

Lab Notes on the exceptional Lie group E_8 at the prime 2

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Dedicated to Morton L. Curtis (1921-1989).

ABSTRACT. This is an account of the author's use of computer algebra tools to explore the structure of the maximal elementary abelian 2-subgroups of the exceptional Lie group E_8 . The principal result obtained thus far by these methods is that any rank 8 connected 2-compact group (BX, X) with Weyl group isomorphic to that of the exceptional Lie group E_8 has its normalizer of the maximal torus isomorphic to that of E_8 at the prime 2. Similar results hold for the comparison of possible exotic forms of G_2 , $DI(4)$, F_4 , and $E_7/\mathcal{Z}(E_7)$ to the standard forms.

Corollaries of this result include that the Krull dimension of the mod 2 cohomology of such BX is 9 and that the cohomology ring is not Cohen-Macaulay.

The proof follows from

- (1) calculations of the ambient cohomology group $H^2(BW(G), T_G)$ that classifies extensions

$$1 \rightarrow T_G \rightarrow N_G(T_G) \rightarrow W(G) \rightarrow 1$$

and the

- (2) discovery of subgroups in common between the "real" E_8 and possible W -clones X which are large enough to detect the k -invariant.

Thanks to the National Science Foundation, Purdue University, Johns Hopkins University, and Fukuoka University for financial support during this research and the 2000 sabbatical of the author. Thanks to the Clay Foundation for travel support during this research.

1. Introduction and Outline

E_8 is one of the few remaining simple exceptional Lie groups G for which the complete structure of $H^*(BG, F_2)$ is not known. In 1987, using details about the Lie representation theory of the semi-spin (i.e. half-spin) group $SSpin(16)$ and E_8 , Frank Adams [1] determined that the maximal elementary abelian 2-subgroups of E_8 fall into two conjugacy classes, of ranks 8 and 9 respectively. This answered a question raised by Borel in the notes to his collected works [5] – namely, whether the 2-rank of E_8 is 9 or 10. Work of Quillen [22] allows us to interpret this maximal elementary abelian 2-subgroup information in terms of the commutative algebra structure of $H^*(BE_8, F_2)$ – it has Krull dimension nine and (see our last section) is not Cohen-Macaulay.

Classification efforts for 2-compact groups have narrowed to those with the Weyl groups of standard Lie groups but with possibly exotic multiplicative structures. In this setting the tools of representation theory are not available. This note is a test case to explore the extent to which the abstract reflection group structure of the Weyl group forces the overall structure of the connected 2-compact group. It is labwork both in the sense that it studies the special case of E_8 and that it is experimental – the facts were first obtained using the symbolic computational tools **GAP**, **CoCoA**, **Magma**, and **Macaulay 2**.

These experiments were successful in that we show that any connected rank eight 2-compact group X with the same Weyl group as E_8 must also have its normalizer of the maximal torus isomorphic to the fiberwise-2-completion of the normalizer in the standard E_8 .

THEOREM 1.1. *If X be a connected 2-compact group of rank eight with $W(X)$ abstractly isomorphic to $W(E_8)$. Then $BN_X(T_X)$ is homotopy equivalent to the fiberwise 2-completion of $BN_{E_8}(T_{E_8})$. In particular, $H^*(BN_X(T_X), \mathbb{F}_2)$ and $H^*(BX, \mathbb{F}_2)$ have Krull dimension nine and $H^*(BX, \mathbb{F}_2)$ does not have the Cohen-Macaulay property.*

One can explain in general terms why the 2-rank of E_8 and its clones must have Krull dimension at least nine:

THEOREM 1.2. *Let X be a 2-compact group such $W(X)$ contains a central element c that acts as -1 on $\pi_1(T_X)$. If T^W is trivial, then there is an elementary abelian 2-subgroup $E \rightarrow X$ with $2\text{-rank}(E) = \dim T_X + 1$.*

The above is applicable to 2-compact groups with the Weyl groups of type D_{2k} , $\text{adj}(C_n)$, $\text{adj}(E_7)$, G_2 , and F_4 as well as that of E_8 . The methods of calculation of this paper produce analogues of Theorem 1.1 in the G_2 , $DI(4)$, F_4 , and $\text{adj}(E_7)$ cases.

Recall that the classifying map (k-invariant) for the extension $T_X \rightarrow N_X(T_X) \rightarrow W(X)$ can be viewed as a member of the cohomology group $H^3(W(G), \pi_1(T_G))$. The strategy is rather straight forward. First, we use computer algebra programs to calculate $H^3(W(G), \pi_1(T_G))$ in the cases of exceptional 2-compact groups.

Results using Derek Holt's *cohomolo* routine in GAP3:

CALCULATION 1.3.

Weyl group	Order	$H^3(W(G), \pi_1(T_G))$
$W(G_2)$	12	0
$W(DI(4))$	336	$\mathbb{Z}/2\mathbb{Z}$
$W(F_4)$	1152	$\mathbb{Z}/2\mathbb{Z}$
$W(\text{adj}(E_6))$	$2^7 3^4 5$	$\mathbb{Z}/2^k \mathbb{Z}$, $k \geq 1$
$W(E_7)$	$2^{10} 3^4 5^1 7$	$(\mathbb{Z}/2\mathbb{Z})^3$
$W(\text{adj}(E_7))$	$2^{10} 3^4 5^1 7$	$\mathbb{Z}/2\mathbb{Z}$
$W(E_8)$	$2^{14} 3^5 5^2 7$	$\mathbb{Z}/2\mathbb{Z}$
$W(\text{Spin}(12))$	$2^9 3^1 5^1$	$(\mathbb{Z}/2\mathbb{Z})^6$

Results using Kasper Andersen's implementation of the De Concini-Salvetti algorithm in Magma 2.8:

CALCULATION 1.4.

Weyl group	Order	$H^3(W(G), \pi_1(T_G))$
$W(G_2)$	12	0
$W(DI(4))$	336	$\mathbb{Z}/2\mathbb{Z}$
$W(F_4)$	1152	$\mathbb{Z}/2\mathbb{Z}$
$W(\text{adj}(E_6))$	50840	$\mathbb{Z}/2\mathbb{Z}$
$W(E_7)$	2,903,040	$(\mathbb{Z}/2\mathbb{Z})^3$
$W(\text{adj}(E_7))$	2,903,040	$\mathbb{Z}/2\mathbb{Z}$
$W(E_8)$	696,729,600	$\mathbb{Z}/2\mathbb{Z}$
$W(\text{Spin}(12))$	23,040	$(\mathbb{Z}/2\mathbb{Z})^6$

Derek Holt's routines work with $\mathbb{Z}/p\mathbb{Z}$ modules over finite groups, while the De Concini-Salvetti algorithm works with integral or modular modules over a Coxeter group. The results agree, except in the case of $adj(E_6)$, where the Holt program can only determine the number of 2-primary summands.

