A HOMOLOGY THEORY FOR MULTIPLY CONNECTED CONTIGUOUS POINT SPACES

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INTRODUCTION

Contiguous point spaces were first studied by R.L.Moore [4] who developed the basic set-theoretic theorems and made applications to hyperspaces obtained from decompositions of continua (such as the decomposition of Peano continua into their cyclic elements). E.C.Klipple [3] discussed certain two-dimensional contiguous point spaces which have curious non-compactness properties(probably due to permitting "triunes" to separate space). A paper [2] by T.Hailperin discussed relations between cettain "T -spaces" and the contiguous point spaces, but whereas in the latter case transitivity of the contiguity is required, this is not the case in the Moore theory and it does not seem to be a desirable property to require in general.

So long as no homology theory is available for contiguous point spaces, their range of application is rather limited, and it is the general purpose of this investigation to lay the foundation of such a theory. Unlike Moore, we assume no metric. The general space theory and its relations to the above-mentioned papers are given in this and the next chapter. In chapter III we describe the homology theory. Once the basic conventions regarding nuclei and the nerves of coverings have been made, much of the development here becomes the same as in the general Cech theory with adaptations to suit the altered conditions. The general theory may be found in a book of R.L.Wilder [5].

In chapter IV locally connected spaces are treated. Here we follow a paper of E.G.Begle [1], the burden of the development being again a matter of adapting to the new conditions due to the existence of contiguous points. Chapters V - VII are devoted to definitions and investigation of basic configurations, particularly the contiguous point manifolds. In the classical combinatorial topology, the study of manifolds and their homology properties provided a central motive. In the present work, we do not go so far as to treat the problem of the duality theorems of the Poincaré and Alexander type. We confine ourselves to the study of the possible types of contiguous point clusters in a manifold, and justification theorems (such as identifying the one-dimensional case with the simple closed curve).

Chapter I

INTRODUCTION

Contiguous point spaces were first studied by R.L. Moore [4]*. He was only interested in spaces where the relation of contiguity between points was never transitive. E.C. Klipple [3] discussed contiguous points in the plane but, as his space was not necessarily compact and in fact the examples which he gives are of non-compact spaces, his space and the two dimensional orientable generalized closed manifold which we shall discuss are not the same. The only other paper on the subject of contiguous point spaces is one by T. Hailperin relating to the axioms [2].

We shall set up a Čech homology theory on a compact multiply connected contiguous point space. Much of the work is the same as the general Čech homology theory. Other theorems are adaptations of general theorems changed to suit the altered conditions. All this theory is contained in a book by R.L. Wilder [5]. The chapter on locally connected spaces is an adaptation of work by E.G. Begle [1].

After setting up the homology theory, we shall define an orientable generalized closed manifold and use the homology theory to derive some results about

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*Numbers in square brackets refer to the bibliography.

the structure of the contiguous points in a manifold. We shall also show that the simple closed curve is an example of a one dimensional orientable generalized closed manifold.

<u>Definition</u>: A <u>contiguous point space</u> is a topological space with an undefined relation called <u>contiguity</u> holding between certain pairs of points in the space with the following three properties:

A. No point is contiguous to itself.

B. If the point p is contiguous to the point q, then q is contiguous to p.

C. If M is a closed point set and every point of some set H is contiguous to some point of M, then no point of S - M is a limit point of H.

The last axiom, axiom C, serves to limit the number of contiguous points in the space. In chapter II we shall give some theorems which show the character of these limitations.

A T_0 space with one additional property can always be considered as a T_1 or Frechet contiguous point space.

A T_0 space is a set of points, S, where the neighbourhoods satisfy the following four properties.

1. If $p \in S$, there exists a neighbourhood of p, U(p), containing p.

2. If $x \in U(p)$, there exists $U(x) \subset U(p)$.

3. Given two neighbourhoods of p, U(p) and V(p), there

exists a neighbourhood $W(p) \subset U(p) \cap V(p)$.

4. If p and q are two distinct points of S, there exists a neighbourhood of one of them which does not contain the other.

For a T_1 or Frechet space our neighbourhoods satisfy 1, 2, 3, and the stronger separation axiom 4'.

4'. If p and q are two distinct points of S, there exists a neighbourhood of each of them which does not contain the other.

In order to see the T_0 space may be considered as a Frechet contiguous point space by the addition of new neighbourhoods we set up the following sets. Let A_p be the set of all points q such that $q \in U(p)$ for all $U(p)$. Let $F_p = \bar{A}_p$.

The point p is said to be contiguous to the point q if p is not equal to q and either p is an element of A_q or q is an element of A_p. In other words p is contiguous to q if either p is a limit point of q or q is a limit point of p.

Let the neighbourhoods of the point p in the T_0 space S be denoted by $U(p)$, $V(p)$, $W(p)$ etc. For each neighbourhood $U(p)$ add a new neighbourhood $U'(p)$ where

$U'(p) = p \cup \{U(p) - F_p\}$.

We shall show that with the addition of these neighbourhoods, the system of neighbourhoods satisfies the axioms for a Frechet space.

1. For every p, there exists $U(p)$ containing p.

2. Suppose x is an element of some neighbourhood of p.

(a) Suppose $x \in U(p)$, then by axiom 2, there exists $U(x) \subset U(p)$.

(b) Suppose $x \in U'(p) = p \cup \{U(p) - F_p\}$. The point x is not an element of the closed set F_p, hence thre exists $V(x)$ which does not intersect F_p. By axiom 2, there exists $W(x) \subset U(p)$ and by axiom 3, there exists $O(x) \subset V(x) \cap W(x)$. Therefore $O(x) \subset U'(p)$.

3. Suppose we are given the neighbourhoods

$$U'(p) = p \cup \{U(p) - F_p\} \text{ and}$$

$$V'(p) = p \cup \{V(p) - F_p\}.$$

Take $W(p) \subset U(p) \cap V(p)$ from axiom 3. Then

$W'(p) = p \cup \{W(p) - F_p\}$ is contained in the intersection of $U'(p)$ and $V'(p)$. The other cases can very easily be seen to hold.

4. Let p and q be two distinct poins in S. If $q \in U(p)$ for all $U(p)$, then $q \in A_p$. Hence $q \notin U'(p)$ for any $U'(p)$.

We would like to have the following theorem true.

<u>Theorem 1.</u> <u>If A is a set and p is a limit point of A in the old sense, then either p is a limit point of A in the new sense or p is contiguous to some point $q \in A$.</u>

For this purpose we add the following strong axiom to our T_0 space.

<u>Axiom 5.</u> If p is a limit point of a set of points A such that p is not a limit point of any point in A and for

each point $a \in A$, there exists a point b (not necessarily in A) such that either a is a limit point of b or b is a limit point of a, then p is also a limit point of the set of points $\{b\}$. (The set of points $\{b\}$ might however consist of a single point.) Such a point p is however not a limit point of any point q in the space nor is q a limit point of p. (A limit point of contiguous points is not a contiguous point.)

The last part of the axiom makes it impossible that $\{b\}$ be a single point but the last part is not needed in the proof of theorem 1, but only in the proof of axiom C.

Some sort of axiom of this type is needed in order to have axiom C satisfied. If the set A is not restricted to sets for which p is not a limit point of any element of A itself, then we make it impossible to have three points p, q, and r such that every neighbourhood of p contains q and everyneighbourhood of q contains r or every neighbourhood of r contains q. To see this suppose such points exist. If every neighbourhood of p contains q and every neighbourhood of q contains r, then every neighbourhood of p contains r; hence tere exists a neighbourhood of r which contains neigher p nor q by axioms 3 and 4. But q is a limit point of r and as p is a limit point of r, then by axiom 3 q must be a limit point of p which is impossible. If every neighbourhood of p contains q and every neighbourhood of r contains q, then there exists a neighbourhood of q which contains neither p nor r. By similar reasoning we much have every neighbourhood of p containing r and we have the above situation with q and r interchanged. These are not cases which we want to rule out.

We shall now give some examples of T_o spaces and show the new neighbourhoods which are added by the above procedure.

Example 1. In the co-ordinate plane let S consist of the points (0,0), and all points (0,1/n)and (1,1/n) for n a positive integer with the following neighbourhoods.
The point (0,1/n) has a single neighbourhood consisting of the point itself.
The point (1,1/n) has a single neighbourhood consisting of the two points (1,1/n) and (0,1/n).
For each positive integer k the set of points (0,0) (0,1/n) for n greater than k form a neighbourhood of the point (0,0).

The set of points with these neighbourhoods forms a T_o space. In the corresponding T_1 space the points (1,1/n) will also have neighbourhoods consisting of the single points. This space does not satisfy axiom C nor does the original space satisfy axiom 5.

Example 2. Let S consist of the points and neighbourhoods given in example 1 with the point (1,1) having the

following neighbourhoods.
For any positive integer k the set of points (0,0), (1,1), (0,1/n) and (1,1/n) for n greater than k form a neighbourhood of the point (1,1). The new neighbourhoods of the point (1,1) will be the old ones without the point (0,0). This space is a Frechet space but not a Hausdorff space.

Example 3. This example shows that allowing {b} to be a single point in axiom 5 is necessary.

Let S consist of the following points and neighbourhoods.
The point (0,0) with its neighbourhoods consisting of the points (0,0) and (1/n,0) for n ≥ k where both n and k are positive integers.
The point (1/n,0) with neighbourhood consisting of the point itself for each positive integer n.
The point (0,1) with a single neighbourhood consisting of all the above points except the point (0,0).

When we change this space into a contiguous point T, space the only change is that the point (0,1) now has a neighbourhood consisting of the point itself. In this space if we take the set M of axiom C to be the point (0,1) and the set N to consist of the points (1/n,0) for all positive integers n, we see that N has a limit point (0,0) which is not an element of M. In axiom 5 take A to be the set N given above, p to be the point (0,0) and b to be the point (0,1). We see easily that the axiom fails to hold.

Example 4. This example shows that we cannot omit the last part of axiom 5. (A limit point of contiguous points is not a contiguous point). Some sort of restriction of this type is necessary. Of course this section of the axiom also eliminates the case where $\{b\}$ consists of a single point.

Let S consist of the following points and neighbourhoods where n always represents a positive integer. The point (0,0) with neighbourhoods consisting of the points (0,0),(0,1) and (1/n,0) for all n greater than some positive integer k, for each k.

The point (1/n,0) with neighbourhood consisting of the point itself.

The point (0,1) with neighbourhood consisting of the point itself.

The point (0,2) with single neighbourhood consisting of all the points in the space.

The new neighbourhood of the point (0,2) will consist of the point itself. The new neighbourhoods of the point (0,0) consist of the old ones without the point (0,1). If M is taken to be the point (0,2) and H to consist of the points (1/n,0), we see that H has a limit point (0,0) which is not an element of M.

We shall now show that if axiom 5 is satisfied, then theorem 1 holds.

<u>Theorem 1. If A is a set and p is a limit point of A in the old sense, then either p is a limit point of A in the new sense or p is contiguous to some point q, an element of A.</u>

Proof: Suppose p is a limit point of A in the old sense but not in the new sense, then there exists a neighbourhood

$$U'(p) = p \cup \{U(p) - F_p\}.$$

such that $U'(p)$ does not intersect $A - p$ but there exists a point $x \in U(p) \cap (A - p)$. Hence $x \in F_p$. Either $x \in A_p$ or $x \in \bar{A}_p - A_p$. If $x \in A_p$, then p is contiguous to the point $x \in A$; if $x \in \bar{A}_p - A_p$, then x is a limit point of points which are contiguous to p and hence x is contiguous to p. This proves the theorem.

<u>Example 5.</u> The following example is another example of a space in which theorem 1 fails to hold.

Let S consist of the following points and neighbourhoods where m and n represent positive integers.

The point $(1/n, 1/m)$ with neighbourhood consisting of the point itself.

The point $(1/n, 0)$ with neighbourhoods consisting of the points $(1/n, 1/m)$ for m greater than some positive integer k for each k.

The point $(0,0)$ with neighbourhoods consisting of the points $(1/n, 0)$ for n greater than some positive integer k and all the points $(1/n, 1/m)$ for each k.

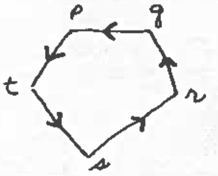
For $p = (0,0)$ the set F_p is equal to the whole space hence a new neighbourhood of the point p is the point p itself. If A is the set of points $(1/n, 0)$, in the old sense they have the limit point $(0,0)$ but in the new sense they do not have this point as a limit point nor is any point of A contiguous to $(0,0)$. The fact that axiom 5 fails to hold here is obvious.

We have still to show that this definition of contiguity satisfies axioms A, B, and C. The first two are obvious from the definition of contiguity. To show axiom C, let M be a closed set and H a set such that every point of H is contiguous to some point of M. Suppose p is a limit point of H; then every old neighbourhood of p intersects H and hence by axiom 5 every old neighbourhood of p intersects M. However also by axiom 5, p is not a contiguous point and hence there are no additional neighbourhoods of p. This means that p is a limit point of M in the new sense. Since M is a closed set, the point p is an element of M.

Hence we have succeeded in showing that a T_0 space with the addition of axiom 5 can always be turned into a T_1 space which is a contiguous point space and the relation between the two topologies on the space is that given by theorem 1.

A contiguous point space cannot however necessarily be turned into a T_0 space as the following example shows.

Example 6. Let S consist of the five points with the contiguities as shown in the diagram below.



The neighbourhoods of the points are the points themselves. Suppose in the neighbourhood system which goes with the desired T_0 space the point p has a neighbourhood consisting of the point p. Hence p is not a limit point of q. As q is contiguous to p, the point q must be a limit point of p and every $U(q) \supset q \cup p$. If every $U(q) = q \cup p$, then since q is contiguous to r, every $U(r) \supset q \cup r$. In this case $q \in U(r)$ and there exists no $U(q) \subset U(r)$. Hence try $U(q) = p \cup q \cup r$. The point q is not contiguous to either s or t and hence there is a neighbourhood of q not containing these points. As $r \in U(q)$ and r is not contiguous to p, there exists $U(r) = r$. Similarly $U(s) = r \cup s \cup t$ and $U(t) = t$. But p and t are contiguous and hence either U(p) contains t or U(t) contains p. In any case we have derived a contradiction under the assumption that there exists one point with a discrete neighbourhood. To show that such a point exists consider the point p. There exists a neighbourhood of p not containing s or r. Suppose U(p) contains the point t, then there exists a neighbourhoodof t contained in this. As there are neighbourhoods of t not containing p or q, U(t) = t. This concludes the proof that a contiguous point space is not necessarily obtainable from a T_0 space by the above procedure.

A contiguous point space cannot in general be turned into a topological space like that of Hailperin as in his space contiguity is a transitive rèlation which it need not be. In fact the spaces which Moore considers satisfy an additional axiom which states that there does not exist a set of three points, each one of which is contiguous to the other two.

Chapter II

POINT SET THEORY

In this chapter we shall give some point set definitions and theorems which will be needed later.

Definition: If A is any set, the set A^c denotes the set of points which are either elements of A or contiguous to points of A.

Theorem 1. If A is a closed set, then the set A^c is also closed.

Proof: The proof follows immediately from axiom C.

Definition: A triune [3] is a set of three points, x_1, x_2, x_3, such that x_i is contiguous to x_j for $i \neq j$.

Definition of boundary. [4,p.] Let M be a point set. A point p will be an element of the boundary $F(M)$ of M if any one of the three following conditions is satisfied.

1. The point p is an element of M and a limit point of $S - M$.

2. The point p is an element of $S - M$ and a limit point of M.

3. The point p is an element of $S - M$ and is contiguous to a point of M.

Remark: If L is a closed set, then $F(L)$ and $F(S - L)$ are not necessarily the same.

Theorem 2. The boundary $F(M)$ of a set M is closed.

[4,p.   ]

Proof: It is easily seen that

$F(M) = A \cup B$

where $A = \overline{S - M} \cap \overline{M}$ and B consists of all points in $S - M$ which are contiguous to points of $\overline{M}$.

The set A is closed. Suppose p is a limit point of B, then by axiom C, the point $p \in \overline{M}$ which is a limit point of $S - M$. Hence $p \in A$. This proves that $F(M)$ is closed.

