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On the Segal Conjecture for Periodic Groups

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Using a hyperext calculation and D. C. Ravenel's "modified Adams spectral sequence" it is shown that the Segal conjecture for $D2^n$ implies the Segal conjecture for $Q2^{n+1}$; in particular, it holds for $Q8$. Also, a homotopy theoretic argument is provided for Ravenel's theorem that the Segal conjecture for $\mathbb{Z}/p^n$ implies the Segal conjecture for $\mathbb{Z}/p^{n+1}$.

In a talk at the Northwestern University Homotopy Theory Conference in March, 1982, Gunnar Carlsson announced a complete affirmative resolution of the Burnside ring conjecture of G. B. Segal. This represented a tremendous advance over previous knowledge, as the reader will see by glancing at J. F. Adams' report [3] at the Adem conference the preceding August. We refer the reader to that report, or to Adams' earlier report [2], for an account of the conjecture itself. The first author had been analyzing the proof by D. C. Ravenel [13] of the implication, "Segal conjecture for $\mathbb{Z}/p^n \Longrightarrow$ Segal conjecture for $\mathbb{Z}/p^{n+1}$," in hopes of constructing an analogous proof for more general group extensions. The starting point for an induction was to have been the validity of the conjecture for elementary Abelian groups, due to Adams, Gunawardena, and Miller [4], as the starting point for Ravenel's induction was its validity for $\mathbb{Z}/p$, due to W. H. Lin [11], [12], and J. H. C. Gunawardena [9]. (Carlsson's proof in fact uses [4] to ground an induction.) In particular, he noticed that Ravenel actually provided the techniques for two distinct proofs of his result, though, at least in the first version of [13] (available in September, 1981), the two were mixed together. One proof relied on a modification of the Adams spectral sequence, designed to make the algebraic result underlying the theorems of Lin and Gunawardena applicable in the presence of higher torsion. The other was entirely homotopy-theoretic, depending instead on a modification of a certain "Atiyah-Hirzebruch-Serre" spectral sequence. Naturally, the latter appeared to the first author more

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promising as a pattern for generalization. However, during a visit in November, 1981, the second author convinced him that the modified Adams spectral sequence approach was worth a try, at least for groups over which one has homological control - such as the quaternion groups. Joint work soon led to

Theorem A. The Segal conjecture is valid for the quaternion group $Q_{2^{n+1}}$ provided it is valid for the dihedral group D_{2^n}.

Since the Segal conjecture was known for $D_4 = \mathbf{Z}/2 \times \mathbf{Z}/2$ (a proof was in fact announced by the first author at the Winter AMS meeting in January, 1981), this constituted a proof of it for Q_8, the first non Abelian p-group for which it had been checked. Of course, this result in itself is now of mainly historical interest, and probably only to us, at that. However, we feel that the techniques used merit an audience.

Recall that if DX denotes the Spanier-Whitehead dual of a spectrum X - i.e., the spectrum representing the contravariant functor $W \longmapsto \pi^0(X \wedge W)$ - then the Segal conjecture describes a map to DBG, for any finite group G, which should be a homotopy equivalence. If G is a p-group (which suffices, by independent work of May and McClure, of Laitinen, and of Segal) then DBG is p-adically complete. Our technique yields a general theorem concerning the p-adic completion of DBG for compact Lie groups G which admit a representation which is free away from 0. The only nondiscrete examples of such groups are $SO(2) = S^1$, $Spin(3) = S^3$, and the normalizer N_2 of the maximal torus in S^3. If G is discrete, it must have periodic cohomology. We restrict attention to p-groups; so [8] the only examples are the cyclic p-groups and quaternion 2-groups. Each of these groups has a unique normal subgroup C of order p. Write $\overline{G}$ for the quotient G/C. In the statement of the following theorem, α denotes the adjoint representation of G on its Lie algebra. Generally, we shall denote by B^λ the Thom space of the vector bundle formed over the base B of a principal G-bundle by mixing with a representation λ of G.

Theorem B. Assume G is a periodic p-group, S^1, S^3, or N_2. There is a canonical map

$$DB\overline{G} \vee BG^\alpha \longrightarrow DBG$$

which is an equivalence after p-adic completion.

