Homotopy fixed point sets of group actions on groups

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Let π be a group acted on by another group G. Then G acts on $B\pi$, and we inquire about the homotopy fixed point set of this action.

The space $(B\pi)^{hG}$ is the space of sections of the projection $EG \times_G B\pi \to BG$. We know that

$$EG \times_G B\pi = B(\pi \tilde{\times} G)$$

where $\pi \times G$ is the semi-direct product: $\pi \times G$ as a set, with product

$$(p,\gamma)(p',\gamma') = (p \cdot \gamma p',\gamma\gamma')$$

Write $q: \pi \tilde{\times} G \to G$ for the projection map.

We know that

$$\max(BG, B(\pi \times G)) = \prod_{[\sigma]} BZ_{\sigma}$$

where σ runs over a set of representatives of conjugacy classes of homomorphisms $G \to \pi \tilde{\times} G$ and Z_{σ} denotes the centralizer of σ ,

$$Z_{\sigma} = \{ p \in \pi : p\sigma(\gamma) = \sigma(\gamma)p \text{ for all } \gamma \in G \}$$

The composite $Bq \circ B\sigma$ is homotopic to the identity map on BG if and only if $q \circ \sigma$ is conjugate to the identity map on G. This condition is invariant under conjugation, and for a unique G-conjugate of σ the composite with qis equal to the identity. For such a σ ,

$$\sigma(\gamma) = (\varphi(\gamma), \gamma)$$

for some function $\varphi : \gamma : G \to \pi$. The fact that σ is a group homomorphism is equivalent to

$$\varphi(\gamma\gamma') = \varphi(\gamma) \cdot \gamma\varphi(\gamma')$$

—i.e. φ is a 1-cocycle on G with values in π .

For $p \in \pi$, the homomorphism $\gamma \mapsto p\sigma(\gamma)p^{-1}$ corresponds to the 1-cocycle $\gamma \mapsto p \cdot \varphi(\gamma) \cdot \gamma p^{-1}$. This gives an action of π on $Z^1(G; \pi)$, with orbit space $H^1(G; \pi)$.

So the set of components of map $(BG, B(\pi \times G))$ lying over the identity component of map(BG, BG) is in bijection with $H^1(G; \pi)$.

The component of the mapping space containing $B\sigma$ is BZ_{σ} . Suppose that σ corresponds to the 1-cocycle φ . Then $(p, \gamma) \in Z_{\sigma}$ if and only if

$$(p,\gamma)(\varphi(\gamma'),\gamma') = (\varphi(\gamma'),\gamma')(p,\gamma)$$

for all $\gamma' \in G$. This is demanding that $\gamma \in Z(G)$ and that p is such that

$$\gamma' p = \varphi(\gamma')^{-1} \cdot p \cdot \gamma \varphi(\gamma') \tag{1}$$

for all $\gamma' \in G$. Write $Z(\varphi)$ for this subgroup of $\pi \times G$.

The space of sections is the disjoint union of the fibers of the maps $BZ(\varphi) \to BZ(G)$ as φ runs over a set of representatives of $H^1(G; \pi)$.

Notice that for any $\gamma \in Z(G)$, $(\phi(\gamma), \gamma) \in Z(G)$ since

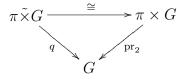
$$\phi(\gamma') \cdot \gamma' \phi(\gamma) = \phi(\gamma' \gamma) = \phi(\gamma \gamma') = \phi(\gamma) \cdot \gamma \phi(\gamma')$$

Let $K(\varphi) = \ker (Z(\varphi) \to Z(G))$. Then

$$(B\pi)^{hG} = \coprod_{[\varphi] \in H^1(G;\pi)} BK(\varphi)$$
(2)

As a first check suppose that the action is trivial. Then $Z^1(G;\pi) = \text{Hom}(G,\pi)$, the group π acts on this by conjugation, and $H^1(G;\pi) = \text{Rep}(G,\pi)$. $Z(\varphi) = Z(G) \times Z_{\varphi} \subseteq G \times \pi, \overline{Z}(\varphi) = Z(G)$, and $K(\varphi) = Z_{\varphi}$, so (2) reduces to the known formula for maps of classifying spaces.

This is actually more general than it seems: Suppose that G acts on π through a homomorphism $f: G \to \pi$: $\gamma p = f(\gamma)pf(\gamma)^{-1}$. Then $(p, \gamma) \mapsto (pf(\gamma), \gamma)$ makes



commute, so $B\pi^{hG} = map(BG, B\pi)$ for any homomorphism f.

For another example suppose |G| = 2, with generator γ . Write $\gamma p = \overline{p}$. Let $\varphi(\gamma) = \tau \in \pi$. Then $1 = \varphi(\gamma^2) = \tau \overline{\tau}$, so

$$Z^1(G;\pi) = \{\tau \in \pi : \tau \overline{\tau} = 1\}.$$

Conjugation by $p \in \pi$ sends the 1-cocycle corresponding to τ to the 1-cocycle corresponding to $p\tau \overline{p}^{-1}$, so

$$H^1(G;\pi) = \{\tau \in \pi : \tau \overline{\tau} = 1\}/\tau \sim p\tau \overline{p}^{-1}$$

The group $\pi \tilde{\times} G$ is given by

$$\pi \times \{1\} \coprod \pi \times \{\gamma\}$$

The centralizer of τ is the subgroup given by

$$Z(\tau) = \{p: \overline{p} = \overline{\tau} p \tau\} \times \{1\} \coprod \{p: \overline{p} = \overline{\tau} p \overline{\tau}\} \times \{\gamma\}$$

It's fun and reassuring to check that this is a subgroup!

The kernel group is

$$K(\tau) = \{ p : \overline{p} = \overline{\tau} p \tau \} \subseteq \pi$$

Thus:

$$(B\pi)^{hG} = \coprod_{[\tau]\in H^1(G;\pi)} BK(\tau)$$