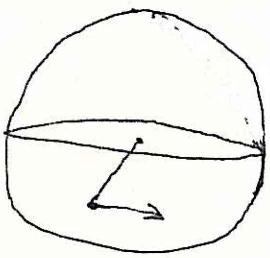
We shall discuss some classical topics in homotopy theory. A great deal of what we will cover owes much to the work of Frank Adams... Today we look at a few of the problems (which turn out to have deep general significance), and begin with

Vector Fields on Spheres,

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$ Here it is:



The idea is to assign to each point $x \in S^{n-1}$ a vector $v(x)$ tangent to x, in a smooth way, i.e. to find a map $v : S^{n-1} \to \mathbb{R}^n$ such that $v(x) \perp x \quad \forall x \in S^{n-1}$!

Now of course there's always the zero vector, which isn't very interesting, so in particular we'll look for nowhere-zero vector fields, which means we can normalize $\|v(x)\|$ to be 1 for all x, then

$$v : S^{n-1} \to S^{n-1} \quad \text{s.t. } v(x) \perp x \ \forall x.$$

n even -- $n = 2k$.

Then S^{n-1} can be thought of as $\{x \in \mathbb{C}^k \mid \|x\| = 1\}$, and $v(x) = ix$ works. So we have a nonvanishing vector field on an odd sphere.

n odd -- then there are none.

Such a $v(x)$ would give a homotopy between the identity and the antipodal map $\alpha(x) = -x$:

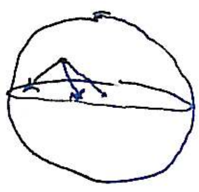
$$h_t(x) = \cos(\pi t)x + \sin(\pi t)\, v(x). \quad \text{So}$$

$H_*(\alpha) = 1$. But α is composite of n reflections in $\mathbb{R}^n$, so

$H_*(\alpha) = (-1)^n = 1 \Rightarrow n$ is even, a contradiction.

Now the next question is how many linearly independent vector fields there are on S^{n-1}.

$v_1, \dots, v_{k-1}$; apply Gramm-Schmidt (which is a continuous process) so we can assume $\underbrace{x}_{\text{call this } v_0}, v_1(x), \dots, v_{k-1}(x)$ are an orthonormal set for all $x \in S^{n-1}$.



This is interesting in its own right; what we shall see in the long run is that it is equivalent to some important problems in homotopy theory.

Anyhow, the way we will will think of this is in terms of

$$V_{n,k} = \{\text{orthonormal } k\text{-frames in } \mathbb{R}^n\} \subset \underbrace{(S^{n-1})^k}_{\text{topologize as subset of this}}$$

$$\begin{tikzcd} V_{n,k} \arrow[d, "\text{proj. onto first factor}"] \\ S^{n-1} \arrow[u, bend left, "v"] \end{tikzcd}$$

You can check that this is locally trivial, and gives a fiber bundle, and our map v is in these terms a section of this bundle.

So now the question has become: what is the largest k for which this bundle has a section?

Thm. (Hurwitz, Radon, Eckmann; Adams)

Write n as
$$n = (\text{odd})\,2^{\nu}$$
Write ν as
$$\nu = 4b + c, \quad 0 \le c \le 3$$
set $$\rho(n) = 8b + 2^c.$$

Then there exist $\rho(n) - 1$ independent vector fields on S^{n-1} (Hurwitz Radon Eckmann)

and no more (Adams).

The first curious fact here is that ρ depends only on the even part of n.

ν	0	1	2	3	4	5	6	7
2^{ν}	1	2	4	8	16	32	64	128
$\rho(n)$	1	2	4	8	9	10	12	16

Two steps to proving this...

1. construct them, which is fairly straightforward; we'll see that next time using Clifford Algebras.

2. show there are no more, which is much harder, and was the first major victory for K-theory.

Before going on, note a corollary which in fact was known before this theorem was proved

Corollary (Kervaire c.1956) Which spheres are parallelizable, i.e. there is a basis for the tangent space, i.e. $\rho(n) = n$? Answer: $n = 1, 2, 4,$ or 8. So the only spheres which have trivial tangent bundles are

S^0, S^1, S^3, and S^7.

A closely related problem is the so-called "Hopf invariant 1" problem. Suppose S^{n-1} is parallelizable, or what is the same, that there is a section

$$\begin{tikzcd} V_{n,n} = O(n) \arrow[d] \\ S^{n-1} \arrow[u, bend left, "v"] \end{tikzcd}$$

Now a point in the "Stiefel manifold" $V_{n,k}$ can be written as an $n \times k$ matrix whose columns are orthogonal and have norm 1:

$$\underbrace{\begin{bmatrix} | & | & | & | \end{bmatrix}}_{k} n$$

, so in particular $V_{n,n} = O(n)$, and then the projection $O(n) \to S^{n-1}$ is given by projection onto the first column. Now $O(n)$ acts on on $\mathbb{R}^n$ and induces an action on S^{n-1}. Combining this with the section v gives

$$\begin{tikzcd} O(n) \times S^{n-1} \arrow[r] & S^{n-1} \\ S^{n-1} \times S^{n-1} \arrow[u] \arrow[ur] & \end{tikzcd}$$

e_1 picks out 1^{st} column of matrix $v(x)$; this is x itself.

$$\begin{tikzcd} (v(x), e_1) \arrow[r, mapsto] & x \\ (x, e_1) \arrow[u, mapsto] \arrow[ur, mapsto] & \end{tikzcd}$$

So e_1 is a right unit of this multiplication on S^{n-1}. e_1 isn't necessarily a left unit; but $v(e_1)$ is of the form $\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$, $\in O(n-1)$

So we can correct the situation by replacing v with $v(e_1)^{-1} v$, which will still be a section.

Now $v(e_1) = I$, so

$$\begin{tikzcd} (I, x) \arrow[r] & x \\ (e_1, x) \arrow[u] \arrow[ur] & \end{tikzcd}$$

In short, a parallelizable $n-1$ sphere has a multiplication with a 2-sided unit. Such a space is called an H-space; to be precise, an <u>H-space</u> is a pointed space $(X, *)$ with a product $X \times X \xrightarrow{\mu} X$ such that $\mu(*, x) = \mu(x, *) = x$.

In our cases,

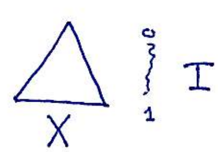
S^0	S^1	S^3	S^7
$\mathbb{Z}/2$	$\mathbb{C}$	quaternions	"Cayley numbers."

In view of the results above, a natural question is when S^{n-1} can be given the structure of an H-space.

To attack this problem it is helpful first to think about the map μ in ridiculous generality -- sort of like thinking about a bilinear form as a tensor product... Namely, consider a map

$$X \times Y \xrightarrow{\mu} Z$$

Now consider the cone over X which comes from $X \times I$ as the quotient $X \times I / \{(x,0) \sim (x';0)\}$

$CX =$ 

μ induces a map

$$CX \times Y \longrightarrow CZ$$
$$([x,t], y) \longmapsto (\mu(x,y), t)$$

and similarly a map $X \times CY \longrightarrow CZ$.

$$(x,(y,t)) \longrightarrow (\mu(x,y),t)$$

$X\times Y$ includes into $CX\times Y$ and $X\times CY$ as $((x,1),y)$ and $(x,(y,1))$ respectively, and Z includes into CZ as $(z,1)$; putting these together we get a diagram:

$$\begin{tikzcd}
& CX\times Y \arrow[rr] & & CZ \\
X\times Y \arrow[ur] \arrow[dr] \arrow[rr, "\mu"] & & Z \arrow[ur] \arrow[dr] & \\
& X\times CY \arrow[rr] & & CZ
\end{tikzcd}$$

The commutativity means that we can take the disjoint union of $CX\times Y$ and $X\times CY$ and identify along the copy of $X\times Y$ in each, and take two copies of CZ and identify along their bases, and get a map

$$\underbrace{X * Y}_{\text{"join" of } X \text{ and } Y} \xrightarrow{H(\mu)} \Sigma Z$$

"join" of X and Y,
meaning $CX\times Y \amalg X\times CY \big/ ((x,1),y) \sim (x,(y,1))$.

This process is called the "Hopf construction" for μ, and so we'll call the resulting map $H(\mu)$.

Facts about the join:

$S^{p-1} * S^{q-1} \cong S^{p+q-1}$ (note first that it's clear this has the right dimension).

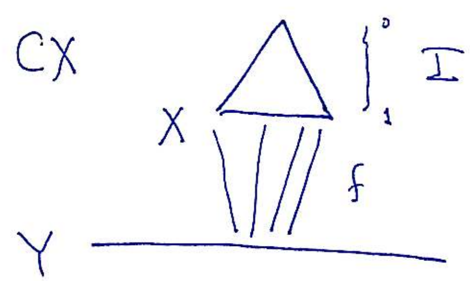
$X * Y \underset{t}{\simeq} \Sigma(X \wedge Y)$ for reasonable X and Y, e.g. CW complexes

i.e. these two have the same homotopy type.

In the case of an H-space, μ induces a map $X * X \to \Sigma X$, and if $X = S^{n-1}$, this is a map $S^{2n-1} \xrightarrow{H(\mu)} S^n$, i.e. a homotopy class of the n-sphere, something to be prized and studied.

There's a duality in homotopy between maps and objects; i.e to every map there is an associated space which contains all the information about the map .. the mapping cone:

$X \xrightarrow{f} Y$



$C(f) = Y \amalg CX \Big/ {(x,1) \sim f(x)}$

Y includes into $C(f)$ as its base, so one has $X \xrightarrow{f} Y \to C(f)$

Applying this to the map $H(\mu): S^{2n-1} \to S^n$, the mapping cone really means just attaching a $2n$-cell via the map $H(\mu)$, so we get

$$S^{2n-1} \xrightarrow{H(\mu)} S^n \longrightarrow S^n \cup_{H(\mu)} e^{2n}.$$

Now, the cohomology is given by (assuming $n>1$ for now)

$$H^*(S^n \cup_f e^{2n}) = \begin{cases} \mathbb{Z} \text{ generated by } y & \dim 2n \\ \mathbb{Z} \text{ generated by } x & \dim n \\ \mathbb{Z} \text{ generated by } 1 & \dim 0 \end{cases}$$

Using the cup product, $x^2 = \alpha y$, $\alpha \in \mathbb{Z}$, where for now α is well-defined only up to sign. α is called the Hopf invariant of f.

<u>Claim</u> $\alpha = \pm 1$ when f the attaching map is $H(\mu)$, i.e. comes from multiplication on the $n-1$-sphere.

<u>Examples</u> $\mu: S^1 \times S^1 \to S^1$

$\Downarrow$ Hopf

$$S^3 \to S^2 \to S^2 \cup e^4 = \mathbb{C}P^2$$

and $H^*(\mathbb{C}P^2) = \dfrac{\mathbb{Z}[x]}{(x^3)}$, $\dim x = 2$.

$$S^7 \to S^4 \to S^4 \cup e^8 = \mathbb{H}P^2$$

$$S^{15} \to S^8 \to \mathbb{O}P^2 \quad \text{"Cayley Projective plane"}$$

non-associativity of Cayley multiplication
$\Rightarrow$ there are no other Cayley projective spaces...

So now the question is: for what spheres is there an element of $\pi_{2n-1}(S^n)$ of Hopf invariant 1?

<u>Theorem</u> (Adams) If $\pi_{2n-1}(S^n)$ contains element of Hopf invariant 1, then $n = 1, 2, 4,$ or 8.

For the time being, take a step back and recall the action $O(n) \times S^{n-1} \to S^{n-1}$.

Now take any $\alpha \in \pi_k(O(n))$ [not necessarily from a section]; this induces a map

$$\begin{array}{ccc} O(n) \times S^{n-1} & \longrightarrow & S^{n-1} \\ \uparrow & & \\ S^k \times S^{n-1} & & \end{array}$$

Doing the Hopf construction here induces a map $S^{k+n} \to S^n$; i.e. we get a homomorphism $\boxed{\pi_k(O(n)) \xrightarrow{J} \pi_{n+k}(S^n)}$

called the "J-homomorphism." For $k < n$, $\pi_k O(n)$ are known by Bott Periodicity; hence there is much interest in the image of J. Note that for $k < n$, $n + k < 2n$, so this is the so-called "Stable Range" where things often seem to work better; one then thinks of J as a map $\pi_* O \to \pi_*^S$.

The Adams conjectures are concerned with the image of this J.

Constructing Vector Fields on Spheres.

In this lecture we will see how many vector fields on spheres we can construct using linear algebra. In particular we will use Clifford Algebras. For more information, see for example the book by Strang.

For $k \geq 0$, C_k is a free associative algebra over $\mathbb{R}$ with generators $e_1, \dots, e_k$, subject to relations

$$e_i e_j + e_j e_i = 0 \quad i \neq j$$
$$e_i^2 = -1$$

For example,

$$C_0 = \mathbb{R}$$

$$C_1 \cong \mathbb{C}$$ { note: you can identify e_1 either with $+i$ or $-i$. The isomorphism is <u>not</u> canonical. This will be true for many of the isomorphisms we construct ~~to~~ today.

$$C_2 \cong \mathbb{H}, \text{ with, say } \begin{array}{l} e_1 \mapsto i \\ e_2 \mapsto j \end{array} \qquad e_1 e_2 \mapsto k.$$

Note that the relations specify that we can give a basis for C_k as the set of words $\{e_{i_1} e_{i_2} \cdots e_{i_m}, m \geq 0\}$, $i_1 < \cdots < i_m\}$ made up of ordered non-repeating sequences of e_i's. So $\dim C_k = 2^k$.

Note also that the collection $\{\pm e_{i_1} \cdots e_{i_m}, i_1 < \cdots < i_m\} = G_m$ is a multiplicative subgroup of C_k.

Also, C_k comes equipped with an antiautomorphism

$C_k \xrightarrow{\ \bar{}\ } C_k$, defined on generators by $\bar{e}_i = -e_i$, and extended to C_k as an antiautomorphism. It is an anti involution in the sense that $\overline{xy} = \bar{y}\,\bar{x}$.

Before describing the C_k further, let's look at how they can be used to construct vector fields on spheres. One gets these by finding representations of C_k: Suppose V is an n-dim vector space with C_k-module structure, i.e. a map $C_k \otimes_{\mathbb{R}} V \longrightarrow V$. V is then a representation for G_k.

Next choose an inner product on V; by averaging over the inner product we can construct a G_k-invariant inner product $(,)$.

Now consider $S(V) =$ unit sphere on V.

$$\begin{array}{c} \cup \\ x \\ \downarrow \\ e_i \cdot x \in S(V) \text{ since inner product is invariant.} \end{array}$$

Claim $\{e_1 x, \dots, e_k x\}$ defines an orthonormal k-frame on $S(V) \cong S^{n-1}$.

Proof i) $(x, e_i x) = (e_i x, e_i e_i x) = -(e_i x, x) = -(x, e_i x)$ so $(x, e_i x) = 0$ and $e_i x$ is tangent to $S(V)$ at x.

ii) $(e_i x, e_j x) = (e_i e_i x, e_i e_j e_j x) = -(e_i^2 e_j x, (-e_i) x)$

$= (e_j x, (-e_i x)) = -(e_i x, e_j x)$. So $(e_i x, e_j x) = 0$, and $e_i x$ and $e_j x$ are orthogonal for $i \neq j$.

Now, we try to find representations of C_k. So what are they? To start with, here is a table.

		dim. of minimal representation $= a_k = 2^{\varphi_k}$	φ_k		
C_0^+	$\mathbb{R}$	1	0	C_0^-	$\mathbb{R}$
C_1^+	$\mathbb{C}$	2	1	C_1^-	$\mathbb{R}\times\mathbb{R}$
C_2^+	$\mathbb{H}$	4	2	C_2^-	$\cancel{\mathbb{C}(2)}$ $\mathbb{R}(2)$
C_3^+	$\mathbb{H}\times\mathbb{H}$	4	2	C_3^-	$\mathbb{C}(2)$
C_4^+	$\mathbb{H}(2)$	8	3	C_4^-	$\mathbb{H}(2)$
C_5^+	$\mathbb{C}(4)$	8	3	C_5^-	$\mathbb{H}(2)\times\mathbb{H}(2)$
C_6^+	$\mathbb{R}(8)$	8	3	C_6^-	$\mathbb{H}(4)$
C_7^+	$\mathbb{R}(8)\times\mathbb{R}(8)$	8	3	C_7^-	$\mathbb{C}(8)$
C_8^+	$\mathbb{R}(16)$	16	4	C_8^-	$\mathbb{R}(16)$

How to get the table.

To begin with, normally a Clifford algebra is associated with a bilinear form, and we'll start by changing the form. So let C_k^- be the free associative algebra over $\mathbb{R}$ on k generators, with relations

$$e_ie_j + e_je_i = 0$$

$$e_i^2 = +1$$

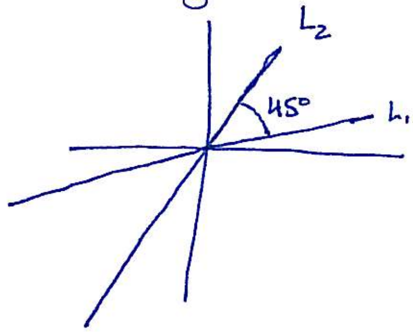
For the same reasons as C_k (which from now on we call C_k^+), ~~this is~~ C_k^- is 2^k dimensional. So what are the C_k^-?

C_1^- has 1 generator whose square is 1.