In Section 1 of this paper we recall Ravenel's method of proof and his "modified Adams spectral sequence," and reduce Theorem B to an algebraic calculation which we carry out in Section 2. In particular, we prove the following Theorem in which A is the Steenrod algebra.

Theorem C. Make $\mathbb{F}_2[z^{\pm 1}]$, $|z| = 4$, into an A-algebra by declaring $Sq\, z = z + z^2$. Make $\mathbb{F}_2[w_2, w_3]$, $|w_i| = i$, into an A-algebra by declaring $Sq\, w_2 = w_2 + w_3$, $Sq\, w_3 = w_3$, and let $\mathbb{F}_2[w_2, w_3]_n$ denote the sub A-module of elements of degree n in the variables. Then there is a canonical isomorphism

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2[z^{\pm 1}]) \cong \bigoplus_{n \geq 0} \mathrm{Ext}^{s-n,t-n}(\mathbb{F}_2, \mathbb{F}_2[w_2, w_3]_n) \quad .$$

The paper ends with an account (due to the first author) of a proof of the inductive step for cyclic groups which does not use Ext.

We wish to thank Doug Ravenel for letting us see an early draft of [13], and the first author thanks Northwestern University for its support and hospitality during the 1981-82 academic year.

Section 1.

We shall set up some machinery valid for any compact Lie group G with a real representation V such that G acts freely off 0. Such groups, and, in fact, such representations, have of course been completely classified [15]. Since G is compact, we are free to give V an equivariant inner product. Let S be the unit sphere and $B_V = G \backslash S$ the orbit space. We begin with a lemma; the notation is as in the introduction.

Lemma 1.1. (i) There is a cofibration sequence

$$B_V^{-V} \longrightarrow BG^{-V} \longrightarrow BG^0$$

in which the first map is induced from the classifying map of $S \longrightarrow B_V$ and the second by including a complement of V (over a finite skeleton of BG) into a trivial bundle.

(ii) ΣB_V^{-V} is Spanier-Whitehead 0-dual to B_V^{α}.

Proof. (i) Ravenel has observed [13] that for any CW complex X and vector bundles α and β over X there is a cofibration sequence

$$S(\alpha)^{\beta} \longrightarrow D(\alpha)^{\beta} \longrightarrow X^{\alpha \oplus \beta}$$

where $S(\alpha)$ and $D(\alpha)$ are the sphere- and disk-bundles of α. Since

$D(\alpha) \cong X$, this may be rewritten, up to homotopy, as

$$S(\alpha)^{\beta} \longrightarrow X^{\beta} \longrightarrow X^{\alpha\oplus\beta} \tag{1.2}$$

where the second map is induced from the inclusion of 0 into α. By restricting to finite subcomplexes and considering complementary bundles, we obtain the same statement for β virtual. Now take $X = BG$, $\alpha = V$, $\beta = -V$, and notice that $S(\alpha) \cong B$.

(ii) Recall (e.g. from [6]) that if G acts freely on a smooth manifold S with orbit manifold B, then $\tau(B) \oplus \alpha \cong G\backslash\tau(S)$. In our situation, $\tau(S) \oplus \underline{\underline{1}} \cong S \times V$ as G-vector bundles. To see this, identify $\tau(S)_s$ with $\{v \in V : v \perp s\}$; then $g \in G$ acts by $g_*(s, v) = (gs, gv)$. The isomorphism is then $(s, v, t) \longmapsto (s, v + ts)$. Dividing by G we find

$$G\backslash(S \times V) \cong G\backslash\tau(S) \oplus \underline{\underline{1}} \cong \tau(B) \oplus \alpha \oplus \underline{\underline{1}} \; .$$

The result is thus Atiyah duality [5] in this instance. □

Let G and $\overline{G}$ be as in Theorem B. Part (i) of Lemma 1.1 gives the top cofibration sequence in the following important diagram, in which X^0 denotes the Thom space of the zero bundle over X - i.e., X with a disjoint base point adjoined.

$$\begin{array}{ccccc} BG^{-V} & \longrightarrow & BG^{0} & \longrightarrow & \Sigma B_V^{-V} \\ & & \big\downarrow & & \\ & & B\overline{G}^{0} & & \end{array} \tag{1.3}$$

Our main technical result is

<u>Theorem</u> 1.4. The composite $BG^{-V} \longrightarrow B\overline{G}^{0}$ induces an isomorphism in $\hat{\pi}^q$ for $q > 1 - \dim V$, where $\hat{\pi}^*$ denotes p-adic cohomology.

