Since $\alpha(k_1, \dots, k_n) \neq 0$, $\psi^*\Phi(y) \neq 0$

Examples of non-trivial Whitehead products

Consider $BSU(k)$

$H^*(BSU(k); Z_p) = Z_p[x_2, \ldots, x_k]$, $\dim x_i = 2i$

If $p \geq n$, $k \geq n$ there is a map $f: S^{2n} \to BSU(k)$ with $f^*(x_n) \neq 0$ (Serre-C Theory paper)

Let $r = [\frac{p}{n}] + 1$ Using the results of Borel and Serre (Am J. - 1953) it is seen that $P_p^1 x_{rn-p+1}$ distinguishes x_n^r in $BSU(k)$ for

$n \leq k < n + \frac{n}{[\frac{p}{n}]}$

$\therefore 0 \notin [\underbrace{f, \ldots, f}_{r \text{ times}}]$ by the above theorem.

For $n = k = 2$, p odd prime so $r = \frac{p+1}{2}$ we get a non-trivial product in $\pi_{2p+1}(BSU(2)) \approx \pi_{2p}(S^3)$

We generalize the usual terminology and refer to this element as a Samuelson product. It is clear from the proof of the theorem that p divides the (group) order of this element. Moreover $\pi_j(S^3)|_p = 0$ for $j < 2p$

Thus the initial p torsion of the homotopy of S^3 is generated by a multiple of an h.o.S. product.

$[f, \dots, f] \neq \emptyset$ if $\exists\, \varphi : T_1\Sigma \longrightarrow BSU(k)$ of type $(f, \dots, f)$. Actually it suffices to find $\beta \not\equiv 0$ mod p such that $\exists\, \varphi : T_1\Sigma \longrightarrow BSU(k)$ of type $(\beta f, \dots, \beta f)$.

Let $\varphi : T_i\Sigma(A_1, \dots, A_n) \longrightarrow X$ with $\varphi \in (f_1, \dots, f_n)$. Let $j_\sigma : T_1\Sigma \longrightarrow T_i\Sigma$ as usual. We know that φ may be extended to $T_{i-1}\Sigma$ $\Longleftrightarrow$ $W(\varphi j_\sigma) = 0$ for all $\sigma \in S$.

Assume that $W(\varphi j_\sigma)$ is of finite group order n_σ with $n_\sigma \not\equiv 0$ mod p

Let N = l.c.m of the n_σ

Let β be the smallest integer $\ni N \mid \beta^{n-i+1}$

Clearly $\beta \not\equiv 0$ mod p

<u>Theorem</u>: There exists $\psi : T_{i-1}\Sigma(A_1, \dots, A_n) \to X$ of type $(\beta f_1, \dots, \beta f_n)$ (We assume each A_j is a suspension)

<u>Proof</u>: Let $\beta_j : \Sigma A_j \longrightarrow \Sigma A_j$ be β times the identity. Use

$$\begin{tikzcd}
T_i\Sigma(A_1, \dots, A_n) \arrow[r, "T_i(\beta_1, \dots, \beta_n)"] & T_i\Sigma(A_1, \dots, A_n) \arrow[r, "\varphi"] & X \\
T_1\Sigma_\sigma \arrow[u, "j_\sigma"'] \arrow[r, "T_1\sigma(\beta_1, \dots, \beta_n)"] & T_1\Sigma_\sigma \arrow[u, "j_\sigma"'] & \\
\Sigma^{n-1}\Lambda \arrow[u, "W(1)"'] \arrow[r, "\beta^{n-i+1}"] & \Sigma^{n-1}\Lambda \arrow[u, "W(1)"'] \arrow[uur, "W(\varphi j_\sigma)"'] &
\end{tikzcd}$$

Prop: Let $f : S^{2n} \longrightarrow BSU(n)$ Let p be any prime $\geq n$, $r = [\frac{p}{n}] + 1$ Then $\exists$ an integer β such that $\beta \neq 0 \mod p$ and $[\underbrace{\beta f, \ldots, \beta f}_{r \text{ times}}]$ is non-empty.

Proof: The obstructions lie in the groups $\pi_{2ni-1}(BSU(n)) \approx \pi_{2ni-2}(SU(n))$ $2 \leq i < r$. For $i < r$ we have $i \leq \frac{p}{n}$ so $2ni - 2 < 2p$. It suffices to show $\pi_{2j}(SU(n))/_p = 0$, $j < p$. James has shown $\pi_{2j}(SU(n))$ is finite and the desired result follows by induction on n. Hence $\exists$ such a β.

Non-Immersions

by Dr. Gitler 24 February 1964
& 2 March

We shall use secondary cohomology operations to obtain non-immersion results for P^m (real proj. sp.) in R^n.

Let A be the Steenrod alg mod 2
Consider a relation $\alpha\beta = \sum \alpha_i \beta_i = 0$ in A where the β_i are homogeneous and the α_i are not necessarily homog.

For example we can form such a relation by adding $Sq^1 Sq^8 + Sq^2 Sq^1 Sq^6 + Sq^8 Sq^1 = 0$ and $Sq^2 Sq^8 + Sq^4 Sq^6 + Sq^8 Sq^2 + Sq^9 Sq^1 = 0$ to get

$$(*) \quad (Sq^1 + Sq^2) Sq^8 + (Sq^2 Sq^1 + Sq^4) Sq^6 + (Sq^8 + Sq^9) Sq^1 + Sq^8 Sq^2 = 0$$

With such a relation $\alpha\beta = 0$ there is assoc. an operation Φ. For $x \in H^q(X)$ with $\beta_i(x) = 0$, $\Phi(x)$ is an element of the quotient of a direct sum of coh groups. Φ is natural and stable but not unique (any two differ by a primary ops)

If Φ is assoc. with $(*)$ then for $x \in H^q(X)$ with $Sq^8 x = Sq^6 x = Sq^1 x = Sq^2 x = 0$

$\Phi(x)$ is an elt of $H^{q+8}(X) + H^{q+9}(X)$ mod $I(\Phi)$,

$$I(\Phi) = (Sq^1 + Sq^2) H^{q+7}(X) + (Sq^2 Sq^1 + Sq^4) H^{q+5}(X) + (Sq^8 + Sq^9) H^q(X) + Sq^8 H^{q+1}(X)$$

Functional operations:

Given $f: X \longrightarrow Y$, $y \in H^q(Y)$, $x = f^* y$ and a rel $\alpha \beta = 0$ s.t. $\beta_i(x) = 0$, then one defines a coset $\alpha_f \beta(y)$.

If $\alpha\beta = 0$ is (*) then $\alpha_f \beta(y)$ lies in the quotient of $H^{q+8}(X) + H^{q+9}(X)$ modulo $I(\phi) + f^*[H^{q+8}(Y) + H^{q+9}(Y)]$

By the Peterson-Stein formula $\Phi(x) = \alpha_f \beta(y)$ mod the larger indeterminacy (r. h. side)

So if one takes $Y = K(Z_2, q)$, knowing $H^*(K(Z_2, q); Z_2)$ we find that $f^* H^*(Y) = \{A(x)\}$ where $A(x) = [\theta x \mid \theta \in A]$ and $\{-\}$ means taking ring generated by $-$.

Let A_q be the homog part of A of dim q.

Take $q = 5$ and the rel (*). If $A_8(x) = 0$, $A_9(x) = 0$ and $Sq^4 x = 0$ then

$f^* H^{13}(Z_3, 5; Z_2) + f^* H^{14}(Z_2, 5; Z_2) = 0$.

We can always obtain such a result by imposing enough conditions of this sort.

Let $B^{(q)}$ be the left ideal in A consisting of elements annihilating cohom classes of dim $\leq q$.

Thm: Let Φ be assoc rel (1) and suppose $\Phi(x)$ admits a functional representation in dim q. Then if for each i, either

$\beta_i \in B^q$ or $\alpha_i \in B^{(q+\dim\beta_i)}$ we have $\Phi(x) = 0$.

This provides conditions for secondary ops. to vanish for dimensional reasons. A more general statement of this is in Adem – Bol Soc. Mat. Mex 1963.

Let P^n be real proj n-space
Let ξ be the canonical line bundle on P^n
Then $\xi^2 = 1$ and if $x = \underset{\uparrow}{\theta}(\xi) - 1 \in \widetilde{KO}(P^n)$
proj to virtual bundle

we have x generates $\widetilde{KO}(P^n)$ and $x^{\varphi(n)} = 0$
where $\varphi(n)$ = number of γ's $1 \leq \gamma \leq n$,
$\gamma \equiv 0, 1, 2, 4 \mod 8$
Also recall that $\tau + 1 = (n+1)\xi$

<u>Theorem</u>: The following conditions on P^n ($n \geq 8$) are equivalent:

a) P^n has an immersion in R^{n+k}

b) $(n+k+1)\xi$ has $(n+1)$-linearly indep non-zero cross-sections.

c) $(2^{\varphi(n)} - n - 1)\xi$ has $(2^{\varphi(n)} - n - k - 1)$ linearly indep non-zero cross sections

<u>Thom space</u>
$(P^n)^{r\xi} = P^{n+r}/P^{r-1}$

Corollary 1: If P^n has an immersion in R^{n+k} then a) P^{2n+k+1}/P^{n+k} desuspends $(n+1)$-times

b) $P^{2^{\phi(n)}-1}/P^{2^{\phi(n)}-n-2}$ desuspends $(2^{\phi(n)}-n-k-1)$ times.

Corollary 2: If P^n has an immersion in R^{n+k} then a) the S-type of P^{2n+k+1}/P^{n+k} has a repr. with fund class in dim k.

b) the S-type of $P^{2^{\phi(n)}-1}/P^{2^{\phi(n)}-n-2}$ has a repr with fund class in dim k.

Known Results (S.W.) $P^n \not\subseteq R^{2n-2}$ if $n=2^r$, $P^n \not\subseteq R^{2n-4}$ if $n = 2^r+1$. These are best possible.

Proof of first one: Suppose $P^{2^r} \subseteq R^{2^{r+1}-2}$. By Cor 1a (with $n=2^r$, $k=2^r-2$) $P^{2^{r+1}+2^r-1}/P^{2^{r+1}-2}$ desuspends 2^r+1 times. But in $H^*(P/P)$ we know that $Sq^{2^r}(x^{2^{r+1}-1}) \neq 0$. Thus if $P^{2^{r+1}+2^r-1}/P^{2^{r+1}-2}$ desuspends d times, then $2^r \leq (2^{r+1}-1)-d$ so $d \leq 2^r-1$. We have a contrad. so $P^{2^r} \not\subseteq R^{2^{r+1}-2}$

Main Results: By using secondary coh ops on proj space and showing their vanishing for dimensional reasons, one can prove:

a) $P^n \not\subseteq R^{2n-5}$ if $n = 2^r+2$ (Baum-Browder) This is best possible

(b) $P^n \not\subseteq R^{2n-7}$ if $n = 2^r+2^s+1$, $r > s+1 \geq 2$. This is best possible if $s=1$ and may well be in general.

c) $\underline{P}^n \not\subseteq R^{2n-9}$ if $n = 2^r + 2^s + 3$, $r > s \geq 2$
This is best possible.

James has proved:
d) $\underline{P}^n \not\subseteq R^{2n - q(n)}$ where $n = 2^r - 1$ and

$$q(n) = \begin{cases} 2r & \text{if } r \equiv 1, 2 \mod 4 \\ 2r+1 & \text{if } r \equiv 0 \mod 4 \\ 2r+2 & \text{if } r \equiv 3 \mod 4 \end{cases}$$

<u>Conjecture</u>: If n is odd then the following results are best possible:

$\underline{P}^n \not\subseteq R^{2n - 2\alpha(n)}$ if $\alpha(n) \equiv 1, 2 \mod 4$

$\underline{P}^n \not\subseteq R^{2n - 2\alpha(n) - 1}$ if $\alpha(n) \equiv 0, 3 \mod 4$

where $\alpha(n)$ = no of terms in dyadic expansion of n.