So $C_1^- \cong \mathbb{R}\times\mathbb{R}$

$e_1 \mapsto (1,-1)$ (or $(-1,1)$).

$C_2^- \cong \mathbb{R}(2)$ algebra of 2×2 matrices over $\mathbb{R}$. This is very noncanonical:



$e_1 \longmapsto r_{L_1}$ reflection through line L_1

$e_2 \longmapsto r_{L_2}$ " L_2,

where the angle between them is $45°$.
Thus $r_{L_1} r_{L_2} = 90°$ rotation (as drawn here, clockwise); $r_{L_2} r_{L_1} = 90°$ rotation (counterclockwise).

This could get very tiring very quickly; fortunately that's as far as we need to go. From now on we use

<u>Lemma</u> $C_k^{\pm} \cong C_2^{\pm} \otimes C_{k-2}^{\mp}$

<u>Proof</u> Again, not very canonical. For example,

$$e_i \longmapsto \left\{ \begin{array}{ll} e_1 \otimes 1 & i=1 \\ e_2 \otimes 1 & i=2 \\ e_1 e_2 \otimes e_{i-2} & i>2 \end{array} \right\} \text{ works.}$$

(Exercise).

The ~~ra~~ remaining $C_k^{\pm}$, $3 \le k \le 8$, in the table can be computed in succession using this Lemma, sort of like "tying up the laces on a skate"; to do this one uses the following isomorphisms:

$(\mathbb{R}\times\mathbb{R}) \otimes A \cong A \times A$

$\mathbb{R}(n) \otimes F \cong F(n)$

$\mathbb{H} \otimes \mathbb{C} \cong \mathbb{C}(2)$

{proof: $\mathbb{H}\otimes\mathbb{C}$ acts on $\mathbb{C}^2 = \mathbb{H}$ {take $1, j$ as $\mathbb{C}$-basis for $\mathbb{H}$} via $(w\otimes z)v = wv\bar{z}$.
Note that in going from left to right action one changes z to $\bar{z}$. Not necessary because $\mathbb{C}$ is abelian, but keep in mind}

$\mathbb{R}(m) \otimes \mathbb{R}(n) \cong \mathbb{R}(mn)$

$\mathbb{H} \otimes \mathbb{H} \cong \mathbb{R}(4)$

{Proof: $\mathbb{H}\otimes\mathbb{H}$ acts on $\mathbb{H} \cong \mathbb{R}^4$ by $(w\otimes z)v = wv\bar{z}$. Now conjugation is important}

From the table we can quickly compute C_k for $k > 8$ if we note the isomorphisms (repeatedly applying the lemma)

$$
\begin{aligned}
C_k^+ \cong C_2^+ \otimes C_{k-2}^- &\cong C_2^+ \otimes C_2^- \otimes C_{k-4}^+ \\
&\cong C_4^+ \otimes C_{k-4}^+ \\
&\cong C_4^+ \otimes (C_4^+ \otimes C_{k-8}^+) \\
&\cong C_8^+ \otimes C_{k-8}^+ \\
&\cong C_{k-8}^+(16)
\end{aligned}
$$

[Similarly $C_k^- \cong C_{k-8}^-(16)$].

Next, we can quickly determine the dimensions of the minimal representations; this is the 3rd column of the table. The fourth column is

$\ell_k = \log_2 a_k$, where a_k is the dim. of the minimal representation.

From $C_k^+ \cong C_{k-8}^+(16)$ we get that $a_{k+8} = 16\, a_k$, or $\ell(k+8) = \ell(k) + 4$.

Applying this to vector fields on spheres: There are k linearly independent vector fields on $S^{a_k - 1}$; by taking copies, there are k linearly independent vector fields on $S^{c a_k - 1}$. Now fix the dimension of the sphere and maximize k; i.e. for S^{n-1}, maximize $\{k \text{ such that } a_k \mid n\}$, so

maximize $\{k \text{ such that } \ell(k) \leq \nu(n)\}$

from last time, largest power of 2 dividing n.

The first few cases are

$\nu(n)$	0	1	2	3	4	5	6
k	0	1	3	7	8	9	11

← and this is exactly $\rho(n) - 1$ from last time.

So we have succeded in constructing $\rho(n)-1$ linearly independent vector fields on the $n-1$ sphere; this is the best we can do using linear algebra. Next we have to show that really is the best one can do.

Today we start attacking the second part of the vector field problem. Recall that the Stiefel manifold $V_{n,k}$ of k-frames on $\mathbb{R}^n$ is given by $n \times k$ matrices with orthonormal columns

$$\begin{array}{cc} n \updownarrow & \begin{bmatrix} \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & \ & \ & \ \end{bmatrix} \\ & \longleftarrow k \longrightarrow \end{array}
\qquad
\begin{tikzcd} V_{n,k} \arrow[d, "\substack{\text{proj. on} \\ 1^{st}\ \text{column}}"] \\ S^{n-1} \arrow[u, dotted, bend left, "s"] \end{tikzcd}$$

and projection onto the first column gave the structure of a fibre bundle $V_{n,k} \to S^{n-1}$. The existence of a section s of this bundle is equivalent to the existence of $k-1$ orthonormal vector fields on S^{n-1}.

Now we will find a consequence for the existence of such a section in the homotopy theory of $\mathbb{RP}^{k-1}$. To start with, there are two important facts about $\mathbb{RP}^{k-1}$:

1^{st} important fact is the fiber bundle with fiber $\mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{c} S^{k-1} \\ \downarrow \\ \mathbb{RP}^{k-1} \end{array} \quad \text{obtained by identifying antipodal points on } S^{k-1};$$

and the 2^{nd} important fact is the existence of the "tautologous line bundle" over $\mathbb{RP}^{k-1}$:

$$\begin{array}{c} L \\ \downarrow \\ \mathbb{RP}^{k-1} \end{array} = \{ (\ell, x) \in \mathbb{RP}^{k-1} \times \mathbb{R}^k \mid x \in \ell \}$$

(where one thinks of $\mathbb{RP}^{k-1}$ as the space of lines through the origin on $\mathbb{R}^k$).

These two constructions are, of course, related: there is a metric on L induced by the metric in $\mathbb{R}^k$; taking the vectors of ~~the~~ length 1 in each fibre gives S^{k-1} back; this is denoted by $S(L)$, the unit sphere bundle of L.

On the other hand, we can recover L as $L = S^{k-1} \times_{\mathbb{Z}/2} \mathbb{R}$;
{i.e. (x, v) $(-x, -v)$}
this is an example of the "Borel construction"; more generally if F is any space with a $\mathbb{Z}/2$ action, we get a bundle with fiber F by taking

$$\begin{array}{ccc} F & \longrightarrow & S^{k-1} \times_{\mathbb{Z}/2} F \\ & & \downarrow \\ & & \mathbb{RP}^{k-1} \end{array}.$$

What follows from the section s of $V_{n,k} \downarrow S^{n-1}$ is:

Lemma If there is a section, then there is a bundle map over $\mathbb{RP}^{k-1}$

$$S(n\epsilon) \xrightarrow{\hat{s}} S(nL)$$

($n\epsilon$ = n times the trivial bundle ϵ, ie trivial bundle of dimension n)

such that over $\pm e_1$, taken as the base point in $\mathbb{RP}^{k-1}$, $\hat{s}$ is the "identity."

Note $\hat{s}$ is a map $\hat{s}: \mathbb{RP}^{k-1} \times S^{n-1} \longrightarrow S^{k-1} \times_{\mathbb{Z}/2} S^{n-1}$

Proof:

For $(x,v) \in \mathbb{R}^k \times S^{n-1}$, $\quad (x,v) \xmapsto{\hat{s}} (x, s(v)x) \in \mathbb{R}^k \times S^{n-1}$.

Now $\quad s(v) = \underbrace{\begin{bmatrix} v & & \\ & & \end{bmatrix}}_{k} \Big\updownarrow n$, and $|(s(v)\,x)| = 1$ if $|x| = 1$,

so $\hat{s}$ maps $S^{k-1} \times S^{n-1} \longrightarrow S^{k-1} \times S^{n-1}$

Moreover $\quad \hat{s}(-x, v) = (-x, -s(v)x) = (x, s(v)x)$ in $S^{k-1} \times_{\mathbb{Z}_2} S^{n-1}$,

so $\hat{s}$ maps $\mathbb{R}P^{k-1} \times S^{n-1} = S(n\epsilon) \longrightarrow S^{k-1} \times_{\mathbb{Z}/2} S^{n-1} = S(nL)$.

Finally, if $x = e_1$, $s(v)e_1 = v$ so $(e_1, v) \mapsto (e_1, v)$.

One point worth mentioning here is that $\hat{s}$ could have been defined as a map $n\epsilon \longrightarrow nL$, but it would not necessarily have been linear on each fiber (since $s(v)$ is not necessarily linear), so the map would not have been in a strict sense a map of vector bundles. Last time we constructed s using Clifford algebras, and so s was in fact linear. In some sense then the point is to find out whether you can get more vector fields by deforming in some clever way.

Now the fact that $\hat{s}$ induces the identity over $\pm e_1$ means that $\hat{s}$ induces a homotopy equivalence on each fiber: for any open $\mathcal{U}$, connected, $\subset \mathbb{R}P^{k-1}$ such that $nL|_{\mathcal{U}}$ is trivial and such that $\hat{s}_b$ is a homotopy equivalence for some $b \in \mathcal{U}$, it follows that $\hat{s}$ induces a map $\hat{s}|_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathrm{Map}_{\pm}(S^{n-1}, S^{n-1})$ which maps $\mathcal{U}$ to the ± 1 degree component of Maps from S^{n-1} to S^{n-1}. So $\hat{s}_x$ for any $x \in \mathcal{U}$ is a homotopy equivalence.

So we have a situation like this: two bundles E and E' over a base B, and a bundle map f

$$\begin{tikzcd} E \arrow[rr, shift left, "f"] \arrow[dr, "p"'] & & E' \arrow[ll, shift left, dotted, "g"] \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

which induces a homotopy equivalence on each fiber. What we really want is a map g going back so that $f \circ g$ and $g \circ f$ are homotopic through bundle maps to the respective identity maps; in that case f is called a "fiber homotopy equivalence." Fortunately in this case there is a very nice theorem due to Dold:

<u>Lemma</u> (Dold) Suppose E and E' are fibrations over B with a bundle map f inducing a homotopy equivalence on each fiber. If E and E' have the homotopy type of CW complexes and B is connected, then f has a fiber homotopy inverse.

(<u>Proof</u>: See attached page from James' "Stiefel Manifolds")

In this context, the lemma gives:

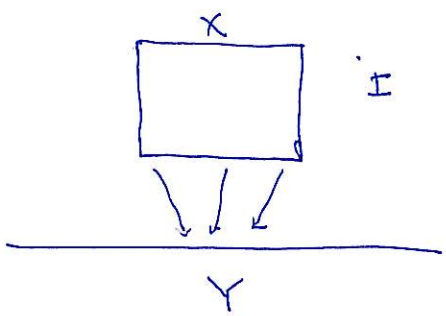
<u>Corollary</u> If $V_{n,k} \longrightarrow S^{n-1}$ sections, then $S(nL) \downarrow \mathbb{RP}^{k-1}$ is fiber homotopy trivial.

Ultimately, we will show the result of Adams that this implies that $k \leq \rho(n)-1$, i.e. that $a_k | n$.

Next we look at the consequences of the discussion so far in terms of the Thom space. Suppose $E \downarrow B$ is a vector bundle with a metric, so there's a nice sphere bundle

$$\begin{array}{c} S(E) \subset E \\ p \downarrow \\ B \end{array}$$

In order to understand the map p, we could look again at the mapping cone $B \cup_p C(S(E))$ of p. If we construct the cone in two stages, first by taking, for $X \xrightarrow{f} Y$, the space we get by taking $X \times I$ (without pinching one end to a point) $\amalg\, Y \big/ {(x,1) \sim f(x)}$,



the resulting space is called the "mapping cylinder;" and it is homotopy equivalent to Y. Moreover there is a nice inclusion $X \hookrightarrow$ mapping cylinder which is a cofibration.

And the mapping ~~cylinder~~ cone is obtained by pinching $X \times \{0\}$ to a point. Now in the case of $S(E) \xrightarrow{f} B$, the mapping cone is called the Thom space of E, $T(E)$. In fact, it doesn't matter how you replace the space Y with a homotopy equivalent space; the point of the mapping cylinder discussion is that you can make the Thom construction on any sphere bundle. In the case of the vector bundle E, one can think of the mapping cylinder of $S(E)$ as being the disk bundle $D(E)$ of vectors of length ≤ 1; the Thom space is the quotient space $D(E)/S(E)$.

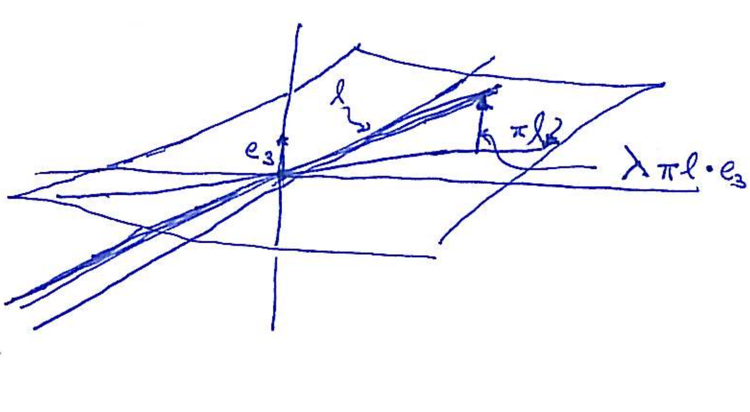
$$\begin{tikzcd}
S^{n-1} \arrow[r, hook] \arrow[d] & D^n \arrow[r] \arrow[d] & S^n \arrow[d] \\
S(E) \arrow[r, hook] & D(E) \arrow[r] \arrow[d, "\simeq"] & T(E) \\
 & B &
\end{tikzcd}$$

$\leftarrow$ unit disk bundle

Remark

Now the fiber over a basepoint $b \in B$ is $S^{n-1} \subset S(E)$, $D^n \subset D(E)$; it pinches to an S^n in $T(E)$. So if B is connected, the Thom space comes with a class $S^n \to T(E)$.

Next, we return to $nL \downarrow \mathbb{RP}^{k-1}$. The first claim is that this bundle is the normal bundle of the inclusion $\mathbb{RP}^{k-1} \hookrightarrow \mathbb{RP}^{n+k-1}$ induced by the inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^{n+k}$. To wit, suppose $l = l(t)$ is a line in $\mathbb{R}^{n+k}$; let πl be the projection of this line to a line in $\mathbb{R}^k$. We can recover the line l by $l(t) = (\pi l(t), \lambda_1 \pi l(t), \ldots, \lambda_n \pi l(t))$ where clearly $\lambda_i(rv) = r\lambda_i(v)$ so $\lambda_i \in L^*$, hence the normal bundle can be identified with nL^*. Since nL comes equipped with a metric, this gives an isomorphism of the normal bundle with nL.



But there's more structure around: using the usual map of the open unit ball in $\mathbb{R}^n$ to $\mathbb{R}^n$, one can construct a map from $D(\nu)$ to $\mathbb{RP}^{n+k-1}$ such that $S(\nu)$ maps to $\mathbb{RP}^{n-1}$, and such that the map is a relative homeomorphism $(D(\nu), S(\nu)) \longrightarrow (\mathbb{RP}^{n+k-1}, \mathbb{RP}^{n-1})$. *

It follows that the Thom space $\boxed{T(nL) \cong \mathbb{RP}^{n+k-1}/\mathbb{RP}^{n-1}}$,

so nL's being fiber homotopy trivial now implies $T(n\epsilon) \simeq T(nL)$

$\cong \mathbb{RP}^{n+k-1}/\mathbb{RP}^{n-1}$.

* The map (or one candidate for the map) is

$$nL \ni \{\underset{\in \mathbb{RP}^{k-1}}{\ell}, \underset{\in \ell}{\xi_1}, \dots, \underset{\in \ell}{\xi_n}\} \longmapsto \begin{cases} [(1-\|\xi\|)z, \langle \xi_1, z\rangle, \dots, \langle \xi_n, z\rangle] \in \mathbb{R}^{n+k}, & z \in \ell,\ \|z\| = 1 \text{ if } \|\xi\| < 1 \\ [0, \xi_1, \dots, \xi_n] & z \in \ell,\ \|z\| = 1 \text{ if } \|\xi\| = 1 \end{cases}$$

Today we'll look at Thom spaces in more detail, and in particular in relation to some other standard constructions on fiber bundles.

First of all, the cross-product: given two bundles

$$\begin{tikzcd} F \arrow[dr] & & & F' \arrow[dr] & \\ & E \arrow[d] & & & E' \arrow[d] \\ * \in & B & & * \in & B' \end{tikzcd}$$

get a bundle

$$\begin{tikzcd} F \times F' \arrow[dr] & \\ & E \times E' \arrow[d] \\ (*,*) \in & B \times B' \end{tikzcd}$$

Now if $B = B'$, you can pull this bundle back by the diagonal map

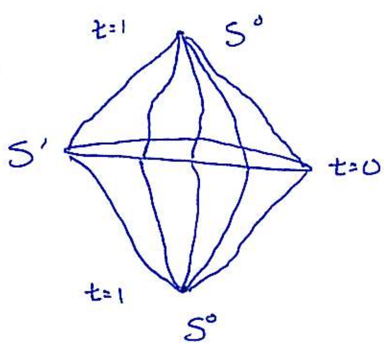
$$\begin{tikzcd} F \times F' \arrow[dr] & & \\ & E \times_B E' \arrow[r] \arrow[d] & E \times E' \arrow[d] \\ & B \arrow[r, "\Delta"] & B \times B. \end{tikzcd}$$

This is the "fiber-wise product." If E and E' are vector bundles, $E \times_B E'$ is the "Whitney Sum" $E \oplus E'$.

