Some local computations in homotopy theory
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Abstract. In 1972 Rafe Zahler noted that complex bordism thinks that the Hopf map eta is non-nilpotent, and asked what the η-localization of the Novikov E_2-term was. Some forty years later, Guillou and Isaksen asked the analogous question about motivic homotopy groups over the complex numbers, where also η is non-nilpotent. I will describe joint work with Michael Andrews in which both questions are resolved.

Thanks for this invitation, it's great to be back in Princeton, and great to have the chance to pay respects to the memory of a great homotopy theorist and a great contributor to the spirit and development of homotopy theory, and to help Martin celebrate his birthday.

I want to talk about several localization theorems in homotopy theory. Some are old - dating back to a time even before Martin and I overlapped in Seattle, in the early 1980s - and some are newer, representing work of Michael Andrews and some joint work I've done with him.

I guess the starting point is Serre's observation that in the stable range – that is, for spectra – rational homotopy can be computed, and it is just the rational homology. (Of course he had a lot to say about arithmetic localization, too.)

The Adams spectral sequence lets us make this more quantitative. This talk is going to be all about the Adams spectral sequence, so let me draw it for you. It has the form

$$H^*(A; H_*(X)) = E_2(X) \Longrightarrow \pi_*(X^\wedge)$$

where A is the dual of the Steenrod algebra, and H^* denotes its cohomology with coefficients in the comodule given by the mod p homology of a spectrum X. It's traditionally displayed with the topological dimension along the horizontal axis and the filtration degree along the vertical, so d_r goes one step left and r steps up. Let me write

$$v_0 \in E_2^{1,1}(\mathbb{S})$$

for the element representing p on the sphere. It acts on $E_2(X)$ for any X, and we can localize by inverting it. The claim is that if M is a bounded below A-comodule then

$$v_0^{-1}H^*(A;M) = H(M;\beta) \otimes \mathbb{F}_p[v_0^{\pm 1}]$$

and, moreover, the localization map

$$H^*(A;M) \to v_0^{-1}H^*(A;M)$$

is an isomorphism above a line of slope $1/(2p-2)$. In fact above this line the entire Adams spectral sequence concides with the homology Bockstein spectral sequence.

Now suppose that p is odd. Then $\mathbb{S}/p$ is a ring spectrum, there's a nonzero class $v_1 \in \pi_{2p-2}(\mathbb{S}/p)$, and we can localize by inverting it. While a postdoc at Northwestern, I proved

Theorem. $\pi_*(v_1^{-1}\mathbb{S}/p) = \mathbb{F}_p[v_1^{\pm 1}] \otimes E[\alpha_1]$.

where α_1 is the first element of order p on the bottom cell of $\mathbb{S}/p$. This was supposed to be obvious from the Novikov spectral sequence, which Doug and Steve and I had been studying under the tutelage of Peter Landweber and Jack Morava. Jack had in effect proven that

$$v_1^{-1}E_2(\mathbb{S}/p;BP) = \mathbb{F}_p[v_1^{\pm 1}] \otimes E[\alpha_1]$$

which is the right result but there's a convergence problem (pointed out to me by a student of Arunas Liulevicius named Ron Ming): the operator v_1 acts horizontally in this spectral sequence, so you could infinitely many torsion bits assembling by extensions to an unexpected v_1-torsion free summand. This is a case of the "telescope conjecture."

An idea of Novikov's lets you have your cake and eat it too: There's a square of spectral sequences

$$\begin{tikzcd} H^*(P;Q/v_0) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & E_2(\mathbb{S}/p;BP) \arrow[d, Rightarrow] \\ E_2(\mathbb{S}/p;H) \arrow[r, Rightarrow] & \pi_*(\mathbb{S}/p) \end{tikzcd}$$

Here

$$P = \mathbb{F}_p[\xi_1, \xi_2, \ldots]$$

is the Milnor dual of the algebra of reduced powers, and Q is the associated graded of the p-adic (or Adams) filtration of BP_*,

$$Q = \mathbb{F}_p[v_0, v_1, v_2, \ldots]$$

The top spectral sequence comes from the p-adic filtration of BP_* and is called the "algebraic Adams-Novikov spectral sequence." The left spectral sequence comes from an extension of Hopf algebras. They are both purely algebraic.

Novikov understood that the left spectral sequence collapses, because the Steenrod algebra at an odd prime is actually bigraded by the Novikov degree or "number of Bocksteins." He hoped that this would show that the right hand spectral sequence would collapse. This was too ambitious, but there is a relation. I proved (and Michael and I now have a clearer proof) that d_2 in the bottom spectral sequence is given up to filtration by d_2 in the top (and purely algebraic) spectral sequence. These are the "BP-theoretic" Adams differentials.

