Spectral Algebra: Some Examples

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The traditional strategy in Algebraic Topology is to start with some topological problem, apply a functor to an algebraic category, and solve the resulting problem there. If it was a non-existence question, you may have the answer. If not, you are at the first stage of a step-by-step process known as "obstruction theory."

The most promising functors are continuous, and since algebra is discrete these functors can't distinguish between homotopic maps. Algebraic Topology is for the most part "Homotopy Theory," and the first step in a problem is often to replace it by a different problem which has homotopical significance. For example, invariance of domain is proven by first removing a point from the Euclidean spaces, at which point you get spaces which are not homotopy equivalent. A topological space thus appears as merely a model of a more intrinsic object, namely a homotopy type.

In recent years a new paradigm has emerged. One can start with a piece of algebra and attempt to find a homotopical enrichment of it. This is akin to the celebrated process of "quantization": you have a "classical" object, often commutative in some sense, and you wish to see it as derived from a much richer "noncommutative" object.

Groups.

I am not refering to the simple process of replacing the study of groups, for example, by the study of topological groups: that is a topological enrichment, not a homotopical one. If a topological group G is homeomorphic to a finite polyhedron, then it is isomorphic as topological group to a compact Lie group. This version of Hilbert's Fifth Problem was solved by von Neumann in the 1930's. For a homotopy theorist the better question is: what can we say of a topological group that is homotopy equivalent to a finite polyhedron? Now the notion of equivalence has to be weakened: two are equivalent if they are connected by a chain of continuous homomorphisms each of which is a homotopy equivalence.

There is a contractible space EG on which G acts freely, and the homotopy type of the orbit space BG is an invariant of G. In fact G can be reconstructed from it: $G \simeq \Omega BG$. It follows that two topological groups are equivalent in our sense exactly when their classifying spaces are homotopy equivalent; so we are asking to classify connected spaces X whose loop spaces are homotopy equivalent to finite complexes.

In around 1970 David Rector showed that there are uncountably many distinct connected homotopy types X such that $\Omega X \simeq S^3$.

We are saved by another homotopical enrichment, the notion of p-adic completion (due in an early form to Dennis Sullivan). It turns out (Dwyer, Miller, Wilkerson) that for each prime p there is only one p-complete homotopy type X such that $\Omega X \simeq S^3{}^{\wedge}_p$.

This observation has led on to a large program to classify p-complete connected spaces X for which $H_*(\Omega X; \mathbb{F}_p)$ is finite: "p-compact groups." Dwyer and Wilkerson proved that there was (in an approprate sense) a “maximal torus,” and a “Weyl group’ which was a p-adic generalized reflection group. The classification of these (by Shephard and Todd) is the starting point for a classification of p-compact groups.

Abelian groups.

What of abelian groups? A topological abelian group A (no longer assumed homotopy equivalent to a finite complex!) has the property that there is a natural isomorphism from $[X, A]$ to a product of cohomology groups of X. The appropriate homotopical weakening is to require that $[X, A]$ should be a generalized homology theory. In effect, we no longer demand that A should be commutative on the nose, but rather only homotopy commutative, but with a homotopically symmetric commuting homotopy, and so on forever. The objects representing homology theories have been part of topology for forty years now, but only in the past couple of years has the theory reached a state at which we may really treat these objects as if they were abelian groups. The new ingredient is a symmetric monoidal tensor product on the category of spectra. Thanks to work of Elmendorff, Kriz, Mandell, and May, and parallel work by Jeff Smith, Mark Hovey, and Brooke Shipley, we now have such a setup. This lets us define rings and commutative rings in the simplest possible way. I would say that the horizons for exploration are now quite broad.

I want to stress that these constructions are occuring on the “pointset” level, not modulo the relation of homotopy. Spectra are hybrid objects, taking some traits from spaces and others from abelian groups. They have underlying homotopy types, and these in turn have a graded homotopy group which may be regarded as an algebraic shadow of the richer homotopy type.

