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THE UNIT THEOREM
FOR THE BURNSIDE RING OF A 2-GROUP

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Introduction.

For a finite group G we let $\Omega(G)$ denote the Burnside ring of G and $\Omega^{\cdot}(G)$ the multiplicative group of units in $\Omega(G)$. The isomorphism between $\Omega(G)$ and ω_G^0 of Segal [4] identifies units in the Burnside ring with stable G-homotopy classes of G-homotopy selfequivalences $S(V) \to S(V)$ of representation spheres. Call a unit $u \in \Omega^{\cdot}(G)$ orthogonal if it is represented by an orthogonal map, i.e. an element of $C_{O(V)}(\bar{G})$, where $\bar{G}$ is the image under the homomorphism $G \to O(V)$. The following partially answers problem 1.5.2 of tom Dieck [1].

Unit theorem. Every unit in the Burnside ring of a 2-group is orthogonal.

This was announced in [6] with an outline of proof. The proof given here is much shorter, but gives less information on the detailed structure of the units. We introduce the Burnside kernel

$$N(G) = \mathrm{Ker}(\Omega(G) \xrightarrow{j} R_{\mathbb{Q}}(G)),$$

where the ring homomorphism j takes a finite G-set X into the permutation representation $\mathbb{Q}X$ based on X. A subgroup $N^{\cdot}(G) \subset \Omega^{\cdot}(G)$ is similarly defined as the kernel of the induced homomorphism $\Omega^{\cdot}(G) \to R_{\mathbb{Q}}^{\cdot}(G)$. In §1 we construct for a given group G a pairing

$$\mathrm{Gal}(K/\mathbb{Q}) \times N(G) \to N^{\bullet}(G)$$

where the numberfield K is generated over $\mathbb{Q}$ by the square roots of the prime divisors of $|G|$. The image in $N^{\bullet}(G)$ consists by construction of ortogonal units. In §2 we show that the pairing maps onto $N^{\bullet}(G)$ for every 2-group G. The final step in §3 handles units in the representation ring over $\mathbb{Q}$ by elementary character theory.

1. Construction of the pairing.

Our first move is to present elements in $N(G)$ in terms of what might be called doubly orthonormally based real inner product G-spaces.

1.1. Proposition. Let G be a finite group. Any element $x \in N(G)$ can be written $x = [X_+] - [X_-]$, where (V, B, X_+, X_-) is as follows:

1) V is a real representation of G.

2) B is a G-invariant positive definite symmetric bilinear form on V.

3) X_+ and X_- are G-invariant subsets of V forming orthonormal bases for V.

Proof. Write the given element as a difference $x = [X_+] - [X_-]$, where X_+ and X_- are finite G-sets. The permutation representations $V_+ = \mathbb{R} X_+$ and $V = \mathbb{R} X_-$ are isomorphic by definition of $N(G)$. We have G-invariant inner products B_+ on V_+ and B_- on V_- determined by the condition that

REFERENCES

[1] T. tom Dieck: Transformation groups and representation theory. Springer Lecture Notes in Mathematics vol.766, (1979).

[2] E. Laitinen: On the Burnside ring and stable cohomotopy of a finite group. Math.Scand.44, (1979), 37-72.

[3] H.R. Miller and S.B. Priddy: On G and the stable Adams conjecture. Springer Lecture Notes in Mathematics, Vol.658, (1978), 331-348.

[4] G.B. Segal: Equivariant stable homotopy theory. Actes Congrès Intern. Math. Tome 2, (1970), 59-63.

[5] J. Tornehave: Equivariant maps of spheres with conjugate orthogonal actions. Canadian Math. Soc. Conf. Proc., Vol.2, part 2, A.M.S., (1982), 275-301.

[6] J. Tornehave: The homotopy Schur lemma. Seminar Notes No.1, (1982), Univ. of Aarhus.

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the orthogonal units needed to hit all such $\pm[L]$. Hence the proof of the unit theorem is completed by:

3.1. Lemma. Let G be a finite group. Any unit in $R_{\mathbb{Q}}(G)$ has the form $\pm[L]$, where L is a 1-dimensional $\mathbb{Q}G$-module.

Proof. The ordinary character defines an injective ring homomorphism from $R_{\mathbb{Q}}(G)$ to the ring of integral class functions $G \to \mathbb{Z}$. A unit $u \in R_{\mathbb{Q}}^{\cdot}(G)$ is mapped into a class function χ_u with $\chi_u(g) = \pm 1$ $(g \in G)$. For the usual inner product $\langle \chi_u, \chi_u \rangle_G = 1$. If $\delta = \chi_u(1)$, $\delta\chi_u$ is a 1-dimensional irreducibel complex character realizable over $\mathbb{Q}$. Hence $u = \delta[L]$, where L has character $\delta\chi_u$.

