# THE ADAMS SPECTRAL SEQUENCE: COURSE NOTES

HAYNES MILLER

## 1. Triangulated categories

**Definition 1.1.** A *triangulated category* is an additive category $\mathcal{C}$ equipped with an additive self-equivalence $\Sigma$ and a class of *distinguished triangles* $X \to Y \to Z \to \Sigma X$, satisfying:

**TR1a.** Any triangle isomorphic to a distinguished triangle is distinguished.

**TR1b.** For any object $X$, $X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow \Sigma X$ is distinguished.

**TR1c.** Any map $X \to Y$ participates in a distinguished triangle

$$X \to Y \to Z \to \Sigma X\,.$$

**TR2.** If $X \xrightarrow{i} Y \xrightarrow{j} Z \xrightarrow{k} \Sigma X$ is distinguished, then so is

$$Y \xrightarrow{j} Z \xrightarrow{k} \Sigma X \xrightarrow{-\Sigma i} \Sigma Y\,.$$

**TR3.** Given two distinguished triangles and maps $f, g$ making the left square below commute, there is a map $h$ making the entire diagram commute.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dotted, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

**TR4.** The *octahedral* or *composition* axiom.

Before giving the octohedral axiom we observe a few corollaries of the first three axioms. These properties were observed in the the homotopy category by Michael Barratt [1] in 1955, and written down as axioms by Dieter Puppe [11] in 1962. The octahedral axiom was added by Jean-Louis Verdier [12] in 1967. Because triangulated categories model stable homotopy categories, we will use the notation $X \vee Y$ for the coproduct, $[A, X]$ for the abelian group $\mathcal{C}(A, X)$, and call $Y \to Z$ (or

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just $Z$) the *cofiber* of $X \to Y$, and $X \to Y$ (or just $X$) the *fiber* of $Y \to Z$, when $X \to Y \to Z \to \Sigma X$ is distinguished.

First notice that the triangle obtained by negating any *two* arrows in a distinguished triangle is isomorphic to the original triangle, using the $-1$ automorphism of one of the objects, and so is again distinguished.

If we apply **TR2** twice to $X \xrightarrow{i} Y \xrightarrow{j} Z \xrightarrow{k} \Sigma X$, we find that

$$Z \xrightarrow{k} \Sigma X \xrightarrow{-\Sigma i} \Sigma Y \xrightarrow{-\Sigma j} \Sigma Z$$

is distinguished, and hence so is $Z \xrightarrow{k} \Sigma X \xrightarrow{\Sigma i} \Sigma Y \xrightarrow{\Sigma j} \Sigma Z$. Shifting again we find that $\Sigma X \xrightarrow{\Sigma i} \Sigma Y \xrightarrow{\Sigma j} \Sigma Z \xrightarrow{-\Sigma k} \Sigma^2 X$ is also distinguished.

**Lemma 1.2.** If $X \xrightarrow{i} Y \xrightarrow{j} Z \xrightarrow{k} \Sigma X$ is distinguished then for any $A \in \mathcal{C}$ the sequences of abelian groups $[A, X] \to [A, Y] \to [A, Z] \to [A, \Sigma X]$ and $[X, A] \leftarrow [Y, A] \leftarrow [Z, A] \leftarrow [\Sigma X, A]$ are exact. $\square$

**Definition 1.3.** A functor from a triangulated category to an abelian category is *homological* if it is additive and sends distinguished triangles to exact sequences.

So we have seen that $[W, -]$ is a homological functor, for any object $W$. In particular, any composite in a distinguished triangle is trivial. Dually, a contravariant functor is *cohomological* if it sends distinguished triangles to exact sequences; and $[-, W]$ is cohomological.

We can use the suspension functor to extend any homological functor $M$ to a functor $M_*$ to graded objects, by defining

$$M_n(X) = M(\Sigma^{-n} X)$$

For example,

$$[W, X]_n = [W, \Sigma^{-n} X] = [\Sigma^n W, X]$$

If we apply Lemma 1.2 to shifts, positive and negative, of the distinguished triangle, we find for any $A$ a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & {[A,\Sigma^{-1}Y]} \arrow[r, "-\Sigma^{-1}j"] & {[A,\Sigma^{-1}Z]} \arrow[dll, "-\Sigma^{-1}k"'] \\
{[A,X]} \arrow[r, "i"] & {[A,Y]} \arrow[r, "j"] & {[A,Z]} \arrow[dll, "k"'] \\
{[A,\Sigma X]} \arrow[r, "-\Sigma i"] & {[A,\Sigma Y]} \arrow[r, "-\Sigma j"] & \cdots
\end{tikzcd}$$

with alternating signs. Of course the sequence remains exact if the signs are changed.

The map $h$ in **TR3** is clearly not unique in general; for example if $Y = 0 = X'$ then $h : \Sigma X \to Y'$ is arbitrary. However, the preceeding lemmas and the 5-lemma show:

**Lemma 1.4.** In the notation of the lemma above, if $f$ and $g$ are isomorphisms then so is $h$. $\square$

As a result, the distinguished triangle completing a map is unique, but only up to a non-unique isomorphism.

The composition axiom is a triangulated analogue of the isomorphism $C/B \cong (C/A)/(B/A)$ arising from inclusions $A \subseteq B \subseteq C$ of abelian groups.

For any composable pair of arrows $X \to Y \to Z$ we may form cofibers of the two arrows and of their composite.

$$\begin{tikzcd}
 & & Y' \arrow[dl, dotted] & & \\
 & X \arrow[rr] \arrow[dr] & & Z \arrow[ul] \arrow[dr] & \\
Z' \arrow[ur, dotted] & & Y \arrow[ll] \arrow[ur] & & X' \arrow[ll, dotted]
\end{tikzcd}$$

Now notice that the composite $Z' \to \Sigma X \to \Sigma Z$ is trivial, so the first map factors though the fiber of the second by a map $f : Z' \to Y'$. Also $X \to Z \to X'$ is trivial, so the second map factors through the cofiber of the first by a map $g : Y' \to X'$. The following depiction of the octahedral axiom is due [6].

