Some ideas about "TQFT"

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A "field," in physics, is a quantity that varies from one point to another across space. Temperature is a field, a "scalar field" because it takes values in "scalars," that is, real numbers. So a scalar field is a function from space to the real numbers (or, sometimes, to the complex numbers). You've also seen "vector fields," that assign a vector to each point in space: like the velocity of a particle of fluid.

"Field theory" is a perspective on physics that regards physical objects as fields. This is inspired by classical quantum mechanics, in which an electron is not thought of as having a definite position, but rather only a probability of being in one place or another. The relevant "field" here is actually a complex-valued function, called the "wave function," and the probability distribution is given by taking the square of the absolute value at each point.

So a field-theoretic description of the electron fills the entire universe. This is not as crazy as it sounds, or as "quantum." After all, an electron exerts an influence on objects at a distance, through its electric charge and through its mass.

A huge conceptual step at this point is to allow ourselves to vary the shape of "space." All that really matters, from this perspective, is that space has some geometric structure. It's almost always modelled as a "manifold" – it looks locally like Euclidean space. So the surface of a sphere, or even a circle, makes a perfectly good thing to look at if we want to develop intuition with a "toy model."

So a field theory model of the electron would begin with a manifold Σ ("space") together with a set whose elements we are to think of as "fields" on the manifold. A particular "state" of the electron will be a particular field. This specifies what can be known about where it is, what its charge and spin are, and whatever other properties it might have. The collection of all fields on Σ (of this type) is (part of) a model for the electron, accommodating all of its possible states. We'll write $A(\Sigma)$ for the collection of all fields on Σ in this model, and φ for a typical field, $\varphi \in A(\Sigma)$. Actually there could be many electrons, or other particles, occupying the same manifold Σ. You'd just have a bigger set of fields $A(\Sigma)$.

The set $A(\Sigma)$ is supposed to have some structure. We're supposed to be able to multiply a field by a scalar, and add two fields. The word for this structure is "vector space." So there are two different kinds of "space" in play. I'll try to say "manifold" for the space Σ.

The notation $A(\Sigma)$ is meant to make you think of the collection of fields on Σ as a "function of" Σ. This is somewhat confusing, because you are also supposed to think of the *elements* of $A(\Sigma)$ as functions *on* Σ.

Naming the manifold you are living in – Σ – allows you to imagine *changing it.* So now you haven't completely specified a theory (of the electron, say) until you have described the relevant space of fields $A(\Sigma)$ on *all possible manifolds* Σ of some type. In our toy case, Σ will be one-dimensional (and "closed"): a disjoint union of circles. More generally Σ might range over closed manifolds of some fixed dimension, say $n-1$.

Note the two different uses of the word "theory." In one use, it means a general area (I almost said "field") of study: "quantum theory." In another use, it means a particular model of how to describe some type of physical object (such as an electron): one speaks of "a topological quantum field theory," or "TQFT." So one TQFT might describe the electron; another might describe the proton. You want to keep your class of manifolds Σ the same, though.

Quantum mechanics talks a lot about measurement. For example the Heisenberg uncertainty principle says that you are not allowed to measure both position and momentum to arbitrarily high accuracy. We won't try to incorporate that, but we will learn from it that we should focus on the concept of measurement. We might measure the temperature at a point, or the value of the wave function at a point, or more complicated things. A measurement is thus a *function* on $A(\Sigma)$.

What sort of function? Well, the simplest ones, building blocks for others, use the vector space structure of $A(\Sigma)$: *linear* functions, that is, functions $f : A(\Sigma) \to \mathbb{R}$ (say) such that

$$f(\varphi + \varphi') = f(\varphi) + f(\varphi')\,, \quad f(c\varphi) = cf(\varphi)\,.$$

These are called *observables.*

Now we have to think about the dynamics of this model. How does it change through time? It turns out to be a good idea to draw separate pictures of the underlying manifold Σ, one for each point in time. So if Σ is a circle, space-time is represented by a cylinder. One end of the cylinder is "incoming," Σ_{in}, the starting position; the other end is "outgoing," Σ_{out}, the ending position. There's a map (or "projection") from it to an interval, representing what time it is. The dynamics represented by this cylinder is supposed to convert a field on Σ_{in} into a field on Σ_{out}: it produces a map

$$A(\Sigma_{in}) \to A(\Sigma_{out}) .$$

In some QFT's this map will depend on the length of the interval. In *topological* QFT's, it does not. This shows how crude a model TQFT's offer. But you have to start somewhere.