The second piece of our strategy is to demonstrate that in all the cases mentioned above (except G_2), $N_X(T_X)$ must have the non-trivial k -invariant! We achieve this by showing that G or its clone X contains a proper maximal rank subgroup H for which the associated k -invariant of its NT is non-trivial. Recall that if H is maximal rank in G , then $W(H)$ is a reflection subgroup of $W(G)$ and $N_H(T_H)$ is the pullback of $N_G(T_G)$ to $W(H)$.

In the Lie case, one has some freedom of choice for such subgroups. For example, for E_8 , there is a copy of $Spin(16)/C_2 = SSpin(16)$. Here C_2 is a $\mathbb{Z}/2\mathbb{Z}$ suitably embedded in the center of $Spin(16)$.

However, from the viewpoint of 2-compact groups $SSpin(16)$ is no better understood than E_8 . Hence this paper seeks simpler maximal rank subgroups which are "well-known" in both settings. One useful collection consists of the 2-compact subgroups which have Weyl groups generated by commuting reflections. We'll discover these subgroups as centralizers of certain elementary abelian 2-subgroups of X . Recall that G is semi-simple if $\pi_1(G)$ is finite.

PROPOSITION 1.5. [11] *Let G be a connected semi-simple 2-compact group such that $W(G)$ is abelian. Then there exists a compact Lie group $H = (SU(2))^n$ and a subgroup $V_G \subset \text{Center}(H) = (\mathbb{Z}/2\mathbb{Z})^k$ such that*

$$BG \approx B(H/V_G),$$

up to 2-completion.

Thus the Weyl group provides part of the recognition algorithm for the desired subgroups. The second ingredient assures that the examples obtained as centralizers are actually connected.

PROPOSITION 1.6. [12] *Let G be a connected 2-compact group and $A \rightarrow T_X$ a 2-abelian subgroup. Suppose further that no reflection in $W(G)$ acts as the identity on the elements of order 2 in T_X . Then the centralizer $C_X(A)$ is connected if and only $W(C_X(A)) = \{w \in W(G) : w|_A = Id_A\}$ is generated by reflections,*

The hypothesis above holds for all simple 2-adic reflection groups except those of type $\{B_n, C_n\}$. The search for suitable centralizers was first carried out by computer algebra:

THEOREM 1.7. *Let G be a compact connected 2-compact group of exceptional type not E_6 , and let T_G be its maximal discrete torus. There exists a subgroup E of ${}_2T_G$ such that the Weyl group of $C_G(E)$ is generated by $rk(G)$ commuting reflections.*

It follows that $C_G(E) \approx (SU(3))^{rk(G)}/V_G$ for some V_G . In the particular case of E_8 or clones, V_G is a 4-dimensional subspace of $\mathcal{Z}(SU(2)^8)$, embedded as the [8, 4] Hamming code. We emphasize that the isomorphism type of this $C_G(E)$ depends only on the action of $W(G)$ on T_G and therefore it is present in both the Lie model and in any clone.

Third, we prove that the extension class for $N_H(T_H)$ of this centralizer is non-zero:

THEOREM 1.8. *For each of the connected 2-compact groups of exceptional type not G_2 , the $C_G(E)$ described above does not have a split normalizer of the maximal torus.*

For example, for E_8 , if the normalizer were split the 2-rank of $C_{E_8}(V_4)$ would be at least 12. However, direct computations show that the 2-rank of $C_{E_8}(V_4)$ is only 9. Hence the NT for $C_{E_8}(V_4)$ is not split, and therefore the NT for E_8 or a clone is not split either.

Two strategies evolved for verifying that $C_G(E)$ has a relatively small 2-rank. Both have successful computational implementations. The first is a Grobner basis attack on the computation of an estimate of the Krull dimension of $H^*(BC_G(E), \mathbb{Z}/2\mathbb{Z})$. The second attack uses maximal clique algorithms to search for maximal elementary abelian 2-subgroups in $C_G(E)$. Both methods work in all the exceptional cases, but require non-trivial resources for the E_8 case.

THEOREM 1.9. *For each simple 2-adic Weyl group W of exceptional type, if G and H are connected centerfree 2-compact groups with W as Weyl group, then $N_G(T_G)$ is isomorphic to $N_H(T_H)$.*

We do not currently have a direct path from knowledge of $N_X T_X$ to the determination of BX , although such a principle holds in the Lie category. However, several authors [19, 25, 13] have established *ad hoc* methods for this in special cases. The following corollary is then a combination of those results together with those of this paper:

COROLLARY 1.10. *For the exceptional types G_2 , $DI(4)$, and F_4 , any connected 2-compact Lie group with this Weyl group is isomorphic to the standard example.*

Parts of the above strategy work for non-exceptional groups. I have included $W(\text{Spin}(12))$ in several of the tables to illustrate this. However, for most Lie groups G , $H^3(W(G), \pi_1(T_G))$ has rank larger than one. Therefore the above strategy in those cases fails to identify the extension class, except to prove that it is non-zero.

One motivation for this paper was a long fascination with the work of Curtis-Wiederhold-Williams, [8], and Tits, [24]. The extension of these ideas to the 2-compact group realm has seemed rather mysterious and this paper at best lifts the veil only slightly. The author dedicates this paper to the memory of Morton L. Curtis, who introduced the beautiful subject of Lie groups to him in 1965.

The author is happy to acknowledge mathematical conversations and e-mails with a long list of people, including K. Andersen, L. Avramov, Dave Benson, Jon Carlson, Bill Dwyer, Dan Grayson, Derek Holt, Jean Lannes, Brad Lucier, and Brend Ulrich. Versions of this material were given in talks at the Oberwolfach Conference on Group Cohomology and Representation Theory(2000), Bonn M.P.I. Workshop on Algebraic Topology (2001), M.I.T.(2001), and Northwestern (2002).

The author would also like to thank the many authors of GAP, Macaulay 2, CoCoA, and Magma. These tools allowed an insight into the structure of E_8 that would have otherwise been impossible.