**TR4.** (Octahedral axiom) The two maps $f$ and $g$ can be chosen in such a way that the outside triangle in the diagram at left is distinguished and the two squares with corners $Y'$ and $Y$ commute.

$$\begin{tikzcd}
 & & Y' \arrow[dl, dotted] \arrow[ddrr, bend left, "g"] & & \\
 & X \arrow[rr] \arrow[dr] & & Z \arrow[ul] \arrow[dr] & \\
Z' \arrow[ur, dotted] \arrow[uurr, bend left, "f"] & & Y \arrow[ll] \arrow[ur] & & X' \arrow[ll, dotted] \arrow[llll, dotted, bend left]
\end{tikzcd}$$

The diagram is the 1-skeleton of an octohedron. If we write $X' = Z/Y$, $Y' = Z/X$, and $Z' = Y/X$, then octahedron and the square ending in $Y'$ represent the isomorphism $Z/Y = (Z/X)/(Y/X)$ and its expected properties. The content of the square ending in $Y$ is in a sense dual to that of the other square. It compares boundary homomorphisms, and as such plays an important role in homotopy theory.

We can assemble most of the statement of the octahedral axiom into a commutative diagram in which the rows and columns are all distinguished:

$$\begin{tikzcd}
X \arrow[r, "="] \arrow[d] & X \arrow[d] & & \\
Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Z/Y \arrow[r, "="] \arrow[d] & \Sigma Y \arrow[d] \\
Y/X \arrow[r, "f"] \arrow[d] & Z/X \arrow[r, "g"] \arrow[d] & Z/Y \arrow[r] & \Sigma Y/X \\
\Sigma X \arrow[r, "="] \arrow[d, "-"] & \Sigma X \arrow[d, "-"] & & \\
\Sigma Y \arrow[r] & \Sigma Z & &
\end{tikzcd}$$

Here I have employed the following convention: If the source and target of two maps differ by a suspension, then the maps do too, unless the arrow is marked by a negative sign, in which case they differ by a suspension and a sign. The diagram contains the four distinguished triangles and the four commuting triangles present in the octahedron, and also the first commuting square.

From this the usual proof of the Meyer-Vietoris theorem leads to a long exact sequence in any homological functor $M_*$

$$\cdots \longrightarrow M_*(Y) \longrightarrow M_*(Y/X) \oplus M_*(Z) \longrightarrow M_*(Z/X) \longrightarrow M_{*-1}(Y) \longrightarrow \cdots$$

The boundary map can be taken to be either $M_*(Z/X) \to M_*(Z/Y) \to M_{*-1}(Y)$ or $M_*(Z/X) \to M_{*-1}(X) \to M_{*-1}(Y)$. The remaining part of the octahedral axiom—commutativity of the second square—shows that the two boundary maps coincide up to sign.

Diagrams of this type may be regarded as providing a triangulated replacement for push-outs or pull-backs.

This admits an interesting generalization.

**Lemma 1.5** (Verdier)**.** Any commutative square

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[d] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

embeds into a diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\
Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & \Sigma Y' \arrow[d] \\
Z' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "-"] & \Sigma Z' \arrow[d, "-"] \\
\Sigma X' \arrow[r] & \Sigma X \arrow[r] & \Sigma X'' \arrow[r, "-"] & \Sigma^2 X'
\end{tikzcd}$$

with distinguished rows and columns, which is commutative except that the bottom right square is anti-commutative. Moreover, there is an object $V$, three morphisms to it, and three morphisms from it, such that in the diagram

$$
\begin{tikzcd}
 & X \arrow[dd] \arrow[drrr] & & & & Y' & \\
 & & X' \arrow[rr] \arrow[urrr] \arrow[dll] & & Y \arrow[dl] \arrow[dr] & & \\
X & X'' \arrow[rr] \arrow[ur, dotted] & & V \arrow[rr] \arrow[ul, dotted] \arrow[dl] & & Y'' \arrow[uu] \arrow[dd] \arrow[dl, dotted] & Y' \arrow[ull] \arrow[dll] \\
 & & Z \arrow[ull, dotted] \arrow[ul, dotted] \arrow[drrr] & & Z' \arrow[ll] \arrow[ul] & & \\
 & Z'' \arrow[uu, dotted] \arrow[urrr, dotted] \arrow[ur, phantom, "-"] & & & & Z'' &
\end{tikzcd}
$$

the triangles in the hexagon are alternately commutative and distinguished; the composites across the middle are the obvious ones; non-obvious maps around the edge are composites of obvious maps; and of the six kites forming the star five commute and one anti-commutes.

*Proof.* Complete arrows to distinguished triangles in the diagram

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr] & X \arrow[r] \arrow[d] & X'' \arrow[r] & \Sigma X' \\
Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] \arrow[dr] & Y'' \arrow[r] & \Sigma Y' \\
Z' \arrow[d] & Z \arrow[d] & V \arrow[dr] & \\
\Sigma X' & \Sigma X & & \Sigma X'
\end{tikzcd}
$$

but resist the temptation to choose fill-ins. Instead, use these five distinguished triangles in two instances of the octahedral axiom:

(1)

$$
\begin{tikzcd}
 & & V \arrow[dl, dotted] \arrow[ddrr, bend left] & & \\
 & X' \arrow[rr] \arrow[dr] & & Y \arrow[ul] \arrow[dr] & \\
X'' \arrow[uurr, bend left] \arrow[ur, dotted] & & X \arrow[ll] \arrow[ur] & & Z \arrow[ll, dotted] \arrow[llll, dotted, bend left]
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
 & & V \arrow[dl, dotted] \arrow[ddrr, bend left] & & \\
 & X' \arrow[rr] \arrow[dr] & & Y \arrow[ul] \arrow[dr] & \\
Z' \arrow[uurr, bend left] \arrow[ur, dotted] & & Y' \arrow[ll] \arrow[ur] & & Y'' \arrow[ll, dotted] \arrow[llll, dotted, bend left]
\end{tikzcd}
$$

Then complete the composite $X'' \to V \to Y''$ to a distinguished triangle $X'' \to Y'' \to Z'' \to \Sigma X''$ and embed this, the outer distinguished triangle in (1), and the shift of the outer distingushed triangle in (2) into the octahedron

(3)

$$\begin{tikzcd}
& & Z'' \arrow[dl, dotted] \arrow[ddrr, bend left] & & \\
& X'' \arrow[rr] \arrow[dr] & & Y'' \arrow[ul] \arrow[dr] & \\
Z \arrow[ur, dotted] \arrow[uurr, bend left] & & V \arrow[ll] \arrow[ur] & & \Sigma Z' \arrow[ll, dotted, "-"'] \arrow[llll, dotted, bend left, "-"']
\end{tikzcd}$$

The outside distinguished triangle is the shift of $Z' \to Z \to Z'' \to \Sigma Z'$, which is therefore distinguished.