Theorem 3. <u>If A is a set of points with limit point p and B is a set of points such that each point of B is contiguous to a point of A and conversely, then p is a limit point of B.</u>

Proof: Suppose p is not a limit point of B; then there exists an open set O containing p such that $\overline{O} \cap \overline{B} = \emptyset$. Let $A' = A \cap \overline{O}$; the set A' has limit point p. Let B' be the set of points of B which are contiguous to points of A'. Then $\overline{B}'$ is a closed set not containing p and A' is a set of points each of which is contiguous to a point of $\overline{B}'$. As A' has limit point p, this is a contradiction to axiom C.

Theorem 4. <u>If x is an element of the open set P, then there exists an open set Q such that $x \in Q \subset P$ and such that Q contains no point contiguous to a point of $S - P$ except possibly x itself.</u>

Proof: As $S - P$ is a closed set, by axiom C, x is not a limit point of the set of points in P which are contiguous to points of S - P; hence there exists an open set Q containing x which contains no point contiguous to a point of S - P except possibly x itself.

<u>Definition:</u> The point x is said to be a <u>proper interior point</u> of the set P if x is not contiguous to a point of S - P.

<u>Theorem 5.</u> <u>If $x \in S$, either there exists an open set $U = x$ or every open set U containing x contains a point y which is a proper interior point of the set U.</u>

Proof: Take any open set U containing x. By theorem 4 there exists an open set Q containing x such that Q contains no points contiguous to points of S - U except possibly x itself. If every point of U is contiguous to a point of S - U, then Q contains only th point x.

<u>Definition of separated or contiguously separated sets.</u> [4,p]. The disjoint sets A and B are said to be separated if the set A does not contain a limit point of B nor B a limit point of A and also there exists no point in A which is contiguous to a point in B.

We will say that there is not contiguity across the sets A and B if there exists no point in A which is contiguous to a point in B. We will say that there is no contiguity across the sets A and B except the contiguity x to y if x and y are points of A and B respectively and x is contiguous to y and there exists no other pair of points with this property.

<u>Definition</u>: A set A will be said to be <u>connected</u> if it is not the union of two separated sets. [4,p.]

<u>Definition</u> A collection of sets $A_1, A_2, \ldots A_n$ will be called a <u>simple chain</u> from the point a to the point b if $a \in A_i$ if and only if $i = 1$, $b \in A_i$ if and only if $i = n$, and the sets A_i and A_j for $i < j$ intersect or there is a contiguity across if and only if $j = i + 1$.

In the same way as in the general theory we can prove the following theorem, the simple chain theorem.

<u>Theorem 6.</u> <u>If M is a connected set and G is a collection of open sets covering M, then there exists a simple chain of sets of G from x to y where x and y are any two points in M.</u>

<u>Theorem 7.</u> <u>In a compact space if A and B are disjoint closed sets there exists only a finite number of pairs of points $\{a_i, b_i\}$ such that $a_i \in A$, $b_i \in B$ and a_i is contiguous to b_i.</u> [4,p]

Proof: Since A and B are closed subsets of a compact space, they are compact. If there exists an infinite number of such pairs of points, then either the set $\{a_i\}$ or the set $\{b_i\}$ is infinite. Suppose the set $\{a_i\}$ is infinite. Then there exists in A an infinite number of points which are contiguous to points of B. By compactness these have a limit point in A. This violates axiom C.

Theorem 8. In a compact space if A and B are separated sets such that $\overline{A} \cap \overline{B} = 0$, then there exist open separated sets U and V such that $A \subset U$ and $B \subset V$.

Proof: By normality there exist open sets U' and V' such that $A \subset U'$, $B \subset V'$ and $\overline{U'} \cap \overline{V'} = 0$. By the preceding theorem there exists only a finite number of pairs of points $\{a_i, b_i\}$ such that $a_i \in U'$, $b_i \in V'$ and a_i is contiguous to b_i. Since A and B are separated, if $a_i \in A$ then $b_i \notin B$. Let A' be the union of the a_i which are not elements of A and B' be the union of the b_i which are not elements of B. Both these sets are closed. Let $U = U' - A'$ and $V = V' - B'$. These are open sets satisfying the conditions of the theorem.

Corollary. If A and B are closed separated subsets of a compact space, then there exist open sets U and V which are separated such that $A \subset U$ and $B \subset V$.

Chapter III

HOMOLOGY THEORY

The purpose of this section will be to set up a homology theory for a compace contiguous point space S. We consider the set Σ of all finite coverings $\mathcal{U}$ of the space S. With each $\mathcal{U}$, we associate a complex called the nerve of the covering in the following way.

1. To each $U_i \epsilon \mathcal{U}$ we associate a vertex.
2. Consider any set of $(n+1)$ different elements of the covering $\mathcal{U}$, say $U_0, U_1, \ldots, U_n$. If it is possible to choose a set of points $p_0, p_1, \ldots, p_n$ such that $p_i \epsilon U_i$ for $i = 0,1,\ldots n$ and either p_i is equal to p_j or p_i is contiguous to p_j for $i \neq j$, then we associate with this set of elements of the covering an n-cell σ^n. If U_{i_0}, $U_{i_1}, \ldots, U_{i_r}$ is a sub-collection of the collection $U_0, U_1, \ldots, U_n$, then $U_{i_0}, \ldots U_{i_r}$ will be associated with a face of σ^n.

If σ^n is the cell associated with the elements of the covering $U_0, U_1, \ldots, U_n$ then we will speak of the cell $\sigma^n = U_0, U_1, \ldots, U_n$. We will also use $\mathcal{U}$ to denote both the covering and the associated complex. The nerve of $\mathcal{U}$ is a closed complex as for any cell in the nerve of $\mathcal{U}$ all the faces also belong to the nerve of $\mathcal{U}$.

The set of points $\{p_i\}$ are called a nucleus of the cell. If it is possible to choose $p_i = p_0$ for all i, then we say that the cell has a <u>proper nucleus</u>, otherwise we say that the cell has a <u>non-proper nucleus</u>. A nucleus of a cell is not a unique set of points in general. In the case of a space with no contiguous points, a nucleus would be any point in the common intersection of the open sets making up the cell. It would be possible to consider the common intersection as the nucleus where it does exist but that would make it necessary to break our considerations into cases.

If the covering $\mathcal{V}$ is a refinement of the covering $\mathcal{U}$, a projection π of $\mathcal{V}$ into $\mathcal{U}$ will in the usual way mean a simplicial mapping of $\mathcal{V}$ into $\mathcal{U}$ such that if $V \epsilon \mathcal{V}$, then $\pi_{\mathcal{V}}^{\mathcal{U}} V = U$ where U is some element of $\mathcal{U}$ which contains V. If $\sigma^n = V_0, \ldots V_n$ is a cell of $\mathcal{V}$ with nucleus $p_0 \ldots, p_n$ then the same points will be a nucleus of $\pi_{\mathcal{V}}^{\mathcal{U}} \sigma^n$.

The concepts of chain, "cycle", etc. are defined on the nerve in the usual way using coefficients from a field $\mathcal{F}$.

<u>Theorem 1.</u> <u>If $\mathcal{V} > \mathcal{U}$ and $Z^n(\mathcal{V})$ is a cycle of $\mathcal{V}$, then for every pair of projections π_1, and π_2 of $\mathcal{V}$ into $\mathcal{U}$, $\pi_1 Z^n(\mathcal{V}) \sim \pi_2 Z^n(\mathcal{V})$.</u> [5]

Proof: The proof of this theorem is the same as in the general theory but it is included here as it enables us to bring out a fact which will be used later.

Let the elements of $\mathcal{V}$ be ordered $V_1, V_2, \ldots V_k$, and let $\pi_1 V$, $\pi_2 V$ be denoted by $U^{(1)}$ and $U^{(2)}$ respectively. If $\sigma^n = V_{i_0}, V_{i_1}, \ldots V_{i_n}$ is a cell of $\mathcal{V}$, let $P(\sigma^n)$ be the chain of $\mathcal{U}$ defined by

$$P(\sigma^n) = P(V_{i_0}, V_{i_1}, \ldots, V_{i_n}) = \sum_j (-1)^j U^{(1)}_{i_0}, \ldots, U^{(1)}_{i_j}, U^{(2)}_{i_j}, \ldots, U^{(2)}_{i_n}.$$

If P is extended linearly to arbitrary chains of $\mathcal{V}$ there is obtained a chain mapping of $C^n(\mathcal{V})$ into $C^{n+1}(\mathcal{U})$. This is a homomorphism such that

$$\partial P(C^n) = \pi_2 C^n - \pi_1 C^n - P(\partial C^n).$$

The proof of this relation is obtained by direct calculation. Consequently if $Z^n(\mathcal{V})$ is a cycle of $\mathcal{V}$,

$$\partial P(Z^n) = \pi_2 Z^n - \pi_1 Z^n.$$

Hence $\pi_2 Z^n(\mathcal{V}) \sim \pi_1 Z^n(\mathcal{V})$ on $\mathcal{U}$.

<u>Remark</u>: We notice that if $p_0, \ldots, p_n$ is a nucleus of the cell σ^n, then the cell $U^{(1)}_{i_0}, \ldots, U^{(1)}_{i_j}, U^{(2)}_{i_j}, \ldots, U^{(2)}_{i_n}$ in $P(\sigma^n)$ will have a nucleus $p_0 \ldots, p_j, p_j, \ldots, p_n$.

Analogous to the chain mapping P given above we have associated with any simplicial mapping f of the complex $\mathcal{U}$ into the complex $\mathcal{U}'$ a chain mapping called the deformation mapping $\mathcal{D}$ of the complex $\mathcal{U}$ into the complex $\mathcal{U} \cup \mathcal{U}'$ given by

$$\mathcal{D}(\sigma^n) = \mathcal{D}(u_0 \ldots u_n) = \sum_j (-1)^j U_0 \ldots, U_j\, f(U_j), \ldots, f(U_n)$$

This mapping obeys the analogous relation

$$\partial \mathcal{D}(C^n) = f(C^n) - C^n - \mathcal{D}\partial(C^n).$$

An n-dimensional Čech-cycle of the space S is defined in the usual way as a collection $\{Z^n(\mathcal{U})\}$ where for each $\mathcal{U} \in \Sigma$, $Z^n(\mathcal{U})$ is a cycle and if $\mathcal{V} > \mathcal{U}$, then

$\pi_v^{\mathcal{U}} Z^n(\mathcal{V}) \sim Z^n(\mathcal{U})$ on $\mathcal{U}$. By theorem 1, this homology is independent of the particular choice of projection.

The definition of addition of Cech-cycles and of bounding cycles is the usual one and we obtain the n-dimensional homology group, $H^n(S, \mathcal{F})$ of the space S over the field $\mathcal{F}$.

When we come to the definition of a chain being on or in a set we find that some change is necessary due to the way we have defined a nucleus and the presence of the non-proper nuclei. We find it most convenient to define two types of "on" and two types of "in".

<u>Definition of "on"</u>: A cell σ^n of a covering $\mathcal{U}$ of the space S will be said to be on the set M, if there exists a nucleus $p_0, \ldots, p_n$ of σ^n such that $p_i \in M$ for all i.

<u>Definition of "contiguously on"</u>: A cell σ^n of a covering $\mathcal{U}$ of the space S will be said to be contiguous on the set M, if there exists a nucleus $p_0, \ldots, p_n$ of σ^n such that either $p_i \in M$ or p_i is contiguous to a point $q_i \in M$ for each i.

<u>Definition of "in"</u>: A cell σ^n of a covering $\mathcal{U}$ of the space S will be said to be in the set M, if for every choice of nucleus $p_0, \ldots, p_n$ the points p_i are elements of M.

<u>Definition of "contiguously in"</u>: A cell σ^n of a covering of the space S will be said to be contiguously in the set M, if for any choice of nucleus $p_0, \ldots, p_n$ at least

one $p_i \in M$.

This last definition is equivalent to the statement. A cell σ of a covering $\mathcal{U}$ of the space S will be said to be contiguously in the set M, if it does not lie on the set $S - M$.

In definitions and theorems usually the notions of on and in, in the general theory, are here replaced respectively by the notions of "on" and "contiguously in". However the other two notions are also used. It is usually easy to pass from a theorem stated in terms of "on" to one stated in terms of "contiguously on" and from one in terms of "in" to one in terms of "contiguously in" and conversely. We shall see some examples of this later. It might be noted that a cell may be both contiguously on the set M and at the same time in the set $S - M$.

<u>Definition:</u> A set Σ' of coverings of S, will be called a <u>complete family of coverings</u> of S if for every $\mathcal{U} \in \Sigma$ there exists a covering $\mathcal{V} \in \Sigma'$ such that $\mathcal{V} > \mathcal{U}$.

The natural isomorphism between $H^{\sim}(S)$ and the n-dimensional homology group determined by a complete family of coverings is obtained in the same way as in the general theory. The following theorem is also identically the same as in the general theory.

<u>Theorem 2. The existence theorem.</u> [5]

<u>Let the flat $F(\mathcal{U})$ be a coset in the vector space $Z^{\sim}(\mathcal{U})$</u>

of cycles of $\mathcal{U}$ modulo a closed subspace E of S, modulo some subspace of $Z^n(\mathcal{U})$, such that if $Z \in Z^n(\mathcal{U})$ and $Z \sim F \in F(\mathcal{U})$ mod E, then $Z \in F(\mathcal{U})$. If for each $\mathcal{U} \in \Sigma$ we have a non-empty flat $F(\mathcal{U})$ such that if $\mathcal{V} > \mathcal{U}$, $\pi^{\mathcal{U}}_{\mathcal{V}} F(\mathcal{V}) \subset F(\mathcal{U})$, then there exists a C-cycle mod E, $\{Z^n \mathcal{U}\}$ such that for each $\mathcal{U} \in \Sigma$, $Z^n(\mathcal{U}) \in F(\mathcal{U})$.

Definition of dimension: A covering $\mathcal{U}$ of a space S is said to be n-dimensional if n is the dimension of the nerve of $\mathcal{U}$. A compact space S is said to be n-dimensional if n is the smallest integer such that every covering has a refinement of dimension n.

The set of all n-dimensional coverings form a complete system of coverings of an n-dimensional space.

We shall now consider some special refinements of a covering which are useful when working with cycles.

Theorem 3. Closure Refinement.

If $U_1, U_2, \ldots, U_m$ is a covering of S, then each U_i contains a set U_i' such that $U_1', U_2', \ldots, U_m'$ is a covering of S and each U_i' is closure contained in U_i, and, if x is a proper interior point of some U_i then x is a proper interior point of some U_j'.

Proof: The proof is by induction. First consider the set $B_1 = \overline{U}_1 - U_1$. By theorem 4 chapter II, each $x \in B_1$ is contained in an open set V_x which lies in every U_i that contains x and there exists no set of points p, q, r, all distinct from x, such that $p \in V_x$, $r \in S - U_i$ and where

$x \in U_i$, p is contiguous to q, q is contiguous to r. Let $V_1 = \bigcup_{x \in B_1} V_x$ and $U_1'' = U_1 - V_1$. The set U_1'' is a closed subset of U_1. Take an open set U_1' such that $U_1 \gg U_1' \supset U_1''$.

The sets $U_1', U_2, \ldots, U_m$ cover S. If y is a proper interior point of the set U_1, then $y^c \subset U_1$. If y^c is contained in some other U_i, then the new covering has y as a proper interior point of one of its elements. If y^c is not contained in any other U_i, then for every $x \in B_1$, V_x does not intersect y^c and y^c is still contained in $U_1' \ll U_1$.

Suppose $U_1', U_2', \ldots, U_{k-1}'$ for $k \leq m$ have been defined so that $\overline{U}_i \subset U_i$ for $i = 1, \ldots k-1$ and such that the sets $U_1', U_2', \ldots, U_{k-1}', U_k, \ldots U_m$ form a covering $\mathcal{U}_{k-1}$ of S. To obtain U_k' we let $B_k = \overline{U}_k - U_k$ and for each $x \in B_k$ let V_x be an open set containing x and lying in every element of $\mathcal{U}_{k-1}$ that contains x and such that no chain of points p,q,r all distinct from x similar to that given before. Let $U_k'' = U_k - \bigcup_{x \in B_k} V_x$ and U_k' be an open set such that $U_k \gg U_k' \supset U_k''$. If we continue in this way we see that the collection $U_1', U_2', \ldots, U_m'$ is the desired covering of the space. This covering is called a closure refinement of the original covering.