2. The cohomology computation

Two independent algorithms are available for the calculations. The first is the *cohomolo* package for GAP3 by Derek Holt, [17]. It has the advantage of being applicable to general finite groups. Because it uses reduction to the p -Sylow subgroup, the modules need to be modules over $\mathbb{Z}/p\mathbb{Z}$. Luckily, for all of the groups in question, except $W(E_6)$, the it suffices to use $\mathbb{Z}/2\mathbb{Z}$ calculations, [14]. For the $W(E_6)$ case, we learn that $H^3(W(E_6), \pi_1(E_6))$ consists of a single cyclic 2-group, but the method does not yield its order.

The relatively small resolutions described by De Concini and Salvetti, [9] provide the second method. This algorithm uses the Coxeter structure and seems rather effective. Kasper Andersen [3] has implemented the De Concini-Savetti algorithm in MAGMA and permitted me to use it. The algorithm works for modules with base ring \mathbb{Z} and verifies the GAP3 calculations as well as providing the definitive answer for the $W(E_6)$ case.

Source files for the scripts used in these calculations are available from the author's website,
<http://www.math.purdue.edu/~wilker/labnotes>.

THEOREM 2.1. *Let $W(E_8)$ be the Weyl group of the exceptional compact Lie group E_8 and $\sigma : W(E_8) \rightarrow GL(8, \mathbb{Z})$ the standard reflection group action on $L = \pi_1(T_{E_8})$.*

- (1) *Up to inner automorphism in $GL(8, \mathbb{Z}_2)$, this is the unique 8-dimensional reflection representation of $W(E_8)$.*
- (2) *This induces an action on $T_{E_8} = (L \otimes \mathbb{R})/L$ such that $H^*(W(E_8), L)$, $H_*(W(E_8), L)$, $H^*(W(E_8), T_{E_8})$ and $H_*(W(E_8), T_{E_8})$ are elementary abelian 2-groups.*
- (3) *Also, $H^3(W(E_8), L) = H^2(W(E_8), T_{E_8}) = H^2(W(E_8), L/2L) = \mathbb{Z}/2\mathbb{Z}$.*

Similar calculations can be performed for the other exceptional 2-compact groups. Let G be a connected simple 2-compact of exceptional type, with $L = \pi_1(T_G)$ the $W(G)$ -lattice and $M = L/2L$. We summarize the results of the GAP/*cohomolo* calculations:

PSEUDO-CODE 2.2.

- (1) *Define $W(G)$ and action of $W(G)$ on M in GAP3.*
- (2) *Use these to define a cohomolo data structure, **CHR**.*

- (3) Call the functions **FirstCohomologyDimension(CHR)** and **SecondCohomologyDimension(CHR)** from the package **cohomolo** from within *GAP3*.

The procedure for Andersen's MAGMA routines is similar:

PSEUDO-CODE 2.3.

- (1) Specify Weyl group to used.
- (2) Specify which lattice to use as the module.
- (3) Call the routines **BuildMatrix**.
- (4) Use these matrices to calculate kernel/image.
- (5) Format and print results.

CALCULATION 2.4. $H_0(W, L)$, $H^*(W, L/2L)$, and $H^3(W, L)$

Weyl group	H_0	H^0	H^1	H^2	$H^3(W, L)$
$W(G_2)$	0	0	0	0	0
$W(DI(4))$	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$W(F_4)$	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$W(\text{adj}(E_6))$	$\mathbb{Z}/3\mathbb{Z}$	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$W(E_7)$	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^4$	$(\mathbb{Z}/2\mathbb{Z})^3$
$W(\text{adj}(E_7))$	$\mathbb{Z}/2\mathbb{Z}$	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$W(E_8)$	0	0	0	0	$\mathbb{Z}/2\mathbb{Z}$
$W(\text{Spin}(12))$	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^4$	$(\mathbb{Z}/2\mathbb{Z})^8$	$(\mathbb{Z}/2\mathbb{Z})^6$

REMARK 2.5. The rightmost column is deduced from the long exact sequence corresponding to the SES $0 \rightarrow L \rightarrow L \rightarrow L/2L \rightarrow 0$, and the fact the integral cohomology groups (except in the $W(E_6)$ case) are known to be elementary abelian 2-groups. Notice that these methods do not determine $H^3(W(E_6), L)$ completely. However, the calculations with Andersen's program verifies the above calculations in all cases, and supplies the result for E_6 . The $\text{Spin}(12)$ group example is included to show that the strategy of this note will not completely determine $N_G(T_G)$ in general.

REMARK 2.6. The E_8 calculation by the Holt routines requires about 3 minutes on a 850 mhz Pentium III machine. The same calculation using Andersen's code for the De Concini-Salvetti resolutions, requires about 30 seconds. Calculations seem to be feasible by these methods for ranks less than 12.

3. Finding a $(SU(2))^8/V_4$ within E_8 and its W -clones

In the cases of G_2 , $DI(4)$, and F_4 there are maximal rank subgroups of the form $SU(2)^k/\Delta(\mathbb{Z}/2\mathbb{Z})$ that arise as centralizers of elementary abelian 2-subgroups. Here $\Delta(\mathbb{Z}/2\mathbb{Z})$ is a $\mathbb{Z}/2\mathbb{Z}$ embedded diagonally in $(\mathbb{Z}/2\mathbb{Z})^k$ with respect to the product coordinates. One must generalize somewhat to understand E_8 .

REMARK 3.1. Cohen-Seitz, [7] used subgroups similar to the ones below to find the 2-ranks of the exceptional groups.

If $\phi : SU(2)^{rk(G)} \rightarrow G$ is a surjective homomorphism with finite kernel then

- (1) $\ker \phi \subseteq \mathcal{Z}(SU(2)^{rk(G)}) \approx (\mathbb{Z}/2\mathbb{Z})^{rk(G)}$.
- (2) $W(\text{image } \phi)$ is generated by $rk(G)$ pairwise commuting reflections from $W(G)$.
- (3) For $V_\phi = \text{Image } \mathcal{Z}(SU(2)^{rk(G)})$, V_ϕ is centralized by at least $W(\text{image } \phi) \subseteq W(G)$.

To find a subgroup of E_8 of the form $(SU(2)^8)/V_G$, first perform a search for 8 pairwise commuting reflections within $W(E_8)$. In GAP3 such a search is not difficult:

PSEUDO-CODE 3.2.