Now we fill in the $4 \times 4$ diagram with the maps available. Note that final arrow of the right edge is the negative of the suspension of $Z' \to \Sigma X'$, and the final arrow of the botton edge is the negative of the suspension of $X'' \to \Sigma X'$.

$$\begin{tikzcd}
X' \arrow[rr] \arrow[dd] & & X \arrow[rr] \arrow[dd] & & X'' \arrow[rr] \arrow[dd] \arrow[dl] \arrow[dr] & & \Sigma X' \arrow[dd] \\
& & & V \arrow[dr] & & V \arrow[ur] \arrow[dl] & \\
Y' \arrow[rr] \arrow[dd] & & Y \arrow[rr] \arrow[dd] \arrow[ur] \arrow[dl] \arrow[dr] & & Y'' \arrow[rr] \arrow[dd] \arrow[ddrr] & & \Sigma Y' \arrow[dd] \\
& V \arrow[dr] & & V \arrow[ur] \arrow[dl] & & & \\
Z' \arrow[rr] \arrow[dd] \arrow[ur] \arrow[dr] & & Z \arrow[rr] \arrow[dd] \arrow[ddrr] & & Z'' \arrow[rr] \arrow[dd] & & \Sigma Z' \arrow[dd, "-"] \arrow[dl, "-"'] \\
& V \arrow[ur] \arrow[dl] & & & & \Sigma V \arrow[dr] & \\
\Sigma X' \arrow[rr] & & \Sigma X \arrow[rr] & & \Sigma X'' \arrow[rr, "-"] \arrow[ur] & & \Sigma^2 X'
\end{tikzcd}$$

Every square and every triangle commutes by reference to one of the three octahedra, except for the triangle at the right end of the bottom edge, which anti-commutes.

The properties of the object $V$ are captured by the remaining diagram in the statement. $\square$

**Remarks 1.6.** (1) The hexagon expresses the fact that the cofibers of parallel diagonals coincide, up to suspension, in a very coherent way. We will call the structure occuring in the conclusion of Verdier's lemma a *Verdier system*.

(2) Observe that the proof did not use **TR3**; in fact it shows that **TR3** is a consequence of the other axioms.

(3) The hexagon may be replaced by

$$
\begin{tikzcd}
 & Y' \arrow[drrr] \arrow[dd] & & & & X & \\
 & & X' \arrow[urrr] \arrow[rr] \arrow[dll] & & Y \arrow[dl] \arrow[dr] & & \\
Y' & Z' \arrow[rr] \arrow[ur, dotted] & & V \arrow[rr] \arrow[ul, dotted] \arrow[dl] & & Z \arrow[uu] \arrow[dd] \arrow[dl, dotted] & X \arrow[ull] \arrow[dll] \\
 & \arrow[dr, phantom, "-"] & Y'' \arrow[ull] \arrow[ul, dotted] \arrow[drrr] & & X'' \arrow[ul] \arrow[ll] & & \\
 & Z'' \arrow[uu, dotted] \arrow[urrr, dotted] & & & & Z'' &
\end{tikzcd}
$$

in which the roles played by the composites through the center and the non-obvious edges are reversed.

Here is an interesting corollary of the existence of a Verdier structure. Let $M$ be a homological functor, and suppose that $x' \in M(X')$ maps to zero in $M(Y)$ under the diagonal map in the commutative square

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[d] \\
Y' \arrow[r] & Y
\end{tikzcd}
$$

Let $x'$ map to $x \in M(X)$ and to $y' \in M(Y')$. Suppose we have an enlargement of the $2 \times 2$ square to a $4 \times 4$ square. Then $x$ maps to $0 \in M(Y)$ so it lifts to an element $z \in M_1(Z)$, and $y'$ maps to $0 \in M(Y)$ so it lifts to an element $y'' \in M_1(Y'')$.

**Proposition 1.7.** If the $4 \times 4$ square admits a Verdier stucture, then we can pick the lifts $z$ and $y''$ so that they map to the same element of $M_1(Z'')$.

*Proof.* One of the large triangles in the Verdier structure is

$$
\begin{tikzcd}
 & & & Y' \\
 & X' \arrow[urr] \arrow[dl] & & \\
X & & V \arrow[ul, dotted] \arrow[r] \arrow[dl] & Y'' \arrow[uu] \arrow[dd] \\
 & Z \arrow[ul] \arrow[drr] & & \\
 & & & Z''
\end{tikzcd}
$$

Since $x'$ maps to zero in $M(Y)$, it pulls back to an element $v \in M_1(V)$. If we choose the images of $v$ in $M_1(Z)$ and $M_1(Y'')$, the lower right kite gives us what we want. $\square$

## 2. Symmetric monoidal structure

Very frequently one has both a triangulated structure and a symmetric monoidal structure. These two should interact well. Hovey, Palmieri, and Strickland [6] proposed some axioms, and Peter May [7] proposed more. Our axioms form a subset of May's. We will write $\wedge$ for the symmetric monoidal structure, $S$ for the unit object, and $T : X \wedge Y \to Y \wedge X$ for the symmetry. In the presence of both structures, we will write $S^n = \Sigma^n S$.

We will assume that a symmetric monoidal structure on an additive category is *additive*: the natural maps at the top and bottom of the commutative diagram

$$\begin{tikzcd}
(W \wedge X) \vee (W \wedge Y) \arrow[r] \arrow[d, "\cong"] & W \wedge (X \vee Y) \arrow[d, "\cong"] \\
(W \wedge X) \times (W \wedge Y) & W \wedge (X \times Y) \arrow[l]
\end{tikzcd}$$

are isomorphisms. This implies that the functors $W \wedge - : \mathcal{C} \to \mathcal{C}$ and $- \wedge W : \mathcal{C} \to \mathcal{C}$ are additive.