To prove the result (a) $P^n \not\subseteq R^{2n-5}$ if $n = 2^r + 2$ we use the following secondary cohom function Φ:

$$(Sq^1 + Sq^2)\, Sq^{8k} + (Sq^2 Sq^1 + Sq^4)\, Sq^{8k-2} + Sq^{8k} Sq^2$$
$$+ (Sq^{8k} + Sq^{8k+1})\, Sq^1 = 0$$

Note that for $k=1$ this is the relation (*).

$$\Phi_{8k} : H^q(X) \longrightarrow H^{q+8k}(X) + H^{q+8k+1}(X) / \text{Indeterminacy}$$

Suppose $P^{2^r+2} \subseteq R^{2^{r+1}-1}$. Then $P^{2^{r+1}+2^r+2} / P^{2^{r+1}-1}$ desuspends 2^r+3 times by Cor 1a.

We shall operate on $x^{2^{r+1}} \in H^{2^{r+1}}(P^\infty)$ with Φ_{2^r}.

$\Phi_{2^r}(x^{2^{r+1}})$ is defined and lies in

$$H^{2^{r+1}+2^r}(P^\infty) + H^{2^{r+1}+2^r+1}(P^\infty) \mod I$$

where the indet. I is generated by

$$(Sq^1 + Sq^2)\, H^{2^{r+1}+2^r-1}(P^\infty) + (Sq^2 Sq^1 + Sq^4)\, H^{2^{r+1}+2^r-3}(P^\infty)$$
$$+ Sq^{2^r} H^{2^{r+1}+1}(P^\infty) + (Sq^{2^r} + Sq^{2^r+1})\, H^{2^{r+1}}(P^\infty)$$

Since $Sq^i(x^k) = \binom{k}{i} x^{k+i}$ in $H^*(P^\infty)$ we have

$$I = \{ x^{2^{r+1}+2^r},\ x^{2^{r+1}+2^r+1} \}$$

Look now at $H^*(CP^\infty)$ and the generator $y \in H^{2^{r+1}}(CP^\infty)$. $\Phi_{8k}(y) = (\Phi'_{2^r}(y), 0)$ where $\Phi'_{2^r}(y) \neq 0$ was computed by Adams and has 0 indet.

Take the map $f: P^\infty \longrightarrow CP^\infty$ where the classifying map f induces cohom. iso. in even dims.

By naturality $f^*\Phi_{2^r}(y) = \Phi_{2^r}(x^{2^{r+1}}) = (x^{2^{r+1}+2^r}, 0)$

To complete the proof we need the following props which hold in arbitrary spaces.

Prop: If $A_{8k}(X) = 0$, $A_{8k+1}(X) = 0$, $Sq^3 X = 0$ and $\Phi(X)$ is defined, then $\Phi_{8k}(X) = 0$ for $\dim X \leq 8k-3$

Prop: If $P(x)$ is a polynomial alg. on a single gen. of dim 1 then $A_q(x^n) = 0$ for $\alpha(q+n) > \alpha(n)$ where $\alpha(m) = \#$ of terms in dyadic exp of m.

Thus $A_{2^r}(x^{2^{r+1}}) = 0$

By pulling our operation from P^∞ to $P^{2^{r+1}+2^r+2}/P^{2^{r+1}-2}$ we find that this desuspends only $2^{r+1}-(2^r+3) = 2^r - 3$ times. This contradicts the figure 2^r+3 obtained assuming $P^{2^r+2} \subseteq R^{2^{r+1}-1}$.

$\therefore$ $P^{2^r+2} \not\subseteq R^{2^{r+1}-1}$ This is result a.

Result b is proved similarly to a.

For result c one uses a Thom cx $T(u)$ $\overset{\sim}{\text{S-type}}$ reduced proj. sp.

$u \in H^*(T(u))$
$\uparrow \delta$
$v' \in H^*(\text{sphere bundle})$

$\delta v' = u$, $\delta\Phi(v') = \Phi(u) \neq 0$

Get contrad. to immersion by showing that $\Phi(u)$ trivial.

Relations among Characteristic Classes

by F.P. Peterson 9 March 1964

This talk presents joint work with E.H. Brown Jr.

For the classifying space $BO(n)$ we have $H^*(BO(n), Z_2) = Z_2[w_1, \ldots, w_n]$ the polynomial ring on the S.W. classes w_i of the universal bundle over $BO(n)$.

There is a 1:1 corresp. between equiv. classes of n-plane bundles over a complex K and homot. classes of maps $K \longrightarrow BO(n)$. One simply takes induced bundles.

We shall restrict attention to the case of tangent bundles to closed C^∞ n-dim'l manifolds. To such a manifold M^n there thus corresponds a classifying map $\tau_M : M \longrightarrow BO(n)$ where τ_M is determined up to homotopy.

$$\tau_M^* : Z_2[w_1, \ldots, w_n] \longrightarrow H^*(M; Z_2)$$

We wish to compute

$$\bigcap_{\text{all } M^n} \operatorname{Ker} \tau_{M^n}^* \stackrel{\text{def}}{=} I_n(O,2)$$

<u>Fact</u>: $I_n(O,2)^g = 0$ for $g \leq \frac{n}{2}$

This is shown by constructing a manifold with no such relations.

In higher dims we shall show $I_n(O,2)$ has many non-trivial elts.

Algebraic Background (with Z_2 coeffs)

Let X be a space, ξ a vector bundle of dim n over X. We shall define a right action (depending on ξ) of the Steenrod alg. $\mathcal{A}$ on $H^*(X)$.

Let $T(\xi)$ = Thom space of ξ, that is $T(\xi)$ = 1 pt. compactification of total space of ξ

Let $\varphi : \bar{H}^k(X) \xrightarrow{\approx} \bar{H}^{n+k}(T(\xi))$ be the iso of Thom

Def: For $x \in H^*(X)$, $a \in \mathcal{A}$ the right action of a on x is $(x)a = \varphi^{-1}(\chi(a)\varphi(x))$ where $\chi : \mathcal{A} \to \mathcal{A}$ is the canonical anti-automorphism

One shows $(x)(aa') = ((x)a)a'$

Examples: 1) For $X = M^n$ a manifold, let $\xi = \nu$ the normal bundle (corresponding to an imbedding $M^n \subset R^{2n+1}$)

2) For $X = BO(n)$, $\xi = \nu$ the inverse bundle to the classifying bundle γ ($\gamma \oplus \nu$ = trivial bundle)

The right action of $\mathcal{A}$ on M or on $H^*(BO(n))$

will implicitly corresp. to these bundles.

Properties of rt. action

(1) $(x_1 x_2)a = \sum (x_1)a_i' \cdot \chi(a_i'')(x_2)$ where $\psi(a) = \sum a_i' \otimes a_i''$, ψ being the usual diagonal map.

(2) Setting $x_1 = 1$ we get

$(x)a = \sum (1)a_i' \cdot \chi(a_i'')(x)$

(3) In $H^*(BO(n))$ or in $H^*(M)$

(1) $\chi(Sq^i) = \overline{w}_i$

(4) In $H^*(BO(n)$ or in $H^*(M)$

$(x)\chi(Sq^r) = \sum_{i=0}^{r} \overline{w}_i \cdot Sq^{r-i}(x)$ This follows from 2), 3) and the fact that

$\psi\chi(Sq^r) = \sum_i \chi Sq^i \otimes \chi Sq^{r-i}$ (since $\chi : \mathcal{a} \to \mathcal{a}$ is a coalgebra homom.)

(5) In $H^*(M^n)$ let $x \in H^j(M)$, $y \in H^{n-i-j}(M)$, $a \in \mathcal{a}^i$ Then $(x)a \cdot y = x \cdot a(y)$ assuming M connected. This crucial property could be used as a definition of the rt. action

(6) $\tau_M^* : H^*(BO(n)) \longrightarrow H^*(M)$ is a right $\mathcal{a}$ module homom.

Lemma: Let $x \in H^j(BO(n))$ For $i > \frac{n-j}{2}$ take $(x) Sq^i \in H^{j+i}(BO(n))$ Then $(x) Sq^i \in I_n(O,2)$

Proof Let M be an n-manif

To show $\tau_M^*((x) Sq^i) = 0$ it suffices by Poincaré duality to show $\tau_M^*((x) Sq^i) \cdot y = 0$ for all $y \in H^{n-i-j}(M)$ But $\tau_M^*((x) Sq^i) \cdot y = \tau_M^*(x) Sq^i \cdot y$ $= \tau_M^*(x) \cdot Sq^i(y) = 0$ by (6), (5) and the fact that $i > n-i-j$. Hence $\tau_{M^n}^*((x) Sq^i) = 0$ for all M^n, so $(x) Sq^i \in I_n(O,2)$

Main Theorem: $I_n(O,2)$ is the Z_2-module generated by $H^j(BO(n)) Sq^i$ for $i > \frac{n-j}{2}$

We now set up the proof of this theorem.

A homology theory $\mathcal{N}_*$ on CW complexes is obtained by taking bordism groups. The abelian group $\mathcal{N}_n(X)$ consists of $\{[M^n, f] \mid f : M^n \longrightarrow X,\ M^n \text{ a } C^\infty \text{ manif}\}$

Let $\mathcal{N}_*(X)^*$ be the Z_2 dual of $\mathcal{N}_*(X)$ This gives a cohomology theory.

Define $\bar{\Theta}$: $H^*(BO) \otimes H^*(X) \xrightarrow{\bar{\Theta}} \mathcal{N}_*(X)^*$ by

$\bar{\Theta}(u \otimes x)[M^n, f] = \tau_M^*(u) \cdot f^*(x) \in Z_2$

Thus $\Theta(u \otimes x)[M^n, f] = 0$ unless $\dim u + \dim x = n$ since we are here identifying $H^n(M^n) \approx Z_2$

It follows from properties 5) and 6) of the rt. action that $\bar{\Theta}((u)a \otimes x) = \bar{\Theta}(u \otimes a(x))$

Thus we can define $\Theta : H^*(BO) \otimes_{\mathcal{a}} H^*(X) \to \mathcal{N}_*(X)^*$

<u>Th</u> $H^*(BO)$ is a free rt. $\mathcal{a}$-module

This result of Thom (1954) gives us the structure of $H^*(BO) \otimes_{\mathcal{a}} H^*(X)$

<u>Th</u> Θ is an iso for each CW cx X

This is first proved for X a point. The general case then follows since $H^*(BO) \otimes_{\mathcal{a}} H^*(X)$ and $\mathcal{N}_*(X)^*$ are cohomology theories

To compute $I_n(0, 2)^q$ take $X = K(Z_2, n-q)$
$\iota \in H^{n-q}(Z_2, n-q)$ the canonical generator

Consider

$$H^*(BO) \xrightarrow{\lambda} H^*(BO) \otimes_{\mathcal{a}} H^*(Z_2, n-q) \underset{\approx}{\xrightarrow{\Theta}} \mathcal{N}_*(K(Z_2, n-$$

where $\lambda(u) = u \otimes \iota$

<u>Lemma</u>: $(\operatorname{Ker} \Theta\lambda)^q = I_n(O,2)^q$

<u>Proof</u>: For $u \in H^q(BO)$ we have $u \in \operatorname{Ker}(\Theta\lambda)^q \iff \Theta(u \otimes \iota) = 0 \iff$ for all $[M^n, f]$, $f: M^n \longrightarrow K(Z_2, n-q)$ we have $\tau^*_{M^n}(u) \cdot f^*(\iota) = 0 \iff$ for all M^n and $v \in H^{n-q}(M^n)$ we have $\tau^*_{M^n}(u) \cdot v = 0 \iff$ for all M^n, $\tau^*_{M^n}(u) = 0$ $\iff u \in I_n(O,2)^q$

<u>Consequence</u>: $I_n(O,2)^q = (\operatorname{Ker}\lambda)^q$ since Θ is an iso

Let $\{\mu_i\}$ be a basis for $H^*(BO)$ as a (free) right $\mathcal{a}$-module. Each $u \in H^*(BO)$ is expressed as $u = \Sigma(\mu_i) a_i$ Thus for $u \in H^q(BO)$ we have $u \in (\operatorname{Ker}\lambda)^q \iff$

$0 = \lambda(u) = \Sigma(\mu_i) a_i \otimes \iota = \Sigma \mu_i \otimes a_i(\iota)$

$\iff a_i(\iota) = 0$ for all i

In $H^*(Z_2, n-q)$ one shows that $Sq^I(\iota) = 0 \iff e(I) > n-q$

The Main Theorem stated on page 4 now follows.