- (1) Define $G = W(E_8)$.
- (2) Define $R = \text{Reflections}(G)$.
- (3) Define $S = \text{SylowSubgroup}(G, 2)$.
- (4) Define $\text{Ref}s = R \cap S$.
- (5) Define $A = \text{Group}(\text{Ref}s)$.
- (6) Test for A abelian.
- (7) Find $\text{card}(A)$.

Given the large size of $W(E_8)$ this worked rather quickly – a few seconds in GAP3.

CALCULATION 3.3. *Each 2-Sylow subgroup of $W(E_8)$ contains eight reflections. These commute and thus generate an elementary abelian 2-subgroup A_W of rank 8.*

The next step is to realize A_W as the Weyl group of the centralizer of some subgroup of T_{E_8} .

THEOREM 3.4. *Let G be a connected 2-compact group and $P \subset T_G$ a 2-toral subgroup. Then $C_G(P)$ and $N_G(P)$ are defined, and $W(C_G(P)) =$*

$\{w \in W(G) | w|P = Id_P\}$ and $W(N_G(P)) = \{w \in W(G) | w(P) \subset P\}$. $C_G(P)$ and $N_G(P)$ have a common identity component. If $W(C_G(P))$ is generated by reflections which do not act as the identity mod 2 on L_G , then $C_G(P)$ is connected.

We need to find suitable subgroups of T_G to centralize. The obvious candidate is $T_G^{A_W}$:

PSEUDO-CODE 3.5.

- (1) Define the action of $W(G)$ and A_W on $L/2L = V = {}_2T_G$.
- (2) Compute V^{A_W} .
- (3) Compute $Centralizer(W, V^{A_W})$.
- (4) Compute $Normalizer(W, V^{A_W})$.
- (5) Compute $Normalizer(W, V^{A_W})/Centralizer(W, V^{A_W})$.

CALCULATION 3.6.

For the A_W above $V^{A_W} \subset V$ has dimension 4. $C_W(V^{A_W}) = A_W$, and $N_W(V^{A_W}) = N_W(A_W)$ has 344064 elements and acts by permutations on the reflection generators of A . $\Gamma_{A_W} = \Gamma_4 = N_W(V^{A_W})/C_W(V^{A_W})$ also acts by permutations on the reflections in V_W and has order 1344.

Similar results hold for other 2-compact groups of exceptional type:

CALCULATION 3.7.

Weyl group	A_W	$\dim V^{A_W}$	$\#N_W(A_W)$	$\#\Gamma_{A_W}$
$W(G_2)$	$(\mathbb{Z}/2\mathbb{Z})^2$	1	4	3
$W(DI(4))$	$(\mathbb{Z}/2\mathbb{Z})^3$	1	48	6
$W(F_4)$	(a) $(\mathbb{Z}/2\mathbb{Z})^4$	3	384	24
$W(F_4)$	(b) $(\mathbb{Z}/2\mathbb{Z})^4$	1	384	24
$W(E_6)$	$(\mathbb{Z}/2\mathbb{Z})^4$	3	384	24
$W(E_7)$	$(\mathbb{Z}/2\mathbb{Z})^7$	4	21504	168
$W(\text{adj}(E_7))$	$(\mathbb{Z}/2\mathbb{Z})^7$	3	21504	168
$W(E_8)$	$(\mathbb{Z}/2\mathbb{Z})^8$	4	344064	1344
$W(\text{Spin}(12))$	$(\mathbb{Z}/2\mathbb{Z})^6$	4	1536	24

THEOREM 3.8. Γ_4 permutes the 8 reflections in $A_W \subset W(E_8)$. This permutation action can be identified with the action of

$$AGL_3(\mathbb{F}_2) = (\mathbb{Z}/2\mathbb{Z})^3 \rtimes GL_3(\mathbb{F}_2)$$

on the 8 points of affine 3-space over \mathbb{F}_2 . The action of Γ_4 on $V = \mathbb{Z}(SU(2)^8)$ can be identified with this action on A . Then the action of

Γ_4 on V^A is that of Γ_4 on V/V_4 . That is, there is a short exact sequence of Γ_4 modules

$$0 \rightarrow V_4 \rightarrow V \rightarrow V/V_4 = \mathcal{Z}(SU(2)^8/V_4) = V^A \rightarrow 0.$$

We can use this action to determine V_4 more explicitly. The action of Γ_4 on $(SU(2)^8)$ induces an action on $\mathcal{Z}((SU(2)^8) \approx (\mathbb{Z}/2\mathbb{Z})^8)$.

THEOREM 3.9. *Given the permutation of coordinates action of Γ_4 on $V = (\mathbb{Z}/2\mathbb{Z})^8$, there is a unique rank 4 Γ_4 -submodule V_4 of V . It is spanned by the row vectors of the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

With respect to this coordinate system, each element of V_4 has either 0, 4, or 8 non-zero coordinates. As Γ_4 -modules, there is the S.E.S.

$$0 \rightarrow V_4 \rightarrow V \rightarrow V_4^* \approx V_4 \rightarrow 0.$$

Jean Lannes has pointed out to me that this mod 2 vector space is the extension of the very first binary linear code, the Hamming code [7, 4], to an [8, 4] code obtained by adding a parity bit to each 7-bit word, e.g., [20].

PROPOSITION 3.10. *If X is a connected 2-compact group with discrete maximal torus T_X and a sub-2-compact group $E \subseteq T_X$, then if $N_X(T_X)$ has trivial k -invariant, the same is true for $N_{C_X(E)}(T_X)$.*

In the next sections we explore two computational paths to exploit this proposition. In each case, the finite 2-group E is the center of the $(SU(2)^8)/V_4$ found above. If the normalizer of the torus within this group were split, then the 2-rank would be at least 12. Hence any calculation or estimate that establishes a smaller 2-rank proves that the normalizer is not split.

For reference, the same strategy can be employed in the $DI(4)$, F_4 , E_7 , and $E_7/\mathcal{Z}(E_7)$ cases to show that the normalizer is not split. For G_2 however, it is split.