X_+ and X should form orthonormal bases for V_+ and V_- respectively. The isomorphism $V_+ \to V_-$ of $\mathbb{R}G$-modules can by a simple application the spectral theorem be improved to an isomorphism of quadratic $\mathbb{R}G$-modules. Finally, we need only identify V_+ and V_- by the improved isomorphism.

Let $\gamma \in \mathrm{Aut}(\mathbb{C})$ be a field automorphism to be kept fixed in the following discussion.

To a quadruple (V,B,X_+,X_-) with the properties of proposition 1.1 is associated an element $[\varphi_-^{-1} \circ \varphi_+] \in KO_G^{-1}(pt)$ now to be defined. We complexify V and extend B to a symmetric bilinear form on $V \otimes \mathbb{C}$ still called B. Define $\varphi_+ : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$ by

$$\varphi_+\Big(\sum_{x \in X_+} a_x x\Big) = \sum_{x \in X_+} \gamma(a_x) x$$

and let φ_- be similarly defined using X_-. These maps are $\mathbb{Q}$-linear, G-equivariant and satisfy

$$\varphi_\pm(av) = \gamma(a)\varphi_\pm(v)$$

$$B(\varphi_\pm(v),\varphi_\pm(w)) = \gamma(B(\varphi_\pm(v),\varphi_\pm(w))).$$

Hence $\varphi_-^{-1} \circ \varphi_+$ is in the orthogonal group $O(V,\mathbb{C})$ of $(V \otimes \mathbb{C}, B)$, so $\varphi_-^{-1} \circ \varphi_+ \in C_{O(V,\mathbb{C})}(\bar{G})$, where $\bar{G}$ is the image of G in $O(V,\mathbb{C})$.

Actually $\bar{G} \subseteq O(V)$ and $C_{O(V)}(\bar{G})$ is a maximal compact subgroup of $C_{=(V,\mathbb{C})}(\bar{G})$ - in particular $C_{O(V)}(\bar{G})$ is a deformation retract of $C_{O(V,\mathbb{C})}(\bar{G})$ (compare Lemma 1.8 of [5]). Hence $\varphi_-^{-1} \circ \varphi_+$ determines a pathcomponent of $C_{O(V)}(\bar{G})$, which in turn determines the class $[\varphi_-^{-1} \circ \varphi_+] \in KO_G^{-1}(pt)$.

The image $J([\varphi_-^{-1} \circ \varphi_+])$ under the equivariant J-homomorphism $J\colon KO_G^{-1}(pt) \to (\omega_G^0)^{\cdot}$ can be identified with an orthogonal unit $u \in \Omega^{\cdot}(G)$. The character formula of Proposition 1.2 below will show that u depends only on γ and $x = [X_+] - [X_-]$.

For a subgroup $H \leqq G$ we let $d_H\colon \Omega(G) \to \mathbb{Z}$ denote the ring homomorphism, which counts the number of H-fixed points in a finite G-set. Let $P_H\colon \Omega(G) \to \mathbb{Q}_+^{\cdot}$ be the homomorphism into the multiplicative groups of positive rational numbers that carries a finite G-set X into the product of the cardinalities of the H-orbits in X. Furthermore we shall use the standard pairing of Kummer theory

$$\mathrm{Aut}(\mathbb{C}) \times \mathbb{Q}^{\cdot} \to \{\pm 1\}$$

given by $(\gamma, a) = \gamma(\sqrt{a})/\sqrt{a}$.

<u>1.2. Proposition</u>. Let $x \in N(G)$ be written $x = [X_+]-[X_-]$, where (V,B,X_+,X_-) has the properties of 1.1. Let $u \in \Omega^{\cdot}(G)$ be the orthogonal unit given by the construction above. Then u is determined by the character formula

$$d_H(u) = (\gamma, P_H(x)) \qquad (H \leqq G).$$

<u>Proof</u>. Connect $\varphi_-^{-1} \circ \varphi_+$ by a path in $C_{O(V,\mathbb{C})}(\bar{G})$ to $f \in C_{O(V)}(\bar{G})$. Then $d_H(u) = \deg(f^H)$, where f^H is the restriction of f to H-fixed points $f^H\colon S(V)^H \to S(V)^H$. The path restricts to a path in $O(V^H, \mathbb{C})$, and the determinant $\det\colon O(V^H,\mathbb{C}) \to \{\pm 1\}$ is constant along the restricted path. Therefore $d_H(u) = \det((\varphi_-^{-1} \circ \varphi_+)^H)$, where $(\varphi_-^{-1} \circ \varphi_+)^H$ is the restriction to $V^H \otimes \mathbb{C}$ of $\varphi_-^{-1} \circ \varphi_+$.

