Milnor K-theory and motivic homotopy

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Milnor. Milnor, Inv. math. 9 (1970) 318–344:

k a field. $K_*^M(k)$ is a graded ring; it comes with a group homomorphism $k^\times \to K_1^M(k)$; and is universal as such with the relation

$$\rho_a\rho_b = 0 \quad \text{if} \quad a+b=1$$

So $\rho_{ab} = \rho_a + \rho_b$ and $\rho_1 = 0$. Milnor gives a beautiful proof that the ring is graded-commutative, which makes you think that $K_*^M(k)$ really does carry deep information about k. In particular $2\rho_a = 0$. Also $\rho_a^2 = \rho\rho_a$ where $\rho = \rho_{-1}$. And if $a_1 + \cdots + a_n = 1$ then $\rho_{a_1}\cdots\rho_{a_n} = 0$.

The relations are all quadratic, so $K_1^M(k) = k^\times$.

An order on k determines a surjective ring homomorphism

$$\text{sgn} : K_*^M(k) \to \mathbb{F}_2[t]$$

The order determines a group homomorphism $k^\times \to \{\pm 1\}$, sending x to 1 if $x > 0$ and to -1 if $x < 0$. Regard this as a map $f : k^\times \to \mathbb{F}_2\langle t\rangle$, so positive numbers go to 0 and negative numbers go to t. Then note that since either x or $1-x$ is positive, the map satisfies $f(x)f(1-x) = 0$, and determines sgn as shown. It is surjective since $\rho \mapsto t$.

Milnor asserts that $K_n^M(\mathbb{R})$ is the direct sum of a cyclic group of order 2 generated by ρ^n (nonzero since $\text{sgn}(\rho^n) = t^n$) with an infinitely generated divisible group.

Morel ("Motivic π_0"; see also Dugger and Isaksen, "Motivic Hopf elements and relations") defines the Milnor-Witt K-groups as the graded ring accepting a map $k^\times \to K_1^{MW}(k)$ (no longer assumed to be a homomorphism) and containing an element $\eta \in K_{-1}^{MW}(k)$, and universal subject to:

(i) $\eta\rho_a = \rho_a\eta$

(ii) $\rho_a\rho_b = 0$ if $a+b=1$

(iii) $\eta h = 0$, where $h = 2 + \eta\rho$

(iv) $\rho_{ab} = \rho_a + \rho_b + \eta\rho_a\rho_b$

(v) $\rho_1 = 0$

Note that (iv) implies that $\rho h = 0$ also.

The element $\epsilon = -1 - \rho\eta = 1 - h$ plays a special role; regarded as an element of the homotopy of the smash square of the zero sphere, it swaps the factors. It's central in $K_*^{MW}(k)$, $\epsilon^2 = 1$, and In his book, p 51, Cor 3.8, Morel shows that $\rho_a \rho_b = \epsilon \rho_b \rho_a$. This noncommutativity is subtle. (iv) shows that it's killed by η.

Filter $K_*^{MW}(k)$ by powers of the central element η. Then $\mathrm{gr}^0 = K_*^M(k)$. The relation (iii) shows that 2 kills gr^s for $s > 0$, and indeed

$$\mathrm{gr}^* K_*^{MW}(k) = K_*^M(k)[\eta]/(2\eta)$$

Look at $K_0^{MW}(k)$. It is generated by elements (following Morel, "Motivic π_0") $\langle a \rangle = 1 + \eta\rho_a$, $a \in k^\times$, and these elements satisfy:
(ii) $1 + \langle a \rangle \langle b \rangle = \langle a \rangle + \langle b \rangle$ if $a + b = 1$
(iii) $h\langle a \rangle = h$ where $h = 1 + \langle -1 \rangle$
(iv) $\langle ab \rangle = \langle a \rangle \langle b \rangle$
(v) $\langle 1 \rangle = 1$.
At least when the characteristic of k is not 2, this is the "Grothendieck-Witt ring" $GW(k)$, the group completion of the commutative monoid of quadratic forms over k. The class $\langle a \rangle$ corresponds to the quadratic form ax^2.

Multiplying by η kills h but does no other damage. So

$$K_{\leq 0}^{MW}(k) = GW(k)[\eta]/(\eta h)$$

The quotient $GW(k)/(h)$ is the "Witt ring" of k; h represents the hyperbolic form. Also $h = 1 - \epsilon$, which Dugger and Isaksen ("Motivic Hopf elements and relations") pick out as the first element of Hopf invariant one.

Morel proves that Milnor-Witt K-theory gives the motivic stable homtopy ring in coweight zero:

$$K_n^{MW}(k) = \pi_{-n,-n}(S)$$

So in coweight zero and positive dimension we see precisely the Witt group of the field.