The table below summarizes the outcomes of this strategy for the exceptional groups:

Weyl Group	V_G 2-rank	$C_G(E_G)$ 2-rank	Γ_G	Code	χ
G_2	1	3	Σ_2	[2 : 1]	3
$DI(4)$	1	4	Σ_3	[3 : 1]	3
F_4	1	5	Σ_4	[4 : 1]	3
$E_7/Z(E_7)$	4	8	$GL_3(\mathbb{F}_2)$	[7 : 4]	135
E_8	4	9	$AGL_3(\mathbb{F}_2)$	[8 : 4]	2025
$Spin(12)$	2	6	Σ_4	[6 : 2]	15

REMARK 3.11. $W(E_6)$ is anomalous in that it does not have a non-trivial center. Thus the analogous computations are slightly different. There are only four commuting reflections in each 2-Sylow subgroup of $W(E_6)$. These reflections centralize a rank 2 subspace E of $L/2L$. Then

$$C_{E_6}(E) = \frac{SU(2)^4 \times U(1)^2}{V_4}.$$

Similar computations can be done here to establish that the 2-rank of $C_{E_6}(E)$ is six.

4. Estimating 2-ranks using CoCoA

For $G = (SU(2))^k/V_G$ where $V_G \subset V = \mathcal{Z}(SU(2)^k)$. Choose the basis $\{e_j\}$ for V corresponding to the center of each factor, and let $\{x_j\}$ be the corresponding dual basis of $V^\#$. Let the forms $\{f_i = \sum_j a_{ij}x_j\}$ be a basis for the linear forms that vanish on V_G .

The fibration

$$\mathcal{Z}(G) \rightarrow G \rightarrow (SU(2)^k/\mathcal{Z}(SU(2)^k) = SO(3)^k.$$

after converting to classifying spaces, yields

$$B\mathcal{Z}(G) \rightarrow BG \rightarrow BSO(3)^k \rightarrow B^2\mathcal{Z}(G) = K(\mathcal{Z}(G), 2).$$

We want to estimate

$$\text{Image } H^*(BSO(3)^k, \mathbb{F}_2) \subset H^*(BG, \mathbb{F}_2).$$

This is smaller than the quotient of $H^*(BSO(3)^k, \mathbb{F}_2)$ by the ideal J generated by $\text{Image } H^*(B^2\mathcal{Z}(G), \mathbb{F}_2)$. In concrete terms, J is the Steenrod closure of the ideal generated by

$$\{g_i = \sum_j a_{ij}w_2(j) \in H^2(BSO(3)^k, \mathbb{F}_2)\},$$

where the $\{w_2(j)\}$ are the second Steifel Whitney classes of the j -th term in the product. From the work of Quillen, [22], it is not hard to see that

$$2\text{-rank}(G) = 2\text{-rank}(\mathcal{Z}(G)) + K.D.(\text{Image}(H^*(BSO(3)^k, \mathbb{F}_2))).$$

In turn, it's clear that

$$K.D.(\text{Image}(H^*(BSO(3)^k, \mathbb{F}_2))) \leq K.D.(H^*(BSO(3)^k, \mathbb{F}_2)/J).$$

Since each $SO(3)$ contains a unique up to conjugacy $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, one can convert to computations in $H^*(B(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^k, \mathbb{F}_2)$.

COMPUTATIONAL PROBLEM

- (1) Given an ideal I_G generated by a set of linear combinations of the quadratic polynomials $q(x_i, y_i) = x_i^2 + x_i y_i + y_i^2$ from the ring $R = \mathbb{Z}/2\mathbb{Z}[(x_1, y_1), \dots, (x_k, y_k)]$, calculate the Krull dimension of the quotient ring R/J_G , where J_G is the Steenrod closure of I_G .
- (2) Equivalently, do the computation in the ring $\mathbb{Z}/2\mathbb{Z}[(w_2(1), w_3(1)), \dots, (w_2(8), w_3(8))]$.

The Steenrod operations needed on w_2 are of the form $\{Sq^1, Sq^2Sq^1, \dots\}$, which also can be calculated as the Milnor derivations $\{D_j\}$ of degree $2^j - 1$ on w_2 . If one works with $w_2 = x^2 + xy + y^2$, then $D_j w_2 = x^{2^j} y + x y^{2^j}$, while if one works with $\mathbb{Z}/2\mathbb{Z}[w_2, w_3]$, one can easily calculate that

- (1) $D_1 w_2 = w_3$,
- (2) $D_2 w_2 = w_2 w_3$,
- (3) $D_3 w_2 = w_2^3 w_3 + w_3^3$,
- (4) $D_4 w_2 = w_2^7 w_3 + w_2^4 w_3^3 + w_2 w_3^5$, etc.

PSEUDO-CODE 4.1. *Sketch of CoCoA or Macaulay 2 code:*

- (1) *Setup the polynomial ring $R = \mathbb{Z}/2\mathbb{Z}[x_1, y_1, \dots, x_8, y_8]$.*
- (2) *Define the Milnor operations on R .*
- (3) *Define the generators $\{f_1, f_2, f_3, f_4\}$ of the ideal J .*
- (4) *Define the Milnor derivations $\{D_k f_j\}$ on these generators, and the ideals J_1, J_2, \dots containing the D_k acting on the generators.*
- (5) *Calculate the Krull dimension and Poincare series for the quotients R/J_k for $k = 1, \dots$ until the answers stabilize.*

CALCULATION 4.2. *Results of calculation:*

- (1) *Calculation in $\mathbb{Z}/2\mathbb{Z}[(x_1, y_1), \dots, (x_8, y_8)]$ did not complete in 24 hours.*
- (2) *Calculation in $\mathbb{Z}/2\mathbb{Z}[(w_2(1), w_3(1)), \dots, (w_2(8), w_3(8))]$ took less than 10 seconds on a Pentium III 850mhz computer with CoCoA 4.2 to show that $\text{Dim}(R/J_3) = 5$. Here J_3 is generated by $\{g_1, \dots, g_4, D_1 g_1, \dots, D_4 g_1, \dots, D_4 g_4\}$.*

REMARK 4.3. Since $J_3 \subset J$, we conclude that $\text{Dim}(R/J) \leq 5$. Hence the 2-rank of $(SU(2)^8)/V_G$ is less than or equal 9. But $(SU(2)^8)/V_G$ contains an elementary abelian 2-group of rank 9, namely the -1 from the Weyl group together with the rank 8 from the maximal torus. Hence the 2-rank of $(SU(2)^8)/V_G$ is 9. If $N_G T$ for $(SU(2)^8)/V_G$ were split, then its 2-rank would be at least $8 + 4 = 12$. Hence it is not split, and therefore $N_{E_8} T$ is also not split.