<u>Proof</u>. Since both sides have trivial restrictions to proper subgroups, it suffices that $d_G(U(t_G)) = -1$ to show the formula.

An element $t_G \in \Omega(G)$ satisfying (2.3) appears in Laitinen [2], Proposition 3.6. Clearly $t_G \in N(G)$, when G is non-cyclic. As observed in [2] t_G is a pull back from the Frattini quotient $V = G/FrG$ of the corresponding element t_V. By naturality it suffices to show that $d_V(U(t_V)) = -1$.

Regard V as a $\mathbb{F}_2$-vectorspace of dimension $n \geqq 2$. From [2]

$$t_V = \sum_{d=0}^{n} (-1)^d \, 2^{\frac{1}{2}(d-1)(d-2)} \, \eta_d,$$

where

$$\eta_d = \sum_{\operatorname{codim} W = d} [V/W].$$

Terms with $d \geqq 3$ have even coefficients and V-orbits in η_2 have cardinality 4. By (2.1) $d_V(U(t_V)) = -1$, since the number of hyperplanes in V is odd.

3. The final step.

The results of the preceeding two sections show that all units in $N^{\cdot}(G)$ for a 2-group G are orthogonal. By the exactness of

$$1 \to N^{\cdot}(G) \to \Omega^{\cdot}(G) \to R_{\mathbb{Q}}^{\cdot}(G),$$

it suffices to find sufficiently many orthogonal units in $\Omega^{\cdot}(G)$ to hit all of $R_{\mathbb{Q}}^{\cdot}(G)$ in order to prove the unit theorem.

We have units $\pm[L] \in R_{\mathbb{Q}}^{\cdot}(G)$ given by 1-dimensional $\mathbb{Q}G$-modules L. Note that these correspond to homomorphisms $G \to \{\pm 1\}$. Antipodal maps for 1-dimensional $\mathbb{R}G$-modules provide

$x = [X_+] - [X_-]$ of cardinality an odd power of 2 is even or odd.

The next result shows that all units in $N^\bullet(G)$ for a 2-group G are orthogonal.

2.2. Theorem. U is surjective for every 2-group G.

Proof. Define $s: N^\bullet(G) \to N(G)$ by

$$s(u) = u - 1.$$

We claim that $U \circ s: N^\bullet(G) \to N^\bullet(G)$ is bijective. It suffices to show injectivity, because $N^\bullet(G)$ is finite.

Assume G to be a minimal counterexample to the injectivity of $U \circ s$. Let $u_1, u_2 \in N^\bullet(G)$ be distinct units with $Us(u_1) = Us(u_2)$. By minimality u_1 and u_2 have the same restriction to any proper subgroup $H < G$. Interchanging u_1 and u_2 if necessary, we may assume that $t_G = u_1 - u_2$ is determined by

$$(2.3) \qquad d_H(t_G) = \begin{cases} 2 & \text{if } H = G \\ 0 & \text{if } H \neq G \end{cases}$$

The case $H = G$ implies that G is not cyclic. Furthermore

$$U(t_G) = U(s(u_1) - s(u_2)) = 1.$$

reduces the proof to the following explicit calculation.

2.4. Lemma. For any non-cyclic 2-group G there is an element $t_G \in N(G)$ given by (2.3) and

$$U(t_G) = 1 - t_G.$$

We have an orthonormal basis for V^H consisting of the elements

$$e_S = \frac{1}{\sqrt{|S|}} \sum_{x \in S} x$$

where S runs through the H-orbits of X_+. Observe that

$$\varphi_+(e_S) = (\gamma, |S|) e_S.$$

If $n = \dim V^H$ we have a symmetric bilinear form on the 1-dimensional space $\Lambda^n(V^H \otimes \mathbb{C})$ given by

$$B_1(v_1 \wedge \ldots \wedge v_n, w_1 \wedge \ldots \wedge w_n) = \det(B(v_i, w_j)).$$