Further results:

We know that the free rt. $\mathcal{A}$ module $H^*(BO)$ contains $I_n(O,2)$ as an ideal closed under left and right $\mathcal{A}$ operations. $I_n^q(O,2) = 0$ for $q \leq n/2$

Theorem: $I_n(O,2)$ is a free right $\mathcal{A}$-module in dim $\leq \frac{3}{4}n$

Conjecture (quite sure): $I_n(O,2)$ in dims $\leq n$ has homological dim $\leq [\log_2 n]$ as a rt. $\mathcal{A}$ module

In the notation $I_n(O,2)$
O = orthogonal group   2 = prime 2

Define $I_n(SO,p)$ = relations among the $\left\{\begin{array}{l}\text{SW classes} \\ \text{or Pontryagin classes mod } p\end{array}\right\}$ of orientable n-manifolds

Define $I_n(U,p)$ = relations among Chern classes mod p of weakly almost complex n-manifolds.

"Theorem" We can compute all these

$I_{2n}(U,p)$ = rels among Chern classes of n-dim'l complex manifolds

Approximating Stable Homeomorphisms by Piecewise Linear Ones

E.H. Connell

6 April 1964

<u>Reference</u>: Connell, <u>Same title</u>, Ann. of Math, 78 (1963), 326–338

Our main result is that if $h: R^n \to R^n$ is an orientation preserving homeomorphism of euclidean n-space $(n \geq 7)$, then h is stable $\iff$ h can be approximated by a p.w. linear homeomorphism.

This is also true if we replace "p.w. l. homeom" by "diffeomorphism."

<u>Def</u> A homeom. $h: R^n \to R^n$
$(S^n \to S^n)$
is stable if $\exists$ non-empty open sets $U_1, U_2, \ldots, U_k$ and homeoms $h_i: R^n \to R^n$ $(S^n \to S^n)$ such that $h = h_1 h_2 \cdots h_k$ and $h_i | U_i = I$

<u>Convention</u>: homeom = orientation preserving homeom.

Background Theorems

1. Every homeom of R^n (or S^n) is stable if $n = 1, 2, 3$. The conjecture that every homeom. of R^n is stable is equivalent to the annulus conjecture.

2. The stable homeoms form a normal subgroup of the group of all homeoms.

3. For S^n this subgroup is simple, ie. contains no proper normal subgroup.

4. Every stable homeom is isotopic to the ident.

5. Every $\left\{ \begin{matrix} \text{p.w. linear homeom} \\ \text{diffeom} \end{matrix} \right\}$ is stable

6. Simply connected manifolds have stable structures.

[One can give a local definition of stability for homeoms f of open subsets of R^n. Roughly, f is stable at a point, if when f is restricted to a nbhd of that pt and then extended to $R^n \to R^n$ the extended map is stable. A manifold has a stable structure if its coordinate transfs are stable.]

7. For $n = 1, 2, 3$ a homeom. $h: R^n \longrightarrow R^n$ can be approximated by a p.w. linear one.

Notation: $O = O^n = \{x \in R^n : \|x\| < 1\}$

If $B \subset R^n$ $aB = \{x : \exists\, b \in B$ with $x = ab\}$

Ba = complement in R^n of aB

For $x, y \in R^n$, $\Theta(x, y)$ = angle between $\overrightarrow{Ox}$ and $\overrightarrow{Oy}$ measured in radians.

I = identity function.

We now give the modified engulfing lemma. It adds the requirement $\Theta\{h(x), x\} < \varepsilon$ to the conclusion of John Stallings engulfing lemma.

Lemma 1: Suppose R^n $(n \geq 4)$ has an arbitrary p.w.l. structure T; K is a finite subcomplex of T with $\dim K \leq n-4$; $0 < a < b$, $\varepsilon > 0$ and $K \subset bO$ $(= bO_n)$. Then $\exists$ a p.w.l homeom $h: R^n \longrightarrow R^n$ such that $h/(a-\varepsilon)O = I$, $h(Ob) = I$, $h(aO) \supset K$ and $\Theta\{h(x), x\} < \varepsilon$ for $x \in R^n$.

Lemma 2: Identical to Lemma 1 except that the expansion is in toward the origin rather than away from it. [$K \subset \bar{O}a$, $h | O(b+\varepsilon) = I$, $h | aO = I$, $h(\bar{O}b) \supset K$]

Lemmas 1 and 2 are applied to prove:
Lemma 3: Suppose R^n ($n \geqq 7$) has an arbitrary p.w.l. structure T and $0 < a < b$, $\varepsilon > 0$. Then $\exists$ a homeom $h: R^n \longrightarrow R^n$ such that h is p.w.l. relative to T, $h | (a-\varepsilon)O = I$, $h | O(b+\varepsilon) = I$, $h(aO) \supset bO$, and $\Theta\{h(x), x\} < \varepsilon$ for $x \in R^n$.

We restate Lemma 3 to give a "controlled expanding theorem."
Theorem 2: Suppose R^n ($n \geqq 7$) has an arbitrary p.w.l. structure T and $0 < a < b$, $\varepsilon > 0$. If $f: R^n \longrightarrow R^n$ is any homeom $\exists$ a p.w.l. (rel T) homeom $g: R^n \longrightarrow R^n$ such that $g | f((a-\varepsilon)O) = I$ $g | f(O(b+\varepsilon)) = I$, $g(f(aO)) = f(bO)$ $\Theta\{f^{-1}[g(y)], f^{-1}(y)\} < \varepsilon$

The key to approximating stable homeoms of R^n is in the following:

Lemma 4: Let T be an arbitrary p.w.l. structure on R^n $(n \geq 7)$. Let $h: O \longrightarrow R^n$ be a homeom such that $h(0) = 0$, $\Theta\{h(x), x\} = 0$ for $x \in O$ and if $0 < r < 1$ $\exists$ a number $u(r) > r$ such that $h[r(\bar{O} - O)] = u(r)(\bar{O} - O)$. Then if $\varepsilon(x): O \longrightarrow (0, \infty)$ is continuous $\exists$ a homeom $f: O \longrightarrow R^n$ which is p.w.l. rel. T and such that $|f(x) - h(x)| < \varepsilon(x)$ for $x \in O$.

The proof calls for a p.w.l. expansion of O into R^n which is nearly radial. This expansion is obtained through a sequence of steps each using Theorem 2. One takes precautions to prevent an accumulation of the angle error.

We now give the main result:

Theorem 3: Let R^n $(n \geq 7)$ have an arbitrary p.w.l. structure T. If $g: R^n \longrightarrow R^n$ is a stable homeomorphism and $\varepsilon(x): R^n \longrightarrow (0, \infty)$ is continuous then $\exists$ a p.w.l. (rel T) homeom $f: R^n \longrightarrow R^n$ such that $|f(x) - g(x)| < \varepsilon(x)$ for $x \in R^n$.

<u>Sketch of Proof</u>: Assume $g|O = I$

Let $\delta(z) : R^n \longrightarrow (0, \infty)$ be continuous such that if $z, b, c \in R^n$, $|b - z| < \delta(z)$, $|c - z| < \delta(z)$ then $|g(b) - g(c)| < \varepsilon(c)$.

Let $h : O \longrightarrow R^n$ be as in Lemma 4, $h(x) = u(\|x\|)x$. Then $\exists$ a homeom. $f_1 : O \longrightarrow R^n$ which is p.w.l. rel T and such that $|f_1(y) - h(y)| < \delta[h(y)]$ for $y \in O$.

Let T_1 be the p.w.l. structure on R^n given by $T_1 = g^{-1}(T)$. Since $g|O = I$ we have that T_1 and T agree on O. Thus $f_1^{-1} : R^n \longrightarrow O$ is p.w.l. from T to T_1.

Using Lemma 4 again, $\exists$ a homeom $f_2 : O \longrightarrow R^n$ which is p.w.l from T_1 to T_1 and such that $|f_2(y) - h(y)| < \delta[h(y)]$

Setting $f = g\, f_2\, f_1^{-1} : R^n \longrightarrow R^n$ we find f is p.w.l. from T to T because $f_2 f_1^{-1}$ is p.w.l. from T to T_1 and g is p.w.l. from T_1 to T. One further shows that $|f(x) - g(x)| < \varepsilon(x)$ for $x \in R^n$.

Thus $f : R^n \longrightarrow R^n$ is the desired approximation to g.

Some Relations between Homology and Homotopy

by Edward Curtis 13 April 1964

We shall use Kan's semi-simplicial approach to this problem.

<u>Combinatorial</u> <u>calculation</u> <u>of</u> <u>homotopy</u>

Let X be a connected simplicial cx with ordered vertices. There is associated to X a semi-simplicial complex K.

$$K_n = \{ (\iota_0, \iota_1, \dots, \iota_n) \mid \iota_0 \le \iota_1 \le \dots \le \iota_n \text{ and all } \iota_j \text{ lie in a simplex of } X$$

$\partial_j : K_n \longrightarrow K_{n-1}$, $j = 0, \dots, n$ is $\partial_j(\iota_0, \dots, \iota_n) = (\iota_0, \dots, \hat{\iota}_j, \dots$

$s_j : K_n \longrightarrow K_{n+1}$, $j = 0, \dots, n$ is $s_j(\iota_0, \dots, \iota_n) = \iota_0, \dots, \iota_j, \iota_j, \dots, \iota_n$

Take a maximal tree T of X

Define free groups:

F_n = free gp on K_n with relations

$\sigma = 1$ if σ spans a simplex of T or

if $\sigma = (\iota_0, \dots, \iota_{n-1}, \iota_{n-1})$ (i.e. $\sigma = s_{n-1}\tau$ for $\tau \in K_{n-1}$)

Define $d_j : F_n \longrightarrow F_{n-1}$ for $j = 0, \dots, n-1$

$d_j \sigma = \partial_j \sigma$, $j \ne n-1$

$d_{n-1}\sigma = (\partial_{n-1}\sigma)(\partial_n \sigma)^{-1}$

Take $\tilde{F}_n = \bigcap_{i \ne 0} \operatorname{Ker} d_i \subset F_n$

Then $\{\tilde{F}_n\}$ forms a (non-abelian) chain complex with boundary $d_0 : \tilde{F}_n \longrightarrow \tilde{F}_{n-1}$

That is, $\operatorname{Im} d_0 (\subset \tilde{F}_n)$ is a normal subgroup of $\operatorname{Ker} d_0 (\subset \tilde{F}_n)$ for all n

<u>Theorem</u> $\pi_n(X) \approx H_n(\tilde{F})$ for all n

<u>Remark</u>: For $\pi_1(X)$, this gives the classical result.

<u>Remark</u>: Essentially $F_{n+1} = (GK)_n$ where GK is the semi-simplicial loop complex (Kan) of K. The above isom. follows from a more general theorem that $\pi_n(|K|) \approx \pi_{n-1}(GK)$ for any reduced semi-simpl. cx K.

<u>Semi-simplicial version of th Hurewicz Thm</u>

The homology $H_n(X)$ is obtained as follows. Let $A_n = F_n/[F_n, F_n]$ and let $d_j : A_n \longrightarrow A_{n-1}$, $j = 0, \ldots, n-1$ be the homos induced by $d_j : F_n \to F_{n-1}$

Let (A, ∂) be the free abelian chain complex with $\partial : A_n \longrightarrow A_{n-1}$ given by $\partial \alpha_n = \sum_{j=0}^{n-1} (-1)^j d_j \alpha_n$

Then $H_n(X) = H_n(A)$

Just as we formed $\tilde{F}$ from F we can form another chain complex $\tilde{A}$ from A. Let $\tilde{A}_n = \bigcap_{i \neq 0} \operatorname{Ker} d_i$ and $d_0 : \tilde{A}_n \longrightarrow \tilde{A}_{n-1}$ be the boundary.

Then the inclusion $\tilde{A} \longrightarrow A$ is a chain map and induces isomorphisms $H_n(\tilde{A}) \approx H_n(A)$.
Hence $H_n(\tilde{A}) \approx H_n(X)$.

The natural abelianization maps $F_n \longrightarrow A_n$ induce a chain map $\tilde{F} \xrightarrow{h} \tilde{A}$
Then $\pi_n(X) \approx \pi_n(\tilde{F}) \xrightarrow{h_*} H_n(\tilde{A}) \approx H_n(X)$
gives the Hurewicz map.

<u>Remark</u>: This generalizes Poincaré's Theorem that $H_1(X)$ is $\pi_1(X)$ made abelian, except that in our theorem abelianization comes first.