**Definition 2.1.** A symmetric monoidal structure and a triangulated structure on an additive category are *compatible* provided:

**TC1.** We are given a natural isomorphism $X \wedge S^1 \to \Sigma X$ such that (with $X = S^1$)

$$\begin{tikzcd}
S^1 \wedge S^1 \arrow[r, "T"] \arrow[d] & S^1 \wedge S^1 \arrow[d] \\
\Sigma S^1 \arrow[r, "-1"] & \Sigma S^1
\end{tikzcd}$$

commutes.

**TC2.** For any distinguished triangle $X \to Y \to Z \to \Sigma X$ and any object $W$,

$$W \wedge X \longrightarrow W \wedge Y \longrightarrow W \wedge Z \xrightarrow{k} \Sigma(W \wedge X)$$

is distinguished, where $k$ is the composite along the top in

$$\begin{tikzcd}
W \wedge Z \arrow[r] & W \wedge \Sigma X \arrow[r] & \Sigma(W \wedge X) \\
& W \wedge (X \wedge S^1) \arrow[u, "\cong"] \arrow[r, "\cong"] & (W \wedge X) \wedge S^1 \arrow[u, "\cong"]
\end{tikzcd}$$

**TC3.** Smashing two distinguished triangles together extends to a Verdier system.

Before we elucidate **TC3**, we mention some easy consequences of the first two axioms. The isomorphism $X \wedge S^1 \to \Sigma X$ determines a natural isomorphism $X \wedge \Sigma Y \to \Sigma(X \wedge Y)$, defined so that

$$\begin{tikzcd}
X \wedge \Sigma Y \arrow[r] & \Sigma(X \wedge Y) \\
X \wedge (Y \wedge S^1) \arrow[u, "\cong"] \arrow[r, "\cong"] & (X \wedge Y) \wedge S^1 \arrow[u, "\cong"']
\end{tikzcd}$$

commutes. Using the symmetry, we also obtain a natural isomorphism $(\Sigma X) \wedge Y \to \Sigma(X \wedge Y)$ such that

$$\begin{tikzcd}
(\Sigma X) \wedge Y \arrow[r] \arrow[d, "T"] & \Sigma(X \wedge Y) \arrow[d, "\Sigma T"] \\
Y \wedge \Sigma X \arrow[r] & \Sigma(Y \wedge X)
\end{tikzcd}$$

commutes.

Of course we also receive a natural isomorphism $S^1 \wedge Y \to \Sigma Y$ by composing with the transposition $T$. Similarly, if $X \to Y \to Z \to \Sigma X$ is distinguished then so is $X \wedge W \longrightarrow Y \wedge W \longrightarrow Z \wedge W \xrightarrow{TkT} \Sigma(X \wedge W)$, with $k$ as in **TC2**. The natural isomorphism $S^p \wedge S^q \to \Sigma^q S^p = \Sigma^{p+q} S = S^{p+q}$ is "skew-commutative":

$$\begin{tikzcd}
S^p \wedge S^q \arrow[r] \arrow[d, "T"] & S^{p+q} \arrow[d, "(-1)^{pq}"] \\
S^q \wedge S^p \arrow[r] & S^{p+q}
\end{tikzcd}$$

The meaning of **TC3** is this: Given two distinguished triangles, $A \to B \to C \to \Sigma A$ and $X \to Y \to Z \to \Sigma X$, we may form the $4 \times 4$ commutative diagram

$$\begin{tikzcd}
A \wedge X \arrow[r] \arrow[d] & A \wedge Y \arrow[r] \arrow[d] & A \wedge Z \arrow[r] \arrow[d] & \Sigma(A \wedge X) \arrow[d] \\
B \wedge X \arrow[r] \arrow[d] & B \wedge Y \arrow[r] \arrow[d] & B \wedge Z \arrow[r] \arrow[d] & \Sigma(B \wedge X) \arrow[d] \\
C \wedge X \arrow[r] \arrow[d] & C \wedge Y \arrow[r] \arrow[d] & C \wedge Z \arrow[r] \arrow[d] \arrow[dr, phantom, "-"] & \Sigma(C \wedge X) \arrow[d, "-"] \\
\Sigma(A \wedge X) \arrow[r] & \Sigma(A \wedge Y) \arrow[r] & \Sigma(A \wedge Z) \arrow[r, "-"] & \Sigma^2(A \wedge X)
\end{tikzcd}$$

in which the rows and columns are distinguished. (Check the sign of the lower right square!) The axiom asserts that this $4 \times 4$ diagram may be completed to a Verdier system, as in the conclusion of Verdier's lemma 1.5. The diligent reader may check that this is precisely the content of May's axiom **TC3$'$**.

## 3. Ideals in triangulated categories

This material is taken largely from [4], which improved on [5] and [9].

**Definition 3.1.** Let $\mathcal{I}$ be a class of objects and $\mathcal{N}$ a class of morphisms in a pointed category $\mathcal{C}$. The pair $(\mathcal{I}, \mathcal{N})$ is *orthogonal* provided that for all $I \in \mathcal{I}$, all $j : X \to Y$ in $\mathcal{N}$, and all $f : Y \to I$, the composite $fj$ is trivial. The pair $(\mathcal{I}, \mathcal{N})$ is *complementary* if it is an orthogonal pair such that

- An object $I$ is in $\mathcal{I}$ whenever $fj = 0$ for all $j : X \to Y$ in $\mathcal{N}$ and all $f : Y \to I$, and
- A morphism $j : X \to Y$ is in $\mathcal{N}$ whenever $fj = 0$ for all $I \in \mathcal{I}$ and all $f : Y \to I$.

The classes $\mathcal{I}$ and $\mathcal{N}$ in a complementary pair determine each other and each is closed under retracts and in particular isomorphisms.

**Lemma 3.2.** The class of morphisms in a complementary pair is an *ideal*: if $j : X \to Y$ is in $\mathcal{N}$ then so are $jh$ and $gj$ for any $h : W \to X$ and any $g : Y \to Z$.