I was able to calculate the v_1-localization of the left E_2 terms:

$$v_1^{-1}E_2(\mathbb{S}/p; H) = \mathbb{F}_p[v_1^{\pm 1}] \otimes E[h_{n,0} : n \geq 1] \otimes \mathbb{F}_p[b_{n,0} : n \geq 1]$$

where $h_{n,0}$ is a class related to ξ_n and $b_{n,0}$ is its transpotence.

Invert v_1 and write $K = \mathbb{F}_p[v_1^{\pm 1}]$. Then:

$$\begin{tikzcd}
K \otimes E[h_{n,0}] \otimes \mathbb{F}_p[b_{n,0}] \arrow[r, Rightarrow, "d_2 h_{n+1,0} = v_1 b_{n,0}"] \arrow[d, Rightarrow, "="'] & K \otimes E[\alpha_1] \arrow[d, Rightarrow] \\
v_1^{-1}E_2(\mathbb{S}/p; H) \arrow[r, Rightarrow, "d_2 h_{n+1,0} \equiv v_1 b_{n,0}"] & \pi_*(v_1^{-1}\mathbb{S}/p)
\end{tikzcd}$$

The differential in the algebraic Novikov spectral sequence comes from a fairly elementary fact from the theory of formal groups. Novikov's comparison principle then implies the Adams differential, up to a certain filtration. But that's enough to conclude that

$$E_3(\mathbb{S}/p; H) = E_\infty(\mathbb{S}/p; H) = K \otimes E[h_{1,0}]$$

So there can be no Adams differentials, and the localizaton theorem follows.

The localization map

$$E_2(\mathbb{S}/p; H) \to v_1^{-1}E_2(\mathbb{S}/p; H)$$

is an isomorphism above a line of slope $1/(p^2 - p - 1)$. The E_2 term for the sphere in that range looks like a complete mess. But the cofiber sequence $\mathbb{S} \to \mathbb{S} \to \mathbb{S}/p$ gives us a "Bockstein" spectral sequence starting with $E_2(\mathbb{S}/p)$ and converging to $E_2(\mathbb{S})$. This spectral sequence can be localized, and using it Michael Andrews, in his thesis, was able to compute the odd-prime Adams E_2 page for the sphere above a line of slope given by powers of b_0; that is, a fraction of the E_2 term converging to 1 as $p \to \infty$. I have them written down here; perhaps you will take my word for it when I tell you that they are very simple.

$$p^{[n]} = p^{n-1} + \cdots + 1 \quad , \quad a_n = v_1^{-p^{[n]}} h_{n,0}.$$

Then

$$d_{p^{[n]}} v_1^{p^{n-1}} = v_1^{p^{n-1}} a_n \quad , \quad d_{p^n - 1} a_n = v_1^{-p \cdot p^{[n]}} b_{n,0}$$

He had to understand the structure of Bockstein spectral sequences such as this one in detail. Andrews was then able to go on to analyze the Adams spectral sequence in this range. The first differential, d_2, kills off everything except what's needed to produce the known representation of the Image of J. To produce ImJ requires differentials of arbitrarily high order; as far as I know this is the first verification of the failure of the Adams spectral sequence for the sphere to collapse at any finite stage.

Now, what about $p = 2$? The BP-based Adams spectral sequence seems much dumber than the classical H-based Adams spectral sequence at the prime 2. Right off the bat, it thinks that η is non-nilpotent. In fact, Novikov knew that the 1-line was cyclic in each odd topological degree (and used that to prove the Hopf invariant one theorem, so BP wasn't so dumb). Write $\overline{\alpha}_n$ for those generators, so $\overline{\alpha}_1$ represents η. Doug, Steve, and I showed that $\overline{\alpha}_3, \overline{\alpha}_4$, etc, all support η-towers. Rafe Zahler asked what $\eta^{-1} E_2(\mathbb{S}; BP)$ was. But the question was left open, because we knew that $\eta^4 = 0$ in homotopy. (This is accomplished by the differential $d_3 \overline{\alpha}_2 = \eta^4$.)

Andrews and I can now answer this question: there are no more towers. The proof is pretty simple: Localize the algebraic Novikov spectral sequence by inverting η:

$$\eta^{-1} H^*(P; Q) \Longrightarrow \eta^{-1} E_2(\mathbb{S}; BP).$$

But at $p = 2$, P is just A with degrees doubled. Under this doubling operation, η corresponds to v_0, and I explained at the outset how to compute the

v_0-localization of $H^*(A; M)$. Then it's a simple matter to compute

$$\eta^{-1}H^*(P;Q) = L[v_1^2, v_2, v_3, \ldots] \quad , \quad L = \mathbb{F}_2[\eta^{\pm 1}].$$

Then some elementary facts about formal groups (which actually go right back to my very first paper, with Steve Wilson!) give

$$d_2 v_{n+1} = \eta v_n^2 \quad , n \geq 2$$

and so

$$\eta^{-1}E_2(\mathbb{S}; BP) = L[v_1^2, v_2]/v_2^2$$

You can see the monomials in v_1 and v_2 along the zero line, generating $\overline{\alpha}_n$ by multpling by η. Still not so interesting; the Adams differential

$$d_3 v_2 = \eta^3$$

kills a unit, so $E_4 = 0$.