We pause to explain about p-adic completion. Let $\tilde{S}^0$ be the fiber of $S^0 \longrightarrow S^0[\frac{1}{p}]$; it is a Moore spectrum with $H_{-1}(\tilde{S}^0) = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. Following A. K. Bousfield [7], we define the p-adic completion of a spectrum X as the spectrum $\hat{X}$ representing the contravariant functor $W \longmapsto [\tilde{S}^0 \wedge W, X]$. It comes equipped with a canonical map $X \longrightarrow \hat{X}$; and it clearly satisfies

$$D(\tilde{S}^0 \wedge X) \cong (DX)\hat{}$$

where D is Spanier-Whitehead duality as in the introduction. Define

$\hat{\pi}^q(X) = [X, \hat{S}^q]$; then

(1.5) $$\hat{\pi}^{-q}(X) = \pi_q((DX)\hat{\ }) \quad .$$

Now D carries cofibration sequences to cofibration sequences [1], so by Lemma 1.1(ii) we obtain from (1.3) the diagram

$$\begin{tikzcd} DBG^{-V} & DBG^0 \arrow[l] & B_V^{\alpha} \arrow[l] \\ & D B\overline{G}^0 \arrow[u] & \end{tikzcd}$$

We may now p-adically complete and take a direct limit over increasing V, using the maps induced by the inclusions $V \longrightarrow V \oplus W$. We get:

$$\begin{tikzcd} \varinjlim_V (DBG^{-V})\hat{\ } & (DBG^0)\hat{\ } \arrow[l] & (BG^{\alpha})\hat{\ } \arrow[l] \\ & (DB\overline{G}^0)\hat{\ } \arrow[u] \arrow[ul] & \end{tikzcd}$$

Theorem 1.4 implies that the diagonal map is a homotopy equivalence, using (1.5) and the Whitehead theorem. Consequently the cofibration splits, and we have obtained Theorem B.

We remark that Segal's conjecture specifies a map $BG^{\alpha} \longrightarrow DBG^0$, constructed from the transfer, which should enter into a description of the homotopy-type of DBG^0. The appendix to [13] shows that the present map agrees with Segal's; so Theorem A follows.

The proof of Theorem 1.4 is achieved by exhibiting spectral sequences converging to the two completed cohomotopy groups involved, and checking that the map at E_2 is an isomorphism in a suitable range of dimensions. Following Ravenel, we use a modification of the Adams spectral sequence, constructed as follows. Assume given a diagram of cofibration sequences

(1.6) $$\begin{tikzcd} X = X_0 \arrow[r] & X_1 \arrow[r] & X_2 \cdots \\ E_0^0 X \arrow[u] & E_1^0 X \arrow[u] & \end{tikzcd}$$

The fiber terms are named $E_s^0 X$ because we choose to think of $X \longrightarrow X_s$ as the quotient of X by the s^{th} stage in an increasing filtration, and of $E_s^0 X$ as the associated quotient. Pick an Adams resolution [1]

$$\begin{tikzcd} \hat{S}^0 = Y^0 \arrow[d] & Y^1 \arrow[l] \arrow[d] & Y^2 \arrow[l] \cdots \\ K^0 & K^1 & \end{tikzcd}$$

for the p-adic sphere. By means of mapping telescopes (as in [13]) we may take $F(X_m, Y^n)$ to be a subspectrum of $F(X, \hat{S}^0)$. Filter $F(X, \hat{S}^0)$ by

$$F_s = \bigcup_{m+n \leq s} F(X_m, Y^n) \quad .$$

To describe the E_2-term of the associated homotopy spectral sequence, let $H^*(E_\bullet^0 X)$ denote the cochain complex of A-modules obtained from (1.6) using the boundary maps. Then [13]

$$E_2 = \mathbf{Ext}_A(\mathbb{F}_2, \bar{H}^*(E_\bullet^0 X)) \quad .$$

Here $\mathbf{Ext}_A(\mathbb{F}_2, C^\bullet)$ denotes the hyperext module associated to a cochain complex $C^\bullet$ over A; it is computed by means of a projective resolution $P_\bullet$ of $\mathbb{F}_2$:

(1.7) $$\mathbf{Ext}_A(\mathbb{F}_2, C^\bullet) \cong H(\text{tot Hom}_A(P_\bullet, C^\bullet)) \quad .$$

To prove Theorem 1.4 we construct filtrations of BG^{-V} and of $B\bar{G}^0$, compatible with the map in question, and verify:

(1.8) $$\text{Ext}_A(\mathbb{F}_p, \bar{H}^*(BG^{-V})) \xrightarrow{\cong} \mathbf{Ext}_A(\mathbb{F}_p, \bar{H}^*(E_\bullet^0 BG^{-V}))$$

(1.9) $$\text{Ext}_A(\mathbb{F}_p, e^{-1}\bar{H}^*(BG^0)) \xrightarrow{\cong} \mathbf{Ext}_A(\mathbb{F}_p, \bar{H}^*(E_\bullet^0 B\bar{G}^0))$$

where e is the mod p reduction of the period of G. Notice that since the mod p Euler class of V is a nonzero scalar multiple of some power of e, we have

(1.10) $$\bar{H}^q(BG^{-V}) \xrightarrow{\cong} e^{-1}\bar{H}^q(BG^0) \qquad \text{iso for } q \geq -\dim V \quad .$$

Using (1.8) and (1.9) it follows that the map $BG^{-V} \longrightarrow B\bar{G}^0$ at $E_2^{s,t}$ is iso for $s-t > -\dim V$. According to [13], the two spectral sequences are convergent in the sense that such an E_2-iso induces an isomorphism in $\hat{\pi}^q$ for $q > 1 - \dim V$; and Theorem 1.4 follows.

Section 2.

In this section we construct filtrations satisfying (1.8) and (1.9). We actually do this explicitly only for $p = 2$ and $G = Q_2n+1$. The modifications required for the other cases may safely be left to the reader, and of course the cases $G = \mathbf{Z}/p^n$ and $G = S^1$ were done by Ravenel [13], though from a slightly different perspective. We shall begin with a proof of Theorem C of the introduction.

If $C^{\bullet}$ is a cochain complex over A, there are two spectral sequences useful in studying $\mathrm{Ext}_A(\mathbf{F}_p, C^{\bullet})$:

$$'E_2 = \mathrm{Ext}_A(\mathbf{F}_p, H(C^{\bullet})) \Longrightarrow \mathrm{Ext}_A(\mathbf{F}_p, C^{\bullet})$$

$$''E_1 = \mathrm{Ext}_A(\mathbf{F}_p, \natural C^{\bullet}) \Longrightarrow \mathrm{Ext}_A(\mathbf{F}_p, C^{\bullet}) \quad .$$

Here $\natural C^{\bullet}$ denotes $C^{\bullet}$ without its differential. These are obtained from the usual filtrations of the double complex defining $\mathrm{Ext}_A(\mathbf{F}_p, C^{\bullet})$, and result in the following:

Lemma 2.1. Let $f : C^{\bullet} \longrightarrow D^{\bullet}$ be a map of bounded below cochain complexes over A. Then $\mathrm{Ext}_A(\mathbf{F}_p, f)$ is an isomorphism if either $H(f)$ or $\mathrm{Ext}_A(\mathbf{F}_p, \natural f)$ is an isomorphism. $\square$