If E and E' are sphere bundles, this construction isn't very satisfying because a sphere cross a sphere isn't another sphere. One product for which the result is a sphere, though, is the join $*$. Here's another way of looking at the join (we will return to the join often, and try as often as possible to give a different definition of it!):

$$X * Y = \frac{X \times I \times Y}{\begin{matrix}(x,1,y) \sim (x,1,y') \\ (x,0,y) \sim (x',0,y)\end{matrix}}$$

e.g. $S^0 * S'$



Now we define a fiberwise join by doing this on each fiber:

$$\begin{matrix} F \searrow & & F' \searrow & & F * F' \searrow & & \\ & E & & E' & & E \hat{*} E' & \\ & \downarrow p & & \downarrow p' & \rightsquigarrow & \downarrow & \\ & B & & B' & & B \times B' & \end{matrix} \qquad E \hat{*} E' = \frac{E \times I \times E'}{\begin{matrix}(x,1,y_0) \sim (x,1,y_1) \text{ if } \\ p'y_0 = p'y_1; \\ (x_0,0,y) \sim (x_1,0,y) \text{ if } \\ px_0 = px_1.\end{matrix}}$$

One checks that the construction yields something locally trivial.

<u>Remark</u> Define $X * \phi = X$ and $\phi * Y = Y$.

This fiberwise join for sphere bundles is related to the fiberwise cross-product of vector bundles: if V and W are vector bundles, then $S(V \times W) = SV \hat{*} SW$. Moreover, if $B = B'$ we can once again pull back by the diagonal map

$$\begin{array}{ccc} E *_B E' & \longrightarrow & E \hat{*} E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\Delta} & B \times B \end{array},$$

and $S(V \oplus W) = SV *_B SW$.

In the case $Y = \{*\}$ is a point, $X * Y = CX$ is the cone over X. Applying this in the fiberwise join,

$$\begin{array}{ccc} F \searrow & & CF \searrow \\ E & \rightsquigarrow & E *_B B = C_B E \\ \downarrow & & \downarrow \\ B & & B \end{array}$$

gives us a fiber-wise cone construction, a bundle $C_B E$ whose fiber is the cone over the old fiber F.

Applying this to a sphere bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ gives a disk bundle $\begin{array}{c} C_B E \\ \downarrow \\ B \end{array}$,

and it's easy to check that if V is a vector bundle, then $D(V) = C_B S(V)$. So we have a way of recovering the disk bundle from the sphere bundle.

Back to Thom Spaces. If $E \downarrow B$ is a sphere bundle, its Thom space is $T(E) = C_B E/E$. If the fiber over the basepoint of B in E is S^{n-1}, then $CS^{n-1} = D^n \subset C_B E$, which becomes an $S^n \subset T(E)$. $T(E)$ comes equipped with a canonical basepoint $*$ which is the image of the E which has been pinched to a point.

NB In case $E = \phi$, $C_B \phi = B$, and $T(\phi) = B/\phi \overset{\text{(by convention)}}{=} B \amalg \{*\}$

distinct basepoint.

If $V \downarrow B$ is a vector bundle, then $T(V) \overset{\text{def}}{=} T(S(V))$, and also $T(0) = B \amalg \{*\}$.

To understand better, let's look at the trivial sphere bundle $B \times S^{n-1} \to B$.

$$T(B \times S^{n-1}) = \frac{B \times D^n}{B \times S^{n-1}}.$$

Now thinking of S^n as D^n with $S^{n-1} \subset D^n$ smashed to a point, it follows that $\frac{B \times D^n}{B \times S^{n-1}} \cong \frac{B \times S^n}{B \times *}$. If we let $B_+ = B \amalg \{*\}$ be B together with a disjoint base point, it follows that $T(B \times S^{n-1}) = \frac{B \times D^n}{B \times S^{n-1}} \cong \frac{B \times S^n}{B \times *} \cong B_+ \wedge S^n = \Sigma^n B_+$.

So the Thom space of a trivial bundle is this twisted sort of suspension.

Important **Fact**

$$\boxed{T(E \hat{*} E') = T(E) \wedge T(E').}$$

A rough proof in the special case $E = \begin{matrix} SV \\ \downarrow \\ B \end{matrix}$, $E' = \begin{matrix} SW \\ \downarrow \\ B' \end{matrix}$:

$$TV \wedge TW = \frac{DV}{SV} \wedge \frac{DW}{SW}$$

$$= \frac{\frac{DV}{SV} \times \frac{DW}{SW}}{\begin{matrix} * \times \text{anything} \\ \text{anything} \times * \end{matrix}}$$ (where base points $*$ are images of SV and SW in upper quotients)

$$= \frac{DV \times DW}{\begin{matrix} SV \times DW \\ DV \times SW \end{matrix}} = \frac{D(V \times W)}{S(V \times W)}$$ $\left\{ \text{since bdry. of } D^p \times D^q \text{ is } D^p \times S^{q-1} \cup S^{p-1} \times D^q \right\}$

$$= T(V \times W).$$

In the special case that $\begin{matrix} E' \\ \downarrow \\ B' \end{matrix}$ is $\begin{matrix} \mathbb{R}^n \\ \downarrow \\ * \end{matrix}$, we get that

$$\boxed{T(V \oplus n\epsilon) \simeq T(V) \wedge S^n = \Sigma^n T(V)}$$

The Thom space is useful for deciding whether a fiber bundle is trivial.

Fact: If E and E' are fibre homotopy equivalent S^{n-1} bundles, then

$T(E) \simeq T(E')$ relative to the copy of S^n in $T(E)$ resp. $T(E')$ coming from the fiber over the base point in E resp. E'!

S^n rel.

For example, suppose $\begin{matrix} S^{n-1} \searrow \\ E \\ \downarrow \\ B \end{matrix}$ is fiber-homotopy trivial, and

B is connected. I.e. we have

$$\begin{tikzcd} E \arrow[r, "\text{f.h.e.}"] \arrow[dr] & B \times S^{n-1} \arrow[r, "\text{proj.}"] \arrow[d] & S^{n-1} \\ & B & \end{tikzcd}$$

Now by the nice theorem of Dold (last time), the composite of the horizontal arrows contains all the information about the fiber homotopy equivalence, i.e. this equivalence is equivalent to a map $E \xrightarrow{f} S^{n-1}$ which is a homotopy equivalence on each fiber.

Another way to think about this is as a bundle map

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & S^{n-1} \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

which induces a map

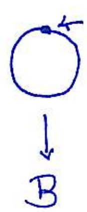
$$\begin{tikzcd} T(E) \arrow[r] & S^n \\ S^n \arrow[u] \arrow[ur, "\simeq"'] & \end{tikzcd}$$ "coreduction"

and the homotopy equivalence of f restricted to each fiber implies that the map $S^n \to S^n$ is a homotopy equivalence. Such a triangle is called a "coreduction." Now the game is to show obstructions to the splitting off of this n-sphere, which is in fact the bottom-dimensional cell in $T(E)$.

Of course, what we really want is an obstruction to a f.h.e. between nL and $n\epsilon$, and a coreduction of $T(E)$ need not imply that E itself is f.h.t. To find out what we do get from a coreduction of $T(E)$, we look at another construction of $T(E)$ [. and think of the name Bott when thinking about $T(E)$ this way]:

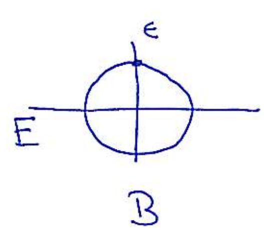
We build $T(E)$ in two steps.

1) collapse to sphere in each fiber, i.e. take $D(F)/S(F)$ in each fiber F of $E \downarrow B$. This gives a sphere bundle.



"section at ∞", image of $S(F)$ in each fiber.

2) collapse section at ∞ to a point.



But this exhibits $T(E)$ as the sphere bundle $S(E\oplus\epsilon)$ with the 1-section of ϵ, which is homeomorphic to B, smashed to a point, so

$$T(E) = S(E\oplus\epsilon)\Big/ B_{\text{embedded as 1-section of } \epsilon}$$

Now the fiber in $S(E\oplus\epsilon)$ is the same as the S^n sitting on $T(E)$, so with this construction, a coreduction for $T(E)$ gives

$$\begin{tikzcd} S(E\oplus\epsilon) \arrow[r] & T(E) \arrow[r] & S^n \\ & S^n \arrow[ul] \arrow[u] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

If the base B is connected, then from Dold's theorem it follows that $E\oplus\epsilon$ is fiber homotopy trivial. So a coreduction of $T(E) \Rightarrow$ that $E\oplus\epsilon$ is f.h.t.

This is a process of stabilization: the lesson of today's discussion is that we have to play back and forth with various processes of this kind: Σ, Thom space, $V\oplus\epsilon$, and so on. Hence the natural role of K-theory.

A few words on K-theory. Let X be a finite complex (it is at least necessary that X be a compact Hausdorff space). Let $\mathrm{Vect}(X)$ = the set of isomorphism classes of vector bundles over X {why is $\mathrm{Vect}(X)$ a set?}. The Whitney sum gives a monoid structure to $\mathrm{Vect}(X)$, with zero given by the 0-dim vector bundle.

Now one applies the "Grothendieck Construction" which simply means to add formal inverses, making a group:

take equivalence classes of pairs $(V, W) \sim (V', W')$ [where (V, W) is supposed to represent the formal difference "$V - W$"] if $E + V + W' \cong V' + W + E$. This is an abelian group called $KO(X)$.

Now it may not seem that $KO(X)$ has topological significance, but in view of the above, it quickly becomes apparent that it does. For if $V \oplus n\epsilon \cong W \oplus n\epsilon$ (V and W are "stably isomorphic") then the classes $[V]$ and $[W]$ are equal in $KO(X)$. Moreover, if X is a compact Hausdorff space, it goes both ways. Suppose $[V] = [W]$, so $V + E \cong W + E$ for some E. Over a compact Hausdorff space, any vector bundle E is a subbundle of a sufficiently big trivial bundle

$$\begin{tikzcd} E \arrow[r, hook] \arrow[dr] & X \times \mathbb{R}^N \arrow[d] \\ & X \end{tikzcd}$$

as we shall see next time. By choosing a metric on $\mathbb{R}^N$, we get an orthogonal complement $F \longrightarrow X$ such that $E \oplus F$ is trivial. Then $V \oplus E \oplus F \cong W \oplus E \oplus F$, so V and W are stably isomorphic!

On with K-theory (and J-theory).

X = ptd. cpct. Hausdorff space.

Recall that $KO(X)$ was defined to be the abelian group given by the Grothendieck construction on the monoid $\{\mathrm{Vect}(X)\}$.

If $X \xrightarrow{f} Y$, and $E \downarrow Y$ is a vector bundle, then the vector bundle E pulls back by f^* to a vector bundle over X, inducing a map $KO(Y) \xrightarrow{f^*} KO(X)$. Pull-backs by homotopic maps induce isomorphic bundles, so $f \simeq g \Rightarrow f^* = g^*$.

Now the canonical maps $* \to X \to *$ induce maps $KO(*) \xrightarrow{\eta} KO(X) \xrightarrow{\epsilon} KO(*)$. It is clear that $KO(*) \simeq \mathbb{Z}$, that $\mathrm{Im}\,\eta$ = trivial bundles, and that ϵ takes the dimension of the fibre over the basepoint. Reduced KO-theory is defined in terms of these maps; there are two definitions:

$$\widetilde{KO}(X) = \begin{cases} \ker \epsilon & \text{-- "virtual vector bundles, i.e. formal differences } V-W \text{ with } \dim_* V = \dim_* W\text{"} \\ \mathrm{coker}\,\eta & \text{-- } [V]=[W] \text{ iff. } V \oplus m\epsilon \cong W \oplus n\epsilon. \end{cases}$$

Next let's see, as advertised last time, how to embed a vector bundle in a high-dimensional trivial bundle. Suppose $\begin{array}{c} E \\ p\downarrow \\ X \end{array}$ is an n-dimensional vector bundle over a compact Hausdorff space X.

Then X has a finite open cover $U_1, \dots, U_k$ that trivializes E, that is, there are homeomorphisms $f_i : E|_{\mathcal{U}_i} \cong \mathcal{U}_i \times \mathbb{R}^n$

$$\begin{tikzcd} E|_{\mathcal{U}_i} \arrow[rr, "\cong"] \arrow[dr, "p"'] & & \mathcal{U}_i \times \mathbb{R}^n \arrow[dl, "\pi_i"] \\ & \mathcal{U}_i & \end{tikzcd}$$

such that the triangle commutes and f_i is linear on each fiber.

Let $\varphi_1, \dots, \varphi_k$ be a partition of unity subordinate to the $\mathcal{U}_i$'s, and define maps $g_1, \dots, g_k : E \longrightarrow \mathbb{R}^n$ by

$$g_i : \begin{cases} E|_{\mathcal{U}_i} \xrightarrow[\cong]{f_i} \mathcal{U}_i \times \mathbb{R}^n \xrightarrow{(x,v) \mapsto \varphi_i(x)\cdot v} \mathbb{R}^n \\ 0 \text{ on } E \text{ away from } \mathcal{U}_i \end{cases}$$

The g_i give a linear embedding $f : E \hookrightarrow X \times (\mathbb{R}^n)^k$ defined by $f(e) = (p(e), g_1(e), \dots, g_k(e))$.

$$\begin{tikzcd} E \arrow[rr, hook, "f"] \arrow[dr] & & X \times (\mathbb{R}^n)^k \arrow[dl] \\ & X & \end{tikzcd}$$

But this map gives more: to each $x \in X$ we associate via f the n-dimensional subspace $f(E_x)$ of $\mathbb{R}^{nk = N}$; this induces a map

$g : X \longrightarrow G_{N,n}$ = Grassmanian of n-planes in $\mathbb{R}^N$. Over $G_{N,n}$ is the "canonical n-plane bundle $E_{N,n}$," and we have in fact expressed $E \downarrow X$ as the pull-back of

$$\begin{tikzcd} & E_{N,n} \arrow[d] \\ X \arrow[r, "g"] & G_{N,n} \end{tikzcd}$$

From this point of view it is clear that the choice of N is somewhat arbitrary; there is an obvious inclusion

$$G_{N,n} \subsetneq G_{N+1,n} \subsetneq G_{N+2,n} \cdots$$

and E can be induced from the canonical bundles over $G_{N,n}$ for any sufficiently large N. Thus $BO(n) = \bigcup_N G_{N,n}$,

and $\mathrm{Vect}_n(X) \xrightarrow{\cong} [X, BO(n)]$.

What happens when we go to $KO(X)$? The equivalence $V \sim V \oplus \epsilon$ gives us $\widetilde{KO}$, so we get $\widetilde{KO}$ by identifying an element of $[X, BO(n)]$ with an element of $[X, BO(n+1)]$ when all it does is to add a trivial bundle. Now if $EO(n)$ is the canonical n-plane bundle over $BO(n)$, then $EO(n) \oplus \epsilon$ is an $n+1$-plane bundle, and there is a classifying map

$$\begin{array}{ccccc} EO(n)\oplus\epsilon & & EO(n+1) & & \\ \downarrow & & \downarrow & & \\ BO(n) & \longrightarrow & BO(n+1) & \longrightarrow & \cdots \end{array}$$

The limit $\bigcup_n BO(n)$ is called BO, and $\widetilde{KO}(X) = [X, BO]$.

Finally, KO remembers dimension, so $KO(X) = [X, \mathbb{Z} \times BO]$.

J-theory.

If V and W are vector bundles over X, then we write $V \underset{J}{\simeq} W$ iff $S(V \oplus n\epsilon) \underset{\text{f.h.e.}}{\simeq} S(W \oplus \epsilon n)$;

i.e. the idea is analogous to K on the level of sphere bundles.

Claim This defines an additive equivalence relation on $KO(X)$.

(for example, suppose $V \underset{J}{\simeq} W$, and V' is any vector bundle.
Then $S(V' \oplus V \oplus n\epsilon) \cong SV' *_X S(V \oplus n\epsilon)$

$$\underset{\text{f.h.e.}}{\simeq} SV' * S(W \oplus n\epsilon) \cong S(V' \oplus W \oplus n\epsilon))$$

$J(X)$ is defined to be the quotient subgroup of $KO(X)$ by this relation, and $\tilde{J}(X)$ the image of $\widetilde{KO}(X)$. Now from the Clifford algebra story we know: over $\mathbb{RP}^k$, the canonical line bundle L over $\mathbb{RP}^k$ has $a_k L \cong a_k \epsilon$. Considering the class $[L]-1 \in \widetilde{KO}(\mathbb{RP}^k)$, this means $a_k([L]-1) = 0$.

Thm (Adams) $\widetilde{KO}(\mathbb{RP}^k) = \mathbb{Z}/a_k\mathbb{Z}$ generated by $[L]-1$,

and $\widetilde{KO}(\mathbb{RP}^k) \xrightarrow{\cong} \tilde{J}(\mathbb{RP}^k)$ is an isomorphism.