*Proof.* Let $I \in \mathcal{I}$. If $f : Y \to I$, then $fj = 0$, so $f(jh) = (fj)h = 0$, so $jh \in \mathcal{N}$. If $f : Z \to I$, then $f(gj) = (fg)j = 0$, so $gj \in \mathcal{N}$. □

If $\mathcal{C}$ is additive, then $\mathcal{N}(X, Y)$ is a subgroup of $[X, Y]$, and both classes are closed under formation of direct sum.

**Lemma 3.3.** Suppose that $(\mathcal{I}, \mathcal{N})$ is an orthogonal pair in a triangulated category. Assume that

(a) $\mathcal{I}$ is closed under retracts.

(b) $\mathcal{N}$ is a right ideal.

(c) Any object $X$ embeds in a distinguished triangle $X' \xrightarrow{j} X \longrightarrow I$ with $j \in \mathcal{N}$ and $I \in \mathcal{I}$.

Then $(\mathcal{I}, \mathcal{N})$ is a complementary pair.

*Proof.* Let $I$ be an object and assume that $fj = 0$ for any $j : X \to Y$ in $\mathcal{N}$ and any $f : Y \to I$. Take for example $j$ to be the map in the distinguished triangle $X \xrightarrow{j} I \xrightarrow{g} Z$ with $j \in \mathcal{N}$ and $Z \in \mathcal{I}$ guaranteed by (c), and take $f : I \to I$ to be the identity. Then orthogonality implies that $j = 0$, so $g$ is a split monomorphism, and $I \in \mathcal{I}$ by (a).

Let $j : X \to Y$ be a morphism and assume that $fj = 0$ for any $I \in \mathcal{I}$ and any $f : Y \to I$. Let $W \xrightarrow{i} Y \xrightarrow{f} Z$ be the distinguished triangle with $i \in \mathcal{N}$ and $Z \in \mathcal{I}$ guaranteed by (c). Since $fj = 0$, the map $j$ factors as $j = ik$ for some $k : X \to W$. (b) then implies that $j \in \mathcal{N}$. $\square$

**Definition 3.4.** Let $\mathcal{C}$ be a triangulated category. A complementary pair is an *injective class* if every object $X$ embeds in a distinguished triangle $X' \xrightarrow{j} X \longrightarrow I$ in which $j \in \mathcal{N}$ and $I \in \mathcal{I}$. An injective class $(\mathcal{I}, \mathcal{N})$ is *stable* if $\mathcal{I}$ and $\mathcal{N}$ are closed under suspension and desuspension.

We will refer to objects in $\mathcal{I}$ as *injectives*, morphisms in $\mathcal{I}$ as *null maps*, and maps with injective target and null fiber as *injective envelopes*. We will write $\mathcal{N}(X, Y)$ for the abelian group of null maps from $X$ to $Y$.

We will assume stability without further mention.

**Example 3.5.** Assume that $\mathcal{C}$ has a compatible symmetric monoidal structure, and let suppose we have an object $R$ with a two-sided unit $\eta : S \to R$:

$$\begin{tikzcd}
S \wedge R \arrow[r, "\eta\wedge 1"] \arrow[dr, "="'] & R \wedge R \arrow[d, "\mu"] & R \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "="] \\
& R &
\end{tikzcd}$$

The unit map $\eta$ determines, for any $X$, a distinguished triangle,

$$X \wedge \overline{R} \longrightarrow X \xrightarrow{\eta} X \wedge R \longrightarrow \Sigma X \wedge \overline{R} \tag{4}$$

Say that an object $I$ is *$R$-injective* if $I \wedge \overline{R} \to I$ is trivial, or equivalently $X \to X \wedge R$ is a split monomorphism, or $X \wedge R \to \Sigma X \wedge \overline{R}$ is a split epimorphism. This is equivalent to requiring that $I$ is a retract of some spectrum of the form $W \wedge R$, as shown by the commutative diagram

$$\begin{tikzcd}
I \arrow[r, "\eta\wedge 1"] \arrow[d, "\iota"'] & I \wedge R \arrow[r] \arrow[d, "\iota\wedge 1"] & I \\
X \wedge R \arrow[r, "\eta\wedge 1"] \arrow[dr, "1"'] & X \wedge R \wedge R \arrow[d, "1\wedge\mu"] & \\
& X \wedge R \arrow[uur, "\pi"'] &
\end{tikzcd}$$

Say that a map $j : X \to Y$ is *$R$-null* if $1 \wedge j : X \wedge R \to Y \wedge R$ is trivial. This equivalent to requiring that the composist $X \to X \wedge R \to Y \wedge R$

is trivial, as shown by the commutative diagram

$$\begin{tikzcd}
X \wedge R \arrow[r, "1\wedge\eta\wedge 1"] \arrow[dr, "1"'] & X \wedge R \wedge R \arrow[r, "f\wedge 1"] \arrow[d, "1\wedge\mu"] & Y \wedge R \wedge R \arrow[d, "1\wedge\mu"] \\
& X \wedge R \arrow[r, "f\wedge 1"] & Y \wedge R
\end{tikzcd}$$

**Lemma 3.6.** These form an injective class.

*Proof.* These classes are orthogonal: Let $j : X \to Y$ be $R$-null, let $I$ be an $R$-injective, and let $f : Y \to I$ be any map. Let $s : I \wedge R \to I$ split the natural map. Then

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d] & Y \arrow[r, "f"] \arrow[d] & I \arrow[d, "\eta"] \arrow[dr, "1"] & \\
X \wedge R \arrow[r, "0"] & Y \wedge R \arrow[r] & I \wedge R \arrow[r, "s"] & I
\end{tikzcd}$$

shows that $fj = 0$.

The classes clearly satisfy (a) and (b) of Lemma 3.3, and (4) provides the distinguished triangle required by (c). □

This injective class is *multiplicative*, in the sense that for any object $K$ if $j$ is $R$-null then so is $K \wedge j$ and if $I$ is $R$-injective then so is $K \wedge I$.