<u>Definition</u>: If M is a subset of S and $\mathcal{U}$ is a covering of S, then the <u>star</u> of M with respect to $\mathcal{U}$, $ST(M, \mathcal{U})$ is the union of all elements of $\mathcal{U}$ that meet M. The

The <u>contiguous star</u> of M with respect to $\mathcal{U}$, $ST^c(M,\mathcal{U})$ is the union of all elements of $\mathcal{U}$ that either meet M or contain a point contiguous to a point of M.

<u>Definition</u>: If $\mathcal{U}$ and $\mathcal{V}$ are coverings of S, then by $ST^c(\mathcal{V},\mathcal{U})$ we mean the covering whose elements are the sets $ST^c(V,\mathcal{U})$ for each $V\in\mathcal{V}$.

In the case of a contiguous point space a covering may not have a refinement analogous to the star refinement. This can be easily seen in the case of example 6 chapter I if we take the covering which consists of the points themselves as open sets. However we can get the following covering which will be called a star refinement.

<u>Theorem 4.</u> <u>Star Refinement.</u>

<u>If $\mathcal{U}$ is a covering of S, then there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that each $ST^c(V,\mathcal{V})$ for $V\in\mathcal{V}$, is contained in one element U of $\mathcal{U}$ unless V contains a point which is a proper interior point of no element of the covering $\mathcal{U}$. In this case the elements of $\mathcal{V}$ which contain points contiguous to this point may lie outside U.</u>

Proof: Using the symbols employed in the proof of the closure refinement, denote by V_x, $x\in S$ an open set such that for each i,

1. If $x\in U_i'$, then V_x is contained in U_i' and V_x contains no points contiguous to $S-U_i'$ except possibly x itself (theorem 4 chapter II)

2. If $x\in U_i$, then V_x is contained in U_i and V_x contains

no points contiguous to $S - U_i$ except possibly x itself.

3. If $x \in S - U_i$, then V_x is contained in $S - \overline{U_i'}$ and contains no points contiguous to points of $\overline{U_i'}$ except possibly x itself.

Let $x_1, \ldots, x_k \in S$ such that the sets $V_{x_1}, \ldots V_{x_k}$ form a covering of S. This is the desired covering.

To see this consider V_{x_1}. If x_1 is a proper interior point of some element of $\mathcal{U}$, then it is a proper interior point of some U_i'. Hence by 1. $V_{x_1} \subset U_i' \subset U_i$. Consider any other element of the covering V_{x_j}. If x_j is contained in $S - U_i$, then $V_{x_j} \cap \overline{U_i'} = 0$ and hence V_{x_j} does not intersect V_{x_1}. By 1 and 3 the sets V_{x_1} and V_{x_j} are contiguously separated. Therefore if V_{x_j} either intersects or has a contiguity with V_{x_1}, the point x_j is an element of U_i, and V_{x_j} lies in U_i. If x_1 is not a proper interior point of any element of the covering $\mathcal{U}$, then it is easy to see that the only elements V_{x_j} which have a contiguity with V_{x_1} and do not lie in U_i are the ones where x_j is contiguous to x_1. This completes the proof of the theorem.

Before we leave the star refinement we want to consider in more detail exactly what we have. If $ST^c(V_{x_1}, \mathcal{V})$ is not contained in U_i and in fact V_{x_2} is one of the cells in the contiguous star which is not contained in U_i, then x_2 is an element of $S - U_i$.

By 1. and 3 we see that V_{x_1} and V_{x_2} have no other contiguity across them except the contiguity x_1 to x_2. If x_3 is contiguous to both x_1 and x_2, then the only element of the covering which contains x_3 is V_{x_3}. To see this suppose $x_3 \in V_x$.

1. If $x \in U_i'$, then V_x contains no points contiguous to points of $S - U_i'$ except x. As it must contain x_3 which is contiguous to x_2, the point x_3 must equal x.
2. If $x \in U_i$, the same reasoning holds.
3. If $x \in S - U_i$, since x_3 is contiguous to $x_1 \in U_i'$, we must have $x = x_3$.

It may also be noticed that for any $V \in \mathcal{V}$, the set $ST(V, \mathcal{V})$ is contained in some element of $\mathcal{U}$.

Theorem 5. <u>If F is a compact subset of a locally compact space S, and $\mathcal{U}$ is a covering of S, there exists $\mathcal{V} > \mathcal{U}$ and an open set Q containing F such that only a finite number of elements of $\mathcal{V}$ lie on F and if the nucleus of a cell lies on Q, then it lies on F.</u>

Proof: Since S is locally compact and F is compact, there exists an open set P containing F such that $\overline{P}$ is compact. Consider the elements of $\mathcal{U}$ that meet $\overline{P}$ and reduce to a finite covering $\mathcal{U}'$ of $\overline{P}$.

For every cell $\tau^* = U_0', U_1', \ldots, U_n'$ let $A = \bigcap_j U_j'$. If A intersects F, take a point $p_i^n \in A \cap F$, otherwise take $p_i^n \in A$. Let $M = \bigcup_{i,r} p_i^n$.

Build up a new covering of $\overline{F}$ as follows

1. $U_r' \supset U_r''$.

2. $U_r'' \supset M \cap U_r'$.

3. $\mathcal{U}'' = \{U_r''\}$ is a covering of $\overline{F}$.

This is like the closure refinement, theorem 3 and is obtained in the same way.

Take the covering $\mathcal{V} = \mathcal{U}'' \cup \mathcal{U}'''$ where $\mathcal{U}'''$ consists of the open sets $(S - \overline{F}) \cap U$ for all $U \in \mathcal{U}$. This is a covering of S.

Consider any cell $\sigma_i^\nu = U_0'', U_1'', \ldots, U_\nu''$. Let $A' = \bigcap_j U_j''$ and $A = \bigcap_j U_j'$ where U_j' corresponds to U_j'' in the refinement. It is easily seen that if $A \cap F \neq 0$, then the set $A' \cap F \neq 0$.

We shall show that either σ_i^ν lies on F or there exists an open set Q_i^ν containing F such that σ_i^ν does not lie on Q_i^ν.

Suppose σ_i^ν does not lie on F. Then for any choice of nuclei points $p_0, p_1, \ldots, p_\nu$, one at least lies off F. There may exist many possible choices of nuclei points. Consider the set $\{p_0'\}$ of points p_0 which lie off F. If this set $\{p_0'\}$ does not have a limit point on F, then there exists an open set Q_0 containing F which does not intersect the set $\{p_0'\}$. If the set $\{p_0'\}$ has a limit point p_0 on F, then by theorem 3 chapter II, the set $\{p_j'\}$ of points p_j which are contiguous to points of $\{p_0'\}$ have the same limit point p_0. Hence $p_0 \in \overline{U_j''} \subset U_j' \cap F$ for all j. Hence the cell $\sigma_i^{\sim}$ lies on F. Since this is a contradiction, the set Q_o exists. There exist similar open set Q_j for each j. Let $Q_i^{\sim} = \bigcap_j Q_j$. The cell $\sigma_i^{\sim}$ does not lie on $Q_i^{\sim}$.

For every cell $\sigma_i^{\sim}$ there exists an open set $Q_i^{\sim}$ containing F such that if the nucleus of $\sigma_i^{\sim}$ lies on $Q_i^{\sim}$, then the cell lies on F. As there exists only a finite number of such cells, there is an open set $Q = \bigcap_{i,\sim} Q_i^{\sim}$ such that if the nucleus of any cell lies on Q, then the cell lies on F. This proves the theorem.

The next theorem is analogous to theorem 5 but concerns cells contiguously on a set instead of on a set.

Theorem 6. If F is a compact subset of a locally compact space S and $\mathcal{U}$ is a covering of S, there exists $\mathcal{V} > \mathcal{U}$ and Q an open set containing F such that only a finite number of elements of $\mathcal{V}$ lie contiguously on F and if the nucleus of a cell lies contiguously on Q, then it lies contiguously on F.

Proof: Let $B = F^c$. By theorem 1 chapter II, the set B is closed. Hence by theorem 5, there exists a covering $\mathcal{V} > \mathcal{U}$ and an open set Q' containing B such that if the nucleus of a cell lies on Q', then it lies on B. The sets $S - Q'$ and F are contiguously separated closed set, hence by theorem 8 chapter II, there exists an open set Q containing F such that Q and $S - Q'$ are contiguously separated.

If a cell $\tilde{\sigma}$ of $\mathcal{V}$ lies contiguously on Q, then there exists a nucleus $p_0, \ldots, p_N$ such that either $p_i \in Q$ or p_i is contiguous to a point $y_i \in Q$. In either case $p_i \in Q'$. Therefore the cell lies on Q' and hence on B. However if a cell lies on B, it lies contiguously on F.

<u>Theorem 7. If L is a closed subset of S compact and $\mathcal{U}$ is a covering of S, then there exists a covering $\mathcal{W} > \mathcal{U}$ such that if the nucleus of a cell of $\mathcal{W}$ lies on both L and S - L, then it lies on F(L).</u>

Proof: By theorem 2 chapter II, the set F(L) is closed. By theorem 5 there exists an open set Q containing F(L) and a covering $\mathcal{V} > \mathcal{U}$ such that if a nucleus of a cell lies on Q, then the cell lies on F(L).

If $V_i \in \mathcal{V}$, let

$$V_{i1} = V_i \cap (L \cup Q)$$

$$V_{i2} = V_i \cap (S - L)$$

Let $\mathcal{W}$ be the covering made up of the set V_{ij} given above.

Let $\tilde{\sigma} = W_0, \ldots, W_N$ be a cell of $\mathcal{W}$ which lies on both L and S - L. Then there exists a nucleus $p_0, \ldots, p_N$ such that $p_i \in L$. Since $V_{i2} \cap L = 0$, the set W_i = some V_{j1} for all i. Also $\tilde{\sigma}$ is on S - L, there exists a nucleus $q_0, \ldots, q_N$ such that $q_i \in S - L$. The points q_i must be elements of Q and therefore $\tilde{\sigma}$ has a nucleus lying on Q. This proves that the cell $\tilde{\sigma}$ lies on F(L).

This theorem shows that the set of coverings where if a cell lies on L and on S - L, then it lies on F(L), is a complete system of coverings of the space.

We shall now prove two theorems which are useful when we desire to work with cycles.

<u>Theorem 8. If $\gamma^{\sim}$ is a cycle mod K on M for two closed</u> K and M, then the collection $\{\partial\gamma^{\sim}(u)\}$ is an (r - 1) cycle on K which we denote by $\partial\gamma^{\sim}_{,}(r>0)$. Evidently $\partial\gamma^{\sim}\sim 0$ on M. [5]

Proof: That $\gamma^{\sim}$ is a cycle mod K implies not only that $\partial\gamma^{\sim}(u)$ is on K, but also that for every pair of coverings u and v such that $v>u$ there exists a chain $C^{\sim+1}(u)$ such that

(1) $\partial C^{\sim+1}(u) = \gamma^{\sim}(u) - \pi^{u}_{v}\gamma^{\sim}(v) - A^{\sim}(u)$

where $A^{\sim}(u)$ is on K. Applying the operator ∂ to relation (1) and transposing, we get

$$\partial\gamma^{\sim}(u) - \pi^{u}_{v}\partial\gamma^{\sim}(v) = \partial A^{\sim}(u).$$

Hence $\{\partial\gamma^{\sim}(u)\}$ is a cycle on K which we denote by $\partial\gamma^{\sim}$.

<u>Definition:</u> If $C^{\sim}(u)$ is a chain and L is any set, then by $C^{\sim}(u)\wedge L$ we denote the portion of $C^{\sim}(u)$ which lies on L.

If $v>u$, and $C^{\sim+1}(v)$ is a chain of v, we can immediately conclude the relation

(2) $\pi^{u}_{v}[C^{\sim+1}(v)\wedge L] \subset [\pi^{u}_{v}C^{\sim+1}(v)]\wedge L.$

<u>Theorem 9. Let K and L be disjoint closed subsets of S,</u>

<u>and suppose that γ^n is a cycle on A which bounds on S. Then there exists a cycle Z^n on F(L) such that $\gamma^n \sim Z^n$ on $\overline{S - L}$.</u> [5]

Proof: If $v > u$, then

(3) $\pi_v^u \gamma^n(v) \sim \gamma^n(u)$ on K.

And by hypothesis, for each covering v there exists a relation

$$\partial C^{n+1}(v) = \gamma^n(v)$$

Consequently we have

(4) $\partial \pi_v^u C^{n+1}(v) = \pi_v^u \gamma^n(v)$

Relations (3) and (4) imply that $\partial \pi_v^u C^{n+1}(v) \in [\gamma^n(u)]^*$, where $[\gamma^n(u)]^*$ denotes the coset of $H^n(K, u, \mathcal{F})$ determined by $\gamma^n(u)$.

Let $C_\gamma^{n+1}(u, \mathcal{F})$ be the subspace of $C^{n+1}(u, \mathcal{F})$ consisting of those chains whose boundaries are on K. Let ϕ be the homomorphism of $Z^n(K, u, \mathcal{F})$ into $H^n(K, u, \mathcal{F})$ - the natural one mapping $B^n(K, u, \mathcal{F})$ into the element 0. Then, applying $\phi\partial$ to $C_\gamma^{n+1}(u, \mathcal{F})$ we get a homomorphism

$$\phi\partial : C_\gamma^{n+1}(u, \mathcal{F}) \rightarrow H^n(K, u, \mathcal{F}).$$

Evidently $[\gamma^n(u)]^*$ is the image of some coset of $C_\gamma^{n+1}(u, \mathcal{F})$ under this homomorphism, and we denote this coset by $\{C^{n+1}(u)\}$, since it contains the given chain $C^{n+1}(u)$.

We have, then, for each u, a flat $\{C^{n+1}(u)\}$ in the space $C_\gamma^{n+1}(u, \mathcal{F})$, such that for each $v > u$, $\pi_v^u C^{n+1}(v) \in \{C^{n+1}(u)\}$. Consequently by theorem 2, there exists a cycle γ^{n+1} mod K such that $\gamma^{n+1}(u) \in \{C^{n+1}(u)\}$ and therefore $\partial\gamma^{n+1}(u) \sim \gamma^{n}(u)$ on K. Moreover, by theorem 8, $\{\partial\gamma^{n+1}(u)\}$ constitutes a cycle on K, and for each u and $v > u$, there is a relation

(5) $\partial C^{n+2}(u) = \gamma^{n+1}(u) - \pi_v^u \gamma^{n+1}(v) - A^{n+1}(u)$

where $A^{n+1}(u)$ is on K.

From now on we shall restrict our coverings to the complete family of coverings which satisfy theorem 7. Also no element of the coverings meets both K and L.

From relation (5) we have

(6) $\partial[C^{n+2}(u) \wedge L] = \gamma^{n+1}(u) \wedge L - \pi_v^u[\gamma^{n+1}(v) \wedge L] - B^{n+1}(u)$

where $B^{n+1}(u)$ is a chain on L due to the restriction of $C^{n+2}(u)$ to L. Also the chain $B^{n+1}(u)$ lies on $\overline{S - L}$ and hence lies on F(L).

Applying the operator ∂ to this relation (6) we obtain the relation

$$\partial[\gamma^{n+1}(u) \wedge L] - \pi_v^u \partial[\gamma^{n+1}(v) \wedge L] = \partial B^{n+1}(u).$$

Hence $\partial[\gamma^{n+1}(u) \wedge L]$ is a cycle of L.

Since the chain $B^{n+1}(u)$ is on F(L) we have that

$\partial[\gamma^{n+1}(u) \wedge L] = Z^{n}$ is a cycle of F(L).

Finally, since for each u the chain $\gamma^{n+1}(u) - \gamma^{n+1}(u) \wedge L$ is on $\overline{S - L}$ and

$$\partial[\gamma^{n+1}(u) - \gamma^{n+1}(u) \wedge L] = \gamma^{n}(u) - Z^{n}(u),$$

we have that $\gamma^{n} \sim Z^{n}$ on $\overline{S - L}$.