<u>Remark</u>: Actually, the short exact sequence

$$1 \longrightarrow [F, F] \longrightarrow F \longrightarrow A \longrightarrow 1$$

gives rise to a long exact sequence

$$\cdots \longrightarrow \gamma_n(X) \longrightarrow \pi_n(X) \xrightarrow{h_*} H_n(X) \longrightarrow \gamma_{n-1}(X) \longrightarrow \cdots$$

If $\pi_i(X) = 0$ for $0 \leq i \leq n$ then
$\gamma_i(X) = 0$ for $0 \leq i \leq n+1$
<u>This yields</u> the full Hurewicz theorem

If G is any group we define inductively $\Gamma_1 G \supset \Gamma_2 G \supset \Gamma_3 G \supset \cdots$ as follows

$\Gamma_1 G = G$

$\Gamma_r G = [\Gamma_{r-1} G, G]$ = the subgroup of G generated by elements $x^{-1}y^{-1}xy$ where $x \in \Gamma_{r-1} G$, $y \in G$

Then $\Gamma_r G$ is a normal subgroup of G and of $\Gamma_{r-1} G$ and $\Gamma_{r-1} G / \Gamma_r G$ is abelian.

If G is a free group then $G / \Gamma_r G$ is called the free nilpotent group of class $r-1$

Let X, F be as before. We form $F / \Gamma_r F$ (i.e. $F_n / \Gamma_r F_n$ in each dim n) and take the natural homom $F \xrightarrow{\varphi} F / \Gamma_r F$

<u>Theorem</u>: If $\pi_i(X) = 0$, $1 \leq i \leq n$ then $\varphi_q : H_q(\tilde{F}) \longrightarrow H_q(\widetilde{F/\Gamma_r F})$ is an isom for $1 \leq q \leq n + \log_2 r$

<u>Theorem</u>: There is a spectral sequence (homology exact couple)

$$E^2_{p,q} = H_{p+q}(\widetilde{\Gamma_p F / \Gamma_{p+1} F})$$

converges $\Big\downarrow\Big\downarrow$ ss.

$$E^\infty_{p,q} = \mathcal{G}^p \pi_{p+q}(X)$$

$E^2_{p,q}$ depends only on $H_*(X)$

<u>The free Lie ring over an abelian gp</u>

For any abelian gp A let AM be the free non-associative ring on M, i.e. $$AM = \sum_{r=0}^{\infty} A^r M \quad \text{where } A^1 M = M$$

and $$A^r M = \sum_{i=1}^{r-1} (A^i M) \otimes (A^{r-i} M) \quad \text{for } r > 1$$

Let IM be the two sided ideal in AM generated by the elements $x \otimes x$ and $(x \otimes y) \otimes z + (y \otimes z) \otimes x + (z \otimes x) \otimes y$ for all $x, y, z \in AM$.

Then the *free Lie ring* $L(M)$ over M is

$$L(M) = AM/IM = \sum_{r=0}^{\infty} L^r M$$

L and L^r are functors from abelian gps to ab. gps.

L has the following universal property:

$$\begin{tikzcd} M \arrow[r, "\subset"] \arrow[dr, "f"'] & L(M) \arrow[d, dashed, "g"] \\ & R \end{tikzcd}$$

For any abelian gp homom $f : M \to R$ where R is a Lie ring, $\exists$ a unique extension of f to a Lie map $g : L(M) \to R$

Theorem (Witt): If G is a free group there is a natural isom $L^p(G/\Gamma_2 G) \xrightarrow{\cong} \Gamma_p G/\Gamma_{p+1} G$

Theorem (Dold): If A is a semi-simpl free abelian gp complex and T a functor from ab. gps to ab. gps, then $H_*(\tilde{T}A)$ depends only on $H_*(\tilde{A})$

The case $X = S^{n+1}$

Let GK be the loop complex of K where K is the s.s. complex associated with X. (as remarked earlier $(GK)_j = F_{j+1}$)

Then we have $\cdots \subset \Gamma_3 GK \subset \Gamma_2 GK \subset GK$ and $H_j(\widetilde{\Gamma_r/\Gamma_{r+1}} GK) \approx H_j(L^r(\widetilde{\Gamma_1/\Gamma_2} GK))$

Now $H_j(L^r(\widetilde{\Gamma_1/\Gamma_2} GK))$ depends only on the groups $H_j(\widetilde{\Gamma_1/\Gamma_2} GK) \approx H_{j+1}(K)$

Theorem (Schlesinger): For p prime

$$H_s(L^p(\Gamma_1/\Gamma_2 GK)) = \begin{cases} Z_p & s = n-1+2k(p-1) \\ & \text{where } 1 \leq k \leq [n/2] \\ Z & p=2,\ n \text{ odd},\ s=2n \\ 0 & \text{otherwise} \end{cases}$$

Action of Γ_n on Concordance Classes of Differentiable Manifolds

J. R. Munkres 21 April 1964

I. Definition of the problem

Let M be a compact, non-bounded, connected C^∞ n-manifold

Γ_n = group of diffeoms of S^{n-1} / those extendable to B^n

Γ_n is abelian

Action of Γ_n:

Dig out from M the interior of an imbedded ball B^n to get M_*. If B^n were pasted back in by $\Psi: S^{n-1} \longrightarrow M_*$ we would get a diffeomorph of M again. If we first apply $\varphi: S^{n-1} \longrightarrow S^{n-1}$ and then Ψ we get another manifold $N = M_* \cup_{\Psi\varphi} B^n$. Its diffeom class depends only on $[\varphi] \in \Gamma_n$ and the diffeom class of M. This gives the action of Γ_n.

Question: When is $N \approx M$? Tamura, Brown + Steer, Novikov, Browder, Kosinski.

Def The *inertia group* $I(M)$ is the subgroup of Γ_n which leaves the diffeom class of M unchanged.

Remark: (Smale) Γ_n acts on S^n to give all possible diff structures $(n \neq 4)$ and $I(S^n) = 0$. Hence $\Gamma_n \longleftrightarrow$ the set of diffeom classes of diff manifolds with S^n as underlying space.

Concordance of Δted manifolds:

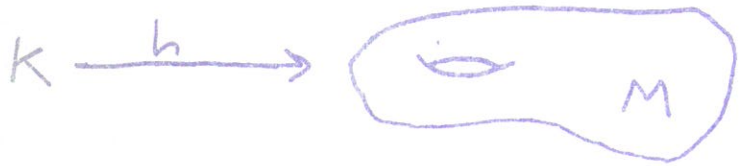
$$\begin{tikzcd} & K \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ M \arrow[rr, "f"] & & N \end{tikzcd} \qquad \begin{tikzcd} & K \times I \arrow[dl, "H_1"'] \arrow[dr, "H_2"] & \\ M \times I \arrow[rr, "F"] & & N \times I \end{tikzcd}$$

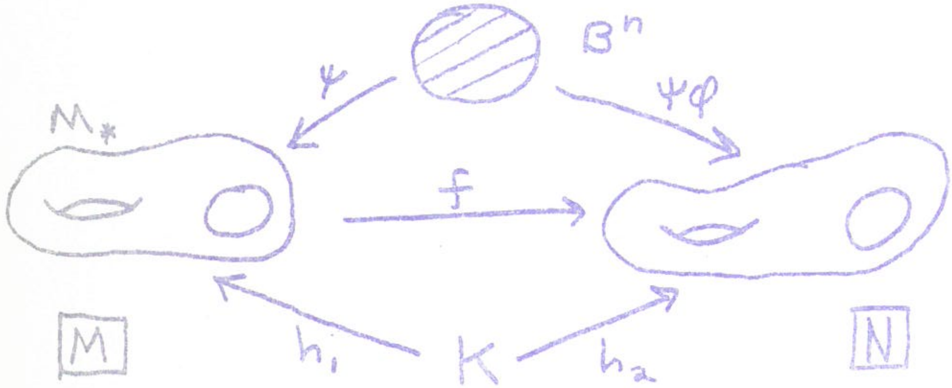
Given two manifolds M, N piecewise smoothly Δted by the same complex K, they are *concordant* if there is a diagram of p.w. smooth Δtions $\ni$ $H_i \mid K \times 0 = h_i$ and $F / M \times 0$ is a diffeom. H_i, F need not be level preserving.

Remark: This is equiv. to hypothesizing a compatible diff structure on $K \times I$ whose restrictions to $K \times 0$ and $K \times 1$ are those induced by h_1 and h_2 resp.

(Proof uses Munkres' obstr. theory)

action of Γ_n on concordance classes





Within the concordance class of (M, h, K) there is $h_1 : K \longrightarrow M$ which induces a Δtion of M_*. There is a Δtion $h_2 : K \longrightarrow N$ which equals h_1 on the subcomplex Δting M_*.

<u>Theorem</u> (a) $(M, h_1, K), [\phi] \longrightarrow (N, h_2, K)$ is well defined on the concordance class level.
b) The triviality of the action of $[\phi]$ is <u>independent</u> of the choices of K, h

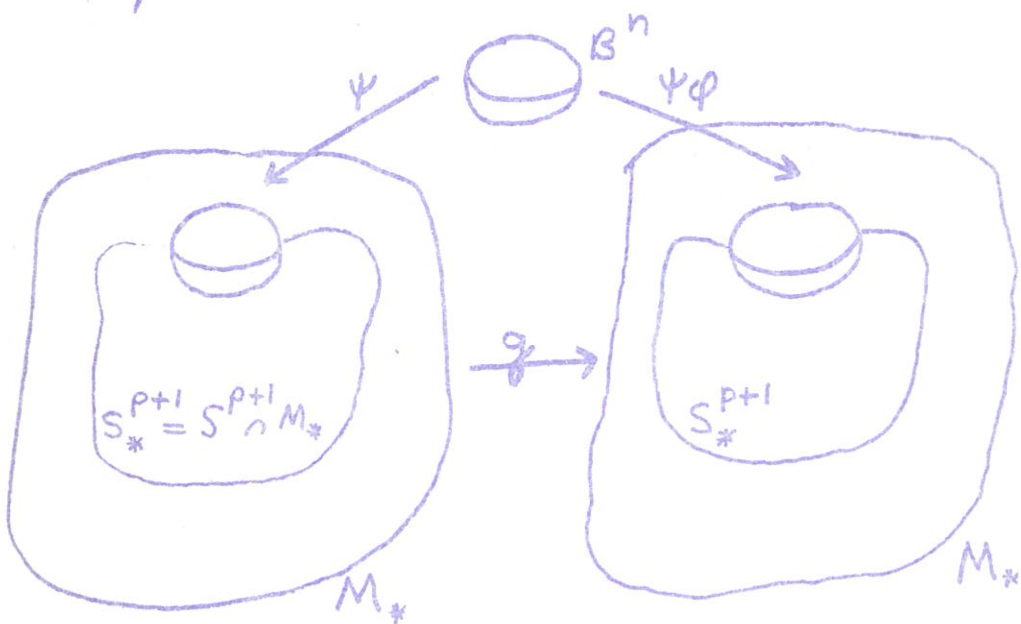
Proof uses techniques of J.H.C. Whitehead.

<u>Notation</u>: $I_c(M)$ = concordance inertia group of M.

$I_c(M) \subset I(M)$

II. Finding an element of Γ_n which acts trivially on (M, h, K)

<u>Construction</u> Let M contain a diff embedded S^{p+1}. We shall construct from this an element of Γ_n which acts trivially on the concordance class in question.



A diffeom $g : M_* \longrightarrow M_*$ determines $\varphi : S^{n-1} \longrightarrow S^{n-1}$ by the rule

$(\Psi\varphi)\Psi^{-1} \mid Bd\, M_* = g \mid Bd\, M_*$ This φ automatically acts trivially on the diffeom class of M.

We now construct such a g.

Let $h : S^{p+1}_* \times R^{q-1} \longrightarrow \eta(S^{p+1}_*)$ a tubular nbhd of $S^{p+1}_* \subset M_*$ $(p+q=n)$

Let $\gamma : R^{q-1} \longrightarrow R^{q-1}$ be a diffeom of compact support and define $1 \times \gamma : S^{p+1}_* \times R^{q-1} \longrightarrow S^{p+1}_* \times R^{q-1}$

Then define $g: M_* \longrightarrow M_*$ by $g | \mathcal{N}(S_*^{p+1}) = h \circ (1 \times \gamma) \circ h^{-1}$ and $g = 1$ elsewhere. g is a diffeom.