REMARK 4.4. Since the ideal

$$J \subset \ker\{H^*(BSO(3)^k, \mathbb{F}_2) \rightarrow H^*(BG, \mathbb{F}_2)\},$$

this method overestimates $K.D.(\text{Image}(H^*(BSO(3))))$ and hence provides an upper bound for the 2-rank of $SU(2)^k/V_G$.

5. Exact calculation of 2-rank $((SU(2))^k/V_G)$ using GAP3 and the Dimacs maximal clique finder

Given the setup of the previous sections, one can deal with the combinatorial problem, neglecting the algebraic context. This turns out to be computationally feasible even for the example of E_8 .

DEFINITION 5.1. Let $X \subseteq (\mathbb{Z}/2\mathbb{Z})^k$. Define 2-rank (X) as the maximum of the ranks of $\mathbb{Z}/2\mathbb{Z}$ -subspaces totally within X .

REMARK 5.2. 2-rank (X) depends on the embedding of X into $(\mathbb{Z}/2\mathbb{Z})^k$ dramatically.

DEFINITION 5.3. Let $\mathbf{q}(x, y) = x^2 + xy + y^2$. Then extend coordinate wise to the **quadratic map**

$$\mathbf{Q} : (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^k \rightarrow (\mathbb{Z}/2\mathbb{Z})^k$$

by

$$((a_1, b_1), (a_2, b_2), \dots) \rightarrow (\mathbf{q}(a_1, b_1), \mathbf{q}(a_2, b_2), \dots).$$

In particular, given a subset $Y \subseteq (\mathbb{Z}/2\mathbb{Z})^k$, one can form $\mathbf{Q}^{-1}(Y)$. While it is clear that $2\text{-rank}(\mathbf{Q}^{-1}(Y)) \geq 2\text{-rank}(Y)$, exact formulas are not obvious:

COMPUTATIONAL QUESTION: Given the linear subspace V_G of $(\mathbb{Z}/2\mathbb{Z})^k$, compute the 2-rank of $\mathbf{Q}^{-1}(V_G) \subseteq (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^k$.

Given $X \subseteq (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^k$ one can define a non-directed graph $Gr(X)$ with one node for each point of X and an edge between x and y if and only if $x + y$ is in X . (I thank Bill Dwyer for this observation.) Then the computational question can be rephrased:

GRAPH PROBLEM: Find the maximal cliques (complete subgraphs) in $Gr(\mathbf{Q}^{-1}(V_G))$.

At first glance, this translation is not so promising. The maximal clique problem classically is a difficult combinatorial problem. Also, one might feel that important special features of the topological problem have been ignored. Nonetheless, the baseline clique solver code *dfmax.c* published by the Dimacs group [10] when applied to the E_8 data finds the maximal size clique in a computationally practical amount of machine time.

PSEUDO-CODE 5.4.

- (1) Find a basis for the vector space V_G .
- (2) Use GAP3 to print out the set $\mathbf{Q}^{-1}(V_G)$.
- (3) Use a custom 'C' program to massage this into a graph data structure suitable for input into the solver, *dfmax.c*.
- (4) Run the executable of *dfmax* on this data to find the maximum size of a clique.

For the case of E_8 , *dfmax* on a 850 mhz Pentium 3 requires about 5 minutes to find the maximal clique.

LEMMA 5.5. For the particular subspace $V_4 \subset (\mathbb{Z}/2\mathbb{Z})^8$ arising from $W(E_8)$, the number of points in $\mathbf{Q}^{-1}(V_4)$ is $3^0 + 3^8 + (16 - 2)3^4 = 1 + 1134 + 6561 = 7698$.

CALCULATION 5.6. For the particular subspace $V_4 \subset (\mathbb{Z}/2\mathbb{Z})^8$ arising from $W(E_8)$, the maximum sized cliques in $C_{\mathbf{Q}^{-1}(V_4)}$ have 32 nodes. That is $2\text{-rank}(\mathbf{Q}^{-1}(V_4)) = 5$.

COROLLARY 5.7. For any W -clone X of E_8 , $BN_X(T_X)$ is h.e. to the fiberwise 2-completion of $BN_{E_8}(T_{E_8})$.

For the other exceptional Lie groups the results are similar:

CALCULATION 5.8.

Weyl Group	$W(C_G(E))$	$Z(C_G(E))$	$\mathbf{Q}^{-1}(V_G)$	2-rank $\mathbf{Q}^{-1}(V_G)$	2-rank $C_G(E)$
$W(G_2)$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	10	2	3
$W(DI(4))$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^2$	28	2	4
$W(F_4)$	$(\mathbb{Z}/2\mathbb{Z})^4$	$(\mathbb{Z}/2\mathbb{Z})^3$	82	2	5
$W(\text{adj}(E_7))$	$(\mathbb{Z}/2\mathbb{Z})^7$	$(\mathbb{Z}/2\mathbb{Z})^3$	2944	5	8
$W(E_8)$	$(\mathbb{Z}/2\mathbb{Z})^8$	$(\mathbb{Z}/2\mathbb{Z})^4$	7698	5	9
$W(\text{Spin}(12))$	$(\mathbb{Z}/2\mathbb{Z})^6$	$(\mathbb{Z}/2\mathbb{Z})^4$	244	2	6

REMARK 5.9. These methods of computation could be avoided if the symbolic algebra programs had effective routines for the calculation of the lattice of elementary abelian subgroups of a finite group. Magma has such a routine, but it is not adequate to treat the group of cardinality 2^{22} that would arise for $W(E_8)$.

6. $H^*(BE_8, \mathbb{F}_2)$ is not Cohen-Macaulay.

The best behaved cohomology of classifying spaces are the polynomial algebras. These arise for many of the classical groups, as well as for BG_2 , $BDI(4)$, and BF_4 . In a hierarchy of tractable algebra types, one has

- (1) polynomial algebras
- (2) complete intersections
- (3) Gorenstein rings
- (4) Cohen-Macaulay rings

In this hierarchy, membership in class n implies membership in class $n + i$, for $i > 0$. For example, $H^*(BSpin(n), \mathbb{F}_2)$ is polynomial for $n < 10$, but Quillen [21] demonstrated that $H^*(BSpin(n), \mathbb{F}_2)$ is a complete intersection, since it has the form $H^*(BSO(n), \mathbb{F}_2)$ modulo an ideal J generated by a regular sequence, tensored with an extra polynomial algebra of rank one. Here J is generated by Steenrod operations on the second Steifel-Whitney class w_2 . The Cohen-Macaulay property is weaker, requiring only a set parameters over which the algebra is a free module. Benson and Greenlees have shown that if $H^*(BG, \mathbb{F}_2)$ is Cohen-Macaulay and G is connected, then H^* is also Gorenstein and its Poincare series has special properties, [4]. In this section, we'll show that the cohomology algebras of BE_6 , $B(\text{adj}(E_7))$, BE_8 , and their clones, fail to have the Cohen-Macaulay property.