In other words, there are no other trivializations, and so

Corollary If $S(nL) \downarrow \mathbb{RP}^k$ is fiber homotopy trivial, then $a_k \mid n$, so there are at most $\rho(n)-1$ vector fields on S^{n-1}.

One of the things we get out of this theorem is that $\widetilde{KO}(\mathbb{RP}^n)$ is finite; in general, $\tilde{J}X$ is finite whenever:

Theorem (Atiyah) If X is a finite connected complex, then $\tilde{J}(X)$ is finite.

Sketch of proof:

A vector bundle over X is classified by a map $X \to BO(n)$. Similarly a sphere bundle should have a similar classifying procedure. In general, the structure group would have to be taken to be $\mathrm{Homeo}(S^{n-1})$, but since we are only concerned with fiber homotopy equivalences, we should be able to use the monoid G_n of homotopy self-equivalences of S^{n-1}! Such a classifying procedure exists; call the corresponding space BG_n. Now there is a map $O_n \hookrightarrow G_n$, and hence maps

$$\begin{tikzcd} BO_n \arrow[r] \arrow[d] & BG_n \arrow[d] \\ BO \arrow[r] & BG \end{tikzcd}$$

Claim 1 $\tilde{J}(X) = \mathrm{Im}\,([X,BO]_* \longrightarrow [X,BG])$.

Claim 2 $[X,BG]$ is finite. We prove this cell-by-cell, so what we really need to show is that $\pi_i(BG_n)$ is finite for $i \ll n$. Since X is finite, $[X,BG] = [X,BG_n]$ for some n sufficiently larger than the top-dimensional cell of X, so this is enough.

But there is a fibration

$$\begin{tikzcd} G_n \arrow[r] & \boxed{/\!/\!/} \arrow[d] \simeq * \\ & BG_n \end{tikzcd}$$

so the homotopy groups of G_n are the same as those of BG_n, with a shift. So we are left with showing that $\pi_i G_n$ is finite for $n \gg i$. Now an element of G_n is a map $S^{n-1} \to S^{n-1}$; by evaluating at the basepoint $* \in S^{n-1}$, we get a map $G_n \to S^{n-1}$. This is a fibration with fiber

$$\begin{tikzcd} \mathrm{map}_*(S^{n-1}, S^{n-1})_{\pm 1} \arrow[r] & G_n \arrow[d] \\ & S^{n-1} \end{tikzcd}$$

ptd. maps

degree ± 1 (because htpy self equivalence)

Now $\mathrm{map}_*(S^{n-1}, S^{n-1})_{\pm} = \Omega^{n-1}_{\pm} S^{n-1}$.

For $i \ll n$, $\pi_i G_n \cong \pi_i(\Omega^{n-1}_{\pm}, S^{n-1})$

and $\pi_i(\Omega^{n-1}_{\pm}, S^{n-1}) \subset \pi_{i+n-1} S^{n-1} = \pi_i^S$ for $i \ll n$, which is known to be finite (Serre).

Today begins a several-days' blitz on Steenrod operations, from a somewhat geometric point of view.

To begin, remember that ordinary cohomology is representable:

$$\tilde{H}^q(X;\pi) = \underbrace{[X, K(\pi,q)]_+}_{\text{basepoint-preserving}}. \quad \text{For example, } \tilde{H}^q(S^n) = \pi_n(K(\pi,q)) = \begin{cases} \pi & q=n \\ 0 & q\neq n. \end{cases}$$

A basepoint gives a lot more than you might think at first. For example, a basepoint in X gives a filtration of X^n:

$$* \in X$$

$$X^n \supset F_k X^n = \{(x_1,\dots,x_n) \mid \text{at least } n-k \text{ of the } x_i \text{ are } *\}.$$

So

$$F_0 X^n = \{(*,\dots,*)\}$$
$$\cap\!|$$
$$F_1 X^n = \{\text{union of axes}\} = \bigvee^n X$$
$$\cap\!|$$
$$\vdots$$
$$\cap\!|$$
$$F_{n-1} X^n = \text{“Fat Wedge”}$$
$$\cap\!|$$
$$F_n X^n = X^n.$$

Suppose $\pi \subset \Sigma_n$. π acts on X^n by permuting the coordinates, and this action respects our filtration (in this discussion, we will be concerned mostly with the case $\pi = \mathbb{Z}/2$.).

Consider the space $E\pi$, a contractible CW-complex with a free π-action. For example, if $\pi = \mathbb{Z}/2$, then π acts antipodally on S^{n-1}. S^{n-1} is not contractible, but its image is contractible in S^n; that is, the inclusion $S^{n-1} \hookrightarrow S^n$ is null-homotopic. So the direct limit of the inclusions $\ldots S^{n-1} \subseteq S^n \subset \ldots \;\; = \bigcup_n S^n = S^\infty$ is contractible and has a free π-action. The orbit space $E\pi/\pi = B\pi$ is clearly seen to be $\mathbb{RP}^\infty$.

Okay, so here's the key construction: on the universal π-bundle $E\pi \to B\pi$, do the Borel construction to mix in X^n, where π acts on X^n by permuting the factors;

$$\begin{tikzcd} X^n \arrow[r] & E\pi \times_\pi X^n \arrow[d] \\ & B\pi . \end{tikzcd}$$

Recall that π acts diagonally on $E\pi \times X^n$, and $E\pi \times_\pi X^n$ is obtained as the quotient space of this action. This construction is called the "π-extended power of X."

Now the fact that the π-action on X^n respects our filtration means that we have a <u>sub-bundle</u>

$$\begin{tikzcd} E\pi \times_\pi F_{n-1}X^n \arrow[r, phantom, "\subseteq"] \arrow[d] & E\pi \times_\pi X^n \arrow[d] \\ B\pi \arrow[r, equal] & B\pi \end{tikzcd}$$

We want to pinch this subbundle to a point. First, $X^n / F_{n-1}X^n$ is the n-fold smash product of X with itself, $X^{(n)}$.

$$\frac{E\pi \times_\pi X^n}{E\pi \times_\pi F_{n-1}X^n} = \frac{E\pi \times_\pi X^{(n)}}{E\pi \times_\pi \{*\}}.$$ This is almost the smash product, but $E\pi$ doesn't have a base point. But we can just add one in and then take the smash product, and this has no effect -- so

$$\frac{E\pi \times_\pi X^n}{E\pi \times_\pi F_{n-1}X^n} = E\pi_+ \wedge_\pi X^{(n)},$$ where the π means to take the orbit space of the diagonal action on $E\pi_+ \wedge X^{(n)}$. This is called the "π-adic construction" on X (for lack, really, of a better name), and will be written $D_\pi(X)$.

This space is of some concern; first we want to know its cohomology.

<u>Lemma</u> Let $\bar{H}^i(X) = 0$ for $i < q$, where coefficients are taken in a field; and $\bar{H}^q(X)$ is finite dimensional. Then

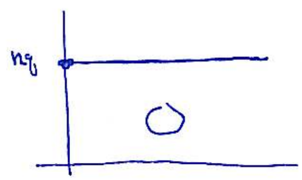
$$\bar{H}^i(D_\pi X) = 0 \quad \text{for } i < nq$$

$$\bar{H}^{nq}(D_\pi X) = \left(H^q(X)^{\otimes n}\right)^\pi \leftarrow \text{"invariants under } \pi \text{ action."}$$

Proof: One could produce a CW-complex for $D_\pi X$ having no cells of dimension $< nq$. Alternatively, one can use a relative Serre spectral sequence, with twisted coefficients since $B\pi$ is not simply connected; in fact $B\pi$ is a $K(\pi,1)$.

$$H^*(B\pi; \{H^*(X^n, F_{n-1}X^n)\}) \Rightarrow \bar{H}^*(D_\pi X).$$

Now $H^*(X^n, F_{n-1}X) \cong \bar{H}^*(X^{(n)}) \cong \bar{H}^*(X)^{\otimes n}$, and this last isomorphism is equivariant w.r.t. permutations, a fact that is not trivial and likely to be false with other cohomology theories. It follows that in the spectral sequence everything is zero below the horizontal line at nq in the E^2 term:



So $\bar{H}^{nq}(D_\pi X) = H^0(B\pi; \{\bar{H}^q(X)^{\otimes n}\}) = (\bar{H}^q(X)^{\otimes n})^\pi$.

This is the key fact that gives the Steenrod operations; all else follows more or less from it. Take for example the case $K = K(\mathbb{F}_2, q)$.

$$\left.\begin{aligned} H^i(K) &= 0 \quad i < q \\ H^q(K) &= \mathbb{F}_2 \end{aligned}\right\} \text{ by the Hurewicz theorem.}$$

So $\bar{H}^{nq}(D_\pi K) = \mathbb{F}_2$ (taking coeffs. in $\mathbb{F}_2$). Moreover the spectral sequence shows that the map

$$\mathbb{F}_2 = \bar{H}^{nq}(D_\pi K) \xrightarrow[\cong]{i^*} \bar{H}^{nq}(K^{(n)}) \quad \text{is an isomorphism.}$$

Now $\bar{H}^{nq}(K^{(n)})$ contains an element $\iota_q^{\wedge n}$, the n-fold smash power of the fundamental class $\iota_q \in \bar{H}^q(K)$ ~~$K(\mathbb{F}_2, q)$~~, and we have

Corollary There is a unique class $P_\pi \iota_q \in \bar{H}^{nq}(D_\pi K)$ such that $i^* P_\pi \iota_q = \iota_q^{\wedge n}$.

So if X is any pointed space, $u \in \bar{H}^q(X) \longleftrightarrow [X, K]$

$$f^* \iota_q \longleftrightarrow [f]$$

Representing u as a (homotopy class) of maps $X \xrightarrow{u} K$, we have an induced map $X^{(n)} \xrightarrow{u^{\wedge n}} K^{(n)} \xrightarrow{\iota_q^{\wedge n}} K(\mathbb{F}_2, nq)$, and then

$$\begin{tikzcd} X^{(n)} \arrow[r, "u^{\wedge n}"] \arrow[d, "i"'] & K^{(n)} \arrow[r, "\iota_q^{\wedge n}"] \arrow[d, "i"] & K(\mathbb{F}_2, nq) \\ D_\pi X \arrow[r, "D_\pi u^{\wedge n}"] & D_\pi K \arrow[ur, "P_\pi \iota_q"'] & \end{tikzcd}$$

shows that there is a unique class $P_\pi u \in \tilde{H}^{nq}(D_\pi X)$ such that $i^* P_\pi u = u^{\wedge n}$. $P_\pi u$ is called the "Steenrod power" of u.

Finally, the diagonal map $X \xrightarrow{\Delta} X^{(n)}$ is equivariant (π acts trivially on the single factor X), and this induces

$$\begin{array}{ccc} E\pi_+ \wedge_\pi X & \xrightarrow{\Delta} & E\pi_+ \wedge X^{(n)} \\ \| & & \| \\ B\pi_+ \wedge X & \xrightarrow[j]{} & D_\pi X . \end{array}$$

Now, take the case $\pi = \mathbb{Z}/2\mathbb{Z}$, $n = 2$; write D_2 for D_π and P for P_π.

- $B\pi = \mathbb{RP}^\infty$, as we saw above, and $H^*(B\pi) = \mathbb{F}_2[x]$, $|x| = 1$

$$H^*(B\pi) = \tilde{H}^*(B\pi_+)$$

Hence $\tilde{H}^*(B\pi_+ \wedge X) \cong H^*(B\pi) \otimes \tilde{H}^*(X)$. Now in $\tilde{H}^*(B\pi_+ \wedge X)$ we have the class $j^* Pu$ for any $u \in \tilde{H}^q(X)$. This isomorphism means we can write

$$\begin{array}{ccc} \tilde{H}^*(B\pi_+ \wedge X) & \cong & H^*(B\pi) \otimes \tilde{H}^*(X) \\ \rotatebox{90}{$\in$} & & \\ \underbrace{j^* Pu}_{\text{degree } 2q} & = & \sum_{i=0}^{q} x^{q-i} \otimes \underbrace{Sq^i u}_{\text{degree } q+i} \end{array}$$

and take this to be the definition of $Sq^i u$.

Note that the above can be to a large extent carried out in other theories, giving similar operations. Mostly one needs to have computed $H^*(B\pi \wedge X)$.

Now let's start finding properties of the Squares. First of all, we don't even know that they're not all zero yet. But in the diagram, if $|u| = q$,

$$\begin{tikzcd} \sum_{i=0}^{q} x^{q-i} \otimes Sq^{i} u & & Pu \arrow[ll, mapsto] \arrow[dd, mapsto, bend left] \\ B\pi_+ \wedge X \arrow[rr, "j"] & & D_\pi X \\ X \arrow[u, "k"] \arrow[rr, "\Delta"] & & X^{(2)} \arrow[u, "i"] \\ u \cup u & & u^{\wedge 2} \arrow[ll, mapsto, bend left] \end{tikzcd}$$

Now by the pull-back $k^*: H^*(B\pi) \otimes \bar{H}^*(X) \to \bar{H}^*(X)$, all powers of x, the generator of $H^*(B\pi)$, except 0 are killed. So $Sq^q u = u^2$.

Next, the Cartan Formula.

$$D_\pi(X \wedge Y) = E\pi_+ \wedge_\pi (X \wedge Y)^{(2)} \xrightarrow{\delta} E\pi_+ \wedge X^{(2)} \wedge E\pi_+ \wedge Y^{(2)}$$

$$(e, (x_1, y_1), (x_2, y_2)) \overset{\delta}{\longmapsto} (e, (x_1, x_2); e, (y_1, y_2)).$$

And moreover the diagram

$$\begin{tikzcd} (X \wedge Y)^{(2)} \arrow[r, "i"] \arrow[d, "T"] & D_\pi(X \wedge Y) \arrow[d, "\delta"] & B\pi_+ \wedge (X \wedge Y) \arrow[l, "j"'] \arrow[d, "\tilde{\Delta}"] \\ X^{(2)} \wedge Y^{(2)} \arrow[r, "i \wedge i"] & D_\pi X \wedge D_\pi Y & B\pi_+ \wedge X \wedge B\pi_+ \wedge Y \arrow[l, "j \wedge j"'] \end{tikzcd}$$

commutes.

<u>Claim</u> $\delta^*(P(u) \wedge P(v)) = P(u \wedge v)$

<u>Pf.</u> We can assume $X = K(\pi; p)$, $Y = K(\pi; q)$, $u = \imath_p$, and $v = \imath_q$.

Then $D_\pi(X \wedge Y)$ has lowest dimensional cohomology in dimension $2(p+q)$, and here it is $\mathbb{F}_2$. But $i^*(P(u) \wedge P(v)) = (\imath_p \wedge \imath_q)^{\wedge 2}$; $P(u \wedge v)$ is by definition the unique class in $H^{2(p+q)}(D_\pi(X \wedge Y))$ for which this holds.

Corollary $Sq^k(u \wedge v) = \sum_{i+j=k} Sq^i u \wedge Sq^j v$ (Cartan Formula)

Proof: $j^* P(u \wedge v) = \sum_k x^{p+q-k} \otimes Sq^k(u \wedge v)$

$$\|$$

$j^* \delta^* (P(u) \times P(v)) = \tilde{\Delta}^* (j \times j)^* (P(u) \wedge P(v))$

$$= \tilde{\Delta}^* \left[\left(\sum x^{p-i} \otimes Sq^i u \right) \wedge \left(\sum x^{q-j} \otimes Sq^j v \right) \right]$$

$$= \sum_{i,j} x^{p+q-i-j} \otimes Sq^i u \wedge Sq^j v.$$

Corollary (Internal version). Taking $X = Y$, and using $X \xrightarrow{\Delta} X \times X$,

$$Sq^k(uv) = \sum_{i+j=k} Sq^i u \, Sq^j v.$$

Problem What are possible operations on S^1? Show that $Sq^0 e = e$, where e is the generator of $\tilde{H}^1(S^1) = \mathbb{F}_2$.

We continue to derive properties of the Steenrod squares using the construction from last time. Recall from last time:

$\pi \subset \Sigma_n$ and a space X gave us $D_\pi(X) = E\pi_+ \wedge_\pi X^{(n)}$, and we had inclusions

$$\begin{tikzcd} B\pi_+ \wedge X \arrow[r, "j"] & E\pi_+ \wedge_\pi X^{(n)} = D_\pi(X) \\ & X^{(n)} \arrow[u, "i"] \end{tikzcd}$$

Working with field coefficients, we found there was a unique natural transformation $P_\pi : \tilde{H}^q(X) \longrightarrow \tilde{H}^{nq}(D_\pi X)$ such that

$$i^* P_\pi u = u^{\wedge n} \qquad \text{for all } u \in \tilde{H}^q(X).$$

In the case $\pi = \mathbb{Z}/2$, $n = 2$, we use the Künneth isomorphism $\tilde{H}^*(B\pi_+ \wedge X) \cong \tilde{H}^*(B\pi_+) \otimes \tilde{H}^*(X)$, and the fact that $\tilde{H}^*(B\pi_+) = H^*(B\pi) = \mathbb{F}_2[x]$ $(|x| = 1)$, to write, for each $u \in \tilde{H}^q(X)$,

$$\tilde{H}^*(B\pi_+ \wedge X) \ni j^* P_\pi u = \sum_{i=0}^{q} x^{q-i} \otimes Sq^i u.$$

This was the definition of $Sq^i u$. Since $P_\pi u$ has degree $2q$, $Sq^i u$ has degree $q+i$.