<u>Fact</u>: The element $[\phi] \in \Gamma_n$ determined by this g lies in $I_c(M)$ (we know $[\phi] \in I(M)$)

We must show that g may be connected to a comb. equiv. f which equals the identity on M_*. Here use JHC. W. again.

The idea is that we can deform g so that it becomes PL and still preserves fibres in $\mathcal{N}(S_*^{p+1})$. Then we have a PL homeom of $Bd(B^n \times I)$ onto itself, which we may extend.

III. Identifying this element of Γ_n

$$\phi = \psi^{-1}(g | Bd M_*)\psi : S^{n-1} \longrightarrow S^{n-1}$$

Let $\mathcal{N}(S^p)$ be a standard tubular nbhd of the "equator" S^p in $S^{n-1} \subset B^n$.

By abuse of language let $S^p \times R^{q-1} = \mathcal{N}(S^p)$

We assume that S^{p+1} is nicely imbedded in M, so that $\psi(S^p) = Bd(S_*^{p+1})$ and $\psi(S^p \times R^{q-1}) = Bd\, S_*^{p+1} \times R^{q-1}$ where $Bd S_*^{p+1} \times R^{q-1} \subset S_*^{p+1} \times R^{q-1}$ is the tubular nbhd considered earlier.

We also assume that the part of S^{p+1} outside of M_* corresponds to the disk in B^n with boundary S^p.

The map $\Psi: S^p \times R^{q-1} \longrightarrow (\operatorname{Bd} S_*^{p+1}) \times R^{q-1}$ determines (assuming Ψ is nice) a map $A: S^p \longrightarrow SO(q-1)$
$x \longrightarrow A_x$

A can be regarded as the characteristic map of the normal bundle of $S^{p+1} \subset M$.

Now $\varphi = \Psi^{-1}(g \mid \operatorname{Bd} M_*)\Psi : S^{n-1} \longrightarrow S^{n-1}$ is the identity outside of $\eta(S^p) = S^p \times R^{q-1}$

Also

$$\begin{tikzcd}
(x,y) \in S^p \times R^{q-1} \arrow[d, "\Psi"] & & (x, A_x^{-1}\gamma A_x y) \\
(\Psi(x), A_x y) \in \operatorname{Bd} S_*^{p+1} \times R^{q-1} \arrow[rr, "1 \times \gamma"] & & (\Psi(x), \gamma A_x y) \arrow[u, "\Psi^{-1}"]
\end{tikzcd}$$

so on $S^p \times R^{q-1}$, φ sends $(x,y) \longrightarrow (x, A_x^{-1}\gamma A_x y)$

<u>Theorem</u>: Given $\gamma: R^{q-1} \longrightarrow R^{q-1}$ having compact support and $A: S^p \longrightarrow SO(q-1)$ differentiable. Define $S^p \times R^{q-1} \longrightarrow S^p \times R^{q-1}$ by $(x,y) \longrightarrow (x, A_x^{-1}\gamma A_x y)$ and use this to obtain a diffeom $\varphi: S^{n-1} \longrightarrow S^{n-1}$ $(n = p+q)$. Then $[\varphi]$ depends only on the class of γ in Γ_q (γ extends to a diffeom $R^{q-1} \cup \infty \longrightarrow R^{q-1} \cup \infty$)
The function $\tau_p : \pi_p(SO(q-1)) \otimes \Gamma_q \longrightarrow \Gamma_n$ where $\tau_p([A_x],[\gamma]) = [\varphi]$ is a homom.

Theorem: Let M contain a diff. imbedded S^{p+1} with normal bundle having ch. class $\sigma \in \pi_p(SO(q-1))$. If $\Sigma \in \Gamma_n$ equals $\tau_p(\sigma, [\gamma])$ for some $[\gamma] \in \Gamma_q$ then $\Sigma \in I_c(M)$.

Theorem [Milnor) τ_p is non-trivial in some dimensions. (It is related to compositions in homotopy gps of spheres) τ_1 has a non-trivial image in Γ_9, Γ_{10}, and Γ_{15}, for example; τ_3 has a non-trivial image in Γ_{17}

IV Obstruction theory interpretation

Take comb. equiv. f again. From it may be obtained (by radial extension) a diffeom f_0 with a single singularity. This is a "smoothing" of f in a technical sense. It gives us a 0-dim obstruction

$\lambda_0 f_0 = [\phi] \otimes p \in H_0(M; \Gamma_n)$

This obstruction might be inessential, ie. arise from some wrong choices in smoothing f. The measure of difference of choice is determined by obstruction operators Λ_i carrying homology classes

$H_m(M; \Gamma_{n-m+1})$ into $H_{m-i}(M; \Gamma_{n-m+i})$

difference chains obstruction chains

<u>Theorem</u>: Assume the hypotheses of the preceding theorem. If z is fund cycle of S^{p+1}, then $[\gamma] \otimes z \in H_{p+1}(M; \Gamma_q)$ and

$$\Lambda_{p+1}([\gamma] \otimes z) = (\tau_p(\sigma, [\gamma]) \otimes \rho) \in H_0(M; \Gamma_n)$$

Now concordance $\Rightarrow$ smoothability, so if $[\phi] \in I_c(M)$ then $[\phi] \otimes \rho$ must be inessential in the obstruction theory sense. If all the homology of M is generated by such ρ^m's we can say then precisely what $I_c(M)$ is.

<u>Theorem</u>: Let $\sigma_i \in \pi_{p_i}(SO(n-p_i-1))$ $(i=1,\dots,m)$
Let G be the subgroups of Γ_n gen by the groups $\tau_{p_i}(\sigma_i \otimes \Gamma_{n-p_i})$.
$\exists$ n-manifold $M \ni I_c(M) = G$

<u>Proof</u>: M = connected sum of sphere bundles over spheres. By 1st thm, $I_c(M) \supset G$. By second theorem (<u>all</u> homology is thus generated), this is an equality.

Unknotting in Manifolds

by E. M. Brown 11 May 1964

Manifold = comb. manif with boundary

Map = piecewise linear map

Def: given manifolds $N \subset P$ we say an embedding $f: N \longrightarrow P$ unknots relative to (P, N) if $\exists$ a homeom $h: P \longrightarrow P$ st $h(f(N)) = N \subset P$

The classical case is $P = S^3$, $N = S^1$. If $S^1 \subset S^3$ is an equator then the unknotting problem has been "solved," but for bad embeddings it is far from solved.

Def An embedding $f: N \longrightarrow P$ is regular provided $f(\partial N) = f(N) \cap (\partial P)$, ie f carries ∂N + only ∂N into ∂P.

We assume henceforth that M is a compact 2-manifold with boundary.

We will prove the following theorems.

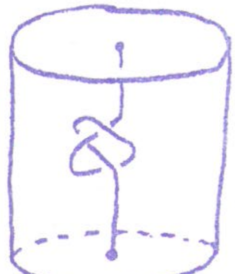
Theorem 1: Let $f: M \longrightarrow M \times I$ be a regular embedding. There exists a homeo $h: M \times I \longrightarrow M \times I$ s.t. $h | M \times 0 \cup M \times 1$ = ident $h f(M) = M \times \frac{1}{2}$ if and only if $f(M)$ separates $M \times 0$ from $M \times 1$ in $M \times I$.

This was proved simultaneously but indep by C. H. Edwards, Ross Finney, and E. M. Brown.

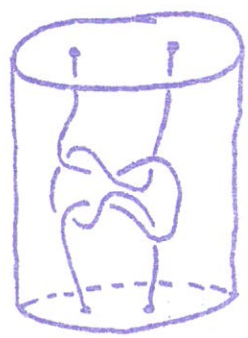
(2)

Theorem 2: Let M be connected, $p \in \operatorname{int} M$.
Let $\alpha : I \longrightarrow M \times I$ a regular embedding with
$\alpha(0) = (p, 0)$, $\alpha(1) \in \operatorname{int} M \times 1$
Then there exists a homeom. $h : M \times I \longrightarrow M \times I$
such that $h / M \times 0 \cup (\partial M) \times I = id$
$h\,\alpha(t) = (p, t)$
if and only if for $i = 0, 1$ the natural homos
$$\pi_1(M \times i - \alpha(i)) \longrightarrow \pi_1(M \times I - \alpha(I))$$
are isos.

Remark: Let M be a disk



The overhand knot is of course knotted



Here neither curve is knotted but they are linked

We can extend thm 2 to the case of a finite # $\alpha_1 \ldots \alpha_n$ of p.w. disj arcs

Replace the homotopy assertion by
$$\pi_1 \left(M \times i - \bigcup_{j=1}^{n} \alpha_j(i) \right) \longrightarrow \pi_1 \left(M \times I - \bigcup_{j=1}^{n} \alpha_j(I) \right)$$
are isom for $i = 0, 1$ and the theorem again holds.

Theorem 3: Let $\lambda : S^1 \longrightarrow M$ be a regular embedding. Let $f : S^1 \times I \longrightarrow M \times I$ be a regular embedding. If λ is not homotopic to zero then there exists a homeomorphism $h : M \times I$ onto $M \times I$ such that $h \mid M \times 0 \cup (\partial M) \times I = id$ and $hf(s,t) = (\lambda(s), t)$.
If λ is homotopic to zero then h exists if & only if $f \mid s \times I$ is unknotted where $s \in S^1$

Theorem 4: Let $\alpha : I \longrightarrow M$ be a regular embedding. Let $f : I \times I \longrightarrow M \times I$ be a regular embedding such that

$f \mid I \times 0 = \alpha \times 0$

$f(i,t) = (\alpha(i), t)$ for $i = 0, 1$

$f \mid I \times 1$ is a regular embedding in $M \times 1$

Then $\exists\, h : M \times I \longrightarrow M \times I$ $h \mid M \times 0 \cup \partial M \times I = id$ and $hf(s,t) = (\alpha(s), t)$

The above theorems are essentially corollaries of the following theorem or its proof.

Theorem 0: Let M be a compact connected 2-manif, not the projective plane. Let B be a compact connected "Poincaré" 3-manif (see below). Let $h : M \times 0 \cup \partial M \times I \to \partial B$ be an embedding such that

(1) $M_1 = \partial B - (h(M \times 0) \cup (\partial M) \times [0,1))$ is a non-vacuous connected 2-manifold.

(2) If $M_0 = h(M\times 0)$ then for $i=0,1$ the natural homomorphisms $\pi_1(M_i) \longrightarrow \pi_1(B)$ are isoms.

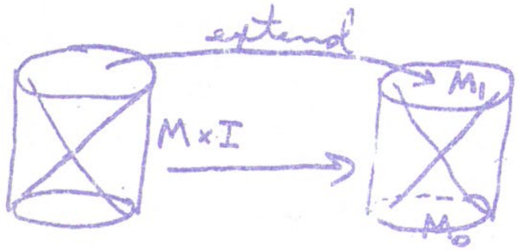
Then h can be extended to a homeom of $M\times I$ onto B.

Def: A 3-manif B is a Poincaré manif if every compact contract submanif bndd by a 2-sphere is a 3-cell.

Example: $M\times I$ is a Poincaré manif for any compact 2-manif M.

Theorem 0 was first proved by J. Stallings. We sketch another proof (by E.M. Brown) because Thms 3 + 4 are corollaries of this proof.

Proof (Thm 0): Consider first the case where M is a disk. Then so is M_1



so we extend h radially over $M\times 1$ to M_1. Now B is bounded by a 2-sphere, compact + contractable. Thus B is a 3-cell because it is a Poincaré manif. So we can extend h over $M\times I$ to B radially.

We reduce to this case by induction.

We know the compact conn. 2-manifolds. For example

Let λ be a simple separating loop on M, $\lambda \times 0$. Choose an annulus $A \subset B$ so that one end of A is $h(\lambda \times 0)$ the other end is in int M, Extend h over $\lambda \times I$ to A any way you please. Cut M apart along λ and cut B apart along A. M breaks into N, P, B into C, D. Check hypotheses of theorem for N, $h | N \times 0 \cup (\partial N) \times I$ and C and for P, $h | P \times 0 \cup \partial P \times I$ and D. This is how one proves Thm 3 for λ sep loop not homotopic to zero.