I thank Lucho Avramov for showing me the result below in the finite group case. Versions of the theorem hold for compact Lie groups, 2-compact groups, and noetherian connected graded commutative algebras with an unstable action of the Steenrod algebra. The proof below relies on the unstable algebra case.

LEMMA 6.1. *Let G be a compact Lie group or 2-compact group and $i : E \rightarrow G$ an elementary abelian 2-subgroup. Then $\mathfrak{p}_E = \ker(Bi^*)$ is a prime ideal of $R = H^*(BG, \mathbb{F}_2)$ and has $K.D.(R/\mathfrak{p}_E) = 2\text{-rank}(E)$. E is maximal among the elementary abelian 2-subgroups of G if and only if \mathfrak{p}_E is a minimal prime ideal of R .*

LEMMA 6.2. *Let R be a connected graded commutative \mathbb{F}_2 -algebra with an unstable action of the Steenrod algebra \mathfrak{A}_2 . Then each homogeneous prime ideal \mathfrak{q} of R contains a maximal homogeneous prime ideal \mathfrak{p} closed under the action of the Steenrod algebra, i.e. a maximal invariant homogeneous prime ideal. In particular, each minimal homogeneous prime ideal \mathfrak{p}_0 is "invariant".*

THEOREM 6.3. *Let G be a compact Lie group or 2-compact group and E and E' be elementary abelian 2-subgroups of G such that each is maximal and $2\text{-rank}(E') \neq 2\text{-rank}(E)$. Then $H^*(BG, \mathbb{F}_2)$ is not a Cohen-Macaulay commutative ring.*

It is perhaps relevant to state the contra-positive of this:

PROPOSITION 6.4. *Let G be a compact Lie group or 2-compact group and E and E' be elementary abelian 2-subgroups of G such that each is maximal and $2\text{-rank}(E') \neq 2\text{-rank}(E)$. Then $H^*(BG, \mathbb{F}_2)$ is not a Cohen-Macaulay commutative ring.*

REMARK 6.5. This contra-positive applies to $H^*(BSpin(n), \mathbb{F}_2)$ as well as to the classical polynomial algebra cases. That is, in these cases the maximal elementary abelian 2-subgroups have the same rank, even if not conjugate.

REMARK 6.6. One can not in general proceed directly from the non-C.M. behavior of $H^*(BN_G(T), \mathbb{F}_2)$ to results about $H^*(BG, \mathbb{F}_2)$. However, the following result of Larry Smith [23] allows some of the work of this note to give evidence:

THEOREM 6.7. *Let R^* be unstable noetherian connected commutative algebra over the Steenrod algebra \mathbf{A}_2 . Let E be an elementary abelian p -group. If R^* is C.M., then for each $f \in \text{Hom}_{\mathbf{A}_2}(R^*, H^*(BE, \mathbb{F}_2))$, the component of the Lannes T -function $T_f(R^*)$ is C.M. Therefore, if G is a 2-compact group, and $\phi : BE \rightarrow BG$ a morphism, then if $H^*(BG, \mathbb{F}_2)$ is C.M., the centralizer $C_G(E)$ has $H^*(BC_G(E), \mathbb{F}_2)$ C.M. Here $BC_G(E) \approx \text{Map}_{\text{unpt}}(BE, BG)_\phi$.*

It follows that if $H^*(BC_G(E), \mathbb{F}_2)$ is not C.M. for some E , then $H^*(BG, \mathbb{F}_2)$ is not C.M. Here is the result of the analysis of the centralizers in the case of 2-compact groups of exceptional type (here $\Delta(\mathbb{Z}/2\mathbb{Z})$ is a diagonally embedded $\mathbb{Z}/2\mathbb{Z}$):

CALCULATION 6.8.

Weyl Group	$C_G(V)$	maximal elem. 2-subgroup ranks	$H^*(BC_G(E), \mathbb{F}_2)$ Structure
$W(G_2)$	$SU(2)^2/\Delta(\mathbb{Z}/2\mathbb{Z})$	3	polynomial
$W(DI(4))$	$SU(2)^3/\Delta(\mathbb{Z}/2\mathbb{Z})$	4	polynomial
$W(F_4)$	$SU(2)^4/\Delta(\mathbb{Z}/2\mathbb{Z})$	5	polynomial
$W(\text{adj}(E_7))$	$SU(2)^7/V_4$	7, 8	Non C.M.
$W(E_8)$	$SU(2)^8/V_4$	8, 9	Non C.M.
$W(\text{Spin}(12))$	$SU(2)^6/V_2$	6	C.M. not polynomial

PROOF OF THEOREM. From the above lemma, the prime ideals \mathfrak{p}_E and $\mathfrak{p}_{E'}$ are each minimal prime ideals of R , but $K.D.(R/\mathfrak{p}_E) \neq K.D.(R/\mathfrak{p}_{E'})$. But in a C.M. local ring, for each minimal prime ideal \mathfrak{q} , $K.D.(R/\mathfrak{q}) = K.D.(R)$, by Cor.2.1.4, page 59, [6]. Hence R is not C.M. \square