Consider the Laurent series algebra $\mathbf{F}_2[z^{\pm 1}]$ on a generator z of dimension 4, with its natural A-algebra structure: $Sq\, z = z + z^2$. We begin by replacing $\mathbf{F}_2[z^{\pm 1}]$ by a homologically equivalent cochain complex. Consider the bigraded polynomial algebra $\mathbf{F}_2[w_2, w_3]$, $|w_i| = (1, i-1)$, as a cochain complex over A by declaring $d = 0$ and $Sq\, w_2 = w_2 + w_3$, $Sq\, w_3 = w_3$. Let $|x| = 1$ and consider $\mathbf{F}_2[x]$ with $Sq\, x = x + x^2$, and its quotient Hopf algebra over $A, \mathbf{F}_2[x]/x^4$. The map $\tau : \mathbf{F}_2[x]/x^4 \longrightarrow \mathbf{F}_2[w_2, w_3]$ sending $1 \longmapsto 0$, $x \longmapsto w_2$, $x^2 \longmapsto w_3$, $x^3 \longmapsto 0$, is a twisting morphism [10] which is A-linear and has acyclic total complex. For any $\mathbf{F}_2[x]/x^4$-comodule K over A, τ induces an A-linear differential in $\mathbf{F}_2[w_2, w_3] \otimes K$, indicated by decorating the tensor symbol with a superscript τ. An easy calculation shows that $z \longmapsto x^4$ induces an A-linear homology isomorphism

$$\mathbf{F}_2[z] \longrightarrow \mathbf{F}_2[w_2, w_3] \otimes^{\tau} \mathbf{F}_2[x]$$

where $\mathbf{F}_2[x]$ is an $\mathbf{F}_2[x]/x^4$-comodule via the quotient map. Now x^4 acts on $\mathbb{F}_2[x]$ by comodule maps, so we get a structure of $\mathbb{F}_2[x]/x^4$-comodule over A on $\mathbb{F}_2[x^{\pm 1}]$ and on the $\mathbb{F}_2[x]$-submodule $x^{-4k}\mathbb{F}_2[x]$ generated by x^{-4k}, for any k. Thus we obtain A-linear homology isomorphisms

$$(2.2)\qquad \mathbb{F}_2[z^{\pm 1}] \longrightarrow \mathbb{F}_2[w_2, w_3] \otimes^{\tau} \mathbb{F}_2[x^{\pm 1}]$$

$$(2.3)\qquad z^{-k}\mathbb{F}_2[z] \longrightarrow \mathbb{F}_2[w_2, w_3] \otimes^{\tau} x^{-4k}\mathbb{F}_2[x] \quad .$$

On the other hand, Lin's theorem [11] asserts that $\mathbb{F}_2 \longrightarrow \mathbb{F}_2[x^{\pm 1}]$ is an isomorphism in $\mathrm{Ext}_A(\mathbb{F}_2, -)$. Since each degree of $\mathbb{F}_2[w_2, w_3]$ is a finite A-module, it follows that

$$(2.4)\qquad \mathbb{F}_2[w_2, w_3] \longrightarrow \Sigma\mathbb{F}_2[w_2, w_3] \otimes^{\tau} \mathbb{F}_2[x^{\pm 1}]$$

is an isomorphism in $\mathrm{Ext}_A(\mathbb{F}_2, -)$. Combining (2.2) and (2.4) with Lemma 2.1 yields Theorem C. Notice also that we may tensor everything with a finite A-module without altering the proof.

Given a coalgebra C, there is a universal twisting morphism $\theta : \Omega^{\bullet}C \longrightarrow C$ from the cobar construction [10]. If C is a coalgebra over A then, using the diagonal tensor product to make $\Omega^{\bullet}C$ an algebra over A, θ becomes A-linear. The acyclicity of the total complex of τ thus implies that the natural map

$$(2.5)\qquad \Omega^{\bullet}(\mathbb{F}_2[x]/x^4) \otimes^{\theta} M \longrightarrow \mathbb{F}_2[w_2, w_3] \otimes^{\tau} M$$

is an A-linear homology isomorphism, for any $\mathbb{F}_2[x]/x^4$-comodule M over A.