Chapter IV

LOCAL CONNECTEDNESS

In this chapter we will discuss some local properties of a space, local connectedness, local coconnectedness, and local Betti numbers. These concepts will be used later in discussing the contiguous point manifold. The most important theorem of the section is the complex like character of a locally connected space. Difficulties arise in obtaining the star refinement and the realization theorem due to the presence of open sets containing points contiguous to the exterior of these sets. We shall also prove a generalization of the "wheel and spoke" lemma.

Definition: A space S is n-lc at $x \in S$ if given an open set P containing x and $\mathcal{U}$ a covering of S, there exists an open set Q such that $x \in Q \subset P$ and a covering $\mathcal{V} > \mathcal{U}$ such that if $Z^n(\mathcal{V})$ is a cycle lying on Q, then $\pi_{\mathcal{V}}^{\mathcal{U}} Z^n(\mathcal{V})$ is the boundary of a chain lying on P.

Theorem 1. If a regular space S is n-lc at $x \in S$, then given an open set P containing x and $\mathcal{U}$ a covering of S, there exists an open set R such that $x \in R \subset P$ and a covering $\mathcal{W} > \mathcal{U}$ such that if $Z^n(\mathcal{W})$ is a cycle lying contiguously on R, then $\pi_{\mathcal{W}}^{\mathcal{U}} Z^n(\mathcal{W}) \sim 0$ contiguously on P.

Proof: Given x, P and $\mathcal{U}$ take $Q_0 = Q$ and $\mathcal{V}$ from the definition of n-lc. Choose open sets Q_i for i= 1,2,3,4 such that $x \in Q_i \Subset Q_{i-1}$ and Q_i contains no point contiguous to a point of $S - Q_{i-1}$ except possibly x itself. This may be done by theorem 4 chapter II. Let $\mathcal{V}'$ be the covering $\{Q_3, Q_2 - \overline{Q}_4, Q_1 - \overline{Q}_3, Q_0 - \overline{Q}_2, S - \overline{Q}_1\}$ and let $\mathcal{W}$ be a refinement of $\mathcal{V}$ and $\mathcal{V}'$.

Let W_x be an open set of the covering $\mathcal{W}$ containing x. Consider any $Z^n(\mathcal{W})$ lying contiguously on Q_4. Set up a simplicial mapping of the complex $|Z^n(\mathcal{W})|$ as follows. With $W \in |Z^n(\mathcal{W})|$, if $W \cap (S - Q_0) \neq 0$ set $f(W) = W_x$, otherwise set $f(W) = W$.

To show that this is a simplicial we must show that if σ^k is a cell of $|Z^n(\mathcal{W})|$ then $f(\sigma^k)$ is a cell of the covering $\mathcal{V}$. Suppose σ^k is a cell of $|Z^n(\mathcal{W})|$; $\sigma^k = W_0, \ldots, W_k$ with nucleus $p_0, \ldots, p_k$ where $p_i \in W_i$ and either $p_i \in Q_4$ or p_i is contiguous to $y_i \in Q_4$. Suppose W_0 is not mapped identically, $W_0 \to W_x$. Then $W_0 \subset S - \overline{Q}_1$. Hence p_0 is contiguous to x as this is the only point in Q_4 which could be contiguous to a point of $S - \overline{Q}_1$. Consider W_i and suppose $W_i \to W_i$. The set W_i has a contiguity with $W_0 \subset S - \overline{Q}_1$.

1. If $W_i \subset Q_0 - \overline{Q}_2$, then the only point in Q_4 contiguous to such a point p_i is x; hence p_i is contiguous to x.

2. If $W_i \in Q_1 - \overline{Q}_3$, then again p_i is contiguous to x.

3. If $W \subset Q_2 - \overline{Q}_4$, we derive a contradiction as p_i is contiguous to $p_0 \in S - \overline{Q}_1$. The only point in Q_2 which

is contiguous to a point in $S - \overline{Q_1}$ is x and this is not a point of $Q_2 - \overline{Q_4}$.

4. If $W_i < Q_3$, then p_i must equal x.

In any case the points $p_0, \ldots, p_K$ with p_i replaced by x for each i such that $W_i \to W_2$ may be taken as a nucleus of $f(\sigma^K)$. This proves that the mapping f is a simplicial mapping.

Set up the deformation chains associated with the mapping f. The cells on these chains are on the covering $\mathcal{W}$ as the mapped cells either have nuclei points x or points contiguous to x and adding another open set containing x still gives a cell lying on P.

Since the mapping is simplicial $f(Z^n(\mathcal{W})) = Z_1^n(\mathcal{W})$ is also a cycle lying on Q_0. From the properties of the deformation chain we have

$$\partial \mathcal{D} Z^n(\mathcal{W}) = Z_1^n(\mathcal{W}) - Z^n(\mathcal{W}) .$$

The chain $\mathcal{D}Z^n(\mathcal{W})$ lies contiguously on x.

By the assumption of n-lc, there exists a chain $C^{n+1}(\mathcal{U})$ which lies on P such that

$$\partial C^{n+1}(\mathcal{U}) = \pi_{\mathcal{W}}^{\mathcal{U}} Z_1^n(\mathcal{W}).$$

Combining these statements we obtain

$$\partial\left[-\pi_{\mathcal{W}}^{\mathcal{U}} \mathcal{D} Z^n(\mathcal{W}) - C^{n+1}(\mathcal{U})\right] = \pi_{\mathcal{W}}^{\mathcal{U}} Z^n(\mathcal{W}).$$

Hence $\pi_{\mathcal{W}}^{\mathcal{U}} Z^n(\mathcal{W}) \sim 0$ contiguously on P. This proves the theorem.

Remark. Actually the chain which $\pi_{\mathcal{W}}^{\mathcal{U}} Z^n(\mathcal{W})$ bounds lies on P except for some cells which lie contiguously on x itself.

Notation. The symbol lc^n will be used to indicate that a space is locally connected in all dimensions from 0 up to n.

In using Cech cycles we need a type of uniform local connectedness. This is defined as follows.

Definition: A space is called n-ulc if given two coverings $\mathcal{U}$, and $\mathcal{V}$ of S, there exists two coverings $\mathcal{U}_n'(\mathcal{U})$ and $\mathcal{V}_n'(\mathcal{U}, \mathcal{V})$ such that if $Z^n(\mathcal{V}_n')$ is a cycle lying on one element of $\mathcal{U}_n'(\mathcal{U})$, then $\pi^{\mathcal{V}}_{\mathcal{V}_n'} Z^n(\mathcal{V}_n')$ bounds a chain of $\mathcal{V}$ lying on one element of $\mathcal{U}$.

Remark. The notation $\mathcal{V}_n'(\mathcal{U}, \mathcal{V})$ used above implies not only that the covering $\mathcal{V}_n'$ depends on the coverings $\mathcal{U}$ and $\mathcal{V}$ but also that $\mathcal{V}_n'$ is a refinement of both $\mathcal{U}$ and $\mathcal{V}$.

Theorem 2. If the compact space S is n-lc at every point, then S is n-ulc. [5]

Proof: Let $\mathcal{U}$, $\mathcal{V}$ be given. For each $x \in S$ select $U(x) \in \mathcal{U}$ such that $x \in U(X)$. Then with $U(x)$ as the P of the definition of n-lc and $\mathcal{V}$ in place of its $\mathcal{U}$, there exists by the n-lc assumption an open set $Q(x)$ such that $x \in Q(x) \subset U(x)$ and a covering $\mathcal{V}(x)$ such that if Z^n is a cycle of $\mathcal{V}(x)$ on $Q(x)$, then $\pi^{\mathcal{V}}_{\mathcal{V}(x)} Z^n \sim 0$ on $\mathcal{V} \wedge U(x)$.

Let $x_1, \ldots, x_i, \ldots, x_m \in S$ such that the set $\mathcal{U}'$ consisting of the sets $Q(x_i)$ forms a covering of S. Let $\mathcal{V}' > \{V(x_1), \ldots, V(x_i), \ldots, V(x_m)\}$. Then consider any $z^n(\mathcal{V}')$ of diameter $<\mathcal{U}'$. The cycle $z^n(\mathcal{V}')$ is on some Q, say $Q(x_i)$. Then $\pi_{\mathcal{V}'}^{\mathcal{V}(x_i)} z^n(\mathcal{V}')$ is also on $Q(x_i)$ and therefore $\pi_{\mathcal{V}(x_i)}^{\mathcal{V}} \pi_{\mathcal{V}'}^{\mathcal{V}(x_i)} z^n(\mathcal{V}') \sim 0$ on $\mathcal{V} \cap U(x_i)$. Hence $\pi_{\mathcal{V}'}^{\mathcal{V}} z^n(\mathcal{V}')$ bounds a chain on $\mathcal{V}$ of diameter $<\mathcal{U}$.

Both the definitions of n-lc and n-ulc have equivalent alternative definitions. They are obtained by replacing "$\pi_{\mathcal{V}_n'}^{\mathcal{V}} Z^n(\mathcal{V}_n')$ bounds a chain on $\mathcal{V}$" by "$Z^n(\mathcal{V}_n')$ bounds a chain on $(\mathcal{V}_n' \cup \mathcal{V})$.

The notion of realization and partial realization are used here in much the same way as in the ordinary theory, where the norm is defined as follows.

<u>Definition</u>: A realization τ of a complex K on S will be said to be of <u>norm</u> $<\mathcal{U}$ if $\tau(\sigma^K)$ lies on one element of $\mathcal{U}$ for every cell σ^K of K. Also the norm of a partial realization τ' of K on S will be of norm $<\mathcal{U}$ if for every cell σ^K of K, all the faces of σ^K which are mapped by τ' are realized on one element of $\mathcal{U}$.

<u>Theorem 3. The Chain Realization Theorem.</u>

<u>If a compact space S is lc^n, then for every pair of coverings $\mathcal{U}$ and $\mathcal{V}$ there exist coverings $\mathcal{U}_n^*(\mathcal{U})$ and $\mathcal{V}_n^*(\mathcal{U}, \mathcal{V})$ such that if K is a complex of dimension $\leq n+1$, and τ' is a partial realization of K on $\mathcal{V}_n^*$ of norm $<\mathcal{U}_n^*$, then τ' can be extended to a realization τ of K on $\mathcal{V}_n^* \cup \mathcal{V}$ of norm $<\mathcal{U}$.</u>

Proof: Given the coverings $\mathcal{U}$ and $\mathcal{V}$. Take consecutively the coverings

$$\begin{array}{ll} \mathcal{U}_n \overset{*}{>} (\mathcal{U}_n'(\mathcal{U}) & \mathcal{V}_n > \{\mathcal{V}_n'(\mathcal{U},\mathcal{V}), \mathcal{U}_n\} \\ \mathcal{U}_{n-1} \overset{*}{>} \mathcal{U}_{n-1}'(\mathcal{U}_n) & \mathcal{V}_{n-1} > \{\mathcal{V}_{n-1}'(\mathcal{U}_n, \mathcal{V}_n), \mathcal{U}_{n-1}\} \\ \vdots & \vdots \\ \mathcal{U}_0 \overset{*}{>} \mathcal{U}_0'(\mathcal{U}_1) & \mathcal{V}_0 > \{\mathcal{V}_0'(\mathcal{U}_1, \mathcal{V}_1), \mathcal{U}_0\} \end{array}$$

where these are the coverings given by the ulc^n theorem (theorem 2) and theorem 4 chapter III. Let $\mathcal{U}_n^*(\mathcal{U}) = \mathcal{U}_0$ and $\mathcal{V}_n^*(\mathcal{U}, \mathcal{V}) = \mathcal{V}_0$.

To show that these coverings satisfy the theorem, let τ' be a partial realization of K, a complex of dimension $\leq$ n +1 on $\mathcal{V}_n^*$ of norm $< \mathcal{U}_n^*$.

For any $U_0 \in \mathcal{U}_0$, there exists a sequence of open sets associated with U_0.

(1) $U_0 \subset ST(U_0, \mathcal{U}_0) \subset U_0' \subset U_1 \subset ST(U_1, \mathcal{U}_1) \ldots \subset U_{K-1}' \subset U_K \ldots \subset U$

where $U_i \in \mathcal{U}_i$, $U_i' \in \mathcal{U}_i'$, for all i = 0 ,...,n, $U \in \mathcal{U}$ and such that any i-cycle on U_i' bounds a chain on U_{i+1}.

If σ^K is realized by τ', let $\tau(\sigma^K) = \tau'(\sigma^K)$. Since τ' is defined on all the zero cells so is τ.

We shall realize all the non-realized cells of K inductively as follows. For any cell σ^K take some $U_0 \in \mathcal{U}_0$ such that all the vertices of σ^K are realized on U_0. We will then realize σ^K on U_K which is an element of the sequence (1) associated with U_0.

We will first show that this is possible for k = 1. Take a non-realized 1-cell σ' of K. Take some U_0 on which the chain $\partial\sigma'$ is realized. The chain will then be realized on U_0' of the sequence (1) associated with U_0. Hence $\tau\partial\sigma'$ is a zero cycle of $\mathcal{V}_0$ lying on U_0' and therefore $\tau\partial\sigma'$ bounds a chain C' of $\mathcal{V}_0 \cup \mathcal{V}_1$ lying on U_1 of the sequence (1) associated with U_0. Set $\tau\sigma' = C'$. This shows that it is possible to realize all 1-cells in the desired way.

Suppose we have realized all i-cells in this way where $i \leq k - 1$. Consider σ^k a non-realized k-cell of K. Pick any $U_0 \in \mathcal{U}_0$ such that all the vertices of σ^k are realized on U_0. Take σ^{k-1} any face of σ^k. By the induction we have realized σ^{k-1} by first taking some $U_0^{(1)}$ on which all the vertices of σ^{k-1} are realized and then σ^{k-1} was realized on $\mathcal{U}_{k-1}^{(1)}$, an element of the sequence (1) associated with $U_0^{(1)}$. We now have two sequences, one associated with U_0 and the other with $U_0^{(1)}$.

$$U_0 \subset ST(U_0, \mathcal{U}_0) < U_0' \;\ldots\; \subset U_{k-1} < U_{k-1}'$$
$$U_0^{(1)} \quad ST(U_0^{(1)}, \mathcal{U}_0) < U_0'^{(1)} \;\ldots\; U_{k-1}^{(1)} \quad U_{k-1}'^{(1)}$$

Since $\mathcal{V}_0 > \mathcal{U}_0$ it is easily seen that if all the vertices of σ^{k-1} are realized on both U_0 and $U_0^{(1)}$, then the corresponding U_1 and $U_1^{(1)}$ intersect and hence the corresponding U_{k-1} and $U_{k-1}^{(1)}$ intersect. Every face of σ^k is realized on an element of $\mathcal{U}_{k-1}$ which intersects U_{k-1}; hence $\partial\sigma^k$ is realized on U_{k-1}'. The chain $\tau\partial\sigma^k$ is a k - 1 cycle of $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_{k-1}$ lying on U_{k-1}', and therefore there exists a chain C^k of $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_{k-1} \cup \mathcal{V}_k$.

lying on U_κ such that

$$\partial C^\kappa = \tau \partial \sigma^\kappa.$$

Set $\tau\sigma^\kappa = C^\kappa$.

This proves that all the k-cells can be realized, in the desired way. By induction we see that τ' can be extended to a realization τ of K on the covering $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_n \cup \mathcal{V}$ of norm $< \mathcal{U}$. To finish the theorem we may project all chain lying on the covering $\mathcal{V}_1, \cdots \mathcal{V}_n$ up to $\mathcal{V}$.

We shall now prove the complex like character of a lc^n space.

<u>Theorem 4. If the compact space S is lc^n, then there exists a complex K such that the homology groups $H^r(S, \mathcal{F})$ for $r \le n$ are isomorphic with subgroups of the corresponding groups $H^r(K, \mathcal{F})$.</u>

Proof: Let $\mathcal{U}$ be a covering of S and let $\mathcal{U}_0 >^* \mathcal{U}_n^*(\mathcal{U})$ as defined in theorem 3. The covering $\mathcal{U}_0$ is given by the set $V_{x_1}, V_{x_2}, \ldots V_{x_k}$ (see theorem 4 chapter III). Modify $\mathcal{U}_0$ slightly so that only one element of the covering contains x_i for each $i = 1, \ldots k$. We will prove that the nerve of the covering $\mathcal{U}_0$ satisfies the conditions of the theorem.

