18.781 Problem Set 6: Due Wednesday, April 12, 1995.

1. The use of continued fractions to solve Pell's equation

$$x^2 - my^2 = \pm 1 \tag{1}$$

which Davenport describes in his book is different from the one described in lecture. Davenport starts with

$$\sqrt{m} = \langle q_0, \overline{q_1, \ldots, q_{k-1}, 2q_0} \rangle,$$

(with k minimal) with corresponding numerators and denominators a_n, b_n, and observes that for any $n > 0$,

$$a_{nk-1}^2 - mb_{nk-1}^2 = (-1)^{nk}.$$

Your problem is to prove the claim he makes but does not prove: that this process yields *all* solutions to (**??**) with $x, y > 0$. The first step is show that any such (x, y), $x = a_j, y = b_j$ for *some* j. Use the approximation theorem we proved in class for this purpose: if α is any real irrational and c is any nonzero rational number such that

$$|\alpha - c| < \frac{1/2}{(\operatorname{ht} c)^2}$$

then c is one of the continued fraction convergents for α. Then show that j must be of the form $nk - 1$.

2. (a) Find the continued fraction expansion for $\sqrt{55}$.

(b) Find two integral solutions to $x^2 - 55y^2 = 1$. Does $x^2 - 55y^2 = -1$ have any integral solutions? Explain.

3. (a) Show that if $n \in \mathbb{N}$ is of the form $4^m(8k+7)$ then it cannot be written as the sum of three squares.

(b) Give an example of two sums of three squares, m and n, whose product is not a sum of three squares. This shows that $x^2 + y^2 + z^2$ is not a norm in any reasonable sense, and it is this that makes the problem of representing a number as a sum of three squares much more difficult than the two-squares theorem. Nevertheless, Gauss showed that any number not of the form described in **(a)** *can* be written as a sum of three squares.

4. (a) Show that any degenerate quadratic form is equivalent to the form mx^2, for a unique integer m. (Hint: First assume that the quadratic form $q(x, y) = ax^2 + bxy + cy^2$ is primitive and that $a > 0$. Show that in this case a and b are relatively prime squares. Use this to write q as the square of a linear form. Then pass on to the other cases.)

(b) Show that if the discriminant of the quadratic form q is the perfect square m^2, then q is equivalent to $x(nx + my)$ for some integer n. Which of these are equivalent to one another?

There are right-angled triangles with rational sides and area equal to 157. Among them, the one whose sides a, b, c (with $a^2 + b^2 = c^2$ and $ab/2 = 157$) have smallest height has

$$a = \frac{411340519227716149383203}{21666555693714761309610}, \quad b = \frac{6803298487826435051217540}{411340519227716149383203}.$$

–D. Zagier.