Let me rephrase Quillen’s construction of algebraic K-theory in these terms. Grothendeick began with a ring R, and studied the monoid of isomorphism classes of finitely generated projectives. The group completion is $K_0(R)$. Quillen proposed starting with the category $\mathcal{P}_R$ of finitely generated projectives and their isomorphisms. This has a symmetric monoidal structure, coming from direct sum. Grothendieck had been looking at the set of isomorphism classes. The symmetric monoidal structure turns out to be food for a machine to produce a spectrum $K(R)$, whose group of components is $K_0(R)$. The rest of the homotopy groups constitute the Quillen algebraic K-theory of R. The abelian group $K_0(R)$ has been “quantized” to produce the “homotopy abelian group” (spectrum) $K(R)$.

Formal groups.

A “formal group” is a formal power series $F(x, y)$ such that

$$F(x, 0) = 0 = F(0, y), \qquad F(x, F(y, z)) = F(F(x, y), z), \qquad F(x, y) = F(y, x).$$

$F(x, y) = x + y$ and $F(x, y) = x + y - xy$ are the simplest examples, called the additive and multiplicative groups. Certain multiplicative cohomology theories give rise to formal groups in a natural way. K-theory for example gives rise to G_m, and ordinary cohomology gives G_a.

The assumptions are that E^* is evenly graded and E^2 contains a unit. (This captures Bott periodicity in the case of K-theory.) Under this assumption, one discovers that there is an element e such that

$$E^0(\mathbb{C}P^\infty) = E^0[[e]].$$

A complex line bundle $\lambda \downarrow X$ is represented by a map $X \to \mathbb{C}P^\infty$, and we may pull e back along this map to get an "Euler class" $e(\lambda)$. Then we may ask for $e(\lambda \otimes \mu)$, and discover that there is a formal group $F(x, y)$ such that

$$e(\lambda \otimes \mu) = F(e(\lambda), e(\mu)).$$

One may attempt to "pre-homotopicalize" a formal group, finding a periodic cohomology theory affording it. Peter Landweber has given conditions guaranteeing that this can be done. I use the prefix "pre" because one may also ask for a functorial association of a ring spectrum. This has not been done in general and remains a major question. But it has been done in important special cases.

Elliptic curves and Modular forms.

Both G_a and G_m are formal completions of one-dimensional commutative algebraic groups, and the next commutative algebraic groups to consider are elliptic curves. These are one-dimensional projective varieties with a group structure. They are among the most intensively studied objects in all of mathematics. A typical intersting example is the "Legendre curve"

$$y^2 = x(x-1)(x-\lambda) \qquad \text{over} \qquad \mathbb{Z}[1/2, \lambda^{\pm 1}, (1-\lambda)^{-1}].$$

There is a weakly universal example given by the Weierstrass curve

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \qquad \text{over} \qquad \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}].$$

There are coordinate changes that preserve this form, given by

$$\begin{aligned} x &= x' + r \\ y &= y' + sx' + t \end{aligned}$$

Theorem (Hopkins, Miller). On a sufficiently broad category $\mathcal{E}$ of ellitpic curves (including Legendre but not Weierstrass) there is a functor E to ring-spectra, such that

$$\pi_0(E_{A/R}) = R$$

and the formal group associated to $E_{A/R}$ is the formal completion of A.

Tate and Deligne called the subring of $W = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}]$ fixed by the coordinate changes the ring of "integral modular forms," and computed

$$M_* = \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = 12^3 \Delta).$$

This may be regarded as a graded ring with $|c_{2n}| = 4n$ and $|\Delta| = 24$. Our work provides a "quantization" of this computation. By forming an appropriate homotopical analogue of the invariants, we can construct the ring spectrum TM of "topological modular forms." There is a map

$$\pi_*(TM) \to M_*.$$

This map is neither onto nor one-to-one; indeed, the discriminant Δ itself is not in the image. This homomorphism is the edge homomorphism of a spectral sequence, in which the E_2 term is the derived functors of the functor used to get from W to M_*. This higher cohomology is a natural and purely algebraic object, but it has apparently never been studied by algebraists before. It serves here as the optimal purely algebraic model for our new geometric object, TM.

A dream of long standing is to give these cohomology theories the kind of geometric, analytical, and arithmetic significance that has surrounded K-theory (via vector bundles, index theory and algebraic K-theory). Conformal field theory has some role to play, but it's as yet obscure. Perhaps in the end the analogy with quantization which I have been drawing will turn out to be more than merely an analogy.