For each C-cycle Z^r, let $\phi(Z^r) = Z^r(\mathcal{U}_0)$. Then ϕ is a homomorphism of $H^r(S, \mathcal{F})$ into $H^r(\mathcal{U}_0, \mathcal{F})$. We shall show that if the cycle $Z^r(\mathcal{U}_0) \sim 0$ on $\mathcal{U}_0$, then $Z^r(\mathcal{V}) \sim 0$ for every refinement $\mathcal{V}$ of $\mathcal{U}_0$. Since the refinements of $\mathcal{U}_0$ form a complete system of coverings of the space, it will follow that $Z^n \sim 0$ and that ϕ is an isomorphism into a subgroup of $H^n(\mathcal{U}_0, \mathcal{F})$.

Suppose $Z^n(\mathcal{U}_0) \sim 0$ on $\mathcal{U}_0$, where Z^n is a C-cycle. For any $\mathcal{V} > \mathcal{U}_0$ let $\mathcal{V}_n^*(\mathcal{U}, \mathcal{V})$ be the covering defined in the chain realization theorem, theorem 3. Associated with the projections $\pi_{\mathcal{V}_n^*}^{\mathcal{U}_0}$ are the homorphisms $\mathcal{D}$ into $\mathcal{D}\mathcal{V}_n^*$, such that

$$\partial \mathcal{D} Z^n(\mathcal{V}_n^*) = \pi_{\mathcal{V}_n^*}^{\mathcal{U}_0} Z^n(\mathcal{V}_n^*) - Z^n(\mathcal{V}_n^*)$$

Let $C_1^{n+1}(\mathcal{U}_0)$ and $C_2^{n+1}(\mathcal{U}_0)$ be chains such that

$$\partial C_1^{n+1}(\mathcal{U}_0) = Z^n(\mathcal{U}_0)$$

$$\partial C_2^{n+1}(\mathcal{U}_0) = Z^n(\mathcal{U}_0) - \pi_{\mathcal{V}_n^*}^{\mathcal{U}_0} Z^n(\mathcal{V}_n^*).$$

From these relations we conclude that the chain

$$C^{n+1} = C_1^{n+1}(\mathcal{U}_0) - C_2^{n+1}(\mathcal{U}_0) - \mathcal{D} Z^n(\mathcal{V}_n^*)$$

is a chain of $\mathcal{V}_n^* \cup \mathcal{U}_0$ such that

$$\partial C^{n+1} = Z^n(\mathcal{V}_n^*).$$

We want to realize the complex K, the nerve of the covering $\mathcal{U}_0 \cup \mathcal{V}_n^*$ on $\mathcal{V}_n^*$. This complex canot necessarily be partially realized of norm $< \mathcal{U}_n^*$. However we shall set up a mapping which will realize part of the complex of norm $< \mathcal{U}_n^*$; the mapping on the rest of the complex is defined separately. Set up a mapping τ' which maps every V_{x_i} of $\mathcal{U}_0$ into an element of $\mathcal{V}_n^*$ which contains x_i. If the vertices of a cell σ^k of K map into the vertices of a cell τ^k of the nerve of $\mathcal{V}_n^*$, then let $\tau'(\sigma^k) = \tau^k$. In this mapping all the cells on $\mathcal{V}_n^*$ are mapped identically.

Take a cell of K which contains one element of $\mathcal{U}_o$, say V_{x_1}. Consider the projection of each open set of σ^K from the covering $\mathcal{V}_n^*$ to the coverig $\mathcal{U}_o$. If the union of these projected sets is contained in one element of $\mathcal{U}_n^*$ then the map of all the vertices of the cell which are realized by τ' lie in one element of $\mathcal{U}_n^*$.

If there exists a set whose projection is not contained in the element of $\mathcal{U}_n^*$ containing x_1, then this set is contained in V_{x_2}, say, where x_2 is contiguous to x_1. The nucleus of the original cell must lie on x_1 and x_2. Any open set of $\mathcal{U}_o \cup \mathcal{V}_n^*$ appearing in the cell contains a point contiguous to both x_1 and x_2. If the cell contains another element, say V_{x_3} from $\mathcal{U}_o$, then x_3 is contiguous to both x_1 and x_2 and x_3 forms part of the nucleus since no other point of V_{x_3} can be contiguous to both x_1 and x_2. (See theorem 4 chapter III) The vertices of this cell map into sets which still contain the nucleus points, hence the cell maps into a cell and the mapping is defined on this cell. This shows that if σ^K is a cell in the complex on which the mapping is not defined, then τ' of all the faces on which the mapping is defined lie on one element of $\mathcal{U}_n^*$.

The cells on which the mapping is not defined plus their faces form a closed subcomplex K' of K. The partial realization τ' on K' can be extended to a mapping τ of K on $\mathcal{V}_n^* \cup \mathcal{V}$ of norm $< \mathcal{U}$. Hence the whole complex K is realized on $\mathcal{V}_n^* \cup \mathcal{V}$. Therefore

$$\partial \tau C^{n+1} = \tau \partial C^{n+1} = \tau Z^n(\mathcal{V}_n^*) = Z^n(\mathcal{V}_n^*).$$

Consequently the chain $\pi^{\mathcal{V}}_{\mathcal{V}_n^*} \tau C^{n+1}$ bounds $\pi^{*}_{\mathcal{V}_n^*} Z^n(\mathcal{V}_n^*)$ and since the latter is homologous to $Z^n(\mathcal{V})$ on $\mathcal{V}$, we have that $Z^n(\mathcal{V}) \sim 0$ on $\mathcal{V}$.

In the case of local connectedness we defined the concept in terms of cycles lying on a set and then proved a theorem concerning cycles lying contiguously on a set. We shall now define local coconnectedness in terms of cocycles lying contiguously in a set and then prove a theorem concerning cocycles lying in a set.

<u>Definition</u>: A space is <u>n-colc</u> at $x \in S$ if given an open set P containing x, there exists an open set Q such that $x \in Q \subset P$ and such that for each covering $\mathcal{U}$, there exists a covering $\mathcal{V} > \mathcal{U}$ such that if $Z_n(\mathcal{U})$ is a cocycle lying contiguously in Q, then $\pi^{*\mathcal{U}}_{\mathcal{V}} Z_n(\mathcal{U})$ will cobound a chain on $\mathcal{V}$ lying contiguously in P.

<u>Theorem 5</u>. <u>If a space S is n-colc at $x \in S$, then given an open set P' containing x, there exists an open set Q' such that $x \in Q' \subset P'$ and such that for each covering $\mathcal{U}$ there exists a covering $\mathcal{V} > \mathcal{U}$ such that if $Z_n(\mathcal{U})$ is a cocycle which lies in Q' except for some cells which lie contiguously in the set $\{x\}$ then $\pi^{*\mathcal{U}}_{\mathcal{V}} Z_n(\mathcal{U})$ will cobound a chain of $\mathcal{V}$ which lies in P' except for some cells which lie contiguously in $\{x\}$.</u>

Proof: Take P'' an open set containing x such that P''

and $S - P'$ are contiguously separated except for some points in $S - P'$ which are contiguous to x (see theorem 4 chapter II)

Relative to this P'' take Q from the definition of n-colc. This is the desired Q'. For any covering $\mathcal{U}$ take $\mathcal{V}$ from the definition of n-colc. (relation to P' & Q). If $Z_n(\mathcal{U})$ is a cocycle which lies in Q' except for some cells which lie contiguously in x then $Z_n(\mathcal{U})$ lies contiguously in Q'. Hence $\pi^{*\mathcal{U}}_{\mathcal{V}} Z_n(\mathcal{U})$ cobounds a chain C^{n-1} which lies contiguously in P''. We want to show that this chain lies in P' except for some cells which lie contiguously in x. Let σ^K be a cell of $|C^{n-1}|$.

1. If σ^K has a proper nucleus, then this nucleus lies in P'' and hence in P'.
2. If σ^K has a nonproper nucleus $p_0, \ldots, p_K$, at least one of the points, say p_0, is an element of P''. If p_0 is not equal to x, then $p_i \in P'$ as p_i is contiguous to p_0 for $i = 1, \ldots k$. Hence the cell lies in P'.

There exist four different local Betti numbers. However we shall be interested in only two of these.

1. $p_n(x)$. Let P and Q be open sets containing x such that $x \in Q \subset P$ and let $p_n(x,P,Q)$ be the dimension of the vector space formed by the set of all n-cocycles lying contiguously in Q modulo those which cobound chains lying contiguously in P. By letting first Q and then

P go to zero we obtain the number $p_{\sim}(x)$. By a theorem analogous to theorem 5, we see that we obtain the same number if we define $p_{\sim}(x,P,Q)$ in terms of cocycles lying in Q except for some cells which lie contiguously in x modulo those which cobound chains lying in P except for some cells which lie contiguously in x.

2. $g^{w}(x)$. Let $g^{w}(x,P,Q)$ be the dimension of the vector space formed by the C-cycles on Q modulo those which bound on P.

In the same way as in the ordinary theory we may prove the following theorem.

Theorem 6. In order that S should be n-lc at $x \in S$, it is necessary and sufficient that each open set which contains x, also contains a pair of open set P and Q such that $x \in Q \subset P$ and such that $g^{w}(x,P,Q)$ is finite.

Definition: A space S is called semi-r connected if each point x of S has a neighbourhood U(x) such that all r-cycles in U(x) bound on S.

We shall now prove the general "wheel and spoke" lemma for contiguous point spaces.

Theorem 7. Let S be a semi-r-connected, locally compact space which is not r-lc. Then there exist $x \in S$, and open sets P, Q, and R such that $X \in R \ll Q \ll P$ and such that contiguously on F(Q) there exist infinitely many r-cycles that are linearly independent with respect to homology (lirh) on $\overline{P} - R$. [5]

Proof: Suppose S is not r-lc at x. Then by theorem 6 there exist open sets P and V such that $x \in V \subset P$ and infinitely many r-cycles on V are lirh on P. We may suppose P to be taken so that $\overline{P}$ is compact, and that V is such that all r-cycles on V bound on S. Let Q and R be open sets such that $P \supset\supset Q \supset\supset R \supset\supset V$, and let γ_i^{r}; $i = 1, 2, \ldots$ be an infinite set of cycles on V that are lirh on P.

Let $K = \overline{V}$ and $L = S - Q$. Then by theorem 9 chapter III, there exists for each i a cycle Z_i^r on $F(L)$ such that $Z_i^r \sim \gamma_i^r$ on $S - L = Q$. Then the cycles Z_i^r are lirh on $\overline{P} - R$. For suppose there exists a relation

(1) $\sum c^i Z_i^r \sim 0$ on $\overline{P} - R$

Now $Z_i^r \sim \gamma_i^r$ on Q implies

(2) $\sum c^i \gamma_i^r - \sum c^i Z_i^r \sim 0$ on Q.

Adding relations (1) and (2) we obtain

$\sum c^i \gamma_i^r \sim 0$ on P contradicting the fact that the cycles γ_i^r are lirh on P.

Since the cycle Z_i^r lies on $F(L)$, it lies contiguously on $F(Q)$.

Chapter V

SIMPLE CLOSED CURVE

In this section we will discuss two concepts which will be proved to be identical. The first is that of the simple closed curve which is defined in the natural way as the sum of two arcs with common end points which are separated except for the end points. The second is that of the one sphere which is defined to satisfy the "Jordan curve theorem". The Jordan curve theorem states that any n - 1 dimensional sphere γ divides an n sphere into two separated connected parts of which γ is the common boundary. We shall first give the definition of an arc and a few theorems about arcs. The first definition follows naturally from this discussion. However the second definition lends itself more easily to a definition of higher dimensional spheres. The main theorems of the section are justification theorems. We will show that the simple closed curve is an example of the one dimensional orientable generalized closed manifold to be discussed in Chapter VI.

<u>Definition</u>: A <u>simple continuous arc</u> from a to b is a connected compact set which is disconnected by the omission of any of its points except a and b. [4, p. 2]

Denote an arc with end points a and b by I = (a,b). We can prove that the arc is irreducibly connected about its end points and if $x \in I - a - b$ then

$I - x = A \cup B$ sep where $a \in A$ and $b \in B$. Also $A \cup x$ and $B \cup x$ are connected sets (The set $A \cup x$ is irred conn about $a \cup x$). All these theorems are proved in the usual way.

We also know that if M and N are connected subsets of I each containing a then either $N \subset M$ or $M \subset N$. For every $x \in I$ let $I - x = A(x) \cup B(x)$ sep. where $a \in A(x)$ and $b \in B(x)$. This provides an ordering of the points from a to b if we let $x < y$ if $x \neq y$ and $A(x) \subset A(y)$.

<u>Definition</u>: An <u>interior point</u> of an arc (a,b) is any point of the arc except a and b.

<u>Definition</u>: A <u>proper interior point</u> of an arc (a,b) is a point c such that neither $A(c)$ nor $B(c)$ consists of a single point.

<u>Theorem 1. No proper interior point of an arc can be contiguous to an end point of the arc.</u>

Proof: Suppose c is an interior point of $I = (a,b)$ and c is contiguous to a; then

$I - c = A(c) \cup B(c)$ sep.

The set $A(c) \cup C$ is irreducible connected about $a \cup c$. However $a \cup c$ is a connected subset of $A(c) \cup c$ containing $a \cup c$ and hence $A(c) \cup c = a \cup c$. This shows that $A(c) = a$ and c is not a proper interior point of the arc (a,b).

<u>Theorem 2. If C is a closed subset of an arc $I = (a,b)$, then there exists a first point in C in the order from</u>

<u>a to b on the arc.</u>

Proof: Suppose no such first point exists, If $x \in C$, then there exists a point y preceding x on C in the order from a to b on the arc. Hence $x \in B(y)$. The set of these $B(y)$ covers C. Hence there exists a finite set $B(y_1)$, $B(y_2)$, ..., $B(y_k)$ which covers C. Suppose y_K precedes y_i in the order from a to b for all $i \neq k$. Then $B(y_k) \supset B(y_i)$ for all i. But $B(y_K)$ does not contain the point y_K and hence this point is not covered. This proves the theorem.

We shall now consider the two concepts of simple closed curve and one sphere and prove that they are equivalent.

<u>Definition</u>: A <u>simple closed curve</u> is a compact continuum consisting of at least four points which is irreducible with respect to being the union of two distinct arcs with common end points and which does not consist of two and only two triunes. [4,p.7]

This is the definition which Moore gives with one difference. He has an axiom which prohibits the existence of triunes in his space and hence the last restriction is unnecessary.

It will first be noted that if the space consists of two distinct arcs I_1 and I_2 with common end points a and b, then a and b are not contiguous as by the irreducible connectedness of an arc about its end points the only possible arc with end points a and b which are contiguous is the arc consisting of the two points a and b.

<u>Theorem 3. If we divide a simple closed curve into two distinct arcs I_1 and I_2 with end points a and b, then no point interior to one arc is contiguous to a point interior to the other arc.</u> [4,p.40]

Proof: Suppose there exists a point $c \in I_1$ and a point $d' \in I_2$ such that c is contiguous to d'. We have shown above that the points a and b are not contiguous. If d' is contiguous to both a and b, then c is not contiguous to both a and b as otherwise our simple closed curve must consist of the two arcs acb and $ad'b$. This case has been eliminated by the definition. Suppose c is not contiguous to b. By axiom C and the closure of I_2, there exists only a finite number of points of I_2 which are contiguous to c. Hence there exists a first point on I_2 after b in the order from b to a which is contiguous to c. Call this point d.

$$I_1 - c = A(c) \cup B(c) \text{ sep.}$$

where $a \in A(c)$ and $b \in B(c)$.

$$I_2 - d = A(d) \cup B(d) \text{ sep.}$$

where $a \in A(d)$ and $b \in B(d)$.

We shall show that there exist two distinct arcs from b to c neither of which contain a which is a contradiction to the definition of S.

These arcs are $\{B(c) \cup c\}$ and $R \cdot \{B(d) \cup d \cup c\}$. The first of these is immediately an arc from b to c. It remains to show that R is an arc from b to c.