These properties of Sq^i are immediate or were shown last time:

- Sq^i is a natural transformation of functors (spaces) $\to$ (sets)
  $$\bar{H}^*(-) \longrightarrow \bar{H}^{*+i}(-)$$
- $Sq^i u \equiv 0$ for $i > q$, $u \in \bar{H}^q(X)$
- $Sq^q u = u^2$ for $u \in \bar{H}^q(X)$.
- Cartan formula $Sq^k(uv) = \sum_{i+j=k} Sq^i u \, Sq^j v$.

In addition, it was an exercise to show that $Sq^0(e) = e$, where e is the generator of $\bar{H}^1(S^1)$. This important fact has several consequences.

<u>Corollary</u>. Sq^k commutes with the suspension homomorphism
$$\sigma : \bar{H}^q(X) \longrightarrow \bar{H}^{q+1}(\Sigma X).$$

<u>Proof</u>
$$\begin{tikzcd} \sigma : \bar{H}^q(X) \arrow[r] \arrow[d, phantom, "\wr\|"] & \bar{H}^{q+1}(\Sigma X) \\ \bar{H}^q(X) \otimes \bar{H}^1(S^1) \arrow[ur, "\wedge e"'] & \end{tikzcd}$$

is the suspension homomorphism. So

$$Sq^k \sigma u = Sq^k(u \wedge e) \underset{\text{(Cartan formula)}}{=} Sq^k u \wedge Sq^0 e = \sigma Sq^k u.$$

This means that Sq^k is a "stable operation."

Corollary $Sq^0 : \bar{H}^q(X) \longrightarrow \bar{H}^q(X)$ is the identity.

Proof It suffices to check this on the class $\iota_q \in \bar{H}^q(K_q)$ $\left(K_q = K(\mathbb{F}_2, q)\right)$.

For $q=1$, $Sq^0 \iota_1 \neq 0$ because it wasn't zero for S^1. But $\bar{H}^1(K_1) \cong \mathbb{F}_2$, so Sq^0 is the identity on ι_1. Furthermore $Sq^0(e^{\wedge q}) \in \bar{H}^q(S^q)$ is $e^{\wedge q}$ by the Cartan formula and the fact that Sq^0 commutes with suspensions; it follows that $Sq^0 \iota_q = \iota_q$.

Fact The map
$$\begin{tikzcd} \bar{H}^q(X) \arrow[rr, "\beta"] \arrow[dr, "\delta"'] & & \bar{H}^{q+1}(X) \\ & \bar{H}^{q+1}(X;\mathbb{Z}) \arrow[ur, "\text{reduction}"'] & \end{tikzcd}$$
given by the composite of the two lower arrows, and where δ is the zig-zag map of the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$, is called the Bockstein homomorphism. It is Sq^1.

Fact Sq^k is a homomorphism.

Pf: Letting cohomology classes be represented by maps into $K(\pi, q)$'s, we have
$$\begin{tikzcd} K_q \arrow[r, "Sq^k \iota_q"] & K_{q+k} \\ X \arrow[u, "u"] \arrow[ur, "Sq^k u"'] & \end{tikzcd}$$
That Sq^k commutes with suspension in this context means that the diagram below commutes:
$$\begin{tikzcd} \Sigma K_q \arrow[r, "\Sigma Sq^k \iota_q"] \arrow[d, "\sigma"'] & \Sigma K_{q+k} \arrow[d, "\sigma"] \\ K_{q+1} \arrow[r, "Sq^k \iota_{q+1}"] & K_{q+k+1} \end{tikzcd}$$

Taking the adjoint of each map in the square, we get

$$\begin{tikzcd} K_q \arrow[r, "Sq^k_{2q}"] \arrow[d, "\simeq"'] & K_{q+k} \arrow[d, "\simeq"] \\ \Omega K_{q+1} \arrow[r, "\Omega Sq^k_{2q+1}"] & \Omega K_{q+k+1} \end{tikzcd}$$

So Sq^k is an H-map. The H-space structure on K_q represents the addition on $\bar{H}^q$, so it follows that Sq^k is a homomorphism.

The last basic fact about the squares is the Adem Relations, for which we will take an approach using generating functions, following Bullett - MacDonald (Topology 1982).

First, for $u \in \bar{H}^q(X)$ define

$$Sq_x u = \sum x^{-k} Sq^k u \quad \text{where } |x| = 1, \text{ so}$$

Sq_x is homogeneous of degree 0. For example, if $u \in \bar{H}^1(X)$,

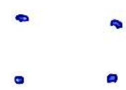
$$Sq_x u = u + x^{-1} u^2 = u^2 (u^{-1} + x^{-1}).$$

This Sq_x occurs naturally in our setting, as our $j^* Pu = x^q Sq_x u$. Sq_x has the nice property that it is a ring homomorphism:

$$Sq_x(u+v) = Sq_x u + Sq_x v$$

$$Sq_x(uv) = (Sq_x u)(Sq_x v) \quad \text{by the Cartan formula.}$$

Second, take Σ_4 to be the symmetric group acting on four letters



and let ω be the subgroup which covers permutations of the rows. If $\pi = \mathbb{Z}/2\mathbb{Z}$, then we have an exact sequence

$$\underset{\substack{\text{permutations}\\ \text{within the}\\ \text{two rows}}}{\pi^2} \longrightarrow \omega \longrightarrow \underset{\substack{\text{interchanging}\\ \text{rows}}}{\pi}$$

ω is sometimes called the "wreath product" $\pi \int \pi$; it is in fact D_8, the 2-sylow subgroup of Σ_4.

What is an ω-space X? X is a π-space with a π-equivariant π^2-action; i.e. we have

$$\underset{\circlearrowleft\,\pi}{\pi^2 \times X} \longrightarrow \underset{\circlearrowleft\,\pi}{X}$$

(the action on this side is the diagonal π action.)

An example of an ω-space is

$$\underset{\substack{\circlearrowleft\\ \pi^2\ \text{antipodal action}}}{\overset{\substack{\pi\ \text{antipodal} \times \text{exchg. of factors.}\\ \circlearrowright}}{E\pi \times (E\pi)^2}}$$

$E\pi \times (E\pi)^2$ is contractible and its ω-action is free; hence it is an $E\omega$, and $B\omega = (E\pi \times (E\pi)^2)/\omega = E\pi \times (B\pi)^2/\pi$

$$= E\pi \times_\pi (B\pi)^2.$$

Now on the one hand, giving a space X a basepoint, we can do the ω-adic construction

$$D_\omega(X) = E\omega_+ \wedge_\omega X^{(4)} = E\omega \times_\omega X^4 / E\omega \times_\omega F_3 X^4 .$$

But this mixing in of X corresponds to mixing in X in the above construction of $E\omega$:

$$E\omega \times_\omega X^4 = E\pi \times_\pi (E\pi \times X^2)^2$$

and this is the iteration of the π-reduced power operation on X.

All this goes to show that

$$D_\omega X = E\omega_+ \wedge_\omega X^{(4)} = D_\pi(D_\pi X),$$

so we have iterated the π-adic construction! That ought to be a good thing, because the Adem relations concern iterated Steenrod operations.

Now we have a diagram

$$\begin{tikzcd}
(X^{(2)})^{(2)} \arrow[d] \arrow[rr, phantom, "\cong"] \arrow[drr, phantom, "\circlearrowright"] & & X^{(4)} \arrow[d] \\
D_\pi(D_\pi X) \arrow[rr, phantom, "\cong"] & & D_\omega X \\
D_\pi(B_{\pi_+}\wedge X) \arrow[u, "D_\pi(j)"] \arrow[rr, phantom, "\circlearrowright"] & & \\
B_{\pi_+}\wedge B_{\pi_+}\wedge X \arrow[u, "j"] \arrow[rr, "\text{induced by inclusion } \pi^2 \subset \omega"'] & & B_{\omega_+}\wedge X \arrow[uu, "j"'] \\
& B_{\pi_+}\wedge B_{\pi_+}\wedge X \arrow[ul, "T\wedge 1"] \arrow[ur, "\text{induced by inclusion } \pi^2 \subset \omega"'] &
\end{tikzcd}$$

flips factors

The claim about this diagram is that the bottom triangle commutes up to homotopy. This is because the flip map T in

$$\begin{tikzcd}
\pi\times\pi \arrow[d, "T"'] \arrow[dr] & & \\
\pi\times\pi \arrow[r] & \omega \arrow[r] & \pi \ni t
\end{tikzcd}$$

$$c_t|_{\pi^2} = T$$

is induced by conjugation by the image of the non-trivial element t of the right-hand π. And it is a basic fact about classifying space that a map $c_t : \omega \to \omega$ induces a map $Bc_t : B\omega \to B\omega$ which is homotopic to the identity (essentially, c_t moves the basepoint). So in some sense the whole point of ω was to convert the

outer automorphism $\pi\times\pi \xrightarrow{I} \pi\times\pi$ to an inner automorphism.
It follows that the map $B\pi_+ \wedge B\pi_+ \wedge X \xrightarrow{j} D_\pi(D_\pi X)$ is up to homotopy π-equivariant, where π flips the factors on the left and leaves the right fixed.

Now if $u \in \bar{H}^q(X)$, we have

$$\begin{array}{ccc} P_\omega u & \in & \bar{H}^{4q}(D_\omega X) \\ \| & & \| \\ P(Pu) & \in & \bar{H}^{4q}(D_\pi D_\pi X) \end{array}$$

Now $j^* D_\pi(j)^* P(Pu) \in \bar{H}^*(B\pi_+ \wedge B\pi_+ \wedge X) \cong \underset{|x|=|y|=1}{\mathbb{F}_2[x,y]} \otimes \bar{H}^*(X)$,

and the homotopy equivariance of the flip action on the $B\pi_+$'s means that this class is symmetric in x any y.

So $\quad j^* D_\pi(j)^* P(Pu) = j^* P(j^* Pu) = j^* P(y^q Sq_y u)$

$$= x^{+2q} Sq_x(y^q Sq_y u)$$

$$= x^{+2q} Sq_x(y^q) \cancel{Sq_x} \; Sq_x Sq_y u$$

$$= x^{2q} y^{2q}(x^{-1}+y^{-1})^q \, Sq_x Sq_y u$$

is symmetric in x and y. This is the shortest statement of the Adem relations.

Last time we got as far as showing $Sq_x Sq_y u = Sq_y Sq_x u$, where $Sq_x = \sum_{i \geq 0} x^{-i} Sq^i$, and $|x| = |y| = 1$, x and y representing 1-dimensional classes in $\tilde{H}^*(\mathbb{RP}^\infty)$. Today we'll see how this gives the more standard form of the Adem Relations.

To begin,

$$Sq_x y = y + x^{-1} y^2 = y^2 (y^{-1} + x^{-1})$$

$$Sq_y x = x + y^{-1} x^2 = x^2 (x^{-1} + y^{-1}).$$

Recall that the Cartan formula implies that Sq_x is a ring homomorphism. So

$$Sq_x Sq_y = Sq_x \Big(\sum_{j \geq 0} y^{-j} Sq^j \Big) = \sum_{j \geq 0} (Sq_x y)^{-j} Sq_x Sq^j$$

$$= \sum_{j \geq 0} y^{-2j} (x^{-1} + y^{-1})^{-j} Sq_x Sq^j .$$

Now this first piece is a mess, so, taking a lesson from calculus, we make a variable substitution $t = y^{-2}(y^{-1} + x^{-1})^{-1}$. So $|t| = -1$, and now

$$Sq_x Sq_y = \sum_{j \geq 0} t^j Sq_x Sq^j = \sum_{j \geq 0} t^j \sum_{i \geq 0} x^{-i} Sq^i Sq^j$$

$$= \sum_{i,j \geq 0} t^j x^{-i} Sq^i Sq^j .$$

Again, letting $s = x^{-1}$, so $|s| = -1$, we have $Sq_x Sq_y = \sum_{i,j \geq 0} s^i t^j Sq^i Sq^j$, a generating function for products of squares.

On the other hand,

$$Sq_y Sq_x = \sum_{j\geq 0} \underbrace{(Sq_y\, x)^{-j}}_{\substack{[x^2(x^{-1}+y^{-1})]^{-j} \\ x^{-2j} y^{2j} t^{j}}} Sq_y Sq^j$$

$$= \sum_{\substack{j\geq 0\\ i\geq 0}} x^{-2j} y^{2j} t^{j}\, y^{-i} Sq^i Sq^j = \sum_{i,j\geq 0} s^{2j} t^{j} y^{2j-i}\, Sq^i Sq^j.$$

We thus need to know how to express y in terms of t and s. Now

$t^{-1} = y^2(x^{-1}+y^{-1}) = y + x^{-1}y^2 = y + sy^2$. At this point we could pull out the quadratic formula, but that seems ill-advised over $\mathbb{F}_2$. Instead, we use the theory of residues: if $f(z)$ is a power series, then the coefficient of z^m in $f(z)$ is the residue of $\frac{f(z)}{z^{m+1}}\,dz$. (Normally, there's a factor of $2\pi i$ or something floating around, but once again this seems ill-advise in $\mathbb{F}_2$). Now we are going to take advantage of the dz to change variables and the claim is that this works for power series over any ring.*

So in our case, we take the coefficient of $(t^{-1})^k$ in y^m:

$$\text{res. } \frac{y^m}{(t^{-1})^{k+1}}\, d(t^{-1}). \quad d(t^{-1}) = dy + s\,dy^2 = dy + \underbrace{2sy\,dy}_{0 \text{ in } \mathbb{F}_2 !}$$

$$= dy.$$

* This is nearly true, nearly false. It does work in this case. See note at end of this lecture.

So we have

$$\operatorname{res} \frac{y^m\,dy}{y^{k+1}(1+sy)^{k+1}} = \operatorname{res} \frac{(1+sy)^{-k-1}\,dy}{y^{k-m+1}} = \text{coefficient of}$$

y^{k-m} in $(1+sy)^{-k-1} = \binom{-k-1}{k-m} s^{k-m}$, where we simply agree that

$(1+z)^m = \sum \binom{m}{k} z^k$ for $m \in \mathbb{Z}$, and for $k<0$, $\binom{m}{k}=0$.

So we know what it is, provided we know what the binomial coefficients are. To sum up, $y^m = \sum_{k\geq m} \binom{-k-1}{k-m} s^{k-m} t^{-k}$.

Hence

$$Sq_y Sq_x = \sum_{i,j\geq 0} s^{2j} t^j y^{2j-i} Sq^i Sq^j$$

$$= \sum_{\substack{i,j,k \\ k\geq 2j-i}} \binom{-k-1}{k-2j+i} s^{k+i} t^{j-k} Sq^i Sq^j.$$

Now in $Sq_x Sq_y$, the coefficient of $s^a t^b$ is $Sq^a Sq^b$. In $Sq_y Sq_x$, set

$k+i=a$,

$j-k=b$. Letting j be the independent variable,

$k=j-b$

$i=a-k= a-j+b$.

So $Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$, and this is the "Adem Relation."

One thing to notice is that if $a\leq 2b$, then since $a\geq 2j$, we have $b\geq j$, so $a+b-j\geq 2j$. So these relations let you express $Sq^a Sq^b$ for $a\leq 2b$ as a sum of $Sq^i Sq^j$ where $i\geq 2j$. Such ~~$Sq^i Sq^j$~~ are $Sq^i Sq^j$ are called <u>admissible</u>.

So the tensor algebra on the squares, $T(Sq^1, Sq^2, \ldots)$, modulo the Adem relations, acts on $H^*(X;\mathbb{Z}/2)$. This algebra $T(Sq^i, \ldots)/\text{Adem Rel'ns}$ is called the "Steenrod Algebra." It is denoted $\mathcal{A}$.

<u>Lemma</u> $\mathcal{A}$ is generated by Sq^{2^i}, and these are indecomposable.

<u>Proof.</u>

① Sq^{2^i} is indecomposable.

After we took all the trouble to maintain homogeneity, we drop it: define $Sq = Sq^0 + Sq^1 + \cdots$. By the Cartan Formula, Sq is a ring homomorphism. In particular, it acts on $H^*(\mathbb{RP}^\infty) = \mathbb{F}_2[x]$, $|x| = 1$.

Now $Sq(x) = x + x^2$, so

$$Sq(x^{2^i}) = [Sq\, x]^{2^i} = (x + x^2)^{2^i} = x^{2^i} + x^{2^{i+1}} \quad \left(\text{because squaring over } \mathbb{F}_2\right)$$

$$= Sq^0\, x^{2^i} + Sq^{2^i} x^{2^i}$$ All lower dimensional squares (except Sq^0) kill x^{2^i}, so Sq^{2^i} is indecomposable.

② Others are decomposable.

The proof is by induction. Suppose Sq^j is decomposable as a $\sum$ product of Sq^{2^i}'s, for $j < c$. Suppose $c \neq 2^i$ for any i, let $k \in \mathbb{Z}$ be so that $2^k < c < 2^{k+1}$. Then

$$Sq^{c-2^k} Sq^{2^k} = \sum_j \binom{2^k - j - 1}{c - 2^k - 2j} Sq^{c-j} Sq^j.$$

$$= \binom{2^k - 1}{c - 2^k} Sq^c + [\text{decomposables}].$$

But

$$\binom{2^k-1}{c-2^k} = \text{coeff. of } z^{c-2^k} \text{ in } (1+z)^{2^k-1}$$

$$= \quad " \quad \frac{(1+z)^{2^k}}{1+z}$$

$$= \quad " \quad \frac{1+z^{2^k}}{1+z} \quad (\text{char. } 2)$$

$$= \quad " \quad 1+\cdots+z^{2^k-1} \quad (\text{char } 2!)$$

$$= \quad 1.$$

So, $$Sq^c = \underset{\substack{\uparrow \\ \text{decomposable}}}{Sq^{c-2^k}} Sq^{2^k} + [\text{decomposables}].$$

$\tilde{H}^*(X)$ is an $\mathcal{Q}$-module, and $Sq^i x = 0$ if $i > \dim x$; such things are called "unstable $\mathcal{Q}$ modules," even though this is bad terminology. Moreover, $x^2 = Sq^q x$, $q = \dim x$, and $Sq(xy) = Sq(x)\,Sq(y)$. Such a thing is called an "unstable $\mathcal{Q}$-algebra." If K is the category of these, then the functor $\tilde{H}^*$: Spaces $\longrightarrow$ graded vector spaces

factors

$$\begin{tikzcd} \tilde{H}^* : \text{Spaces} \arrow[r] \arrow[dr] & K \arrow[d] \\ & \text{graded vector spaces} \end{tikzcd}$$

, and K is the maximal algebraic category through which $\tilde{H}^*$ factors.