Now all the above algebra is closely modelled on geometric constructions. Thus, if G is a compact Lie group, BG comes equipped with a canonical filtration, due to Milnor. By [14],

$$H^*(E^0_{\bullet}BG) \cong \Omega^{\bullet}(H^*G) \quad ,$$

using the Pontrjagin coalgebra structure of H^*G, as cochain complexes over A. Moreover, given a G-space X, the pullback of the Milnor filtration filters $EG \times_G X$, and

$$(2.6)\qquad H^*(E^0_{\bullet}(EG \times_G X)) \cong \Omega^{\bullet}(H^*G) \otimes^{\theta} H^*X \quad .$$

Suppose we have a pullback diagram of compact Lie groups and homomorphisms

(2.7)
$$\begin{tikzcd} H \arrow[r] \arrow[d] & G \arrow[d] \\ \bar{H} \arrow[r] & \bar{G} \end{tikzcd}$$

in which the vertical maps are epic and the horizontal maps are monic. Let

$$N = \ker(H \longrightarrow \bar{H}) = \ker(G \longrightarrow \bar{G})$$
$$M = G/H = \bar{G}/\bar{H} \ .$$

M is a smooth $\bar{G}$-manifold via left translation. On the other hand, recall that in general, the fiber of a fibration $E \longrightarrow B\bar{G}$ is canonically homotopy-equivalent to a principal $\bar{G}$-bundle over E, namely, the pullback over E of $E\bar{G} \longrightarrow B\bar{G}$. Thus we may take BN to be a $\bar{G}$-space, by virtue of the fibration sequence $BN \longrightarrow BG \longrightarrow B\bar{G}$.

Lemma 2.8. There are canonical homotopy equivalences

$$BH \cong E\bar{G} \times_{\bar{G}} (M \times BN)$$
$$B\bar{H} \cong E\bar{G} \times_{\bar{G}} M \ ,$$

under which the map $BH \longrightarrow B\bar{H}$ is induced by $pr_1 : M \times BN \longrightarrow M$.

Proof. If we arrange that $BG \longrightarrow B\bar{G}$ and $B\bar{H} \longrightarrow B\bar{G}$ are fibrations, then BH may be obtained as a pullback in

$$\begin{tikzcd} BH \arrow[r] \arrow[d] & BG \arrow[d] \\ B\bar{H} \arrow[r] & B\bar{G} \end{tikzcd}$$

Thus in the three-dimensional diagram

$$\begin{tikzcd}
M \times BN \arrow[rr] \arrow[dr] \arrow[dd] & & BN \arrow[dr] \arrow[dd] & \\
& M \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
BH \arrow[rr] \arrow[dr] \arrow[dd] & & BG \arrow[dr] \arrow[dd] & \\
& B\bar{H} \arrow[rr] \arrow[dd] & & B\bar{G} \arrow[dd] \\
B\bar{G} \arrow[rr] \arrow[dr] & & B\bar{G} \arrow[dr] & \\
& B\bar{G} \arrow[rr] & & B\bar{G}
\end{tikzcd}$$

each horizontal square is a pullback and the right-most three columns are fibration sequences. It follows that the left-hand column is a fibration sequence too, and this gives the result. □

Using this lemma we obtain filtrations of BH and $B\bar{H}$ from the Milnor filtration of $B\bar{G}$, and the map $BH \longrightarrow B\bar{H}$ is compatible with them. Moreover, if V is a representation of H, then the Thom spectrum BH^{-V} is filtered by the Thom spectra of the pullback of $-V$ to the filtration degrees of BH. Of course the "transfer" map $BH^{-V} \longrightarrow BH^0$ respects filtrations. We obtain:

$$\bar{H}^*(E^0_\bullet BH^{-V}) \cong \Omega^\bullet(H^*G) \otimes^\theta (H^*M \otimes \bar{H}^*BN^{-V}) \tag{2.9}$$

$$\bar{H}^*(E^0_\bullet B\bar{H}^0) \cong \Omega^\bullet(H^*G) \otimes^\theta H^*BN \ . \tag{2.10}$$

We now specialize by taking for (2.7) the diagram

$$\begin{tikzcd}
Q_2n+1 \arrow[r] \arrow[d] & S^3 \arrow[d] \\
D_2n \arrow[r] & SO(3)
\end{tikzcd}$$

Thus $N = \mathbf{Z}/2$, and M is a certain 3-manifold. Take for V a $4k$-dimensional free representation of Q_2n+1; then we claim that (1.8) and (1.9) hold for the filtrations analyzed above in (2.9) and (2.10). We require only the following Lemma whose proof is left to the reader.