Since the sets $R - b = \{B(d) \cup d - b\} \cup c$ and $R - c = B(d) \cup d$ are both connected, neither b nor c disconnects R.

By the choice of d, the set R contains no point contiguous to c except d. Also since $R - c$ is contained in I_2, the point c is not a limit point of $R - c$. Let $x \in R - b - c$.

1. Suppose $x \neq d$.

$R - x - c = R_1 \cup R_2$ sep. where $d \in R_1$. Hence

$R - x = (R_1 \cup c) \cup R_2$ sep.

2. Suppose $x = d$; then

$R - d = c \cup (R - c - d)$ sep.

This completes the proof that R is an arc from b to c and therefore the theorem is proved.

<u>Definition: A zero sphere,</u> S°, is any pair of non contiguous points.

<u>Definition:</u> A <u>one sphere,</u> S', is a connected compact set containing at least one S° and such that $S' - S^\circ$ is the union of exactly two separated connected sets for any S° contained in S'.

<u>Theorem 4. A one sphere, S', is a simple closed curve.</u>

Proof: It is obvious that S' contains at least four points as it contains an S° and $S' - S^\circ$ contains at least two points. Take $S^{\circ} = a \cup b$, and let

$S' - S^{\circ} = M_1 \cup M_2$ sep.

where M_1 and M_2 are connected. Let $I_1 = M_1 \cup a \cup b$. We will first show that I_1 is an arc from a to b.

The first step is to show that I_1 is connected. The set M_1 has a limit point or a point contiguous to either a or b otherwise S' is not connected. Also the set M_2 has a limit point or point contiguous to the same one. Suppose M_1 does not have a limit point or a point contiguous to a, then $M_1 \cup b$ is a continuum. It must have at least two non cut points at least one of which $c \neq b$. The point c is not contiguous to a and therefore

$S' - a - c = (M_1 \cup b - c) \cup M_2$

Both these sets are connected and since M_2 has a limit point or a point contiguous to b, the set $S' - a - c$ is connected. This is a contradiction to the definition and hence I_1 is connected as M_1 has a limit point or a point contiguous to both a and b.

We know that the points a and b do not disconnect I_1. To show that I_1 is an arc from a to b, it remains to show that every other point on I_1 is a cut point. Suppose there exists a point $c \in I_1$ such that $I_1 - c$ is connected.

1. Suppose $I_2 = M_2 \cup a \cup b$ also contains a non cut point d. The set I_2 is also connected.

$S' - c - d = (I_1 - c) \cup (I_2 - d)$

Both the sets $(I_1 - c)$ and $(I_2 - d)$ are connected sets containing a and b and hence $S' - c - d$ is connected.

2. Suppose every point of I_2 is a cut point or I_2 is an arc from a to b. Take $d \in I_2 - a - b$.

$I_2 - d = A \cup B$ sep. where $a \in A$ and $b \in B$. Then

$S' - c - d = (I_1 - c) \cup A \cup B$.

Again we obtain a contradiction and hence we have proved that I_1 is an arc from a to b.

We have shown that S' is the union of two arcs with common end points. We have still to show that it is irreducibly the union of two arcs with common end points. Suppose S' is the union of the two distinct arcs I_1 and I_2 with end points a and b. If a and b are not contiguous, then by the irreducible connectedness of an arc about its end points we know that the two arcs found above are the only possible arcs with end points a and b and these fill out the space. If a and b are contiguous, then the only possible arc with end points a and b is the arc ab. In this case we cannot find two distinct arcs with end points a and b. This concludes the proof of the theorem.

Theorem 5. If D is a connected subset of an S' such that D contains either a or a point contiguous to a and b or a point contiguous to b where a and b are

<u>a pair of non contiguous points then if</u>

$S' - a - b = A \cup B$ sep.

D contains either A or B.

Proof: Suppose $c \in A - D$ and $d \in B - D$. Let I_1 denote the arc $a \cup A \cup b$, and I_2 the arc $a \cup B \cup b$.

$I_1 - c = A_1 \cup A_2$ sep where $a \in A_1$ and $b \in A_2$.

$I_2 - d = B_1 \cup B_2$ sep where $a \in B_1$ and $b \in B_2$.

$S' - c - d = A_1 \cup A_2 \cup B_1 \cup B_2$.

The sets A_1 and B_1 are connected sets containing a; the sets A_2 and B_2 are connected sets containing b; and D is a connected set containing points contiguous or equal to both a and b and contained in $S' - c - d$. Hence $S' - c - d$ is a connected set. This is impossible as c and d are not contiguous by theorem 2.

<u>Corollary: If $D \supset p$ but not a and $p \in A$, then $D \supset A$.</u>

Proof: Suppose $c \in A - D$.

$I_1 - c = A_1 \cup A_2$ sep where $a \cup p \in A_1$ and $b \in A_2$. By theorem 1, the point c is not contiguous to a.

$S' - c - a = A_1 \cup (A_2 \cup b \cup B)$ sep.

By theorem 5, $D \supset B$ and hence $D \subset A_2 \cup b \cup B$. This is impossible as $p \in A_1 \cap D$.

<u>Theorem 6. If M and N are connected subsets of an S', then either $M - N$ is connected or $M \supset N$.</u>

Proof: Suppose $M - N$ is not connected. Take two points x and y which are elements of different components of $M - N$. The points x and y are not contiguous.

$S' - x - y = A_1 \cup A_2$ sep.

Since $M \supset x \cup y$, by theorem 5, $M \supset A_1$ say. Also since $N \subset S' - x - y$, $N \subset A_1$ or $N \subset A_2$. If $N \subset A_2$, then $A_1 \cup x \cup y$ is a connected set contained in $M - N$ and x and y lie in the same component of $M - N$. Hence $N \subset A_1 \subset M$, and the theorem is proved.

<u>Theorem 7.</u> <u>A simple closed curve is a one sphere.</u>

Proof: Let S be a simple closed curve, then S is the sum of two distinct arcs I_1 and I_2 with end points c and d. Let $S^0 = a \cup b$ be any zero sphere on S. This certainly exists as in particular the points c and d form such a pair of non contiguous points. Denote $S' - S^0$ by R. We shall show that R is the sum of two separated connected sets.

1. Suppose $a = c$ and $b = d$, then R is the sum of two separated connected sets, $I_1 - a - b$ and $I_2 - a - b$.

2. Suppose $a = c$ and $b \neq d$ but $b \in I_1$.

$I_1 - b = M_1 \cup M_2$ sep. where $c \in M_1$ and $d \in M_2$.

The sets M_1 and M_2 are separated and therefore since $M_1 - a \subset I_1 - a$, the sets $M_1 - a$ and $I_2 - a$ are separated.

Hence

$S' - a - b = (M_1 - a) \cup (M_2 \cup I_2 - a)$ sep.

Both these sets are connected since the latter is the sum of two connected sets both containing d.

3. Suppose $a \in I_1$, $b \in I_1$ where neither are equal to c or d.

$I_1 - a = M_1 \cup M_2$ sep where $c \in M_1$ and $d \in M_2$.

$I_1 - b = N_1 \cup N_2$ sep where $c \in N_1$ and $d \in N_2$.

Let $M_1 \subset N_1$; then $b \in M_2$. Hence R is the sum of the two separated connected sets, $(M_2 - N_1 - b)$ and $(I_2 \cup M_1 \cup N_2)$. (see theorem 6).

4. Suppose $a \in I_1$ and $b \in I_2$.

$I_1 - a = M_1 \cup M_2$ sep where $c \in M_1$.

$I_2 - b = N_1 \cup N_2$ sep where $c \in N_1$.

Then R is the sum of the two separated connected sets $(M_1 \cup N_1)$ and $(N_2 \cup M_2)$. Since this completes all possible cases, R is always the sum of two separated connected sets.

From theorems 4 and 7 we conclude theorem 8.

<u>Theorem 8.</u> <u>A simple closed curve and a one sphere are identical.</u>

In chapter VII we shall show that a simple closed curve is an example of a one dimensional orientable generalized closed manifold.

Chapter VI

THE GENERALIZED MANIFOLD

A locally compact space S will be called a <u>generalized contiguous point manifold of dimension n</u> if the following axioms are satisfied.

1. The dimension of S is n.
2. S is lc^{n}.
3. S is $colc^{n-1}$.

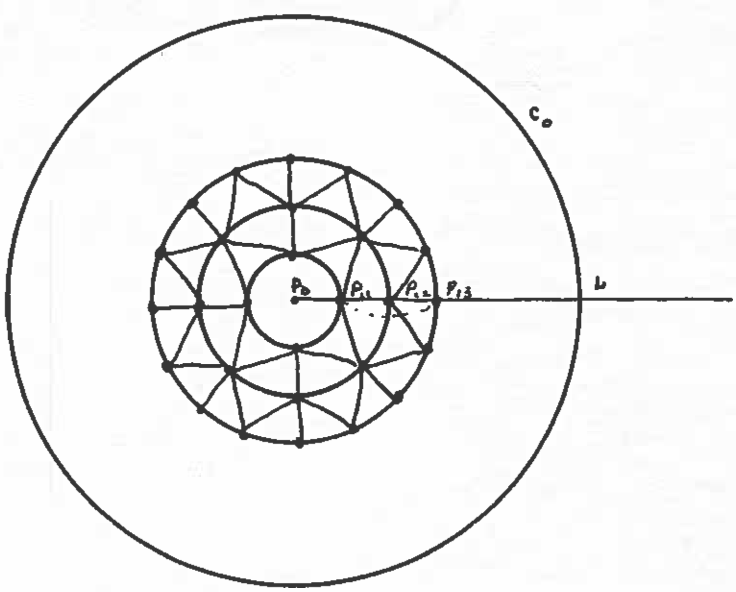
For each $x \in S$, $p_n(x) = 1$

5. If F is a proper closed subset of S or F consists of the same point set as S with the same topology but there exist points of F which are contiguous in S but which are not contiguous in F, then $p^{n}(F, \mathfrak{J}) = 0$

If the space S is compact we call it a generalized closed manifold of dimension n.

An orientable generalized closed manifold is a generalized closed manifold which possesses an n-cycle which does not bound on the space. By axiom 5, this n cycle will be destroyed if we delete from S an open set or a contiguity. The necessity of strengthening axiom 5 to include the case where F is the same as with S with a certain number of contiguities dropped is shown by example 2 which follows. Examples 1 and 3 which follow are examples of an orientable generalized closed manifold.

<u>Example 1.</u> Take a point P_0 in the plane. With centre P_0 and radius $\frac{n}{n+1}$ draw a circle C_n for each natural number n. Through the point P_0 draw a ray L. On the circle C_n put 2^{n+1} points $P_{1,n}, P_{2,n}, \ldots, P_{2^{n+1},n}$, $P_{2^{n+1}+1,n} = P_{1,n}$ spaced equally around the circle startig with $P_{1,n}$ on the ray L and going in the same direction from there each time. Let $P_{j,n}$ be contiguous to $P_{i,n}$ for $j = i + 1$; P_{i1} be contiguous to P_0 for all i; and the points $P_{2K-1,n+1}$, $P_{2K,n+1}$, and $P_{2K-2,n+1}$ on the circle C_{n+1} be contiguous to the point $P_{K,n}$ on the circle C_n. These circles close down on the circle C_0 with centre P_0 and radius 1. This configuration is deformed out slightly to form the top half of a sphere with equator C_0. The bottom half of the sphere may be an ordinary hemisphere or it may be made up of contiguous points in the same way. Let the neighbourhoods of points be the usual metric neighbourhoods. This space will be an orientable generalized closed manifold of dimension 2. (see fig. 1)



<u>Example 2.</u> If in example 1 we also make the point P_{13} be contiguous to the point P_{11}, we have a space which satisfies all the axioms of a manifold except the second part of axiom 5. This example is not a two sphere as thre exists a one sphere, namely the circle C_2 which does not divide it into two separated parts. This shows the necessity of the second part of axiom 5.

<u>Example 3.</u> Through a point P in the plane draw a finite number of rays $L_1, L_2, \ldots, L_k, L_{k+1} = L_1$ arranged cyclically around P where k is any finite number greater than 2. Around P as centre draw circles C_n with radii 1/n for each natural number n. At the intersection of C_n and L_i put the point P_{in}. Let P_{in} be contiguous to $P_{i,n+1}$; $P_{i,n}$ be contiguous to $P_{i+1,n}$ and $P_{i+1,n-1}$. Make two such configurations and put them together at the points P_{i1}. Let the neighbourhoods of points be the usual metric neighbourhoods. This is also a 2-dimensional orientable generalized closed manifold. (see fig. 2)

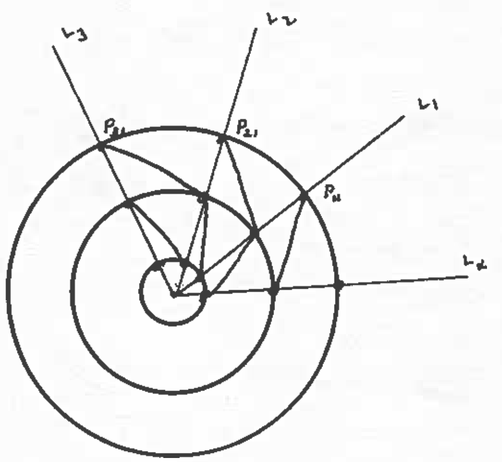


Fig. 2

If S is an orientable generalized closed manifold, then we know from the general theory that if Γ^n is a non-bounding cycle of S and P of an open subset of S, it is impossible that $\Gamma^n \sim 0$ mod $S - P$. By intersection theory we see that for every open set P, there exists a cocycle γ_n contiguously in P such that γ_n does not cobound on the space.

Theorem 1. If P is an open set containing a proper interior point then there exists a cocycle γ_n in P which fails to cobound on the space.

Proof: Let y be a proper interior point of P. By theorem 4, chapter II, there exists an open set Q containing y such that Q and S - P are contiguously separated. From previous considerations we know that there exists a non-cobounding cocycle lying contiguously in Q. This cocycle will lie in P.

We shall now discuss the structure of the contiguous points in an orientable generalized closed manifold of dimension $n \geq 2$. All the following theorems will be for this type of space.

Theorem 2. If x_1 is contiguous to x_2, then on every complete family of coverings Σ' and for every non-bounding cycle Γ^n there exists a covering $\upsilon \in \Sigma'$ such that the complex $|\Gamma^n(\upsilon)|$ contains an n-cell $\sigma^n = V_0, V_1, \ldots, V_n$ with nucleus $p_0, p_1, \ldots, p_n$ where some $p_i = x_1$ and some $p_j = x_2$ for every choice of nuclei points p_i. In other words one of

<u>the coordinates of the cycle Γ^{n} makes use of the contiguity x_1 to x_2.</u>

Proof: We shall first prove the theorem for the case of a complete family of coverings Σ'' such that if $\mathcal{W} \in \Sigma''$ then $\mathcal{W}$ is an n-dimensional covering. Suppose the theorem is not true for Σ''. Then for every $\mathcal{W} \in \Sigma''$ and every σ^{n} of $|\Gamma^{n}(\mathcal{W})|$ it is possible to choose a nucleus $p_0, \ldots, p_n$ so that if $x_1 = p_i$ then no $p_j = x_2$. In other words some of the nuclei may contain the point x_1 and some the point x_2 but for any cell there always exists a nucleus which does not contain both.

Consider the cycle Γ_1^{n} formed as follows. If $\mathcal{W} \in \Sigma''$ take $\Gamma_1^{n}(\mathcal{W}) = \Gamma^{n}(\mathcal{W})$. If $\mathcal{U} \notin \Sigma''$ then there exists a $\mathcal{W} \in \Sigma''$ such that $\mathcal{W} > \mathcal{U}$. Take $\Gamma_1^{n}(\mathcal{U}) = \pi_{\mathcal{W}}^{\mathcal{U}} \Gamma^{n}(\mathcal{W})$. For every cell of $|\Gamma_1^{n}(\mathcal{W})|$ there exists a nucleus which does not contain both the points x_1 and x_2.