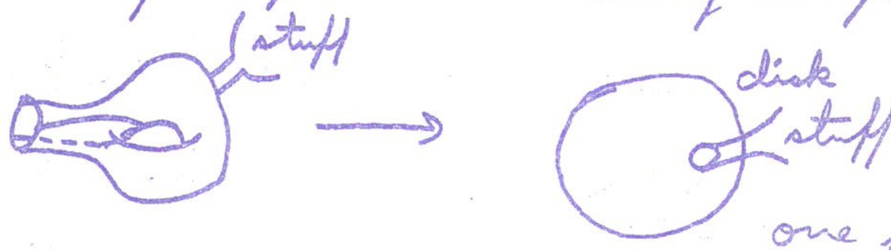
Now if λ is chosen judiciously, and M has at least 2 (handles + cross caps) then each of N and P will have fewer handles + cross caps than M. Thus we reduce to the case where M has at most one handle or one cross-cap.

Next suppose $\partial M \neq \phi$. Let α be a reg. emb. arc in M. We do the same trick as before using α and a disk in B.

e.g.

α joins pts of dist. bdry comps — one fewer bdry comps

or



one fewer handle

or

Moebius

disk
one fewer crosscap

Thus we reduce to the following cases

1) $\partial M \neq \phi$, connected + no handles or crosscaps

2) $\partial M = \phi$ at most one handle or one crosscap.

All other cases reduce to 1)

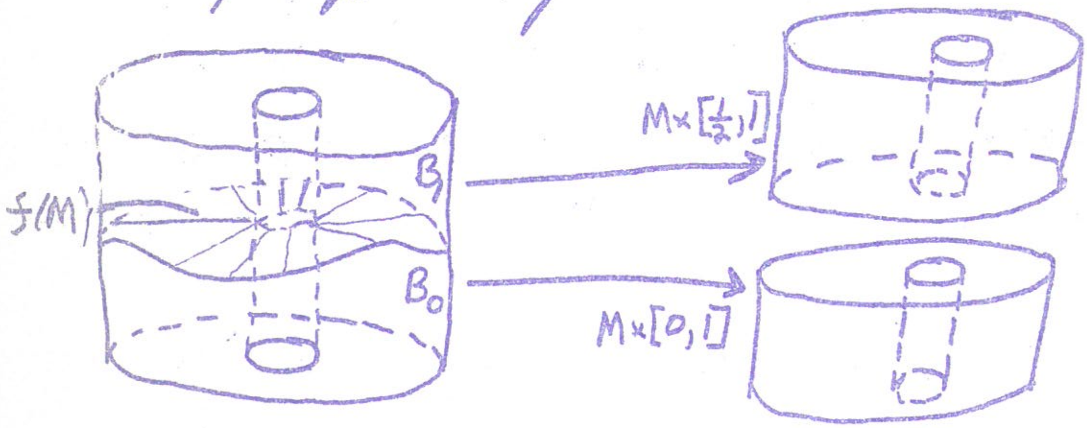
Now case 1) $M =$ disk done

2) $M = S^2, P^2, S^1 \times S^1$

P^2 is out, we use special methods in case $S^2, S^1 \times S^1$

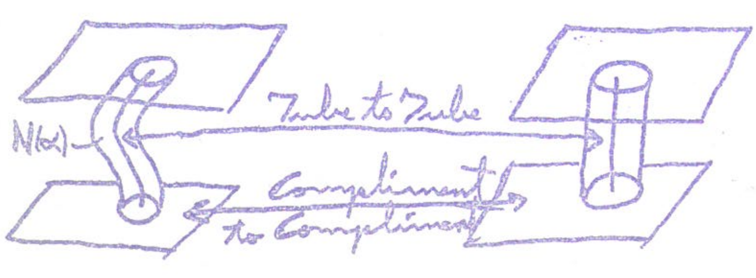
Pf of Theorem 1

If $f(M)$ separates $M \times 0$ from $M \times 1$ and the closures of the complementary domains are B_0 and B_1 then the alg hyp of the Thm are satisfied for B_0, B_1 (This involves cutting type arguments as above.)



Then $B_0, B_1 \cong M \times I$ so map as above + fit together.

Pf of Theorem 2: Bore out a tube around α. What's left is a def retract of $M \times I - \alpha(I)$ So the alg hyps of thm 0 are satisfied for M - disk around p, $M \times I - N(\alpha)$ and any nice map h we construct.



Pf Thm 3 For case $\lambda \sim 0$, λ a separating curve is included in the pf. of thm 0. The case λ a non-sep curve can be reduced to the above (essent.)

Pf Thm 4 Included in pf. of thm 0.

<u>Corollary to thm 2</u>: Let $M = S^2$ α an arc in $S^2 \times I$. Let T be a tube nbhd of α, T' the clos. of its compliment, $A = T \cap T'$ (A is an annulus) Use Van Kampens Thm to compute $1 = \pi_1(S^2 \times I) = \pi_1(T \cup T') =$ $\pi_1(T) \underset{\pi_1(A)}{*} \pi_1(T') = \dfrac{\pi_1(T')}{\pi_1(A)}$ Thus any loop in T' can be deformed into A and hence into $\partial T'$ But $\partial T' = S^2$

$\therefore \pi_1(T') = 1$

$\pi_1(S^2 \times I - \alpha(I)) = \pi_1(T') = 1 = \pi_1(R^3) = \pi_1(S^2 \times i - \alpha(i))$

By thm 2 α is unknotted, ~~so~~ you cannot knot an arc in $S^2 \times I$.

Periodicities in Homotopy Groups of Spheres

by M.G. Barratt   Spring 1964

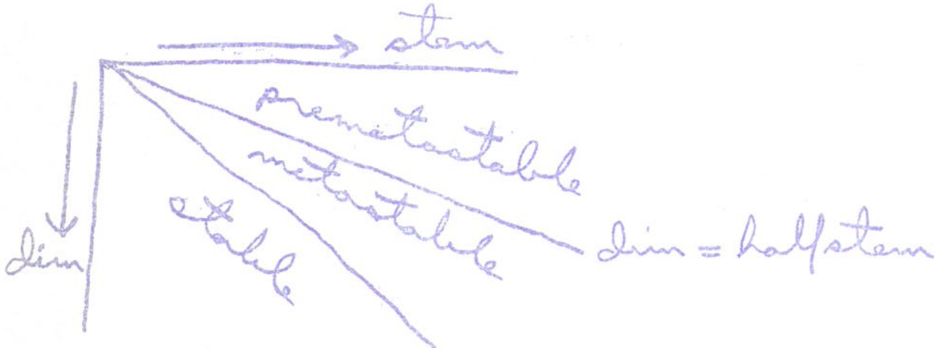
Some of this research was done jointly with M.E. Mahowald.

<u>Theorem</u>: $\pi_{4r}(BO(2r+1)) \longrightarrow \pi_{4r}(BO) = Z$ is onto when $r > 4$

<u>Cor</u>: If $n \geq 13$, $q \leq 2n-3$ then

$$\pi_q(BO(n)) \approx \pi_q(BO) \oplus \pi_q(V_{2n,n})$$

We divide the unstable homotopy groups of spheres into two halves, the "premetastable" and the "metastable" as shown.



The transitions are roughly at $\pi_{3n-2}(S^n)$ (note the Whitehead product $[[\iota_n,\iota_n],\iota_n] \in \pi_{3n-2}(S^n)$) and at $\pi_{2n-1}(S^n)$ (note $[\iota_n,\iota_n] \in \pi_{2n-1}(S^n)$)

Thus the three ranges correspond to the complexity of Whitehead products.

<u>Hope</u>: With a reasonable number of exceptions we have $\pi_{q+n}(S^n) \approx \pi_q^{\$} + J\pi_q(V_{2n,n})$ in the metastable range.

Experiment shows this is true, e.g. in the last 14 groups in stems $\equiv 4 \pmod 8$.

Note that $\pi_q^{\$} = \varinjlim \{\pi_{q+n}(S^n) \xrightarrow{E} \pi_{q+n+1}(S^{n+1}) \to \cdots$ is filtered by the sphere of origin of its elements.

<u>General Philosophy</u>:

(1) Stable elements usually arise from the premetastable range rather than the metastable.

(2) Every stable element should generate a periodic family (elements defined by formula from previous members of the family.

<u>Theorem</u>: $\forall\ k \geq 0$ (with some exceptions for $k \leq 1$) there are stable periodic families as indicated in the following table.

Stem	Filtration	Order	Name
$8k+1$	3	2	$\mu[k]$
$8k+3$	2	2	$\eta\eta\mu[k]$
	3	4	$\bar{\mu}[k]$
	5	8	$\zeta[k]$
$8k-1$	5	2	$a[k]$
	6	4	$b[k]$
	7	8	$c[k]$
	9	16	$d[k]$

Also $e_{r,k}$ of order 2^{r+3} and stem $\equiv -1 \pmod 8$

$e_{1,k} = d[k]$

$e_{r,k} \in \{e_{r-1,k}, 2e_{r-1,k}, 2^{r+1}\iota\}$

Constructions:

$\mu[0] =$ elt of H.I. 1 on S^2
$\mu[k] \in \{\mu[r], 2\iota, a[k-r]\}$

$a[1]$ known as τ'' in 7 stem $\in \{\nu, 8\iota, \nu\}$
$a[k] \in \{a[r], 2\iota, a[k-r]\}$

$b[1]$ known as τ' in 7 stem
$b[k] \in \{b[r], 4\iota, b[k-r]\}$

$c[1]$ known as τ
$c[k] \in \{c[r], 8\iota, c[k-r]\}$

[$d[1]$ known as σ, has odd H.I. on S^8
$d[k] \in \{d[r], 16\iota, d[k-r]\}$

$\bar{\mu}[k] \in \{\eta, 2\iota, \mu[k]\}$ (η is the element of H.I. one in the 1-stem, previously referred to as $\mu[0]$)

[$\rho[k] \in \{\rho[r], d[k-r-1], 16\iota\}$
$\rho[1] = \nu$ has odd H.I. on S^4

Definition of the complex and real adams e-invariants: $e_{\mathbb{C}}$, $e_{\mathbb{R}}$, and $e'_{\mathbb{R}}$

Let φ_r = gen of $\pi_{2r}(BU)$ and

w_r = gen of $\pi_{4r}(BO)$

If θ in odd stem (say in $(2s-1)$-stem) is of order k, define $e_{\mathbb{C}}(\theta)$ by

$\{\varphi_r, \theta, k\iota\} = \phi_{r+s} \cdot k e_{\mathbb{C}}(\theta)$ mod $k \cdot \pi_{2r+2s}(BU)$

If θ in $(4t-1)$-stem is of order k, define $e_{\mathbb{R}}(\theta)$ by $\{w_{2r}, \theta, k\iota\} = w_{2r+t} \cdot k e_{\mathbb{R}}(\theta)$ mod k

and define $e'_{\mathbb{R}}(\theta)$ by

$\{w_{2r+1}, \theta, k\iota\} = w_{2r+t+1} \cdot k e'_{\mathbb{R}}(\theta)$ mod k

$BO \longrightarrow BU \longrightarrow BO$

$w_{2r} \longrightarrow \varphi_{4r} \longrightarrow 2w_{2r}$

$w_{2r+1} \longrightarrow \varphi_{4r+2} \longrightarrow 2w_{2r+1}$

For elements in stems $\equiv -1 \pmod 8$,

$$e_{\mathbb{R}} = \tfrac{1}{2} e_{\mathbb{C}} \text{ and } e'_{\mathbb{R}} = 2 e_{\mathbb{C}}$$

Prove: (1) $e_{\mathbb{C}}(\eta) = \frac{1}{2}$ by hand

(2) $e_{\mathbb{R}}\{\alpha, k\iota, \beta\} = k\, e_{\mathbb{R}}(\alpha)\, e_{\mathbb{R}}(\beta)$
stem $\alpha \equiv -1 \bmod 8$

$e_{\mathbb{R}}(\nu) = \text{odd}/8$ because $2\nu \in \{\eta, 2\iota, \eta\}$

$e_{\mathbb{R}}(\sigma) = \text{odd}/16$ because $8\sigma \in \{\nu, 8\iota, \nu\}$

$4 e_{r,k} \in \{2 e_{r-1,k}, 2^{r+1}\iota, 2 e_{r-1,k}\}$

$\therefore\ e_{\mathbb{R}}(4 e_{r,k}) = 2^{r+3}[e_{\mathbb{R}}(e_{r-1,k})]^2 = \text{odd}/2^{r+1}$

$\therefore\ e_{r,k}$ has order at least 2^{r+3}

On the other hand, a classical Toda bracket argument shows that $4e_{r,k}$ has order at most 2^{r+1}!