An orthonormal basis for $\Lambda^n V^H$ is given by the single element

$$e = e_{S_1} \wedge \ldots \wedge e_{S_n},$$

where $S_1, \ldots, S_n$ are the H-orbits in X_+. Even though φ_+^H is γ-linear rather than linear it induces a γ-linear map $\Lambda^n \varphi_+^H: \Lambda^n(V^H \otimes \mathbb{C}) \to \Lambda^n(V^H \otimes \mathbb{C})$. We find

$$\Lambda^n \varphi_+^H(e) = \left(\prod_{j=1}^{n} (\gamma, |S_j|)\right) e = (\gamma, P_H(X_+)) e.$$

Note that e is up to sign the unique element of $\Lambda^n(V^H \otimes \mathbb{C})$ satisfying $B_1(e,e) = 1$. Hence the corresponding computation for φ_-^H yields

$$\Lambda^n \varphi_-^H(e) = (\gamma, P_H(X_-)) e.$$

The formulas combine to

$$\Lambda^n((\varphi_{-1} \circ \varphi_+)^H) e = (\gamma, P_H(x)) e,$$

showing that

$$d_H(u) = \det((\varphi_{-1} \circ \varphi_+)^H) = (\gamma, P_H(x)).$$

1.3. Theorem. For any field automorphism $\gamma \in \mathrm{Aut}(\mathbb{C})$ there is a natural transformation

$$U(\gamma,-): N(G) \to N^{\cdot}(G)$$

on the category of finite groups and arbitrary group homomorphisms determined by

$$d_H(U(\gamma,x)) = (\gamma, P_H(x)) \qquad (H \leq G) \tag{1.4}$$

Proof. From Proposition 1.2 we have a map

$$U(\gamma,-): N(G) \to \Omega^{\cdot}(G)$$

satisfying (1.4). Recall [1] that $\Omega^{\cdot}(G)$ is embedded in the set of functions from conjugacy classes of subgroups $H \leq G$ to $\{\pm 1\}$ be the map $u \to (d_H(u))_{H \leq G}$. The homomorphism and naturality properties can therefore be read off from (1.4) (alternatively one can directly examine the construction).

It only remains to be proved that $U(\gamma,x) \in N^{\cdot}(G)$. This follows by naturality from

$$N(G) = \bigcap_C \mathrm{Ker}(\Omega(G) \xrightarrow{\mathrm{Res}} \Omega(C))$$

$$N^{\cdot}(G) = \bigcap_C \mathrm{Ker}(\Omega^{\cdot}(G) \xrightarrow{\mathrm{Res}} \Omega^{\cdot}(C)),$$

where C runs through the cyclic subgroups of G.

The above suffices for the proof of the unit theorem in the following sections, but some further remarks are in order. It is apparent from (1.4) that U defines a pairing

$$\mathrm{Aut}(\mathbb{C}) \times N(G) \to N^{\cdot}(G).$$

Since all primdivisors of $P_H(X_+)$ and $P_H(X_-)$ also divide $|G|$, $U(\gamma,x)$ depends only on γ restricted to the field K generated over $\mathbb{Q}$ by the square roots $\sqrt{p}$ of the primes involved in $|G|$. Hence we have the pairing

$$\mathrm{Gal}(K/\mathbb{Q}) \times N(G) \xrightarrow{U} N^{\cdot}(G)$$

as stated in the introduction.

1.5. Remark. The case $H = 1$ of (1.4) shows $\varphi_-^{-1} \circ \varphi_+ \in SO(V,\mathbb{C})$. Actually a natural lift to $\mathrm{Spin}(V,\mathbb{C})$ may be constructed using Clifford algebras as in [5]. The construction which to (V,B,X_+,X_-) assigns this lifting to $\mathrm{Spin}(V,\mathbb{C})$ is a representation theoretic counterpart of the infinite loop map ψ: IBO $\to$ Spin defined in [3]. It may be possible to give a infinite loop space machine construction of ψ using this construction.

The natural transformation of Theorem 1.3 could be constructed using Theorem 4.1 of [5] applied to (V,φ_+) and (V,φ_-).

2. The 2-group case.

Choose $\gamma \in \mathrm{Aut}(\mathbb{C})$ so that $\gamma(\sqrt{2}) = -\sqrt{2}$ and restrict $U(\gamma,-)$ to a natural transformation

$$U: N(G) \to N^{\cdot}(G)$$

on the category of 2-groups. Formula (1.4) may be rewritten

$$d_H(U(x)) = (-1)^{vP_H(x)}, \tag{2.1}$$

where v is the 2-adic valuation on $\mathbb{Q}$. Hence the sign $d_H(U(x)) = \pm 1$ depends on whether the number of H-orbits in