Before we stop, let's take a look again at the formula

$$y^m = \sum_{k \geq m} \binom{-k-1}{k-m} s^{k-m} t^{-k}.$$ It follows that

$$y = \sum_{k \geq 1} \binom{-k-1}{k-1} s^{k-1} t^{-k}.$$ On the other hand, we

know $y = t^{-1} + sy^2$. Since squaring is a homomorphism

in $\mathbb{F}_2$, we get

$$y = t^{-1} + sy^2 = t^{-1} + s(t^{-2} + s^2 y^4) = t^{-1} + st^{-2} + s^3(t^{-4} + s^4 y^8) + \dots$$

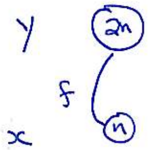
$$= \sum_i t^{-2^i} s^{2^i - 1}$$

which tells you that

$$\binom{-k-1}{k-1} = \begin{cases} 1 & \text{if } k = 2^i \\ 0 & \text{otherwise} \end{cases} \; !$$

Now that we've learned some facts about the Squares, we're in a position to apply them to some of the problems that came up earlier. First, remember from a long time ago the Hopf invariant $\pi_{2n-1} S^n \to \mathbb{Z}$ obtained by

$$S^{2n-1} \xrightarrow{f} S^n \longrightarrow C(f)$$



← the geometry is hard to draw, so high-dimensional cells are written as bubbles!

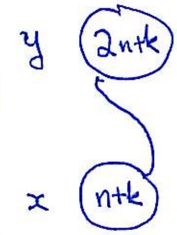
$$\underset{\displaystyle Sq^n x \ \text{(mod 2, anyway)}}{\overset{}{x^2}} = H(f)\, y$$

If $H(f)$ is odd, then reduction mod 2 leaves $H(f)$ non-zero. Now if $n \neq 2^i$, then Sq^n is decomposeable in terms of lower squares, which means there had to be cohomology between n and $2n$ somewhere for these lower squares to hit

, which there certainly isn't. So we get

Theorem (Adem) (He realized as soon as he got the relations that this was a consequence): If there is an element of Hopf invariant odd on S^n, then n is a power of two.

Now if g is any map $S^{2n+k-1} \xrightarrow{g} S^{n+k} \longrightarrow C(g)$



then we no longer have a cup-product that gives possibilities for relations, but we still have Sq^n so we can define $\hat{H}(g)$ by

$$Sq^n x = \hat{H}(g)\, y.$$

The fact that Sq^n commutes with suspensions means

$$\begin{tikzcd} f \arrow[d, mapsto] & \pi_{2n-1} S^n \arrow[r, "H"] \arrow[d] & \mathbb{Z} \arrow[d] \\ \Sigma^k f & \pi_{2n-1+k} S^{n+k} \arrow[r, "\bar{H}"] & \mathbb{Z}/2 \end{tikzcd}$$

commutes. $\pi_{2n-1+k} S^{n+k}$ is independent of k for $k \geq 1$; in other words we have turned the unstable question into a related stable one.

Recall next another fact we had, more directly related to the vector field problem: if S^{n-1} has $k-1$ vector fields, then nL over $\mathbb{RP}^{k-1}$ is fiber homotopy trivial, and this in turn implies the existence of a "coreduction"

$$\begin{tikzcd} T(nL \downarrow \mathbb{RP}^{k-1}) \simeq \Sigma^n \mathbb{RP}^{k-1}_+ \arrow[r] & S^n \\ S^n \arrow[u] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

Now we're in a position to study, using the Squares, when we can split of the S^n from $T(nL \downarrow \mathbb{RP}^{k-1})$. Remember that we found

$$T(nL \downarrow \mathbb{RP}^{k-1}) = \mathbb{RP}^{n+k-1}/\mathbb{RP}^{n-1} \overset{\text{def}}{=} \mathbb{RP}^{n+k-1}_n \quad \text{"stunted projective space."}$$

Its mod 2 cohomology is $\tilde{H}^*(\mathbb{RP}^{n+k-1}_n) = \langle x^n, x^{n+1}, \ldots, x^{n+k-1} \rangle$ where $|x^n| = n$, and it comes in an obvious way from the cohomology of $\mathbb{RP}^{n+k-1}$. Now the coreduction above implies that the class x^n in $\tilde{H}^n(\mathbb{RP}^{n+k-1}_n)$ pulls back from the generator of $\tilde{H}^n(S^n)$. Thus $Sq^{i>0}$ is trivial on x^n since it is on S^n.

Now $x \in \tilde{H}^1(\mathbb{RP}^{n+k-1})$ is a 1-dimensional class, so $Sq\, x = x + x^2$, and we have shown that the fiber homotopy triviality of $nL \downarrow \mathbb{RP}^{k-1}$ implies

$$Sq\, x^n = x^n(1+x^n) \equiv x^n \mod x^{n+k}, \text{ or}$$

$$(1+x)^n \equiv 1 \mod x^k.$$

Write $n = \sum 2^i$, distinct powers of 2. Then

$$(1+x)^n = \prod (1+x)^{2^i} = \prod (1+x^{2^i}) = 1 + x^{2^\nu} + \text{higher powers}$$

← smallest power of 2 occurring in n.

and $(1+x)^n \equiv 1 \mod x^k$ implies that $2^\nu \geq k$.

Looking at a table of $\nu(n)$, $2^{\nu(n)}$, and $\rho(n)$ for small n:

$\nu(n)$	0	1	2	3	4	5
$2^{\nu(n)}$	1	2	4	8	16	32
ρ	1	2	4	8	9	10

where earlier we constructed $\rho(n)-1$ v.fields on S^{n-1}. So we have proved there are at most 15 v.fields on S^{47}, but we've constructed 8.

For $\nu(n) \leq 3$, then, we have obtained an exact answer. But asymptotically, we're doing pretty badly.

Thom Isomorphism and Stiefel-Whitney Classes

There's another approach to this sort of calculation with Thom spaces and squares. Remember that for an S^{n-1}-bundle E, we had

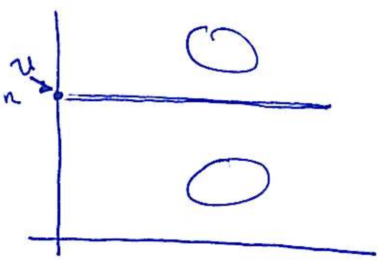
$$\begin{tikzcd} S^{n-1} \arrow[r] & E \arrow[dr] & \subset & C_B E \arrow[dl] \\ & & B & \end{tikzcd}$$

and $T(E) = C_B E / E$. So $\tilde{H}^*(T(E)) = H^*(C_B E, E)$.

Now we can use the relative Serre Spectral Sequence to get a spectral sequence

$$H^*(B, \{H^*(D^n, S^{n-1})\}) \Rightarrow \tilde{H}^*(T(E))$$

But this spectral sequence has E_2 non-zero only at the n-level:



So we get

$$H^*(B, \{H^n(D^n, S^{n-1})\}) \cong \tilde{H}^{*+n}(T(E))$$

Now in general, B is not simply connected, so we need twisted coefficients. But if we have that the system of local coefficients is trivial, then this isomorphism is the Thom isomorphism. An orientation of E is then just that: a choice of trivialization of this local coefficient system.

Now working over $\mathbb{Z}/2$ coefficients, this coeff. system is already canonically trivial, so the concern about orientation doesn't mean anything, and we get $H^*(B; \mathbb{Z}/2) \overset{\text{Thom}}{\cong} \tilde{H}^{*+n}(T(E))$ anyway.

For example, in the case of $nL \downarrow \mathbb{RP}^{k-1}$ we have

$$\begin{aligned} \tilde{H}^*(\mathbb{RP}^{n+k-1}_n) &= \langle x^n, \dots, x^{n+k-1} \rangle \\ & \qquad \updownarrow \qquad\quad \updownarrow \\ H^*(\mathbb{RP}^{k-1}) &= \;\; 1, \dots, x^{k-1} \end{aligned}$$

so the isomorphism comes from mult. by x^n.

This comes out in general from the multiplicative structure of the spectral sequence; namely $H^0(B) \ni 1 \longleftrightarrow \mathcal{U} \in \bar{H}^n(T(E))$ "Thom class."

So $\bar{H}^*(TE) = H^*(C_B E, E)$ is a module over $H^*(B)$ in the sense that

$$\begin{array}{ccc} H^*(B) \otimes H^*(C_B E, E) & \xrightarrow{p^* \otimes 1} & H^*(C_B E) \otimes H^*(C_B E, E) \\ & \searrow & \downarrow \cup \\ & & H^*(C_B E, E) \end{array}$$

$\cup$
this is always called the cup product, i.e. sort of ignoring the pulling back part, since $C_B E$ is a disk bundle.

And the Thom isomorphism says that cup-product by the Thom class $\mathcal{U}$ is an isomorphism

$$\cup \mathcal{U} : H^*(B) \cong \bar{H}^*(TE).$$

This enables us to talk about $\bar{H}^*(TE)$ without much reference to TE except the Thom class. So given a class $y \in \bar{H}^*(TE)$, we write it as $x \cup \mathcal{U}$, $x \in H^*(B)$. Then

$$Sq\, y = Sq\, x \cup Sq\, \mathcal{U}$$

from the Cartan formula, all we need to know is $Sq\, \mathcal{U}$.

Again, in terms of $H^*(B)$, we write

$$Sq^k \mathcal{U} = w_k \cup \mathcal{U}, \quad w_k \in H^k(B).$$

$w_k = w_k(E)$ is called the "kth Stiefel Whitney class of E."

These facts are immediate consequences:

- w_k depends only on the fiber-homotopy type of the sphere bundle.

- $Sq^0 \mathcal{U} = \mathcal{U}$ so $w_0 = 1$.

- $w_k = 0$ for $k > n$.

- Whitney sum formula. (Whitney says this is the hardest theorem he ever proved, but he had the wrong definition of S-W classes).

Namely, $\begin{matrix} S^{p-1} \searrow & \\ & E' \\ & \downarrow \\ & B' \end{matrix}$ $\begin{matrix} S^{q-1} \searrow & \\ & E'' \\ & \downarrow \\ & B'' \end{matrix}$ induce $\begin{matrix} S^{p+q-1} \searrow & \\ & E' * E'' = E \\ & \downarrow \\ & B' \times B'' = B \end{matrix}$

Now $T(E) = T(E') \wedge T(E'')$, and the Thom classes match up because the fibers do, that is

$$\mathcal{U} = \mathcal{U}' \wedge \mathcal{U}''.$$

Thus
$$\underset{\substack{\| \\ w(E) \cup \mathcal{U}}}{Sq\, \mathcal{U}} = \underset{\substack{\| \\ w(E') \wedge w(E'') \cup \mathcal{U}' \wedge \mathcal{U}''}}{Sq(\mathcal{U}' \wedge \mathcal{U}'')} \overset{\text{(Cartan formula)}}{=} Sq\, \mathcal{U}' \wedge Sq\, \mathcal{U}''$$

So $w(E) = w(E') \wedge w(E'')$ by the Cartan Formula.

Finally, $w(n\epsilon) = 1$, so $w(E \oplus n\epsilon) = w(E)$ (now thinking of vector bundles).

Now the Thom class of $nL \downarrow \mathbb{R}P^{k-1}$, as we saw, is x^n.

So $w(L) = 1 + x$, and $w(nL) = (1+x)^n$.

So $nL \downarrow \mathbb{R}P^{k-1}$ is fiber homotopy trivial only if $(1+x)^n \equiv 1 \mod x^k$, which the same result we obtained before.

Now the program is to improve the results so far by pursuing a similar sort of program in KO-theory.

This isn't a course about K-theory; it's a course that uses K-theory. So we're not going to see proofs of lots of basic theorems of K-theory; in particular I'm not going to prove the periodicity theorem. However, we will need some facts which we'll review today.

Recall that we defined the group $\widetilde{KO}(X)$; by doing the same constructions using complex vector bundles, we get $\widetilde{K}(X)$.

Let's see what $\widetilde{K}(S^2)$ is. S^2 splits into contractible hemispheres $D_\pm$ over which any vector bundle is trivial, $D_\pm \times \mathbb{C}^n$. The construction of a vector bundle over S^2 amounts to choosing over each point on the equator S^1 any linear isomorphism $\mathbb{C}^n \to \mathbb{C}^n$ in a continuous manner, i.e. an map $S^1 \longrightarrow GL_n(\mathbb{C})$. So

$$\mathrm{Vect}_{n,\mathbb{C}}(S^2) \cong \pi_1(GL_n(\mathbb{C})).$$

This way of constructing vector bundles is called the "clutching construction." Now the map $GL_n(\mathbb{C}) \xrightarrow{\det.} \mathbb{C}^*$ is an isomorphism on π_1, so $\pi_1(GL_n(\mathbb{C})) = \mathbb{Z}$, and $\widetilde{K}(S^2) \cong \mathbb{Z}$. For $n=1$, the generator of $\pi_1(GL_1(\mathbb{C}))$ can be represented by the embedding of $S^1 \hookrightarrow \mathbb{C}^*$, and from this (with enough thought) it can be seen that the generator of $\widetilde{K}(S^2)$ is $(L-1)$, where L is the tautologous (complex) line bundle over $\mathbb{CP}^1 \cong S^2$, and 1 represents the trivial line bundle.

<u>Claim</u> $\widetilde{KO}(S^8) \cong \mathbb{Z}$. (and in fact, is generated by the tautologous $\mathbb{H}$-line bundle, considered as a 4-dimensional real bundle, over $\mathbb{HP}^2 \cong S^8$.)

Theorem (Bott Periodicity)

$$\tilde{K}(X) \otimes \tilde{K}(S^2) \xrightarrow{\cong} \tilde{K}(X \wedge S^2)$$

$$\widetilde{KO}(X) \otimes \widetilde{KO}(S^8) \xrightarrow{\cong} \widetilde{KO}(X \wedge S^8)$$

A word on this tensor product. An element of $\tilde{K}(X)$ is a "virtual vector bundle" $V - \underline{m}$, where $\underline{m}$ is the trivial bundle of dimension $m = \dim V$. So if $V - \underline{m} \in \tilde{K}(X)$ and $W - \underline{n} \in \tilde{K}(S^2)$, then we defined earlier their exterior tensor product over $X \times S^2$:

$$(V - \underline{m}) \hat{\otimes} (W - \underline{n}) = V \hat{\otimes} W - \underline{m} \hat{\otimes} W - V \hat{\otimes} \underline{n} + \underline{m} \hat{\otimes} \underline{n}.$$

Consider now what this bundle looks like over $X \vee S^2$: on $X \times \{*\}$, W is trivial, so we get $V \hat{\otimes} \underline{n} - \underline{m} \hat{\otimes} \underline{n} - V \hat{\otimes} \underline{n} + \underline{m} \hat{\otimes} \underline{n}$ "$= 0$"; on $\{*\} \times S^2$ V is trivial, so we get $\underline{m} \otimes W - \underline{m} \hat{\otimes} W - \underline{m} \hat{\otimes} \underline{n} + \underline{m} \hat{\otimes} \underline{n}$ "$= 0$". The "$= 0$" here means that in fact, $-V$ is represented by some bundle $+V^\perp$ such that $V \oplus V^\perp$ is trivial, so $V \hat{\otimes} \underline{n} - V \hat{\otimes} n$ $= V \hat{\otimes} n + V^\perp \otimes n = (V + V^\perp) \otimes n =$ trivial. In other words, the bundle $(V - \underline{m}) \hat{\otimes} (W - \underline{n})$ is trivial over $X \vee S^2$, and so pulls back from a vector bundle over $X \wedge S^2$.

This fact means that we have a cohomology theory, which in turn means that we can compute (where "compute" here unfortunately means only that we have spectral sequences). So next let's see how the periodicity theorem can be used to make KO, K the 0th groups of a generalized cohomology theory.

In fact, the first thing we use is that $\widetilde{KO}$ is representable; recall that $\widetilde{KO}(X) = [X, BO\times\mathbb{Z}]_*$ (write B for $BO\times\mathbb{Z}$). One thing we want is long exact sequences. Now it's a general fact that for a map of pointed spaces $A \xrightarrow{f} X$, if we take the mapping cone

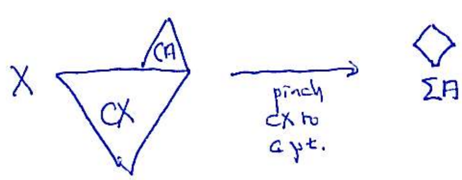
$$A \xrightarrow{f} X \longrightarrow X \cup_f CA = C(f),$$ then the sequence

$$[A,B]_* \longleftarrow [X,B]_* \longleftarrow [X\cup CA, B]_*$$ is exact.