**Lemma 3.7.** Let $(\mathcal{I}', \mathcal{N}')$ and $(\mathcal{I}'', \mathcal{N}'')$ be two injective classes in a triangulated category $\mathcal{C}$. Define $\mathcal{I}$ to be the class of retracts of objects $I$ for which there is a distinguished triangle $I' \to I \to I''$ with $I' \in \mathcal{I}'$ and $I'' \in \mathcal{I}''$. Define $\mathcal{N}$ to be the class of morphisms $k : X \to Z$ which factor as $k = ij$ with $j : X \to Y$ in $\mathcal{N}'$ and $i : Y \to Z$ in $\mathcal{N}''$. Then $(\mathcal{I}, \mathcal{N})$ is an injective class in $\mathcal{C}$.

*Proof.* These classes are orthogonal: Suppose that $k : X \to Z$ is a map which factors as $X \xrightarrow{i'} Y \xrightarrow{i''} Z$ with $i' \in \mathcal{N}'$ and $i'' \in \mathcal{N}''$, and that $I$ is an object which sits in a distinguished triangle $I' \xrightarrow{g'} I \xrightarrow{g''} I''$ with $I' \in \mathcal{I}'$ and $I'' \in \mathcal{I}''$, and let $f : Z \to I$ be any map. Then $g''(fi'') = (g''f)i'' = 0$ since $i'' \in \mathcal{I}''$, so there exists $f' : Y \to I'$ such that $g'f' = gi''$. Then $f'i' = 0$ since $i' \in \mathcal{N}'$, so $g''(i''i') = g''g'(f'i') = 0$, and $(\mathcal{I}, \mathcal{N})$ is orthogonal.

The class $\mathcal{I}$ is closed under retracts by definition, and $\mathcal{N}$ is a right ideal since $\mathcal{N}'$ is.

To check the third condition of Lemma 3.3, let $Y$ be any object. Let $W \xrightarrow{i} Y \longrightarrow I''$ be a distinguished triangle with $i \in \mathcal{N}''$ and $I'' \in \mathcal{I}''$. Let $X \xrightarrow{j} W \longrightarrow I'$ be a distinguished triangle with $j \in \mathcal{N}'$ and $I' \in \mathcal{I}'$.

Let $k = ij : X \to Y$. The octahedral axiom guarantees that there are distinguished triangles $X \xrightarrow{k} Y \longrightarrow I$ and $I' \to I \to I''$. Thus $k \in \mathcal{N}$ and $I \in \mathcal{I}$. $\Box$

We can apply this in particular to a single injective class $(\mathcal{I}, \mathcal{N})$ to produce a sequence of injective classes $(\mathcal{I}_n, \mathcal{N}^n)$, in which $\mathcal{N}^n$ consists of $n$-fold composites of maps in $\mathcal{N}$. $\mathcal{N}^0$ is the class of identity morphisms in $\mathcal{C}$, and $\mathcal{I}_0$ is the zero object; $\mathcal{N}^1 = \mathcal{N}$ and $\mathcal{I}_1 = \mathcal{I}$. $\mathcal{I}_n$ is defined inductively by declaring it to be the class of retracts of objects $I$ fitting into a distinguished triangle $I' \to I \to I''$ with $I' \in \mathcal{I}_i$ and $I'' \in \mathcal{I}_j$ where $i + j = n$; this condition is independent of the choice of $i, j$ with sum $n$.

Clearly composition lifts to maps

$$\mathcal{N}^u(Y, Z) \otimes \mathcal{N}^s(X, Y) \to \mathcal{N}^{u+s}(X, Z)$$

The injective class determines an enrichment of $\mathcal{C}$ over filtered abelian groups.

## 4. Resolutions and Adams systems

**Definition 4.1.** A *sequence* in a pointed category $\mathcal{C}$ is a diagram $X \xrightarrow{j} Y \xrightarrow{k} Z$ such that $kj = 0$. If $\mathcal{I}$ is a class of objects in $\mathcal{C}$, this sequence is $\mathcal{I}$-*exact* if for every $I \in \mathcal{I}$ and every $f : Y \to I$, $fj = 0$ implies that $f$ factors through $k$.

**Lemma 4.2.** Let $(\mathcal{I}, \mathcal{N})$ be an injective class in a triangulated category $\mathcal{C}$, and $X \longrightarrow Y \longrightarrow Z \xrightarrow{j} \Sigma X$ a distinguished triangle. Then $j \in \mathcal{N}$ if and only if $0 \to X \to Y \to Z \to 0$ is $\mathcal{I}$-exact.

*Proof.* Both conditions are equivalent to exactness of

$$0 \longleftarrow [X, J] \longleftarrow [Y, J] \longleftarrow [Z, J] \longleftarrow 0$$

for all $J \in \mathcal{I}$. $\square$

An $\mathcal{I}$-*injective resolution* of an object $X$ is a diagram

$$0 \longrightarrow X \longrightarrow I^0 \xrightarrow{d} \Sigma I^1 \xrightarrow{d} \Sigma^2 I^2 \xrightarrow{d} \cdots$$

such that each $I^n \in \mathcal{I}$ and each two-term sequence is $\mathcal{I}$-exact. (The suspensions are inserted for convenience in what follows.) That is to say, each composite is trivial, and for any $J \in \mathcal{I}$ the sequence

$$0 \longleftarrow [X, J] \longleftarrow [I^0, J] \longleftarrow [\Sigma I^1, J] \longleftarrow [\Sigma^2 I^2, J] \longleftarrow \cdots$$

is exact. With this definition, the usual fundamental theorem of homological algebra holds: Any object admits an $\mathcal{I}$-injective resolution; if $f : X \to Y$ is any map and $0 \to Y \to J^0 \to \Sigma J^1 \to \cdots$ is any injective resolution, then there is a chain map $f^* : I^* \to J^*$ under $f$; and $f^*$ is unique up to chain homotopy.