Note: $e_{r-1,k} \circ e_{r-1,k} = 0$
yet $\{e_{r-1,k}, e_{r-1,k}, 2^{r+2}\iota\}$ may not be same as $e_{r,k}$.

Note: It is plausible that $\theta \to \{\theta, e_{r,1}, 2^{r+3}\}$ is a periodicity among stable classes. This is connected with periodicity in the Adams spectral sequence.

Cobordism Classes of Squares of Orientable Manifolds

by P.G. Anderson 25 May 1964

a reference: C.T.C. Wall, <u>Determination of the cobordism ring</u>, Ann of Math, 1960

We shall prove the following conjecture of Milnor's:

<u>Thm 1</u>: If M^n is an orientable manifold, then there exists a spinor manifold N^{2n} such that $M^n \times M^n$ is non-orientably cobordant to N^{2n}.

M^n is <u>orientable</u> if $Sq^1 : H^{n-1}(M) \longrightarrow H^n(M)$ is the 0-homo.

M^n is <u>spinor</u> if M is orientable and $Sq^2 : H^{n-2}(M) \longrightarrow H^n(M)$ is the 0-homo.

We shall use the cobordism exact Δ of Wall.

$$\begin{tikzcd} \Omega \arrow[rr, "2"] & & \Omega \arrow[dl, "r"] \\ & W \arrow[ul, "\partial"] & \end{tikzcd}$$

Ω is the oriented cobordism ring

W is contained in $\mathfrak{N}$, the unoriented cobordism ring. Explicitly

$W = \{ [M] \in \mathfrak{N} \mid w_1(M) \text{ is induced from a map } M \xrightarrow{f} S^1 \}$

$2 : \Omega \longrightarrow \Omega$ is the doubling homo

$r : \Omega \longrightarrow W$ drops orientation

$\partial : \mathcal{W} \longrightarrow \Omega$ is a homo of degree -1 as follows:
Let $f : M \longrightarrow S^1$ induce $W_1(M)$, where f is t-regular at some $* \in S^1$. Then $\partial[M] = \{f^{-1}(*)\}$

Let λ be a line bundle over a manifold X. Let $P(\lambda \oplus m)$ denote the space of lines thru the origen of each fibre in the $m+1$-plane bundle $\lambda \oplus m$.

We then have a fibration

$$\begin{array}{ccc} P^m & \xrightarrow{i} & P(\lambda \oplus m) \\ & & \downarrow \pi \\ & & X \end{array}$$

<u>Lemma 2A</u>: $W(P(\lambda \oplus m)) =$

$$\pi^*(W(X)) \cdot (1+\alpha)^m (1 + \alpha + \pi^*(W_1(\lambda)))$$

where $i^*(\alpha) \neq 0 \in H^1(P^m)$
($i^*(\alpha)$ is the cohomology gen. of the fibre)

<u>Lemma 2B</u>:
$$H^*(P(\lambda \oplus m)) \approx \frac{H^*(X) \otimes Z_2(\alpha)}{1 \otimes \alpha^{m+1}}$$
$$\approx W_1(\lambda) \otimes \alpha^m$$

Similar results hold for the case where λ is a complex line bundle over simply connected X.

<u>Generators for $\mathcal{W}$</u>

Let λ be the canonical line bundle over P^n and denote $P(\lambda \oplus m)$ by $M(m, n)$.

$$H^*(M(m,n)) = Z_2[\alpha, \beta] / \alpha^{m+1} = \alpha^m \beta \;,\; \beta^{n+1} = 0$$

$$W(M(m,n)) = (1+\alpha)^m (1+\beta)^{n+1} (1+\alpha+\beta)$$

where α and β are 1-dim cohom classes.

Let ξ be a line bundle over $M(m,n)$ $\ni$ $w_1(\xi) = \alpha$.

Denote $P(\xi \oplus r) = M(r, m, n)$

<u>Note</u>: $M(m,n)$ is or. iff m odd & n even and $[M(r,m,1)] \in \mathcal{W}$ iff r odd & m even

$\mathcal{W}$ is a polyn alg. over Z_2 with one generator Y_n in each dim n, where $n \neq 2^j - 1$, $n \neq 2$

If $n = 2k - 1$ with $k \neq 2^j$, write $k = 2^r(2s+1)$ $(s \neq 0)$

Then $Y_n = [M(2^{r+2}s - 1, 2^{r+1})]$

$Y_{n+1} = [M(2^{r+2}s - 1, 2^{r+1}, 1)]$

$Y_{2^{j+1}} = [CP^{2^j}]$

<u>Generators for $r\Omega \subset \mathcal{W}$</u>

Taking all m's odd and n's even the gens of $r\Omega$ are contained in the collection:

$$\left\{ [M(m,n)] ;\; r\partial\Big[\prod_{j=1}^{s} M(m_j, n_j, 1)\Big], \;(s>1) ;\; [CP^{2^j}] \right\}$$

We seek to show that the squares of these generators are represented by spinor manifolds.

Lemma 3: (Wall + Rohlin)

$$w_{2i_1} \cdots w_{2i_r}[M \times M] = w_{i_1} \cdots w_{i_r}[M]$$

and if $w_{j_1} \cdots w_{j_r}$ not of form $w_{2i_1} \cdots w_{2i_r}$ then $w_{j_1} \cdots w_{j_r}[M \times M] = 0$

Using the fact that manifolds with the S-W numbers are cobordant, one then gets by Thom that:

$$P^n \times P^n \sim CP^n$$

$$\boxed{CP^n \times CP^n \sim QP^n}$$

Let λ be a cx line bundle over CP^n such that $w_2(\lambda) \neq 0$

Denote $P(\lambda \oplus m)$ by $CM(m,n)$

We have an iso of Z_2 algebras:

$$H^*(CM(m,n)) \xleftarrow[\approx]{D} H^*(M(m,n))$$

$$H^{2r} \approx H^r$$

$$D(W(M(m,n))) = W(CM(m,n))$$

$$D(w_j(M)) = w_{2j}(CM)$$

Hence by Lemma 3,

$$\boxed{M(m,n) \times M(m,n) \sim CM(m,n)}$$

We now look at the manifold

$$M = \vdash \partial \left(\prod_{j=1}^{s} M(m_j, n_j, 1) \right)$$

(recall m's odd and n's even)

Set $M_j = M(m_j, n_j, 1)$ Let $P_j : M_j \longrightarrow S^1$ where P_j is the composition of projections $M(m_j, n_j, 1) \longrightarrow M(n_j, 1) \longrightarrow S^1$ Actually P_j is the projection of a fibration:

$$\begin{array}{ccc} M(m_j, n_j) & \longrightarrow & M(m_j, n_j, 1) \\ & & \downarrow P_j \\ & & S^1 \end{array}$$

<u>Note</u>: P_j induces $w_1(M(m_j, n_j, 1))$

Let $P : \prod_{j=1}^{s} M_j \longrightarrow S^1$ be

$$P(x_1, \ldots, x_s) = P_1(x_1) \cdots P_s(x_s) \quad , \quad S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$$

P induces $w_1\left(\prod_{j=1}^{s} M_j\right)$

$$M = \{ (x_1, \ldots, x_s) \in \prod M_j \mid P(x_1, \ldots, x_s) = 1 \}$$

If we define $P_1' : M_1 \longrightarrow S^1$ by $P_1'(x) = (P_1(x))^{-1}$ and $P' : \prod_2^s M_j \longrightarrow S^1$ by $P'(x_2, \ldots, x_s) = P_2(x_2) \cdots P_2(x_s)$

then $M = \{ (x, y) \in M_1 \times \prod_2^s M_j \mid P_1'(x) = P'(y) \}$

Thus M is an induced bundle as shown:

$$\begin{tikzcd} M \arrow[d] & M(m_1, n_1) \arrow[r] & M_1 = M(m_1, n_1, 1) \arrow[d] \\ \prod_2^s M_j \arrow[rr, "P'"] & & S^1 \end{tikzcd}$$

Infact we can build the following by taking induced bundles.

$$\begin{tikzcd}
 & & & P^{m_1} \arrow[d] & & P^{m_1} \arrow[d] \\
P^{m_1} \arrow[r] & M \arrow[d, "F"] & P^{n_1} \arrow[dr] & M(m_1, n_1, 1) \arrow[d, "G"] & & M(m_1, k) \arrow[d, "H"] \\
P^{n_1} \arrow[r] & M' \arrow[rr, "\tilde{p}'"] \arrow[d, "f"] & & M(n_1, 1) \arrow[rr, "h"] \arrow[d, "g"] & & P^{K} \\
 & \prod_2^s M_j \arrow[rr, "p'"] & & S' & &
\end{tikzcd}$$

Diagram 1

$$(M', f, \prod_2^s M_j) = p'^*(M(n_1, 1), g, S')$$

$$M(m_1, n_1, 1), G, M(n_1, 1)) = h^*(M(m_1, k), H, P^K)$$

for suff. large K

$$\therefore (M, F, M') = \tilde{p}'^* h^* (M(m_1, k), H, P^K)$$

and $(M, f \circ F, \prod_2^s M_j) = p'^*(M(m_1, n_1, 1); g \circ G, S')$

Using Lemma 2 we can show the following.

$$H^*(M') = \bigotimes_2^s H^*(M_j) \otimes Z_2[\beta_1] / \beta_1^{n_1+1}$$
$$= \beta_1^{n_1}(\gamma_2 + \cdots + \gamma_s)$$

$$H^*(M_j) = Z_2[\alpha_j, \beta_j, \gamma_j] / \alpha_j^{m_j+1}$$
$$= \alpha_j^{m_j} \beta_j, \quad \beta_j^{n_j+1} = \beta_j^{n_j} \gamma_j, \quad \gamma_j^2 = 0$$

$$H^*(M) = H^*(M') \otimes Z_2[\alpha_1] / \alpha_1^{m_1+1} = \alpha_1^{m_1} \beta$$

We define CM, the complex analogue of M, as follows. Define $CM(m,n,1)$ in the obvious way and constr. a diagram similar to Diag. 1 with the modification that P' is replaced by a map $\prod_2^s CM(m_j,n_j,1) \longrightarrow CP^L$ which induces $\gamma_2+\cdots+\gamma_s$ and otherwise complexify everything in sight.

Then the cohomology ring $H^*(CM)$ and SW class can be computed analogously to those for M. A detailed analysis shows that $M\times M$ and CM have the same Stiefel-Whitney numbers. Thus:

$$\boxed{M\times M \sim CM \quad \text{for } M = \Gamma\partial\left(\prod_{j=1}^{s} M(m_j,n_j,1)\right)}$$

The three boxed results show that the squares of generators of $\Gamma\Omega$ are represented by spinor manifolds. Hence Theorem 1.

<u>Note</u> Milnor also conjectured that every spinor manifold is cobordant to the square of an orientable one. F. P. Peterson has found a counterexample in dimension 24.

Coalgebras, Coresolutions, and the Computer Computation of $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$

by Professor A. Liulevicius

May 1964

§1 Graded coalgebras

Let k be a field

Def: A graded coalgebra C over k is

1) a graded vector space over k:

$$C = \{C_n\}_{n \in \mathbb{Z}}$$

2) with linear maps of degree 0

$$\psi : C \longrightarrow C \otimes_k C$$

$$\varepsilon : C \longrightarrow k, \quad \text{where } k_n = \begin{cases} k & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

such that

3)

$$\begin{tikzcd} & C \otimes C \arrow[dr, "\psi \otimes 1"] & \\ C \arrow[ur, "\psi"] \arrow[dr, "\psi"'] & & C \otimes C \otimes C \\ & C \otimes C \arrow[ur, "1 \otimes \psi"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} & C \otimes C \arrow[r, "1 \otimes \varepsilon"] & C \otimes_k k \arrow[d, equal] \\ C \arrow[ur, "\psi"] \arrow[rr, "1"] \arrow[dr, "\psi"'] & & C \arrow[d, equal] \\ & C \otimes C \arrow[r, "\varepsilon \otimes 1"] & k \otimes_k C \end{tikzcd}$$

are commutative diagrams.