PROOF OF GROUP LEMMA In the Lie case, these are consequences of the Quillen Stratification Theorem, [22]. But one can give an argument suitable for either environment. Since $H^*(BE, \mathbb{F}_2)$ is a polynomial algebra on $rank(E)$ variables, $\ker(Bi^*)$ is prime. Since $H^*(BE, \mathbb{F}_2)$ is a f.g. $H^*(BG, \mathbb{F}_2)$ module, $image(i^*) = H^*(BG, \mathbb{F}_2)/\ker(Bi^*)$ has dimension the rank of E . Suppose now that \mathfrak{p} is not a minimal prime ideal. Let $\mathfrak{q} \subseteq \mathfrak{p}_E$ be a minimal prime ideal. Then \mathfrak{q} is closed under the Steenrod algebra action, since it is a minimal prime ideal. Consider the algebra $S = H^*(BG, \mathbb{F}_2)/\mathfrak{q}$. This is a noetherian integral domain with an unstable Steenrod algebra structure. By Adams-Wilkerson, [2], there is an embedding $S \rightarrow H^*(BE', \mathbb{F}_2)$ where $K.D.(S) = 2\text{-rank}(E')$. Hence, by composition, there is $\psi : H^*(BG, \mathbb{F}_2) \rightarrow H^*(BE', \mathbb{F}_2)$ a map respecting the Steenrod algebra structure. By Lannes, [18] in the 2-compact case, this corresponds to $BE' \rightarrow BG$. By Dwyer-Zabrodsky [15] and others, in the compact Lie case, this determines a conjugacy class of embeddings $E' \rightarrow G$. In the 2-compact group case the morphisms are maps of classifying spaces, so that case is covered also. But $2\text{-rank}(E') > 2\text{-rank}(E)$, so this contradicts E being a maximal elementary abelian 2-subgroup in G . Hence \mathfrak{p}_E is a minimal prime ideal of R .

REMARK 6.9. DISCUSSION OF TABLE 8.6

For G_2 , $DI(4)$, and F_4 , the $C_G(V)$ discussed before are of the form $SU(2)^k/\Delta(\mathbb{Z}/2\mathbb{Z})$. It is easily seen that these have polynomial cohomology and thus do not immediately inform us on the C.M. status of $H^*(BG, \mathbb{F}_2)$. However, various authors have shown that any clone of one of these with the same $N_G(T)$ must give an equivalent BX . Hence, indirectly, the cohomology of BG_2 , $BDI(4)$, and BF_4 are each polynomial and hence C.M.

For clones of E_6 , E_7 , and E_8 , the quickest path is to reason indirectly. The Lie examples of $adj(E_6)$, E_7 , and E_8 are known to have maximal elementary abelian 2-subgroups of rank (5, 6), (7, 8), and (8, 9) respectively. It can be shown that these are represented in the $C_G(V)$

centralizers. Hence, the $H^*(BC_G(V), \mathbb{F}_2)$ are non-C.M. But these centralizers in a clone are isomorphic to the centralizers in the Lie case, since their structure depends only on the action of the Weyl group on the torus. Hence we reason that for a clone X to one of E_6 , $adj(E_7)$, or E_8 , one must have that $H^*(BX, \mathbb{F}_2)$ is non-C.M. (because otherwise, the $H^*(BC_X(V), \mathbb{F}_2)$ would be C.M, which it is not).

In principle, one could try to expand the calculation of the earlier sections to include finding all maximal elementary abelian 2-subgroups of $C_G(V)$. I have not done so.

7. Proof of Theorem 1.2

Many Lie groups have the property that $W(G)$ has a non-trivial center. If the reflection action is irreducible, then the non-trivial central element c must act as -1 on the lattice $\pi_1(T_G)$ and as the inversion on T_G . Obviously, c commutes with all elements of order 2 in the maximal torus T_G . The following is similar to Adams, section 2, [1]:

PROPOSITION 7.1. *For G a connected 2-compact group, 1 if $({}_2T_G)^{W(G)} = \{e\}$, then the central extension*

$$\{e\} \rightarrow {}_2T_G \rightarrow N_c \rightarrow \langle c \rangle = \mathbb{Z}/2\mathbb{Z} \rightarrow \{e\}$$

is split. Hence N_c is an elementary abelian 2-group of rank $(G) + 1$.

COROLLARY 7.2. *Under the above hypotheses, the 2-rank of G is greater than or equal to rank $(G) + 1$.*

Proof. Let x be any lift of c to N_c . It suffices to show that $x^2 = e$. We do this by showing that $x^2 \in ({}_2T_G)^{W(G)}$. Certainly x^2 projects to e , so $x^2 \in {}_2T_G$. Let $w \in W(G)$ be any element. Then we need to show that $w(x^2) = x^2$ in ${}_2T_G$. But $w(x) = xt_w$ for some $t_w \in {}_2T_G$. Hence $w(x^2) = w(x)^2 = xt_wxt_w = xt_wx^{-1}xt_w = t_w^{-1}x^2t_w = x^2$, since x^2 and t_w are in T_G . That is x^2 is fixed by each element of $W(G)$ and hence $x^2 = e$ from the hypothesis.

REMARK 7.3. Adams, [1] makes the slightly stronger statement that $x^2 \in \mathcal{Z}(G)$.

REMARK 7.4. This applies to G_2 , $DI(4)$, F_4 , $adj(E_7)$, and E_8 and the adjoint forms of $\{Sp(n), n > 1\}$ and $\{Spin(4n), n \geq 1\}$. The hypothesis fails for E_6 . Indeed, the 2-rank of E_6 is only 6. The 2-rank of the simply connected E_7 is only 7, see Griess, [16] and Cohen-Seitz, [7]. In this case there are non-conjugate elementary abelian 2-groups of rank seven. The author does not know if the cohomology of BE_7 is Cohen-Macaulay.

8. Further Discussion on the structure of $H^*(BE_8, \mathbb{F}_2)$.

Recall that $V_G \subset T_X$ is the particular subspace centralized by 8 commuting reflections from $W(E_8)$. We've identified $C_X(V_G) \approx_2 (SU(2)^8)/V_4$. Our conjectures concern $N_X(V_G)$:

CONJECTURE 8.1. *In any E_8 - W -clone,*

- (1) $N_X(V_G)$ is the semi-direct product of $C_X(V_G)$ by Γ_G
- (2) $H^*(BX, \mathbb{F}_2) \rightarrow H^*(BN_X(V_G), \mathbb{F}_2)^{\Gamma_G}$ is a monomorphism (true by Euler char. considerations).
- (3) $H^*(BX, \mathbb{F}_2) \rightarrow H^*(BN_X(V_G), \mathbb{F}_2)^{\Gamma_G}$ captures a large part of $H^*(BX, \mathbb{F}_2)$ up to Quillen "F"-equivalences. For example, it might be an 'F'-monomorphism or surjection.

If this were true, one could calculate the Quillen diagram of E_8 from $H^*(BC_X(V_G), \mathbb{F}_2)^{\Gamma_G}$ and modulo non-trivial lim^i -calculations establish uniqueness results for E_8 in the context of 2-compact groups.

One of the original motivations for these calculations was to see if one could obtain estimates good enough to force the conclusion that the Rothenberg-Steenrod spectral sequence collapses at E_2 . The above conjecture might be enough to imply this.

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