Moreover when B is an H-space the induced homomorphisms of groups are exact. Now we can continue; that is, take the mapping cone of $X \xrightarrow{g} X\cup CA$:

$$A \xrightarrow{f} X \xrightarrow{g} X\cup CA \longrightarrow (X\cup CA)\cup CX \longrightarrow$$

Now if f is nice enough, $X\cup CA$ will be homotopic to X/A. Certainly $(X\cup CA)\cup CX \simeq X\cup CA/X$:



so the sequence becomes

$$A \to X \to X\cup CA \to \Sigma A \to \Sigma X \to \cdots$$

and moreover the map $\Sigma A \to \Sigma X$ can in fact be given by Σf up to homotopy.

This sequence is called the "Barratt-Puppe" sequence.

So we get a long exact sequence

$$[A,B]_* \leftarrow [X,B]_* \leftarrow [X\cup CA, B]_* \xleftarrow{\delta} [\Sigma A, B]_* \leftarrow [\Sigma X, B]_* \leftarrow \cdots$$

and if this is $\tilde{K}^0(A)$ then this should be $\tilde{K}^{-1}A$. So we get all these facts about a generalized cohomology theory for free as soon as we know it's representable. But by the periodicity theorem we know that

$$\widetilde{KO}^0(A) \cong \widetilde{KO}^0(S^8 \wedge A) = \widetilde{KO}^0(\Sigma^8 A) = \widetilde{KO}^{-8}(A)$$

$$\tilde{K}^0(A) \cong \tilde{K}^0(\Sigma^2 A) = \tilde{K}^{-2}(A).$$

So we define

$$\widetilde{KO}^{-n}(X) = \widetilde{KO}(\Sigma^n X)$$
$$\widetilde{KO}^{n}(X) = \widetilde{KO}(\Sigma^{8k-n} X),\ 8k>n$$
$$KO^*(X) = \widetilde{KO}^*(X_+)$$

and similarly for $\tilde{K}^n(X)$.

Now its an important fact about $\tilde{K}^*(X)$ that it's a commutative ring:

$\tilde{K}(S^2) \cong \mathbb{Z}$ with generator $(L-1)$

$\tilde{K}^{-2}(*) \ni x$ "periodicity element", so $K^*(*) = \mathbb{Z}[x^{\pm 1}]$ Laurent series ring, where x has degree -2.

Similarly, there is a ring structure for $\widetilde{KO}^*$:

n	0	1	2	3	4	5	6	7	8	$\uparrow$
$\widetilde{KO}^{-n}(*)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	
generator	1	η	η^2		q				p	

and $KO^* = \mathbb{Z}[\eta, q, p, p^{-1}] \Big/ \left\{ \begin{matrix} 2\eta = 0 & q^2 = 2p \\ \eta^3 = 0 & \\ \eta q = 0 & \end{matrix} \right\}$.

Unfortunately, we shall need more than just the ring structure; we'll need operations. Now one way to get operations in K-theory is to look for operations on vector spaces, apply these fiber-wise to get operations on vector bundles, and then squeeze out operations on K-theory.

The most useful of these is $V \longrightarrow \Lambda^k(V)$, the k^{th} exterior power. By means of this indeed we get a k^{th} exterior power bundle

$$\begin{matrix} E & & \Lambda^k E \\ \downarrow & \longrightarrow & \downarrow \\ B & & B \end{matrix}.$$

Unfortunately extending to K-theory is hard because Λ^k is not additive. But there is a natural isomorphism $\Lambda^k(V \oplus W) = \bigoplus_{i+j=k} \Lambda^i(V) \otimes \Lambda^j(W)$. This (which looks like a Cartan Formula) inspires the definition

$$\Lambda_t(V) = \sum_{i \geq 0} t^i \Lambda^i(V). \quad \text{Now} \quad \Lambda_t(V \oplus W) = \Lambda_t(V) \cdot \Lambda_t(W).$$

So the operation still isn't additive, but it's exponential -- it takes sums to products; that's good enough, as it turns out. So Λ_t is a homomorphism of abelian groups

$$\Lambda_t : \mathrm{Vect}(X) \longrightarrow \underbrace{1 + KO(X)[[t]]^*}_{\left\{ \begin{matrix} \text{power series in } t \text{ with} \\ \text{leading term 1 and coefficients} \\ \text{in } KO(X) \end{matrix} \right\}}$$

But by definition of KO(X), there is a $\overset{\text{group homomorphism}}{\text{map}}$ $\lambda_t : KO(X) \longrightarrow 1 + KO(X)[[t]]^*$ such that the map Λ_t factors

$$\begin{tikzcd} \mathrm{Vect}(X) \arrow[r, "\Lambda_t"] \arrow[d] & 1 + KO(X)[[t]]^* \\ KO(X) \arrow[ur, "\lambda_t"'] & \end{tikzcd}$$

i.e. for $[E], [F] \in KO(X)$,

$$\lambda_t([E]+[F]) = \lambda_t([E]) \cdot \lambda_t([F]),$$ like total Steenrod operation.

Of course if E is a vector bundle, then $\lambda^j(E) = \Lambda^j(E) = 0$ for $j > \dim E$ (where λ^j is defined, of course, by $\lambda_t(E) = 1 + \sum_{k\geq 1} t^k \lambda^k(E)$.)

The λ operations are all very well, but they are hard to work with because they depend on the ring structure of KO(X). We really want an additive operation, so one thing to try is to take a logarithm, in search of a family of operations ψ^k such that on line bundles, $\psi^k(L) = L^{\otimes k}$. Once again, we start with a generating function $\psi_t(x) = \sum_{k\geq 1} \psi^k(x) t^k$; then our conditions on ψ^k mean:

$$\psi_t(x+y) = \psi_t(x) + \psi_t(y)$$

$$\psi_t(L) = \sum_{k\geq 1} L^k t^k = \frac{Lt}{1-Lt}.$$

The 1st obvious candidate is $\log \Lambda_t$, but unfortunately this has denominators, and we don't know what $\frac{1}{n}$ means in KO(X). So the next possibility is

$$\frac{d}{dt} \log \lambda_t(x) = \frac{\frac{d}{dt}\lambda_t(x)}{\lambda_t(x)},$$ and $\lambda_t(x)^{-1}$ is well-defined in $1 + KO(X)[[t]]^*$,

and so we have the first property. Now for L a line bundle,

$$\frac{d}{dt}\log(1+tL) = \frac{L}{1+tL}.$$

Fix with $-t$ to get $\frac{d}{dt}\log(1-tL) = \frac{-L}{1-tL}$. Finally, multiply by $-t$ to get $\frac{tL}{1+tL}$. So the coefficient of t^k in

$$\frac{-t\frac{d}{dt}\lambda_{-t}(E)}{\lambda_{-t}(E)}$$

is the k^{th} "Adams Operation" ψ^k.

To finish for today, the theorem is:

There exist unique operations $\psi^k : KO(X) \to KO(X)$ such that

- ψ^k is a ring homomorphism for each k.
- $\psi^k\psi^l = \psi^{kl}$
- $\psi^k(L) = L^{\otimes k}$.

Last time we discussed the ring $KO^* = KO^*(pt)$ and said it was

$$KO^* = \mathbb{Z}[\eta, q, p^{\pm 1}], \quad |\eta| = -1, \ |q| = -4, \ |p| = -8,$$

$$\text{and } 2\eta = 0, \ \eta q = 0, \ \eta^3 = 0, \text{ and } q^2 = 4p.$$

Furthermore we defined operations ψ^k on $KO(X)$ which are ring homomorphisms and satisfy

$$\psi^k \psi^l = \psi^{kl}$$

$$\psi^k(L) = L^{\otimes k}, \quad L \text{ a line bundle.}$$

In order to get a better handle on these, we are going to need some character theory, so today will be a few words on representation theory.

Let G be a compact Lie group (e.g. $G = U(n)$, the group of $n \times n$ unitary matrices); then a representation of G is a finite-dimensional complex vector space V together with a linear action of G; two representations V and V' are isomorphic if and only if there is a G-equivariant linear isomorphism $\alpha : V \xrightarrow{\cong} V'$!

Choose a Hermitian inner product on V. By averaging, we can make a G-invariant inner product $\langle , \rangle$: $\langle gx, gy \rangle = \langle x, y \rangle$. So g is unitary. Picking an orthonormal basis gives g as a unitary matrix, and the representation becomes a map $G \xrightarrow{\rho} U(n)$, where ρ is a continuous homomorphism.

With these data, an isomorphism of representations looks like:

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d] & V' \arrow[d] \\ \langle\, , \rangle \arrow[d] & \langle\, , \rangle' \arrow[d] \\ \text{basis,} & \text{basis,} \\ \rho: G \to U(n) & \rho': G \to U(n) \end{tikzcd}$$

α is given by matrix M on these bases, and

$$M\rho(g) = \rho'(g) M$$
$$\Downarrow$$
$$\rho'(g) = M\rho(g)M^{-1} \text{ for all } g.$$

Okay, once we have matrices it makes sense to talk about the trace;

define $tr(\rho(g)) = \chi_V(g) \equiv$ character of V, a function $\chi_V: G \to \mathbb{C}$.

And χ_V is called a "class function" because it is constant on conjugacy classes in G. In particular if two representations V and V' are isomorphic, then $\chi_V = \chi_{V'}$.

χ is by far the easiest way to understand representations, and in fact it tells all: define $Rep_n(G) = Hom(G, U(n))/\text{conjugacy}$.

Then $Rep(G) = \coprod_n Rep_n(G)$ is a monoid, so we get

by $\Big\downarrow$ Grothendieck

$R(G)$, the "representation ring" of G, and the map $V \to \chi_V$ induces a map $\chi: R(G) \to \left\{ \begin{matrix} \text{class functions} \\ G \to \mathbb{C} \end{matrix} \right\}$.

<u>Facts</u>

$$\chi(V \oplus W) = \chi_V + \chi_W$$
$$\chi_{V \otimes W} = \chi_V \chi_W$$

The second fact is true because $\operatorname{tr}(A \otimes B) = \operatorname{tr} A \operatorname{tr} B$ for matrices A, B;

i.e. for $A \in \mathcal{U}(m)$
$B \in \mathcal{U}(n)$, identify $\mathcal{U}(m)$ with $\operatorname{Aut}(\mathbb{C}^m)$
$\mathcal{U}(n)$ with $\operatorname{Aut}(\mathbb{C}^n)$; then

$$A \otimes B \in \operatorname{Aut}(\mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}^n).$$

Now any unitary matrix has a diagonalization ("this is the deepest fact of this lecture"), and if $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ are bases of eigenvectors for A and B, then $\{x_i \otimes y_j\}$ are a basis of eigenvectors for $A \otimes B$, and the eigenvalue for $x_i \otimes y_j$ = $\{$eigenvector for $x_i\}$ $\cdot$ $\{$eigenvector for $y_j\}$.

Fact χ is 1-1; in fact $R(G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong}$ Class functions).

Recall that the 1st construction of the Adams operations used an operation on vector spaces to yield an operation on bundles, and finally, an operation on K-theory. Now we shall use representations to construct new vector bundles out of old ones. First we will need the notion of the principal bundle associated to a vector bundle:

if $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ is an n-dim complex vector bundle and X is paracompact, you can pick a Hermitian metric on E, and get an orthonormal basis on each fiber (~~?~~ meaning, a continuously varying choice of orthonormal basis).

Let $P(E) = \left\{ \begin{array}{c} \text{set of all ordered} \\ \text{orthonormal bases} \\ \text{on fibers} \end{array} \right\}$. Another way to think about it is that each point in $P(E)$ is a linear isometry $\mathbb{C}^n \cong E_x$.

Now there's a projection $P(E) \longrightarrow X$; there is a fiberwise action of $\mathcal{U}(n)$ on $P(E)$ given by composition: if $\alpha \in P(E)$ determines $\mathbb{C}^n \xrightarrow[\cong]{\alpha} E_x$, then we get for $g \in \mathcal{U}(n)$

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\alpha", "\cong"'] & E_x \\ \mathbb{C}^n \arrow[u, "g", "\cong"'] \arrow[ur, "\cong", "\alpha g"'] & \end{tikzcd}$$

Moreover it is clear that this action gives $P(E) \downarrow X$ the structure of a principal $\mathcal{U}(n)$-bundle, the "associated principal bundle to E."

Now the construction of $P(E)$ involved the choice of a Hermitian metric $\langle , \rangle$ for $E \downarrow X$. But if $\langle , \rangle'$ is another Hermitian metric for $E \downarrow X$, then so is $t\langle , \rangle' + (1-t)\langle , \rangle = \langle , \rangle_t^{*}$, for $0 \le t \le 1$, and thus we get a metric for the bundle $E \times I \downarrow X \times I$. If you look at $P(E \times I) \downarrow X \times I$, the restriction to $X \times \{0\}$ is $P_{\langle,\rangle}(E)$, and the restriction to $X \times \{1\}$ is $P_{\langle,\rangle'}(E)$, so $P_{\langle,\rangle}(E) \cong P_{\langle,\rangle'}(E)$ as $\mathcal{U}(n)$-bundles.

Now let V be a representation of $\mathcal{U}(n)$. Then from a complex vector bundle $E \downarrow X$ we can form

$$\begin{array}{ccc} E & & P(E) \times_{\mathcal{U}(n)} V = \alpha_V(E), \\ \downarrow & \longmapsto & \downarrow \\ X & & X \end{array}$$

yielding a new vector bundle with fiber V.

---

* A Hermitian inner product is required to satisfy $\langle x,x \rangle > 0$ for $x \neq 0$; this provides the non-~~de~~degeneracy of $\langle , \rangle_t$.

Clearly $\alpha_{V\oplus W}(E) = \alpha_V(E) \oplus \alpha_W(E)$, so for a fixed vector bundle E, $V \mapsto \alpha_V(E)$ defines an additive homomorphism $\mathrm{Rep}(\mathcal{U}(n)) \longrightarrow \mathrm{Vect}(X) \longrightarrow K(X)$, so by the universality property of the Grothendieck construction this ~~factors~~ extends through $R(\mathcal{U}(n))$:

$$\begin{tikzcd} \mathrm{Rep}(\mathcal{U}(n)) \arrow[r, "\alpha_{-}(E)"] \arrow[d] & K(X) \\ R(\mathcal{U}(n)) \arrow[ur] & \end{tikzcd} \qquad \theta \mapsto \theta(E) \in K(X)$$

For $\theta \in R(\mathcal{U}(n))$, denote its image by $\theta(E)$

Turning this around, a fixed $\theta \in R(\mathcal{U}(n))$ assigns to an n-dimensional vector bundle $E \downarrow X$ a well-defined element $\theta(E)$ of $K(X)$; what we really want is to assign to an element E of $\underline{K(X)}$ another element of $K(X)$. In particular, we have to worry about different values of n; moreover we want to get an additive homomorphism. In other words, we want to choose $\theta_n \in R(\mathcal{U}(n))$ for $n \geq 0$ in such a way that

$$\theta_m(E^m) \oplus \theta_n(F^n) = \theta_{m+n}(E \oplus F) \quad \text{in } K(X).$$

Direct sum gives a homomorphism $R(\mathcal{U}(m)) \times R(\mathcal{U}(n)) \xrightarrow{\oplus} R(\mathcal{U}(m) \times \mathcal{U}(n))$; one way to think of it is that the projections $\mathcal{U}(m) \xleftarrow{pr_1} \mathcal{U}(m) \times \mathcal{U}(n) \xrightarrow{pr_2}$ yield pull-backs of characters, and

$$\theta_m \oplus \theta_n = pr_1^* \theta_m + pr_2^* \theta_n,$$

so $\chi_{\theta_m \oplus \theta_n}(M, N) = \chi_{\theta_m}(M) + \chi_{\theta_n}(N)$.

On the other hand the map $\mathcal{U}(m) \times \mathcal{U}(n) \xrightarrow{\sigma} \mathcal{U}(m+n)$ given by

$$(M, N) \longmapsto \left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right]$$

defines a pull-back $\sigma^* : R(\mathcal{U}(m+n)) \to R(\mathcal{U}(m) \times \mathcal{U}(n))$

defined by

$$\chi_{\sigma^* \theta_{m+n}}(M,N) = \chi_{\theta_{m+n}}\left(\left[\begin{array}{c|c} M & \\ \hline & N \end{array}\right]\right).$$

<u>Definition</u> $\theta = \{\theta_n \in R(\mathcal{U}(n)),\ n \geq 0\}$ is an <u>additive sequence</u> if and only if $\theta_m \oplus \theta_n = \sigma^* \theta_{m+n}$ for all $m,n \geq 0$.

<u>Claim</u> If θ is additive then $\theta_m(E^m) \oplus \theta_n(F^n) = \theta_{m+n}(E \oplus F)$.