Lemma 2.11. (i) The coaction of $H^*SO(3)$ on H^*M is trivial.

(ii) $H^*(BQ_2n+1) \cong \mathbb{F}_2[z] \otimes H^*M$ with the diagonal A-action; $|z| = 4$.

Assembling (2.11), (2.3), (2.5), and (2.9),

$$\begin{aligned}
\bar{H}^* BQ_2^{-V} n+1 &\cong z^{-k}\mathbb{F}_2[z] \otimes H^*M \\
&\longrightarrow \mathbb{F}_2[w_2, w_3] \otimes^{\tau} x^{-4k}\mathbb{F}_2[x] \otimes H^*M \\
&\longleftarrow \Omega^{\bullet}(H^*SO(3)) \otimes^{\theta} \bar{H}^*BN^{-V} \otimes H^*M \\
&\cong \bar{H}^*(E^0_{\bullet}BQ_2^{-V}n+1)
\end{aligned}$$

where both arrows are homology-isomorphisms and hence, by Lemma 2.1, $\mathrm{Ext}_A(\mathbb{F}_2, -)$-isomorphisms. This gives (1.8). Equation (1.9) is immediate from Theorem C tensored with H^*M. The compatibility of everything is immediate from the close parallel between the algebra and the geometry.

Section 3.

In this section we give a proof of Theorem 1.4 (and hence of Theorem B) in case $G = \mathbb{Z}/p^n$ or $G = S^1$ which, while using ideas from [13], is free of Ext calculations. To begin with, we outline a faulty proof; then we indicate how to fix it up. For notational convenience, take p to be 2.

Let G be any compact Lie group and C a normal subgroup of order 2, with quotient group $\bar{G}$. Let V be a representation of G which restricts to $n = \dim V$ times the sign representation of C. Consider the composite

$$BG^{-V} \longrightarrow BG^0 \longrightarrow B\bar{G}^0 \ .$$

Pseudoproposition 3.1. This composite induces an isomorphism in $\hat{\pi}^q$ for $q > 1-n$.

Pseudoproof. If ξ is a (virtual) vector bundle over the total space E of a fibration $p : E \longrightarrow B$, and h^* is any cohomology theory, then we have a relative Atiyah-Hirzebruch-Serre spectral sequence

$$H^*(B; h^*(F^{\xi})) \Longrightarrow h^*(E^{\xi})$$

where F is the fiber of p. Apply this with $p : BG \longrightarrow B\bar{G}$ and $\xi = -V$, so that F^{ξ} is the "stunted real projective space" P_{-n}, and with $h^* = \hat{\pi}^*$. There is a map of spectral sequences

$$\begin{array}{ccc} H^*(B\bar{G};\hat{\pi}^*) & \Longrightarrow & \hat{\pi}^*(B\bar{G}^0) \\ \downarrow & & \downarrow \\ H^*(BG;\hat{\pi}^*P_{-n}) & \Longrightarrow & \hat{\pi}^*(BG^{-V}) \end{array}$$

Lin's theorem asserts that $\hat{\pi}^q \longrightarrow \hat{\pi}^q P_{-n}$ is iso for $q > 1-n$, so the map is iso at E_2 in a range, and hence in the abutment. $\square$

The first author was very happy to have found such a short proof, till the next morning, when Mark Mahowald pointed out that the result contradicted the validity of the Segal conjecture for $\mathbf{Z}/2 \times \mathbf{Z}/2$. Some comfort was gained from the later realization that he was not the first to fall into this trap, which was dubbed the "canonical error."

The trouble, of course, lies in the fact that Lin's theorem gives us an isomorphism above a horizontal line, while what is needed to conclude anything about the abutment is an isomorphism above a line of slope -1. In order to explain how to achieve this, in case G is cyclic or $G = S^1$, we first express the "Serre filtration," inducing the spectral sequence of the pseudo-proof, in terms of Thom spectra. For clarity, we deal only with $G = S^1$; the case of a cyclic p-group follows similar lines.