We have shown that the coordinates of Γ_1^{n} lie on the set F which consists of S without the contiguity x_1 to x_2. We will now show that it is possible to choose the connecting homologies on F. Take any two coverings of the space $\mathcal{U}$ and $\mathcal{V}$ where $\mathcal{V} > \mathcal{U}$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the elements of Σ'' which were used in the definition of $\Gamma_1^{n}(\mathcal{U})$ and $\Gamma_1^{n}(\mathcal{V})$ respectively. Then

$$\Gamma_1^{n}(\mathcal{U}) = \pi_{\mathcal{W}_1}^{\mathcal{U}} \Gamma^{n}(\mathcal{W}_1)$$
$$\Gamma_1^{n}(\mathcal{V}) = \pi_{\mathcal{W}_2}^{\mathcal{V}} \Gamma^{n}(\mathcal{W}_2)$$

Take $\mathcal{W}_3 \in \Sigma''$ where $\mathcal{W}_3 > \mathcal{W}_1, \mathcal{W}_2$. Using the fact that all the elements of Σ'' are n-dimensional we obtain the following relations.

$$\pi_{\mathcal{V}}^{\mathcal{U}} \Gamma_1^{n}(\mathcal{V}) = \pi_{\mathcal{V}}^{\mathcal{U}} \pi_{\mathcal{W}_2}^{\mathcal{V}} \Gamma^{n}(\mathcal{W}_2) = \pi_{\mathcal{V}}^{\mathcal{U}} \pi_{\mathcal{W}_2}^{\mathcal{V}} \pi_{\mathcal{W}_3}^{\mathcal{W}_2} \Gamma^{n} \quad (\mathcal{W}_3)$$

$$\Gamma_1^{n}(\mathcal{U}) = \pi_{\mathcal{W}_1}^{\mathcal{U}} \Gamma^{n}(\mathcal{W}_1) = \pi_{\mathcal{W}_1}^{\mathcal{U}} \pi_{\mathcal{W}_3}^{\mathcal{W}_1} \Gamma^{n} \quad (\mathcal{W}_3)$$

However $\pi_{\mathcal{V}}^{\mathcal{U}} \pi_{\mathcal{W}_2}^{\mathcal{V}} \pi_{\mathcal{W}_3}^{\mathcal{W}_2}$ and $\pi_{\mathcal{W}_1}^{\mathcal{U}} \pi_{\mathcal{W}_3}^{\mathcal{W}_1}$ represent two projections from the covering $\mathcal{W}_3$ to the covering $\mathcal{U}$. By theorem 1 chapter III, we derive the fact that there exists an n-chain $C^{n+1}(\mathcal{U})$ such that

$$\partial C^{n+1}(\mathcal{U}) = \Gamma_1^{n}(\mathcal{U}) - \pi_{\mathcal{V}}^{\mathcal{U}} \Gamma_1^{n}(\mathcal{V})$$

Since $\Gamma^{n}(\mathcal{W}_3)$ consists of cells which possess nuclei not containing both of the points x_1 and x_2, the chain $C^{n+1}(\mathcal{U})$ also consists of cells which possess nuclei not containing both of the points x_1 and x_2. Hence the chain $C^{n+1}(\mathcal{U})$ lies on the set F. This proves that F possesses a non-bounding n-cycle Γ_1^{n}. This is a contradiction to axiom 5. Hence there exists some covering $\mathcal{W}$ such that $|\Gamma^{n}(\mathcal{W})|$ contains an n-cell $\sigma^{n} = W_0, W_1, \ldots, W_n$ with nucleus $p_0, \ldots, p_n$ where some $p_i = x_1$ and some $p_j = x_2$ for every choice of nuclei points p_i.

To prove the theorem for any complete family of coverings Σ', consider the n-dimensional covering $\mathcal{W}$ found above. Let $\mathcal{V} > \mathcal{W}$ such that $\mathcal{V} \in \Sigma'$. Since $\mathcal{W}$ is an n-dimensional covering we have that

$$\Gamma^{n}(\mathcal{W}) = \pi_{\mathcal{V}}^{\mathcal{W}} \Gamma^{n}(\mathcal{V})$$

Hence $\mathcal{V}$ contains a cell σ_1^{n} which projects into σ^{n}. This must also have the property that for every choice of nucleus one of the nuclei points is x_1 and another is

x_λ, or otherwise this property would not hold for σ^n. This completes the proof of the theorem.

Theorem 3. <u>If $x_0, \ldots, x_n$ are a set of $n+1$ points in an n-dimensional manifold such that x_i is contiguous to x_j for each $i \neq j$, and if we have a covering $\mathcal{U}$ of the space in which x_i is contained in U_{x_i} for each i such that $U_{x_i} \cap U_{x_j} = 0$ and there exists no contiguity across U_{x_i} and U_{x_j} except the contiguity x_i to x_j for each $i \neq j$, and no other element of the covering contains x_i except U_{x_i}, then the cell $\sigma^n = U_{x_0}, \ldots, U_{x_n}$ has no super faces.</u>

Proof: The proof is immediate as if there were a cell with σ^n as a face, then there must exist a point x_{n+1} which is contiguous to all the points x_i for $i = 0,1,\ldots,n$. Hence the space is at least $n+1$ dimensional which is a contradiction.

<u>Remark.</u> If we consider a cycle Z^n, such that the coordinate of Z^n on the covering $\mathcal{U}$, $Z^n(\mathcal{U})$, contains the cell σ^n, then if $\mathcal{V} > \mathcal{U}$ the cycle $Z^n(\mathcal{V})$ must contain a cell which projects into σ^n.

Theorem 4. <u>If x has only a finite number of points contiguous to it, then there exist open sets P and Q such that $x \in Q \subset P$ and such that if there exists a cocycle in Q non-cobounding in P, then there exists no non-cobounding n-cocycle lying contiguously in the set x and conversely.</u>

Proof: Let $x_1, \ldots, x_k$ be contiguous to x. Take O an open set containing x and none of the points x_i for $i = 1, \ldots, k$. By theorem 4, chapter II, take O' an open set containing x such that x is the only point in O' contiguous to a point in S - O. Take open sets P and Q such that $x \in Q \subset P \subset O$ and such that $p_n(x;P,Q) = 1$.

Take coverings of the space so that there exists only one element of the covering containing x and this is contained in O'. Also the points $x_1, \ldots, x_k$ are taken in disjoint elements of the covering lying outside O'. No other elements of the covering intersect $x_1, \ldots, x_k$. These coverings form a complete system of coverings of the space. We shall consider only these coverings.

Suppose there exist two cocycles $\gamma_n^1(u)$ and $\gamma_n^2(u)$ such that $\gamma_n^1(u)$ lies in Q and $\gamma_n^2(u)$ lies contiguously in x and neither cobound in P except for some cells which lie contiguously in x. Since $p_n(x;P,Q) = 1$, for some covering v there exists a chain $C^{n-1}(v)$ lying in P except for some cells which lie contiguously in x and such that

$$\delta C^{n-1}(v) = \pi_v^{*u} \gamma_n^1(u) - \pi_v^{*u} \gamma_n^2(u).$$

Express $C^{n-1}(v)$ as $C_1^{n-1}(v) + C_2^{n-1}(v)$ where $C_1^{n-1}(v)$ lies in P and $C_2^{n-1}(v)$ lies contiguously in x. The coboundary of $C_2^{n-1}(v)$ lies contiguously in x, while the coboundary of $C_1^{n-1}(v)$ lies contiguously in P but has no cells lying contiguously in x and hence lies in O'. As there exists no n-cell lying in O' and also lying contiguously in x, we derive the fact that

$$\delta C_1^{n-1}(v) = \pi^{*u}_{v}\gamma_u^1(u)$$

$$\delta C_2^{n-1}(v) = \pi^{*x}_{v}\gamma_v^2(v).$$

This is a contradiction. Hence we have proved the theorem.

<u>Theorem 5. If a point x has only a finite number of points contiguous to it, then there exists a neighbourhood of x which consists of the point x alone.</u>

Proof: Suppose there does not exist a neighbourhood of x which consists of the point x alone. Then take P and Q open sets which satisfy theorem 4. By theorem 1 there exists a cocycle in Q which fails to cobound in P. It follows from theorem 2 that there exists a cocycle lying in x which fails to cobound. By theorem 4 the existence of two such cocycles is impossible. Hence x has a neighbourhood which consists of the point x alone.

The following theorem is an immediate corollary of theorem 5.

<u>Theorem 6. It is impossible to have an open set $P \subsetneqq S$ such that P contains only a finite number of contiguous points and S - P is contiguously closed.</u>

Proof: Since P is a set containing only a finite number of contiguous points and S - P is contiguously closed, each point of P has only a finite number of points contiguous to it and hence has a neighbourhood which consists of the point itself. Therefore P is also contiguously closed and since the space is connected, we have a contradiction.

<u>Theorem 7. If x is a contiguous point and we consider the set x^c, then every point of $x^c - x$ is contiguous to a point $y \in S - x^c$.</u>

Proof: Suppose there exists a point $z \in x^c - x$ such that z is not contiguous to a point $y \in S - x^c$.

Build up coverings of the space as follows. Cover x by U_x such that no element of x^c is an element of $\overline{U}_x - U_x$. If $p \in S - \overline{U}_x$ take $U_p \subset S - \overline{U}_x$ such that $U_p \cap x^c = p$ if $p \notin x^c$ and 0 otherwise. Since $x^c \cap S - U_x$ is a finite set of points, the sets U_p for $p \in x^c$ may be taken to be contiguously separated except for contiguities among elements of x^c. If $p \in \overline{U}_x - U_x$, take U_p such that $U_p \cap x^c = 0$. Reduce this to a finite cover. The set of all such coverings form a complete system of coverings of the space. We shall consider only these coverings.

Consider the fundamental n-cycle Γ^n. By theorem 2 on some one of these coverings $\mathfrak{V}$ there exists a cell $\sigma^n = V_x, V_z, V_{x_2}, \ldots, V_{x_n}$ where $x_2, \ldots, x_n \in x^c$. Consider the cell $V_z, V_{x_2}, \ldots, V_{x_n}$ which is on the boundary of σ^n. If there exists another n-cell with this n - 1 cell on its boundary, then such a cell is given by $V, V_{x_2}, \ldots, V_{x_n}$

where V contains a point x_{n+1} contiguous to $z, x_2, \ldots, x_n$. If x_{n+1} is contiguous to x, we would have a contradiction to the n-dimensional character of the space. Hence x_{n+1} is contiguous to z but not to x. As this is contrary to our assumption, we have proved the theorem.

Corollary. If the set x^c contains an infinite number of points, then the set $(x^c)^c - x^c$ also contains an infinite number of points.

Proof: If the set $(x^c)^c - x^c$ contains only a finite number of points, then as each element of $(x^c)^c - x^c$ can have only a finite number of points in x^c contiguous to it by axiom C, the number of points in x^c which are contiguous to points in $(x^c)^c - x^c$ is finite. By the theorem we see that this is impossible.

Consider the sets $A = (x^c)^c - x^c$ and $B = x^c - x$. If A has any limit points these are included in $B \cup x$. By axiom C we see that x is a limit point of A. The following example shows that it is also possible for A to have limit points in B.

Example 4. In the Cartesian plane take a circle E_0 with radius 1 and centre (0,0). Let $P = (1,0)$ and $Q = (-1,0)$. For each positive integer n take an ellipse E_n with centre (0,0), major axis the line joining the points P and Q and eccentricity $1/(n+1)$. On the ellipse put 2^{n+1} points $P_{n1}, P_{n2}, \ldots, P_{n,2^{n+1}}, P_{n,2^{n+1}+1} = P_{n,1}$

arranged cyclically around the ellipse with $P_{n,1} = P$ and $Q = P_{n,2^n+1}$ and going in the same direction for each n. Let $P_{n,i}$ be contiguous to $P_{n,i+1}$; $P_{n,k}$ be contiguous to the points $P_{n+1,2k-2}$, $P_{n+1,2k-1}$, $P_{n+1,2k}$. Also let P be contiguous to Q. Take this configuration to be the top half of a sphere with equator the circle E_0 . Let the bottom half of the sphere be an ordinary hemisphere. Let the neighbourhoods of points be the usual metric neighbourhoods. (see fig. 3)

If P is the point x, we see that $Q \in B$ is a limit point of A.

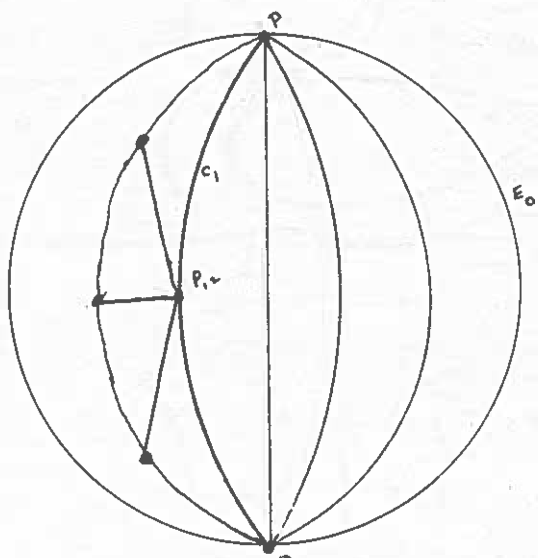


Fig. 3

We have not been able to prove the duality theorems. The proof in the ordinary theory breaks down as some of the open sets in our space may be too small to contain a fundamental cocycle. In the examples of spaces which we have given, the open sets consisting of single points have this property.

There is also the problem of defining a contiguous point two-sphere which would be an example of a two dimensional orientable generalized closed manifold. Klipple [8] discusses a contiguous point plane which is separated by any triune as well as any simple closed curve. Since in our homology theory a triune carries a one dimensional cycle which bounds on the triune itself, we would not want a triune to separate the plane or two sphere.

Chapter VII

SIMPLE CLOSED CURVE CONTINUED

In this chapter we shall show that the simple closed curve defined in Chapter V is an example of a one-dimensional orientable generalized closed manifold. We shall first show that an S' which satisfies the first denumerability axiom is locally connected. In this chapter all simple closed curves will be defined to satisfy the first denumerability axiom.

Theorem 1. Take the component D of a point p in the open U in an S'. If $S' - D$ contains a point y which is not contiguous to p, then

$$S' - p - y = A \cup B \text{ sep.}$$

Two situations are possible; either $D - p$ intersects both A and B or if $D - p \subset A$, then there exists a point x, an element of $B \cap (S' - U)$, such that x is contiguous to p.

Proof: Suppose $D - p \subset A$ and no point of $B \cap (S' - U)$ is contiguous to p. Then B has a limit point p. Suppose there exist only a finite number of components of U which intersect B. The closure of any of these components does not contain p, therefore U - the closure of these components is an open set containing p and not intersecting B. This is impossible as p is a limit point of B. Therefore there are an infinite number of components of U which intersect B.

Take $\{x_n\}$ approaching p such that they are ordered on B and each x_n lies in a different component of U. This is possible by the assumption that S' satisfies the first denumerability axiom. As the x_n's lie in different components, no x_i is contiguous to x_j for $i \neq j$. Consider the arc $I_1 = B \cup p \cup y$.

$I_1 - x_n = E_n \cup F_n$ sep where $p \in E_n$ and $y \in F_n$. The set E_n is not contained in U as otherwise p and x_n lie in the same component. Take $y_n \in E_n - U$. The point y_n is less than x_n in the order from p to y on arc I_1. There exist an infinite number of distinct y_n's, for otherwise there exists a first y_K in the order from p to y on the arc I_1. All x_n are greater than y_K by the choice of y_n's. Let

$I_1 - y_K = P_K \cup Q_K$ sep where $p \in P_K$ and $y \in Q_K$. The set $A \cup P_K$ is an open set containing p which does not intersect any of the x_n's. This is a contradiction as p is a limit point of the x_n's.

The set of y_n's have a limit point c as the space is compact. The point c is an element of I_1. Suppose c is not equal to p. Since $c \in B$, c is not contiguous to p.

$I_1 - c = R_1 \cup T_1$ sep where $p \in R_1$ and $y \in T_1$. Take d an element of R_1 which is not equal to p. This is possible as c is not contiguous to p.

$I_1 - d = R_2 \cup T_2$ sep where $p \in R_2$ and $y \in T_2$.