Note: We shall assume that our coalgebras are connected, that is ε is an isomorphism of C_0 onto k. Furthermore, we assume that either $C_n = 0$ for all $n < 0$ or for all $n > 0$

Def: A (left) <u>comodule M</u> over the coalgebra C is a graded vector space over k with a k-linear map
$$\mu : M \longrightarrow C \otimes_k M$$
of degree zero, making the following diagrams commutative:

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "\mu"] & C \otimes_k M \arrow[d, "\psi \otimes 1"] \\ C \otimes M \arrow[r, "1 \otimes \mu"] & C \otimes C \otimes M \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[dr, "1"'] & C \otimes M \arrow[d, "\varepsilon \otimes 1"] \\ & M \end{tikzcd}$$

Examples: ① k is a left C-module via
$$\mu : k \longrightarrow k \otimes_k k = C_0 \otimes_k k$$
defined by $\mu(1) = 1 \otimes 1$

② If V is a graded vector space over k, we make $C \otimes_k V$ into a left C-module by the mapping
$$\psi \otimes 1 : C \otimes V \longrightarrow C \otimes C \otimes V$$

We shall call such a comodule an injective C-comodule. The name will be justified by Proposition 1.

Def: If (M,μ) and (N,ν) are left C-comodules then a k-linear map $f: M \longrightarrow N$ is said to be a C-comodule map of degree q when $f_n: M_n \longrightarrow N_{n+q}$ and

$$\begin{array}{ccc} M & \xrightarrow{\mu} & C\otimes M \\ \downarrow f & & \downarrow 1\otimes f \\ N & \longrightarrow & C\otimes N \end{array}$$

is commutative.

Proposition 1: If (M,μ) is a C-comodule $C\otimes V$ an injective C-comodule, then there is a 1-1 correspondence

$$Hom_C(M, C\otimes V) \xrightarrow{\lambda} Hom_k(M,V)$$

given by $\lambda(f) = (\varepsilon\otimes 1)\circ f$

Proof: Let $g: M \longrightarrow V$ be a k-linear map, let $w(g): M \longrightarrow C\otimes V$ be $w(g) = (1\otimes g)\mu$.

Claim: $w(g)$ is a C-map and $\lambda(w(g)) = g$

This follows from the commutativity of the following diagrams.

$$\begin{tikzcd}
M \arrow[r, "\mu"] \arrow[d, "\mu"] & C\otimes M \arrow[r, "1\otimes g"] \arrow[d, "\psi\otimes 1"] & C\otimes V \arrow[d, "\psi\otimes 1"] \\
C\otimes M \arrow[r] & C\otimes C\otimes M \arrow[r, "1\otimes 1\otimes g"] & C\otimes C\otimes V
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "\mu"] \arrow[dr, "1"'] & C\otimes M \arrow[r, "1\otimes g"] \arrow[d, "\varepsilon\otimes 1"] & C\otimes V \arrow[d, "\varepsilon\otimes 1"] \\
 & M \arrow[r, "g"] & V
\end{tikzcd}$$

Conversely, if $f : M \longrightarrow C\otimes V$ is a C-comodule map, then $w(\lambda(f)) = f$, which is the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\mu"] & C\otimes V \arrow[d, "\psi\otimes 1"] \arrow[dr, "1"] & \\
C\otimes M \arrow[r, "1\otimes f"] & C\otimes C\otimes V \arrow[r, "1\otimes\varepsilon"] & C\otimes V
\end{tikzcd}$$

If M is a C-comodule, then an injective C-coresolution $\mathcal{J}$ is an exact sequence

$$0 \longrightarrow M \xrightarrow{\eta} C\otimes V_0 \xrightarrow{d_0} C\otimes V_1 \longrightarrow \cdots \longrightarrow C\otimes V_q \xrightarrow{d_q} \cdots$$

of C-maps.

If N is any C-comodule, then $\mathrm{Hom}_C(N, \mathcal{J})$ is a bigraded chain complex,

$\mathrm{Hom}_C(N, \mathcal{I})_{s,t} = \mathrm{Hom}_C(N, C \otimes V_s)$ maps of degree t.

We define $\mathrm{Ext}_C^{s,t}(N, M) = H^s(\mathrm{Hom}_C(N, \mathcal{I}))_t$

<u>Note</u>: If C, M, N are finite dimensional in each grading, then the graded dual C^* is an algebra, M^*, N^* are modules over C^*, and

$$\mathrm{Ext}_C^{s,t}(N, M) \cong \mathrm{Ext}_{C^*}^{s,t}(M^*, N^*)$$

We shall use this idea to compute $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ where A is the Steenrod algebra over Z_2. Henceforth $C = A^* = Z_2[\alpha_1, \dots, \alpha_n, \dots]$ as Hopf algebra with grade $\alpha_i = 2^i - 1$

$$\Psi(\alpha_n) = \sum_{i=0}^{n} \alpha_{n-i}^{2^i} \otimes \alpha_i$$

§2 <u>Minimal injective resolution</u>

We wish to construct an injective resolution $\mathcal{I}$ of Z_2 such that all differentials in $\mathrm{Hom}_C(Z_2, \mathcal{I})$ are zero. Since

$$\mathrm{Hom}_C(Z_2, C \otimes V_q) \cong \mathrm{Hom}_{Z_2}(Z_2, V_q) = V_q$$

this condition reduces to $d_q | V_q = 0$.

Proposition: If A, B are C-comodules and $f: A \longrightarrow B$ is a C-comodule map, then there is an injective C-comodule $C \otimes W$ and a C-map $g: B \longrightarrow C \otimes W$ such that

1) $\operatorname{Ker} g = \operatorname{Im} f$

2) $\operatorname{Im} g \supset 1 \otimes W$

Proof: Let $D = B/_{\operatorname{Im} f}$, comodule structure from B

$$P(D) = \operatorname{Ker}\left(D \longrightarrow C \otimes D \xrightarrow{\ p \otimes 1\ } \bar{C} \otimes D\right)$$

where $p: C \longrightarrow \bar{C}$ is the projection onto elements of positive degree.

Let $W = P(D)$, $r: D \longrightarrow P(D)$ linear retraction onto subspace. If we let $j: B \longrightarrow D$ be the quotient map, then we define $\underline{g}: B \longrightarrow W$ to be just the composite $r \circ j$. We let $g = (1 \otimes \underline{g})\mu : B \longrightarrow C \otimes W$ be the associated C-map. It is clear that $\operatorname{Ker} g \supset \operatorname{Im} f$, for $\operatorname{Ker} \underline{g} \supset \operatorname{Im} f$. Thus g induces a map h from D to $C \otimes W$. If $\operatorname{Ker} h \neq \{0\}$, let us pick an $a \in (\operatorname{Ker} h)_n$, $a \neq 0$, where n is the smallest integer such that $(\operatorname{Ker} h)_n \neq 0$.

Then $a \in P(D)$ and $h(a) = g(a)$ but g is an isomorphism of $P(D)$ onto W, which also proves 2).

It is now clear how to construct a minimal resolution of Z_2 over C: we start with the inclusion $0 \longrightarrow Z_2$, apply the proposition above to get $C \otimes V_0$ and a map $\eta : Z_2 \longrightarrow C \otimes V_0$ such that $1 \otimes V_0 \subset \operatorname{Im} \eta$ (thus $V_0 = Z_2$). Suppose

$$0 \longrightarrow Z_2 \longrightarrow C \otimes V_0 \longrightarrow \cdots \longrightarrow C \otimes V_{g-1} \xrightarrow{d_{g-1}} C \otimes V_g$$

has been constructed and $\operatorname{Im} d_{g-1} \supset 1 \otimes V_g$. We apply the proposition with $A = C \otimes V_{g-1}$, $B = C \otimes V_g$, $f = d_{g-1}$ to get $C \otimes V_{g+1}$ and a map

$$d_g : C \otimes V_g \longrightarrow C \otimes V_{g+1}$$

such that

$$\operatorname{Ker} d_g = \operatorname{Im} d_{g-1}$$

$$\operatorname{Im} d_g \supset 1 \otimes V_{g+1}$$

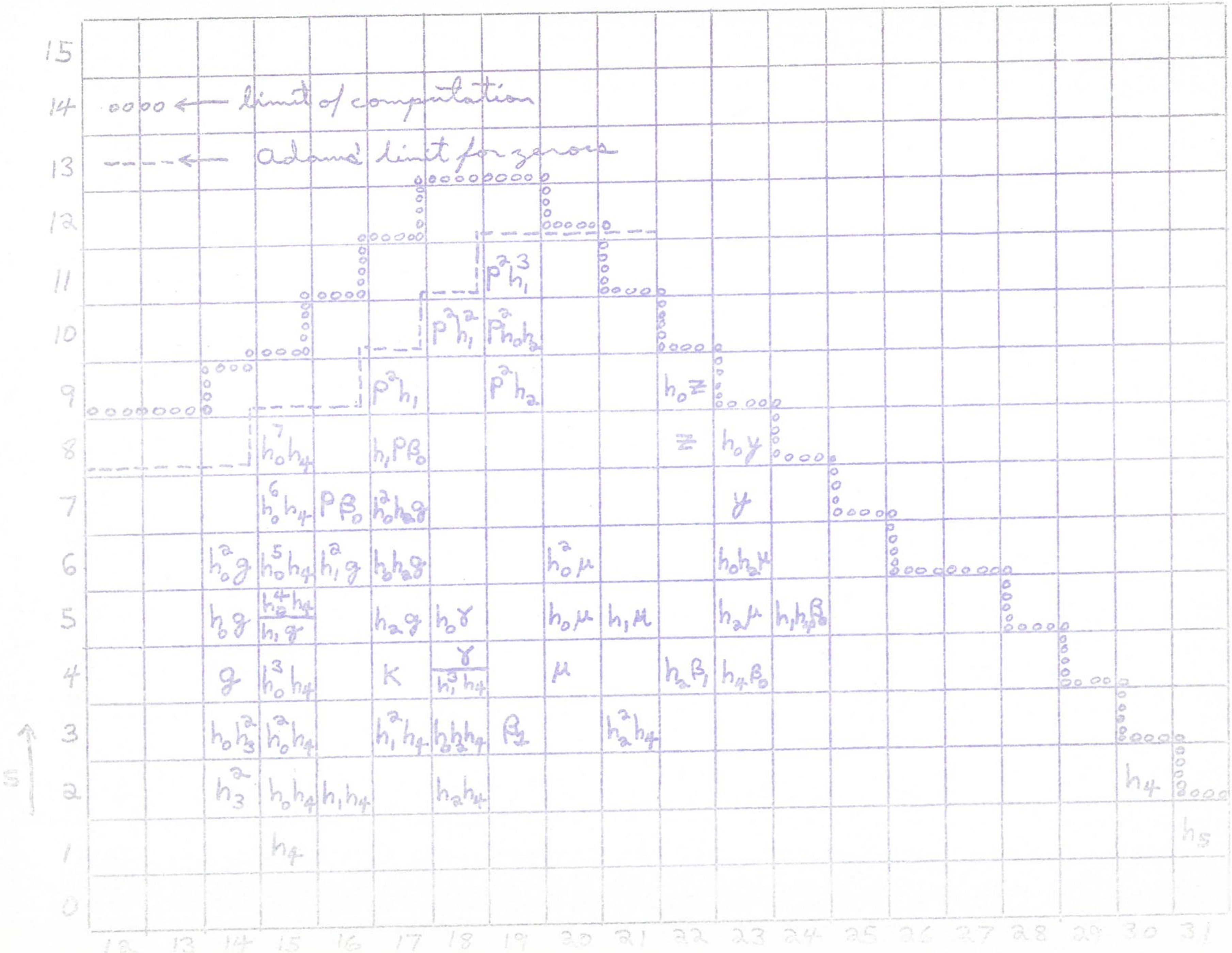
§3 Talking to the machine

The computer used was the IBM 7094: memory of 2^{15} words, each word consisting of 36 bits of information.

A basis for the dual of the mod 2 Steenrod algebra is given by the monomials in the generators $\alpha_1, \dots, \alpha_n, \dots$ Monomials up to grading 254 were represented as words by reserving seven fields of the word for the exponents of the generators.

A program for computing diagonals was designed and the resulting table of diagonals stored on tape. This table was used as data for a second program which computed an actual minimal coresolution of Z_2.

The computation was carried out for $Ext_A^{s,t}(Z_2, Z_2)$ for $t \geq -31$ and all s, and for $t = -32$ and $s \leq 5$ The results are given in the following table (notation as in Adams' Berkeley notes — only the results in stems ≥ 12 are shown).



t − s

Computation of $Ext_A^{s,t}(Z_2, Z_2)$