A resolution of $X$ may be built by inductively constructing the diagram

$$\begin{tikzcd}
X \arrow[r, equal] & X_0 \arrow[dr, "k_0"'] & & X_1 \arrow[ll, "j_0"'] \arrow[dr, "k_1"] & & X_2 \arrow[ll, "j_1"'] \arrow[dr, "k_2"] & \cdots \\
& & I^0 \arrow[ur, dotted, "i_0"] \arrow[rr, dotted, "d"] & & I^1 \arrow[ur, dotted, "i_1"] \arrow[rr, dotted, "d"] & & I^2
\end{tikzcd}$$

in which the triangles are distinguished, the dotted arrows are of degree 1, the objects $I^s$ are in $\mathcal{I}$, and the maps $j_s$ are in $\mathcal{N}$. If we map this diagram into $J \in \mathcal{I}$, the maps along the top of the diagram induce the zero maps since they are in $\mathcal{N}$, so each pair of maps $X_s \to I^s \to \Sigma X_{s+1}$ induces a short exact sequence. These splice together to give a long exact sequence, showing that $0 \to X \to I^0 \to \Sigma I^1 \to \cdots$ is an injective resolution.

We may thus use an injective class to define derived functors of any additive functor $M : \mathcal{C} \to \mathcal{A}$ to an abelian category:

$$R^s_{\mathcal{I}} M(X) = H^s(M(J^*))$$

The large diagram deserves a name.

**Definition 4.3.** Let $(\mathcal{I}, \mathcal{N})$ be an injective class in a triangulated category. An *$\mathcal{I}$-Adams system* for $X \in \mathcal{C}$ is a diagram

$$\begin{tikzcd}
X \arrow[r, equal] & X_0 \arrow[dr, "k_0"'] & & X_1 \arrow[ll, "j_0"'] \arrow[dr, "k_1"'] & & X_2 \arrow[ll, "j_1"'] & \cdots \\
& & I^0 \arrow[ur, dotted, "i_0"'] & & I^1 \arrow[ur, dotted, "i_1"'] & &
\end{tikzcd}$$

in which the triangles are distinguished, the dotted arrows are of degree 1, the objects $I^s$ are in $\mathcal{I}$, and the maps $j_s$ are in $\mathcal{N}$.

The subdiagram

$$0 \to X \to I^0 \to \Sigma I^1 \to \cdots$$

is an $\mathcal{I}$-injective resolution. Conversely:

**Lemma 4.4.** Any $\mathcal{I}$ injective resolution of $X$ embeds in an Adams system.

*Proof.* We use the fact that for any $J \in \mathcal{I}$ the sequence

$$0 \leftarrow [X, J] \leftarrow [I^0, J] \leftarrow [I^1, J] \leftarrow \cdots$$

is exact. Begin by setting $X_0 = X$ and completing $X \to I^0$ to a distinguished triangle $X_1 \xrightarrow{j} X_0 \xrightarrow{k} I^0 \xrightarrow{i} \Sigma X_1$. Exactness at $[X, J]$ implies that $X_1 \to X_0$ lies in $\mathcal{N}$. The composite $X_0 \to I^0 \to \Sigma I^1$ is trivial, so $I^0 \to I^1$ factors through a map $k_1 : I^0 \to X_1$. Then in

$$\begin{tikzcd}
& [X_1, J] \arrow[d] & \\
{[X_0, J]} & {[I^0, J]} \arrow[l] & {[I_1, J]} \arrow[l] \arrow[ul]
\end{tikzcd}$$

$[X_1, J]$ embeds as the kernel of $[X_0, J] \leftarrow [I_0, J]$. It follows from exactness of the bottom row that the map $[X_1, J] \leftarrow [I^1, J]$ is surjective; so $d : I^0 \to \Sigma I^1$ factors as $d = ki$ for some $k : X_1 \to I^1$. Complete this to a distinguished triangle $X_2 \xrightarrow{j} X_1 \xrightarrow{k} I^1 \xrightarrow{i} \Sigma X_2$ and continue. $\square$

**Lemma 4.5.** Suppose given another object $Y$ and an Adams system

$$\begin{tikzcd}
Y \arrow[r, equal] & Y_0 \arrow[dr, "k_0"'] & & Y_1 \arrow[ll, "j_0"'] \arrow[dr, "k_1"'] & & Y_2 \arrow[ll, "j_1"'] & \cdots \\
& & J^0 \arrow[ur, dotted, "i_0"'] & & J^1 \arrow[ur, dotted, "i_1"'] & &
\end{tikzcd}$$

under it, and let $f : X \to Y$. Let $f^* : I^* \to J^*$ be any chain-map under $f$. Then $f^*$ extends to a map of Adams systems.

*Proof.* To see this, define $f_0 : X_0 \to Y_0$ to be $f$. There is a map $f_1 : X_1 \to Y_1$ completing a map of distinguished triangles. We claim that $k_1 f_1 = f^1 k_1$. To see this note that since $j_0 \in \mathcal{N}$

$$[I^0, J^1] \leftarrow [X^1, J^1] \leftarrow 0$$

is exact, so it suffices to show this equality after composing with $j_1$. But $k_1 f_1 j_1 = k_1 j_1 f^1 = df^0 = f^1 d = f^1 k_1 j_1$. The same argument works at later stages. □

If $f^*$ is an isomorphism, then so is each map $f_s : X_s \to Y_s$, by induction and the 5-lemma. So the Adams system is determined up to isomorphism by the resolution.

There is a dual presentation of an Adams system. Apply the octahdral axiom to the composite $X_{s+1} \to X_s \to X$:

$$\begin{tikzcd}
& & X^s \arrow[dl, dotted] \arrow[ddrr, bend left] & & \\
& X_{s+1} \arrow[rr] \arrow[dr] & & X \arrow[ul] \arrow[dr] & \\
I^s \arrow[ur, dotted] \arrow[uurr, bend left] & & X_s \arrow[ll] \arrow[ur] & & X^{s-1} \arrow[ll, dotted] \arrow[llll, dotted, bend left]
\end{tikzcd}$$

This gives us a "tower" under $X$:

$$
\begin{tikzcd}
 & X \arrow[d] \arrow[drr] \arrow[drrrr] & & & & & & \\
\cdots \arrow[r] & X^2 \arrow[rr] & & X^1 \arrow[rr] \arrow[dl, dotted] & & X^0 \arrow[rr] \arrow[dl, dotted] & & 0 \arrow[dl, dotted] \\
 & & I^2 \arrow[ul] & & I^1 \arrow[ul] & & I^0 \arrow[ul] &
\end{tikzcd}
$$

**Remark 4.6.** This notion of Adams system isn't quite as general as I would like. For example, I would like to say that the diagram

$$
\begin{tikzcd}
S \arrow[r, equal] & S \arrow[dr] & & S \arrow[ll, "2"'] \arrow[dr] & & S \arrow[ll, "2"'] & \cdots \\
 & & S/2 \arrow[ur, dotted] & & S/2 \arrow[ur, dotted] & &
\end{tikzcd}
$$

is a Adams system. But there is no injective class that yields it: $2 : S \to S$ would have to be in $\mathcal{N}$, and $S/2$ would have to be in $\mathcal{I}$; but the homotopy of $S/2$ is not killed by 2, so there are maps $f : S \to S/2$ such that $f \circ 2 \neq 0$.