Calculations: $GW(\mathbb{C}) = \mathbb{Z}$ and h corresponds to 2. This is the claim that if $f : \mathbb{C}^\times \to S$ is a map satisfying the properties for which $K_0^{MW}(\mathbb{C})$ is universal, then $f(a) = 1$ for all a.

Let $\sigma : \mathbb{R}^\times \to \mathbb{Z}[C_2]$ by sending a to 1 if $a > 0$ and to the generator T of C_2 if $a < 0$. This map satisfies the properties for which $K_0^{MW}(\mathbb{R})$ is universal, so we get a map $K_0^{MW}(\mathbb{R}) \to \mathbb{Z}[C_2]$ sending $\langle a \rangle$ to 1 if $a > 0$ and to T if $a < 0$. It sends $h = 1 + \langle -1 \rangle$ to $1 + T$. The claim is that this map is an isomorphism; so $W(\mathbb{R}) = \mathbb{Z}$. This is equivalent to claiming if $f : \mathbb{R}^\times \to S$ is any map satisfying these properties, then $f(a)$ depends only on the sign of a.

Morel proves that there is a map

$$K_*^{MW}(k) \to W(k)[\eta^{\pm 1}]$$

sending ρ_a to $\eta^{-1}(\langle a \rangle - 1)$, and this map localizes to an isomorphism

$$\eta^{-1} K_*^{MW}(k) \to W(k)[\eta^{\pm 1}]$$

Motivic gradings. The rank two free abelian group grading motivic homotopy has a variety of coordinate functions defined on it. They are related as follows:

coweight + weight = dimension

coweight − weight = Chow or Novikov degree

	dim	wt	cowt	deg
ϵ	0	0	0	0
ρ_a	-1	-1	0	1
η	1	1	0	-1
ν	3	2	1	-1
τ	0	-1	1	2
τ_i	$2^{i+1} - 1$	$2^i - 1$	2^i	1
ξ_i	$2(2^i - 1)$	$2^i - 1$	$2^i - 1$	0
ζ_i	$2^i - 1$	$2^{i-1} - 1$	2^{i-1}	1

Here ζ_i is the element used by Dugger and Isaksen;

$$A_{\text{Mot}} \to \mathbb{M}_2[\tau^{-1}, \zeta_1, \ldots]$$

with

$$\tau_i \mapsto \zeta_{i+1} \quad , \quad \xi_i \mapsto \tau^{-1}\zeta_i^2$$

Convergence. Bousfield and Kan (Vol 304, p 183) show that if X is a nilpotent space then the $H\mathbb{F}_p$ nilpotent completion tower is pro-isomorphic to the p-completion tower, so we have the Milnor sequence

$$0 \to \text{Ext}(\mathbb{Z}_{p^\infty}, \pi_n(X)) \to \pi_n(X_p^\wedge) \to \text{Hom}(\mathbb{Z}_{p^\infty}, \pi_{n-1}(X)) \to 0$$

This is discussed in the motivic setting by Hu, Kriz, and Ormsby ("Convergence of the motivic Adams spectral sequence"):

$$0 \to \mathrm{Ext}(\mathbb{Z}_{p^\infty}, \pi_{n,q}(X)) \to \pi_{n,q}(X_p^\wedge) \to \mathrm{Hom}(\mathbb{Z}_{p^\infty}, \pi_{n-1,q}(X)) \to 0$$

Generally one needs to complete at η as well as at p, but not over $\mathbb{R}$ or $\mathbb{C}$. In that case then we have

$$0 \to \mathrm{Ext}(\mathbb{Z}_{2^\infty}, \pi_{0,-1}(S)) \to \pi_{0,-1}(S_2^\wedge) \to \mathrm{Hom}(\mathbb{Z}_2, \pi_{-1,-1}(S)) \to 0$$

In his book Morel explains that if every element in k is a square then there's an embedding $k^\times \to K_1^{MW}(k)$. So when $k = \mathbb{C}$ there's an embedding

$$\mathbb{Z}_{2^\infty} \hookrightarrow K_1^{MW}(\mathbb{C}) = \pi_{-1,-1}(\mathbb{C})$$

The mysterious element θ, mapping to $\tau \in \mathbb{M}_{0,-1}$, is a preimage of this map; it does not come from $\pi_{0,-1}(S)$ under the completion map.

Over the reals, $\eta_R \tau = \tau + \tau_0 \rho$: τ is not primitive and does not survive to an element of $\pi_{-1,-1}(S_2^\wedge)$. This corresponds to the fact that $K_1^{MW}(\mathbb{R})$ doesn't contain any infinitely 2-divisible elements.

The short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}_{p^\infty} \to 0$$

gives us the long exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}_{p^\infty}, A) \to \mathrm{Hom}(\mathbb{Z}[1/p], A) \to A \to$$

$$\mathrm{Ext}(\mathbb{Z}_{p^\infty}, A) \to \mathrm{Ext}(\mathbb{Z}[1/p], A) \to 0$$