Let $X = \mathbf{C}P^\infty$, and consider the map $p : X \longrightarrow X$ induced by squaring in S^1. Let λ be the canonical one-dimensional complex representation of S^1. The Serre filtration of $X^{-n\lambda}$, viewed as a succession of quotients, is equivalent to the system of cofibration sequences

$$\begin{array}{ccccccc} X^{-n\lambda} & \longrightarrow & X^{\lambda^2-n\lambda} & \longrightarrow & X^{2\lambda^2-n\lambda} & \longrightarrow & \cdots \\ \nwarrow & & \nwarrow & & \nwarrow & & \\ P_{-2n} & & \Sigma^2 P_{-2n} & & \Sigma^4 P_{-2n} & & \end{array}$$

The horizontal maps are induced from the inclusion $0 \hookrightarrow \lambda^2$, and the identification of the cofibers results from (1.2) together with the observation that $S(\lambda^2) = \mathbf{R}P^\infty$.

We propose to alter this system so as to obtain stunted projective spaces with bottom cell in lower and lower dimension. To achieve this, we employ a device which Ravenel used (needlessly, as shown above), in his $\mathbf{Ext}$ proof. Let λ be any complex line bundle over an arbitrary CW complex X. The map $S(\lambda) \longrightarrow S(\lambda^2)$ by $z \longmapsto z^2$ induces a map $\psi : X^\lambda \longrightarrow X^{\lambda^2}$ of Thom spaces which is trivial in mod 2 cohomology (since it has degree 2 on the Thom class). We may also twist with a virtual vector bundle α to obtain a map $\psi : X^{\alpha+\lambda} \longrightarrow X^{\alpha+\lambda^2}$ of Thom spectra.

Apply this with X and λ as above, and $\alpha = -(k+i+1)\lambda + i\lambda^2$, to obtain a map

$$X^{-(k+i)\lambda+i\lambda^2} \longrightarrow X^{-(k+i+1)\lambda+(i+1)\lambda^2} .$$

It is not hard to check that the cofiber of this map is $\Sigma^{2i-1}P_{-2(k+i)+1}$, and that we receive a commutative diagram

$$\begin{tikzcd}
X^{-k\lambda} \arrow[rr] \arrow[dd] & & X^{-(k+1)\lambda+\lambda^2} \arrow[rr] \arrow[dd] & & X^{-(k+2)\lambda+2\lambda^2} \arrow[rr] \arrow[dd] & & \cdots \\
& P_{-2k-1} \arrow[ul] \arrow[dd] & & \Sigma^2 P_{-2k-3} \arrow[ul] \arrow[dd] & & \Sigma^4 P_{-2k-5} \arrow[ul] \arrow[dd] & \\
\mathbf{CP}_0^\infty \arrow[rr] & & \mathbf{CP}_1^\infty \arrow[rr] & & \mathbf{CP}_2^\infty \arrow[rr] & & \cdots \\
& S^0 \arrow[ul] & & S^2 \arrow[ul] & & S^4 \arrow[ul] &
\end{tikzcd}$$

in which the vertical fiber maps are exactly those guaranteed by Lin's theorem to induce isomorphisms in $\hat{\pi}^q$ for $q > -2k$.

What remains is to prove convergence of each spectral sequence: i.e., that $\varprojlim \hat{\pi}^* = \varprojlim^1 \hat{\pi}^* = 0$ in both cases.

The $\varprojlim^1 \hat{\pi}^*$ terms vanish in both cases since these are inverse systems of compact Abelian groups. Since each horizontal map in the top "exact couple" is trivial in mod 2 cohomology, any string in $\varprojlim \hat{\pi}^* X^{-(k+i)\lambda+i\lambda^2}$ consists entirely of elements of infinite Adams filtration. But the Adams spectral sequences converge [11], [4], so this $\varprojlim$ is zero. For the bottom "exact couple," notice that $\varprojlim \hat{\pi}^*(\mathbf{CP}_i^\infty)$ is contained in the group of phantom completed cohomotopy classes in $\mathbf{CP}^\infty$, i.e., $\varprojlim^1 \hat{\pi}^*(\mathbf{CP}^{i-1})$, and this group is zero, again by compactness.

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