## 5. The Adams filtration and the Adams spectral sequence

Let $M$ be any homological functor $\mathcal{C} \to \mathcal{A}$. Fix an Adams system for $X$, and remember the corresponding tower under $X$. Define

$$F^s M(X) = \operatorname{im}(M(X_s) \to M(X)) = \ker(M(X) \to M(X^{s-1}))$$

Let $Y$ be another object with a chosen Adams system and define $F^s M(Y)$ similarly. Now let $f : X \to Y$ and pick a lift of $f$ to a map of Adams systems. Then

$$\begin{tikzcd} X & X_s \arrow[l] \arrow[d] \\ Y \arrow[from=u] & Y_s \arrow[l] \end{tikzcd}$$

so $f_* : M(X) \to M(Y)$ sends $F^s M(X)$ into $F^s M(Y)$.

When applied to $f = 1 : X \to X$ this observation shows that $F^s M(X)$ is independent of the chosen Adams system; and then when applied to a general map it shows that $F^s M(X)$ is a natural subset of $M(X)$.

Thus an injective class in $\mathcal{C}$ determines a natural filtration on any functor on $\mathcal{C}$. For example, take the functor $[X, -]$. Since $j_u \in \mathcal{N}$ for all $u$ the image of $[X, Y_s] \to [X, Y]$ is a subgroup of $\mathcal{N}^s(X, Y)$. In fact,

**Lemma 5.1.** $F^s[X, Y] = \operatorname{im}([X, Y_s] \to [X, Y]) = \mathcal{N}^s(X, Y)$.

*Proof.* Let $X \to Z_{s-1} \to \cdots \to Z_1 \to Y$ be a sequence of $s$ null maps. Since $J^u \in \mathcal{I}$ for all $u$ and $\mathcal{N}$ is an ideal, we may inductively construct liftings in the following diagram. □

$$\begin{tikzcd}
 & J^0 & J^1 & J^2 & & J^{s-1} \\
Y_0 \arrow[ur] & Y_1 \arrow[l] \arrow[ur] & Y_2 \arrow[l] \arrow[ur] & \cdots \arrow[l] & Y_{s-1} \arrow[l] \arrow[ur] & Y_s \arrow[l] \\
Z_1 \arrow[u] \arrow[ur, dotted] & & & & & \\
Z_2 \arrow[u] \arrow[uurr, dotted] & & & & & \\
\vdots \arrow[u] & & & & & \\
Z_{s-1} \arrow[u] \arrow[uuuurrrr, dotted] & & & & & \\
X \arrow[u] \arrow[uuuuurrrrr, dotted] & & & & &
\end{tikzcd}$$

**Lemma 5.2.** There are associative pairings

$$F^u[X, Y] \otimes F^s M(X) \to F^{s+u} M(Y)$$

compatible with the composition map $[X, Y] \otimes M(X) \to M(Y)$.

*Proof.* Choose Adams systems $(X_\bullet, I^\bullet)$, $(Y_\bullet, J^\bullet)$. Suppose given a map $f : X \to Y$ of Adams filtration $s$. Pick a lift of $f$ to $\overline{f} : X_0 \to Y_s$. Since

$$\begin{tikzcd}
Y_s \arrow[dr] & & Y_{s+1} \arrow[ll] \arrow[dr] & & Y_{s+2} \arrow[ll] \arrow[dr] & & \cdots \arrow[ll] \\
 & J^s \arrow[ur, dotted] & & J^{s+1} \arrow[ur, dotted] & & J^{s+2} \arrow[ur, dotted] &
\end{tikzcd}$$

is an Adams system for $Y_s$, the map $\overline{f}$ can be lifted to a map $f_*$ of Adams systems.

Now let $x \in F^u M_*(X)$. By definition of the Adams filtration, there is $\overline{x} \in M_*(X_u)$ that projects to $x$. $f_*(x) \in M_*(Y)$ is then represented by $f_*(\overline{x}) \in M_*(Y_{s+u})$, so $f_* x \in F^{s+u} M_*(Y)$. □

This filtration is associated to a spectral sequence. If we apply $M_*$ to an Adams system for $X$. we get an exact couple and hence a spectral sequence. In it,

$$E_1^{s,t} = F_t(\Sigma^s I^s) = M_{t-s}(I^s)$$

and $d_1 : M_t(\Sigma^s I^s) \to M_t(\Sigma^{s+1} I^{s+1})$ is induced by $I^s \xrightarrow{i_s} \Sigma X^{s+1} \xrightarrow{k_{s+1}} \Sigma I^{s+1}$. The homology of this complex is by definition the sequence of $\mathcal{I}$-right derived functors of $M_*$:

$$E_2^{s,t} = R^s_{\mathcal{I}} M_t(X)$$

The observations above imply that this isomorphism is natural in $X$ and that the spectral sequence is functorial in $X$ from $E_2$ on, and that the filtration of $M_*$ associated with it is the $\mathcal{I}$-Adams filtation. Without meaning to indicate convergence, we may say that the spectral sequence abuts to $M_*(X)$:

$$E^2_{s,t} \Longrightarrow M_{t-s}(X)$$

For example we can fix an object $W$ and define $M(X) = [W, X]$, and find

$$R^s_{\mathcal{I}}[W, -]_t(X) \Longrightarrow [W, X]_{t-s}$$

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