The set $R_2 \cup A$ is an open set containing all but a finite number of the x_n's and therefore all but a finite number of the y_n's. Hence T_2 contains only a finite number of the y_n's. However $T_2 - y$ is an open set containing c and therefore must contain more than a finite number of the Y_n's as c is a limit point of the y_n's. This means that $c = p$ and p is a limit point of $S' - U$. This also is impossible as U is an open set. We are lead to a contradiction and hence there exists a point x, an element of $B \cap (S' - U)$ such that x is contiguous to p.

<u>Theorem 2.</u> <u>Given a point p contained in an open subset U of S', three situations are possible.</u>

<u>1. There exist two points y and z such that y is contiguous to p and p is contiguous to z and y and z are elements of $S' - U$.</u>

<u>2. There exists an arc lying in U containing p interior to it.</u>

<u>3. There exists an arc lying in U with p as end point and such that p is contiguous to a point $y \in S' - U$.</u>

Proof: Let D be the component of U containing p.

(a) Suppose $D = S'$.

Take a,b a pair of non contiguous points such that neither is equal to p.

$S' - a - b = A \cup B$ sep where $p \in A$.

This gives an arc $a \cup A \cup b$ containing p in the interior.

(b) Suppose $D = p$. Take $x \in S' - D$ such that x is not contiguous to p.

$S' - x - p = A \cup B$ sep.

By theorem 1, there exist $y \in A$ and $z \in B$ such that y is contiguous to p and p is contiguous to z. Since $D = p$, neither y nor z are elements of U.

(c) Suppose $S' \neq D \supsetneq p$. Suppose there exists $x \in S' - D$, such that x is not contiguous to p.

$S' - x - p = A \cup B$ sep.

The set D intersects either A or B, say A. Take $a \in A \cap D$. Let I_1 and I_2 be the two arcs $x \cup A \cup p$ and $x \cup B \cup p$ respectively. If a is contiguous to p, $a \cup p$ is an arc contained in D. If a is not contiguous to p,

$S' - a - p = A_1 \cup A_2$ sep where $x \in A_2$. Since D is a connected set containing a and p but not x, $D \supset A_1$ by theorem 4, chapter V. This gives an arc $a \cup A_1 \cup p$ with end point p contained in D. If $D \cap B = 0$, by theorem 1, we have case 3. If $D \cap B$ is not zero, we may obtain in the same way an arc $b \cup B \cup p$ contained in D with end point p. The union of these two arcs gives an arc with p interior to it contained in D.

If every point in $S' - D$ is contiguous to p, then $S' - D$ consists of the single point x since there are at most two points in S' which are contiguous to p and if these both lie in $S' - D$ then D consists of a single point and we have (b). If we take any point in D which is not contiguous to p and divide the space into two arcs we obtain an arc with p as end point and lying in D and again we obtain case 3. This concludes the proof of the theorem.

This theorem leads immediately to the following theorem 3.

<u>Theorem 3.</u> <u>A simple closed curve which satisfies the first denumerability axiom is locally connected.</u>

<u>Theorem 4.</u> <u>The simple closed curve is one dimensional. i.e. If $\mathcal{U}$ is any covering of S', there exists a covering $\mathcal{V} > \mathcal{U}$ such that the nerve of $\mathcal{V}$ has no two cells.</u>

Proof: As the simple closed curve is locally connected we may take $\mathcal{U}$ consisting of open arcs. (arcs minus the end points which are limit points of the rest of the space).

1. There exists a refinement in which no three elements of the covering cover the whole space. Suppose three elements U_0, U_1, U_2 of the covering cover the space but no two of them cover the space. Take $x \in U_0 - U_1 - U_2$, $y \in U_1 - U_0 - U_2$, $z \in U_2 - U_0 - U_1$. There exists a point p not equal to any of these three points. Suppose $p \in U_0$, such that $p \neq x$. Take U_0' an open set containing p which is closure contained in U_0 and which does not contain x, y or z. Take $U_0'' \subset\subset U_0'$ such that $p \in U_0''$. Take a new covering of the space consisting of the open sets U_0', $U_i - \overline{U_0''}$ for $i = 0,1,2$. This gives a refinement of $\mathcal{U}$ such that no three elements cover S. Take refinement

consisting of arcs.

2. Suppose there exists a two cell formed by the elements of the covering U_0, U_1, U_2. We shall show that the union of two of these open sets contains the third and hence one of them may be dropped and still we will have a covering of the space.

Suppose that the unions of no two of these sets contains the third. Take three points x, y and z such that $x \in U_0 - U_1 - U_2$, $y \in U_1 - U_0 - U_2$, $z \in U_2 - U_0 - U_1$. Since these three points are distinct and the simple closed curve does not contain a triune at least two of them are not contiguous to each other. Suppose x is not contiguous to y. Take p an element of the space which is not covered by any one of these three sets.

$S' - x - y = A \cup B$ sep where $p \in B$.

If $z \in A$ then since U_2 is a connected set contained in $S' - x - y$, the set $U_2 \subset A$. Also the set $U_0 \cup U_1$ is a connected set containing x and y and not containing p and therefore by theorem 4, chapter V, $U_0 \cup U_1 \supset A \supset U_2$. This is a contradiction.

If $z \in B$, then z is not contiguous to both x and y, say z is not contiguous to x.

$S' - z - x = C \cup D$ sep where $p \in C$ and $y \in D$.

In the same way we derive a contradiction and hence we know that one of the open sets U_0, U_1 or U_2 can be dropped and still we have a covering of the space.

Hence if $\mathcal{U}$ is a covering of the space, there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that the nerve of the covering $\mathcal{V}$ is one dimensional.

Definition: A reduced covering will be a covering such that no element of the covering is contained in the union of the other elements of the covering.

We have just shown that the family of reduced coverings is a complete family of coverings of the space such that every element of the family is one dimensional.

Theorem 5. If $\mathcal{U}$ is a covering which cannot be reduced and consists of more than two elements, then each 0-cell of a covering $\mathcal{U}$ is on the boundary of two and only two 1 cells of $\mathcal{U}$.

Proof: 1. Each 0 cell is on the boundary of two 1 cells. Consider U_0 equal to the open arc I_0 with end points a_0 and b_0. The point a_0 is either covered by U_1 or there exists a point a_1, an element of U_1 such that a_1 is contiguous to a_0. Similarly b_0 is either covered by U_2 or there exists a point b_1, an element of U_2 such that b_0 is contiguous to b_1. The set U_1 is not equal to U_2 as otherwise $U_1 \supset a_1 \cup b_1$ and therefore $U_1 \supset U_0$ or $U_1 \supset S' - U_0$. In the first case U_0 is superfluous and in the second case two elements of the coverings cover S' both contrary to the hypothesis. Hence there exist two 1-cells, U_0, U_1 and U_0, U_2 and U_0 is on the boundary of each.

2. Each 0-cell is on the boundary of at most two 1-cells. Suppose also there exists a U_3 such that $U_0 \, U_3$ is a 1-cell. Take $x_i \in U_i - \bigcup_{j \neq i} U_j$ for $i = 0,1,2,3$. Suppose x_1 is not contiguous to x_2.

$$S' - x_1 - x_2 = A_1 \cup A_2 \text{ sep}$$

where $U_0 \cup U_3 \subset A_1$.

The set $U_1 \cup U_2$ is a connected set containing A_2. Let $I = x_2 \cup A_1 \cup x_1$.

$$I - x_3 = B_1 \cup B_2 \text{ sep}$$

where $x_1 \in B_1$ and $x_2 \in B_2$.

Suppose $U_0 \subset B_1$, then x_3 is not contiguous to x_1 by theorem 1, chapter V.

$$S' - x_1 - x_3 = (B_1 - x_3) \cup (B_2 \cup A_2) \text{ sep.}$$

The set U_0 is contained in $B_1 - x_3$, and the set U_2 is contained in $B_2 \cup A_2$ as $x_2 \in B_2$. From this we conclude that U_0 and U_2 are contiguously separated. This is a contradiction and the theorem is proved.

<u>Theorem 6. If $Z'(\mathcal{U})$ is a cycle on a reduced covering $\mathcal{U}$, consisting of more than three open sets, then the set of open sets appearing in the 1-cells of the chain $Z'(\mathcal{U})$ covers S. Hence $|Z'(\mathcal{U})|$ contains all the 1-cells of the nerve of $\mathcal{U}$.</u>

Proof. If $Z'(\mathcal{U})$ contains the terms $U_0 \, U_1$ it also contains the term $U_n \, U_0$ and $U_1 \, U_2$. Eventually it contains the terms (1) $U_0 \, U_1 + U_1 \, U_2 + \dots + U_{n-1} \, U_n + U_n \, U_0$.

Take $x \in U_0$ and a nuclear element of $U_0 \, U_1$, $y \in U_2$

and a nuclear element of $U_1 \cup U_2$. The points x and y are not contiguous as U_0, U_2 is not a 1-cell.

$$S' - x - y = A_1 \cup A_2 \text{ sep.}$$

Suppose the set $U_1 - U_0 - U_2$ which is a connected set lying in $S' - x - y$ (theorem 5, chapter V), is contained in A_1. Consider the set $C = U_0 \cup U_2 \cup U_3 \ldots \cup U_n$. It is a connected set containing x and y and hence containing either A_1 or A_2. If $C \supset A_1$, then $C \supset U_1$ which is impossible. Therefore $C \supset A_2$. The set U_1 is a connected set containing x and y or points contiguous to these. If $U_1 \supset A_2$, then $(U_1 \cap U_0) \cup (U_1 \cap U_2) \supset A_2$. Since we have no two cells $U_1 \cap U_0$ and $U_1 \cap U_2$ are separated sets and A_2 is contained in one of them, say $U_1 \cap U_0$. The set A_2 has a limit point or a point contiguous to y, therefore since $y \in U_2$, thus set U_2 intersects $U_1 \cap U_0$ or has a point contiguous to this and again we have a 2-cell which is impossible. Therefore $U_1 \supset A_1$ and $\bigcup_0^n U_i \supset S'$.

If a 1-cell of the nerve of $\mathcal{U}$ does not appear in (1) then its boundary cells are contained in the sum $\bigcup_{i=0}^{n} U_i$ and neither of them is equal to U_i for any $i = 0 \ldots n$. As this is impossible we conclude that all the 1-cells of the nerve of $\mathcal{U}$ are in the chain $Z'(\mathcal{U})$. All the 1-cycles of $\mathcal{U}$ are just multiples of (1) and $H'(\mathcal{U}) = \mathcal{F}$ and $p'(\mathcal{U}) = 1$.

<u>Theorem 7.</u> <u>$p'(S) = 1$.</u>

Proof: Consider the set Σ' of all connected reduced coverings. This is a complete system of coverings. Set up a C-cycle as follows. For $\mathcal{U} \in \Sigma'$ let $Z'(\mathcal{U})$ be the chain given in (1). Take $\mathcal{V} > \mathcal{U}$ and let $\pi V_i = U_j$ for j the smallest k such that $V_i < U_k$. Consider $\pi_{\mathcal{V}}^{\mathcal{U}} Z'(\mathcal{V})$. We will show that this equals $Z'(\mathcal{U})$. To do this we show that $U_i U_{i+1}$ appears once and only once in $\pi_{\mathcal{V}}^{\mathcal{U}} Z'(\mathcal{V})$.

1. Suppose $U_0 U_1$ appears twice in $\pi_{\mathcal{V}}^{\mathcal{U}} Z'(\mathcal{V})$.

$V_0 V_1 \to U_0 U_1 : V_K V_{K+1} \to U_0 U_1$.

$V_0 \to U_0 : V_1 \to U_1 : V_0 < U_0 : V_1 < U_1 : V_1 \not< U_0$.

$V_K \to U_0 : V_{K+1} \to U_1 : V_K < U_0 : V_{K+1} < U_1 : V_{K+1} \not< U_0$.

The set V_1 contains a point of V_0 or a point contiguous to V_0 and hence V_1 contains a point of U_0 or a point contiguous to U_0. Therefore V_1 and also V_{K+1} contains a boundary point of U_0. Let a and b be the boundary points of U_0 and c and d the boundary points of V_0. Let c be the nearer of the two to a on the arc U_0. As V_1 is not contained in V_0, the set V_1 contains either c or d, say c but not d. If V_1 contains the boundary point b, then as b and c are not contiguous, V_1 must contain a. The set V_{K+1} does not contain c or we would have a two cell; hence it contains d and also b. But $V_{K+1} \cup V_1 < U_1$ and therefore U_1 contains both a and b which is impossible.

2. Suppose $U_0 U_1$ does not appear at all in $\pi_{\mathcal{V}}^{\mathcal{U}} Z'(\mathcal{V})$.

There exists a $V \subset U_0$ since otherwise the U's which do contain V's would cover the space and we would have U_0 covered by the sum of the other elements of the covering. Take any $V_0 \subset U_0$ and consider V_1. If $V_1 \subset U_0$ consider V_2. Finally we reach $V_k \not\subset U_0$ such that $V_{k-1} \subset U_0$. The set V_k is contained in either U_1 or U_n as suppose $V_k \subset U_i$ $i \neq 1, n$. The set V_{k-1}, V_k form a 1-cell hence so do U_0 U_i which is impossible. If $V_k \subset U_n$ take V_n, V_{n-1}, in that order and eventually get V_i $V_{i+1} \to U_0$ U_1. Therefore $\{Z'(\mathcal{U})\}$ is a C-cycle which fails to bound on every covering. Any other C-cycle is a constant multiple of this one. This proves the theorem.

<u>Theorem 8.</u> <u>If F is a closed proper subset of S' or F consists of S' with one contiguity dropped then $p'(F) = 0$.</u>

Proof: 1. Let F be a closed proper subset of S'. Take x an element of $S' - F$. Take P an open set such that $F \subset P \Subset S' - x$. Let $\mathcal{U}$ be a reduced refinement of the covering $(S' - F, P)$. Let U_x be the element of which contains x. Then $U_x \cap F = 0$. However $|Z'(\mathcal{U})|$ contains U_x. This shows that $Z'(\mathcal{U})$ does not lie on F. As there exist no other 1-dimensional cycles on the space $p'(F) = 0$.

2. Let $F = S$ but the contiguity x_1 to x_2 is dropped. Let P_{x_1} and P_{x_2} be open sets containing x_1 and x_2 respectively such that $P_{x_1} \cap P_{x_2} = 0$. Let $\mathcal{X}$ be a reduced

refinement of the covering $(P_{x_1}, P_{x_2}, S' - x_1 - x_2)$. Let U_{x_1} and U_{x_2} be the elements of this coverig which contain x_1 and x_2 respectively. The cycle $Z'(\mathcal{U})$ contains all the 1-cells of the nerve of $\mathcal{U}$ but in P we do not have the 1-cell U_{x_1}, U_{x_2}. Hence $Z'(\mathcal{U})$ is destroyed if we omit the contiguity x_1 to x_2. This proves the theorem.

Theorem 9. The simple closed curve is 1-lc.

The proof of this is immediate as there exist no local 1-dimensional cycles.

Theorem 10. The simple closed curve is 0-colc.

Proof. If x is any point of S', take P an open set containing x such that $\overline{P}^c$ does not contain S'. If $Z_0(\mathcal{U})$ is a cocycle contiguously in P then $Z_0(\mathcal{U}) \cap Z'(\mathcal{U})$ is a 1-cycle contiguously on P. As this is impossible, there exists no 0-cocycle contiguously on P.

Theorem 11. If $x \in S$, then $p'(x) = 1$

Proof. Take P an open set containing x such that $\overline{P}^c \subsetneq S'$. We will show that if σ_1' and σ_2' are two 1-cells of a reduced covering $\mathcal{U}$ which lie contiguously in P, then there exists a 0-chain C^0 such that

$$\delta C^0 = \sigma_1' - \sigma_2'$$

Let $\sigma_1' = U_0 U_1$ with nucleus $x_0 x_1$ and $\sigma_2' = U_0' U_1'$ with nucleus $x_0' x_1'$. Since P is a connected set by the simple chain theorem (theorem 6 chapter II) there exists a simple chain $U_0'', \ldots, U_k''$ from x_0 to x_0'. where $U_0'' \ni x_0$ and $U_k'' \ni x_0'$. If $x_0 = x_1$ and $x_0' = x_1'$ take

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