Once one has this geometric image for space-time, another radical idea occurs: why keep the manifold Σ constant over time? Why not allow it to split in two half way along? Then the space-time model would look like a "pair of pants."

Wow. This is the universe splitting in two bits. Or two bits merging into one! Once you have this perspective, the notion of time itself fades into the background; we've lost the projection to an interval. We still do have the "arrow of time" – time goes from "in" to "out."

That pair of pants represents a transformation, from the universe as a single circle to the universe as two circles. To make it "dynamical," we should also be told how a field on the incoming space gets transformed into a field on the outgoing manifold.

The pair of pants M is a "cobordism" between the incoming manifold Σ_{in} and the outgoing manifold Σ_{out}: Together these two manifolds make up the "boundary" of M. In this model, dynamics are captured by assigning to any cobordism M a function

$$A_M : A(\Sigma_{in}) \to A(\Sigma_{out}) .$$

Now is no time to abandon linearity: we should require that A_M be *linear*,

$$A_M(\varphi + \varphi') = A_M(\varphi) + A_M(\varphi') , \quad A_M(c\varphi) = cA_M(\varphi) .$$

We've really now described the structure of a "topological quantum field theory." An n-dimensional TQFT should assign a vector space $A(\Sigma)$ to every $(n-1)$-manifold Σ, and a linear transformation $A_M : A(\Sigma_{in}) \to A(\Sigma_{out})$ to every cobordism with incoming manifold Σ_{in} and outgoing manifold Σ_{out}.

These should satisfy some properties. The first and most obvious one is "transitivity," which I will let you formulate. The trickiest one relates $A(\Sigma)$ and $A(\Sigma')$ to $A(\Sigma \sqcup \Sigma')$. Here the funny $\sqcup$ symbol (also written $\coprod$) means "disjoint union." So the set of fields on a circle should determine the set of fields on two circles (at least once a specific invertible map from the first circle to each of the other two has been given).

To figure out how that should work, we should go back to thinking about observables.

A scalar observable on $\Sigma \sqcup \Sigma'$ should assign a number to every pair of field $\varphi \in A(\Sigma)$ and $\varphi' \in A(\Sigma')$. The appropriate use of our linear structure in this context is this: If the field stength on Σ doubles, the value of the observable should double; and so on. So:

$$f(\varphi_1+\varphi_2,\varphi') = f(\varphi_1,\varphi')+f(\varphi_2,\varphi')\,,\quad f(\varphi,\varphi'_1+\varphi'_2) = f(\varphi,\varphi'_1)+f(\varphi,\varphi'_2)\,,$$

$$f(c\varphi,\varphi') = cf(\varphi,\varphi')\,,\quad f(\varphi,c\varphi') = cf(\varphi,\varphi')\,.$$

The word for this is: f is *bilinear*. Multiplication is bilinear; we are saying that observables behave multiplicatively for disjoint union.

But this is supposed to be a *linear* function on $A(\Sigma \sqcup \Sigma')$. This pretty much nails down what $A(\Sigma \sqcup \Sigma')$ has to be; the result is called the *tensor product*, written with the funny circle-times symbol:

$$A(\Sigma) \otimes A(\Sigma')\,.$$

So

$$A(\Sigma \sqcup \Sigma') = A(\Sigma) \otimes A(\Sigma')\,.$$

Just a further word about the tensor product of two vector spaces, $V \otimes W$: There is a function $i : V \times W \to V \otimes W$. If $v_1, \ldots, v_m$ is a basis for V and $w_1, \ldots, w_n$ is a basis for W, then $i(v_1, w_1), \ldots, i(v_m, w_n)$ is a basis for $V \otimes W$. The dimension of $V \otimes W$ is the product of the dimensions of V and W.

Realistic field theories are complicated. The spaces of fields are infinite-dimensional (they are "Hilbert spaces"), the manifolds are 3 (or 10) dimensional. But toy models with $n = 2$ and finite dimensional spaces of fields are already nontrivial and mathematically interesting.

Afterword: Physics is always "underdetermined." There are lots of ways things *could* be. Physicists are always trying to add constraints to restrict the possibilities. Allowing the universe to split and merge imposes useful additional constraints on their models, crazy though it may seem.