But now we know that there are other universes, parallel to the standard homotopy theory and sometimes quite close to it, containing the "motivic homotopy theories" over fields. In them, it turns out that η is not nilpotent! So it turns out that far from being blind, BP somehow knows about the motivic world. The element η enters explicitly into Fabien Morel's definition of Milnor-Witt K-theory, which he showed to be isomorphic to the "co-weight zero" part of the motivic stable homotopy ring over any field of characteristic not 2. He observed that

$$\eta^{-1}K_*^{MW}(F) = W(F)[\eta^{\pm 1}].$$

Several years ago Dan Isaksen asked what $\eta^{-1}\pi_*(\mathbb{S}_{\mathrm{Mot}})$ was, at least over $\mathbb{C}$, and he and Bert Guillou made a conjecture and did a lot of computation in that direction. We'll restrict to $F = \mathbb{C}$.

Po Hu, Igor Kriz, and Kyle Ormsby explained that there are analogues of the spectral sequences in play above, and that the motivic BP-based Adams E_2 term is simply a polynomial extension of the classical one. This polynomial generator corresponds to the element τ in the coefficient ring of motivic cohomology. So we have

$$\begin{array}{ccc} H^*(P;Q)[\tau] & \Longrightarrow & E_2(\mathbb{S}_{\mathrm{Mot}}; BP) \\ & & \Downarrow \\ & & \pi_*(\mathbb{S}_{\mathrm{Mot}}) \end{array}$$

The computation Michael and I did gives us the top line in

$$\begin{tikzcd} L[\tau, v_1^2, v_2, \ldots] \arrow[r, Rightarrow, "d_2 v_{n+1} = \eta v_n^2\,,\, n\geq 2"] & L[\tau, v_1^2, v_2]/v_2^2 \arrow[d, Rightarrow, "d_2 v_1^2 = \tau\eta^3"] \\ & L[v_1^4, v_2]/v_2^2 \end{tikzcd}$$

and again it's easy to determine the d_2 in the localized Novikov spectral sequence. This gives us the result predicted by Guillou and Isaksen:

Theorem. (Andrews, Miller) Over $\mathbb{C}$, $\pi_*(\eta^{-1}\mathbb{S}_{\mathrm{Mot}}) = \mathbb{F}_2[\eta^{\pm 1}, v_1^4, v_2]/v_2^2$.

G&I's approach was via the motivic Adams spectral sequence. We can fill that in: Localize

$$\begin{tikzcd} H^*(P;Q)[\tau] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & E_2(\mathbb{S}_{\mathrm{Mot}}; BP) \arrow[d, Rightarrow] \\ E_2(\mathbb{S}_{\mathrm{Mot}}; H) \arrow[r, Rightarrow] & \pi_*(\mathbb{S}_{\mathrm{Mot}}) \end{tikzcd}$$

Again, it's not hard to compute the left hand d_2, and we recover the computation of G&I of $\eta^{-1}E_2(\mathbb{S}_{\mathrm{Mot}}; H)$. Then the transfer of differentials principal lets us verify the conjecture they made about the Adams d_2:

$$\begin{tikzcd} L[\tau, v_1^2, v_2, \ldots] \arrow[r, Rightarrow, "d_2 v_{n+1} = \eta v_n^2\,,\, n\geq 2"] \arrow[d, Rightarrow, "d_2 v_1^2 = \tau\eta^2"'] & L[\tau, v_1^2, v_2]/v_2^2 \arrow[d, Rightarrow, "d_2 v_1^2 = \tau\eta^3"] \\ L[v_1^4, v_2, \ldots] \arrow[r, Rightarrow, "d_2 v_{n+1} = \eta v_n^2\,,\, n\geq 2"'] & L[v_1^4, v_2]/v_2^2 \end{tikzcd}$$

So there are non-nilpotent operators in motivic stable homotopy theory which are not "chromatic": η does not correspond to a generator of BP_*. It corresponds to ξ_1^2 in the dual Steenrod algebra, rather than to one of the generators. It's natural to conjecture that the is a whole series of periodic operators, corresponding to the squares of the ξ_n's. The first one is η. Next one should consider the mapping cone of η and ask for an operator there of topological dimension 5 times a power of 2. This operator occurs in the context of the Steenrod algebra in Margolis's work and was well known to Mahowald. Michael Andrews has in fact constructed such an operator. Motivic homotopy has a second "weight" gradation. η has weight 1. Michael's operator has dimension 20 and weight 12. It's the second "technicolor" operator, and will no doubt be followed by others. But perhaps there are just two rows of operators, rather than the one present in classical homotopy theory. Are there analogues of Morava K-theory whose role in life is to detect them? And there are the other fields to think about.