"Proof:" the "proof" pretends that θ_m corresponds to a genuine representation V_m. Then

$$\theta_{m+n}(E \oplus F) = P(E\oplus F) \times_{\mathcal{U}(m+n)} V_{m+n}.$$

Now it takes some thought, but is in fact true that

$$P(E\oplus F) = \left(P(E) \times_X P(F)\right) \times_{\mathcal{U}(m)\times\mathcal{U}(n)} \mathcal{U}(m+n), \text{ so}$$

$$\theta_{m+n}(E\oplus F) = \left(P(E) \times_X P(F) \times_{\mathcal{U}(m)\times\mathcal{U}(n)} \mathcal{U}(m+n)\right) \times_{\mathcal{U}(m+n)} V_{m+n}$$

$$= P(E) \times_X P(F) \times_{\mathcal{U}(m)\times\mathcal{U}(n)} \underbrace{\sigma^* V_{m+n}}_{V_m \oplus V_n \text{ by additivity.}}$$

$$= \left(P(E) \times_{\mathcal{U}(m)} V_m\right) \oplus \left(P(F) \times_{\mathcal{U}(n)} V_n\right) = \theta_m(E) \oplus \theta_n(F).$$

So an additive sequence θ defines an additive homomorphism $\coprod_n \mathrm{Vect}_n(X) \longrightarrow K(X)$, and so extends to give an operation

$$\begin{tikzcd} \coprod_n \mathrm{Vect}_n(X) \arrow[r, "\theta"] \arrow[d] & K(X) \\ K(X) \arrow[ur, "\theta"'] & \end{tikzcd}$$

This is how we shall present the Adams operations ψ^k;

That is, next time we shall prove

Theorem* For $k \geq 1$, there is a unique additive sequence
$\psi^k = \{\psi^k_n \in R(U(n))\}$ such that ψ^k_1 is the k^{th} power map, i.e.

$$\chi_{\psi^k_1}(z) = z^k, \quad z \in U(1).$$

Remark Note that the additive condition requires $\theta_0 = 0$.

*Warning! The uniqueness assertion is false for real KO-theory. See note at end of Wednesday's lecture.

Recall that for a compact Lie group G we identified a representation of G with a continuous homomorphism $G \xrightarrow{\rho} \mathcal{U}(n)$ which allowed us to speak of the class function χ_ρ, the "character" of the representation. The induced map

$$R(G) \overset{\chi}{\hookrightarrow} \left\{ \begin{matrix} \text{class functions} \\ G \to \mathbb{C} \end{matrix} \right\}$$ is injective, so we identify

ρ with its image under χ.

Recall also that a sequence of virtual representations $\theta_\cdot = \{\theta_n \in R(\mathcal{U}(n))\}$ is <u>additive</u> if

$$\theta_m \oplus \theta_n = \sigma^* \theta_{m+n}.$$

The additional fact to bring to bear now is that any $M \in \mathcal{U}(m+n)$ is conjugate to a diagonal matrix $\begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_{m+n} \end{bmatrix}$, and this diagonal matrix is in the image of $\mathcal{U}(m) \times \mathcal{U}(n) \hookrightarrow \mathcal{U}(m+n)$. It follows that θ_{m+n} (thought of as a class function) is determined by θ_m and θ_n, so the entire additive sequence is determined by θ_1! Now $\theta_1 \in R(\mathcal{U}(1))$ and $R(\mathcal{U}(1)) \cong \mathbb{Z}[z^{\pm 1}]$ is a Laurent series ring on the tautologous representation of $\mathcal{U}(1)$. So we start by defining $\psi_1^k = z^k$, $k > 0$. It is now fairly clear how to go about

<u>Claim</u> There is a unique additive sequence ψ^k with $\psi^k_1 = z^k$.

Namely, what is $\psi_n^k(M)$?

$M \in \mathcal{U}(n)$ is conjugate to a diagonal $\begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}$.

Then $\psi_n^k(M) = \psi_n^k\left(\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}\right) = \sum_j \psi_1^k[z_j] = \sum_{j=1}^n z_j^k = \mathrm{tr}(M^k)$.

The first thing to notice about this expression is that it's symmetric in the z_j's, so we should think about symmetric polynomials:

The n^{th} symmetric group Σ_n has a canonical action on $\mathbb{Z}[z_1, \ldots, z_n]$, and it's a fact that the invariants have the famous form

$\mathbb{Z}[z_1, \ldots, z_n]^{\Sigma_n} = \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$, where the σ_i are

determined by $\prod_{j=1}^n (1 + z_j x) = \sum_{j=0}^n \sigma_j x^j$,

so σ_j is homogeneous of degree j,

$$\sigma_j = \sum_{1 \le i_1 < \cdots < i_j \le n} z_{i_1} \cdots z_{i_j}.$$

In particular

$$\sum_{j=1}^n z_j^k = s_k(\sigma_1, \ldots, \sigma_n)$$

The s_k are the so-called "Newton Polynomials" and it's good entertainment to work out a few of these.

Now we want to show that that these σ_i are real characters, i.e. that they come from genuine representations. In fact,

$\sigma_j = \chi_{\Lambda^j}$, where Λ^j is the jth exterior power of the canonical representation. For if a matrix M has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the above expression gives $\sigma_j(M) = \sum_{1 \le i_1 < \cdots < i_j \le n} \lambda_{i_1} \cdots \lambda_{i_j}$. On the other hand, if $v_1, \ldots, v_n$ are a basis of corresponding eigenvectors, then

$\{ v_{i_1} \wedge \dots \wedge v_{i_j} \mid 1 \le i_1 < \dots < i_j \le n \}$ are a basis of eigenvectors for the induced action of M on the jth exterior power of the representation, and these have eigenvalues $\lambda_{i_1} \cdots \lambda_{i_j}$.

Note that s_k will in general involve subtractions, so it is not true that ψ^k_n comes from a genuine representation, but it does come from $s_k(\Lambda^1, \dots, \Lambda^k)$. This is comforting, since the definition of the Adams operations from before used the exterior product and its properties in an essential way.

Before going on, we should check that ψ^k is additive; that's easy; we only have to do it for diagonal matrices, and

$$\psi^k_{m+n}(M) = \psi^k_{m+n} \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_{m+n} \end{bmatrix} = \sum_{i \le m} z_i^k + \sum_{i > m} z_i^k$$

$$= \psi^k_m \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_m \end{bmatrix} + \psi^k_n \begin{bmatrix} z_{m+1} & & 0 \\ & \ddots & \\ 0 & & z_{m+n} \end{bmatrix}.$$

So now we're where we were with the last definition of ψ^k, but we know ψ^k well enough to attack the product structure of $K(X)$.

$$R(\mathcal{U}(m)) \otimes R(\mathcal{U}(n)) \xrightarrow{\otimes} R(\mathcal{U}(m) \times \mathcal{U}(n))$$ is defined by $\theta_m \otimes \theta_n (M, N) = \theta_m(M)\, \theta_n(N)$. Moreover, there is a map $R(\mathcal{U}(mn)) \xrightarrow{\tau^*} R(\mathcal{U}(m) \times \mathcal{U}(n))$, induced by a map $\tau: \mathcal{U}(m) \times \mathcal{U}(n) \longrightarrow \mathcal{U}(mn)$, by identifying $\mathbb{C}^{mn}$ with $\mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}^n$ and hence $\mathrm{Aut}(\mathbb{C}^{mn})$ with $\mathrm{Aut}(\mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}^n)$. Note that τ is only well-defined up to

conjugacy, since these identifications depend on a choice of ordering of the basis $e_i \otimes f_j$ corresponding to bases e_i for $\mathbb{C}^m$ and f_j for $\mathbb{C}^n$. But τ^* is well-defined on class functions. Note that if

$$\begin{matrix} M \\ N \end{matrix} \text{ have eigenvalues } \begin{matrix} s_1, \dots, s_m \\ t_1, \dots, t_n \end{matrix} \text{ then } \tau(M,N) \text{ has eigenvalues } s_i t_j.$$

A sequence of $\theta_m \in R(\mathcal{U}(m))$ is <u>multiplicative</u> if

$$\theta_m \otimes \theta_n = \tau^*(\theta_{mn}).$$

In other words, we are following a line of argument analogous to that we just did for additive structure; and an analogous result holds:

<u>Theorem</u> If θ is additive and multiplicative, then it defines an operation $K(X) \to K(X)$ which is a ring homomorphism.

<u>Proof</u>: the proof is exactly analogous, depending this time on the identification

$$P(E \otimes F) = \left(P(E) \times_X P(F)\right) \times_{\mathcal{U}(m) \times \mathcal{U}(n)} \mathcal{U}(mn).$$

And, luckily,

<u>Lemma</u> ψ^k is multiplicative

<u>Proof</u>:

$$\tau^* \psi^k_{mn}(M,N) = \psi^k_{mn}(\tau(M,N)) = \sum_{i,j} (s_i t_j)^k = \left(\sum_i s_i^k\right)\left(\sum_j t^k\right)$$
$$= \psi^k_m(M)\, \psi^k_n(N).$$

Summarizing, given an additive and multiplicative sequence $\theta_n \in R(\mathcal{U}(n))$, $n \geq 0$, we get a ring homomorphism $\hat{\theta} : K(X) \longrightarrow K(X)$, defined, when $E \in K(X)$ is in fact an n-dimensional bundle and θ_n corresponds to a genuine representation V_n by $\hat{\theta}(E) = P(E) \times_{\mathcal{U}(n)} V_n$. Moreover we have ψ^k additive ~~seq~~ and multiplicative sequences, defined in terms of their images under $\chi : R(\mathcal{U}(n)) \longrightarrow \left\{ \begin{array}{c} \text{class functions} \\ \mathcal{U}(n) \to \mathbb{C} \end{array} \right\}$, satisfying $\psi_1^k(z) = z^k$. Note that for a line bundle L, it follows that $\hat{\psi}^k(L) = L^{\otimes k}$, since ψ_1^k corresponds to the k^{th} power of the canonical representation of $\mathcal{U}(1)$. So we have nearly all the properties of the Adams Operations listed two times ago; we still need to show the compositional property $\psi^k \psi^l = \psi^{kl}$.

Now if φ and θ are additive sequences they define operations $\hat{\varphi}$ and $\hat{\theta}$ on $K(X)$; certainly $\hat{\varphi} \circ \hat{\theta}$ is another one. But in order to pursue the program here, we must realize $\hat{\varphi} \circ \hat{\theta}$ as $\hat{\xi}$, i.e. induced by some additive sequence ξ. In other words, how can we compose $\varphi(\theta)$ to give another additive sequence ξ?

If we start with an additive sequence $\varphi = \varphi_n$ and a $\theta \in R(G)$, then when θ corresponds to a genuine representation $\rho : G \longrightarrow \mathcal{U}(n)$, there is an induced map

$$
\begin{array}{ccc}
R(G) & \xleftarrow{\rho^*} & R(\mathcal{U}(n)) \\
 & & \cup \\
\rho^* \varphi_n & \longleftarrow & \varphi_n \\
\| & & \\
\text{“}\varphi(\rho)\text{”} & &
\end{array}
$$

<u>Claim</u> This proposal defines an additive natural transformation

$R(G) \to R(G)$.

<u>Proof</u>: it's a matter of working through the definitions. If $\rho_m : G \to \mathcal{U}(m) \in Rep_m(G)$ and $\rho_n : G \to \mathcal{U}(n) \in Rep_n(G)$, then the action on $\mathbb{C}^m \oplus \mathbb{C}^n \cong \mathbb{C}^{m+n}$ defines $\rho_m + \rho_n \in Rep_{m+n}(G)$.

Then

$$\rho_m^* \varphi_m(M) + \rho_n^* \varphi_n(M) = \varphi_m(\underset{\mathcal{U}(m)}{\underbrace{\rho_m M}}) + \varphi_n(\underset{\mathcal{U}(n)}{\underbrace{\rho_n M}})$$

$$= \varphi_{m+n}\left[\begin{array}{c|c} \rho_m M & 0 \\ \hline 0 & \rho_n M \end{array}\right] \quad \text{by additivity of } \varphi$$

$$= (\rho_m + \rho_n)^* \varphi_{m+n}(M).$$

But then φ defines an additive homomorphism, so it extends:

$$\begin{tikzcd} \coprod_n Rep_n(G) \arrow[r, "\varphi"] \arrow[d] & R(G) \\ R(G) \arrow[ur, "\exists \varphi"'] & \end{tikzcd}$$

So if $\varphi = \{\varphi_n\}$ is an additive sequence, and $\theta_* = \{\theta_n\}$ is another additive sequence, then $\varphi(\theta) \overset{\text{def}}{=} \{\varphi(\theta_n) \in R(\mathcal{U}(n))\}$ is another sequence, and we have:

<u>Lemma</u>: Let $\varphi = \{\varphi_n\}$ and $\theta = \{\theta_n\}$ be additive sequences

1) $\varphi(\theta) = \{\varphi(\theta_n) \in R(\mathcal{U}(n))\}$ is an additive sequence.

2) As operators on K-theory, $\widehat{\varphi(\theta)} = \hat{\varphi} \circ \hat{\theta}$.

3) $\psi^k(\psi^l_n) = \psi^{kl}_n$, so $\psi^k(\psi^l) = \psi^{kl}$.

"Proof:"

1) Once again, pretend that θ_n comes from a real representation, and think of it as a Lie group map $\mathcal{U}(n) \longrightarrow \mathcal{U}(n)$.
For $M \in \mathcal{U}(m)$ and $N \in \mathcal{U}(n)$,

$$\sigma^* \varphi(\theta_{m+n})(M,N) = \varphi(\theta_{m+n})\left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right]$$

$$= \varphi_{m+n}\left\{\theta_{m+n}\left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right]\right\}$$

$$= \varphi_{m+n}\left\{\left[\begin{array}{c|c} \theta_m M & 0 \\ \hline 0 & \theta_n N \end{array}\right]\right\} \quad (\text{by additivity of } \theta)$$

$$= \varphi_m \theta_m(M) + \varphi_n \theta_n(N) \quad (\text{by additivity of } \varphi)$$

$$= \varphi(\theta_m) \oplus \varphi(\theta_n)\ (M,N).$$

2) Now, pretend θ_n and φ_n both come from genuine representations, and think of these as Lie group maps $\mathcal{U}(n) \to \mathcal{U}(n)$. Denote by $\theta_n^* \mathbb{C}^n$, for example, the vector space equipped with this action, i.e. think of $\mathbb{C}^n$ as the tautologous representation of $\mathcal{U}(n)$, and for $v \in \theta_n^* \mathbb{C}$, $g \in \mathcal{U}(n)$, $g(v) = \theta(g) \cdot v$. Then by definition, for a vector bundle E, $\hat{\theta}_n(E) = P(E) \times_{\mathcal{U}(n)} \theta_n^* \mathbb{C}^n$.
Now a bundle is determined by its transition functions w.r.t. some open cover; the point of this construction is that the bundle E having transition functions $g_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathcal{U}(n)$ with respect to some open cover $\{V_\alpha\}_{\alpha \in I}$ is replaced by a bundle defined w.r.t. this open cover by transition functions
$V_\alpha \cap V_\beta \xrightarrow{g_{\alpha\beta}} \mathcal{U}(n) \xrightarrow{\theta_n} \mathcal{U}(n)$. From this point of view it's clear that $\hat{\varphi} \circ \hat{\theta}(E) = P(P(E) \times_{\mathcal{U}(n)} \theta_n^* \mathbb{C}^n) \times_{\mathcal{U}(n)} \varphi^* \mathbb{C}^n$ is the bundle with transition functions $V_\alpha \cap V_\beta \xrightarrow{g_{\alpha\beta}} \mathcal{U}(n) \xrightarrow{\theta_n} \mathcal{U}(n) \xrightarrow{\varphi_n} \mathcal{U}(n)$, which is the same as $\widehat{\varphi(\theta)}(E) = P(E) \times_{\mathcal{U}(n)} \varphi_n^* \theta_n^* \mathbb{C}^n$.

Finally, a word on 3). $\psi^k(\psi^\ell_n)$ is hard to compute directly because, for arbitrary n, ψ^ℓ_n does not come from a real representation. However, ψ^ℓ_1 is a representation, and

$$\psi^k(\psi^\ell_1)(z) = \psi^{\ell *}_1 \psi^k_1(z) = z^{\ell k} = \psi^{k\ell}_1(z).$$

Since an additive sequence is determined by its first element, and since $\psi^k(\psi^\ell_n)$ and $\psi^{k\ell}$ are both additive, it follows that $\psi^k \psi^\ell = \psi^{k\ell}$.

So, now we have all the facts about operations in complex K-theory; however, we don't know about the situation for KO yet. In fact, we have to correct a statement from Friday, that the Adams operations are uniquely characterized by

$\psi^k(L) = L^{\otimes k}$ for line bundle

additivity, and $\psi^{k\ell} = \psi^k \psi^\ell$ in KO-theory.

This statement is false: Define $\psi^k(E) = \begin{cases} E & \text{if } k \text{ is odd} \\ \psi^0(E) & k \text{ is even} \end{cases}$

where $\psi^0(E)$ = trivial bundle of dimension = dim E over basept. in X. This works, since for a real line bundle, since $L^* \cong L$, we get

$$L^{\otimes 2} \cong L \otimes L^* \cong \mathrm{Hom}(L, L) \ni 1,$$ which bundle sections, and so $L^{\otimes 2}$ is trivial.

<u>Instead</u>, operations on KO can be obtained from the complex case: use $O(n)$, and for a compact Lie group G, use $RO(G)$ the ring of virtual <u>real</u> representations. Complexification of real representations gives a map $RO(G) \hookrightarrow R(G)$ which is monic, and the diagram

$$\begin{tikzcd} RO(G) \arrow[r, hook, "c"] \arrow[d, "\chi"] & R(G) \arrow[d, "\chi"] \\ \left\{\begin{matrix}\mathbb{R}\text{-class} \\ \text{fns.}\end{matrix}\right\} \arrow[r, hook] & \{\mathbb{C}\text{-class fns}\} \end{tikzcd}$$

commutes. Moreover, there is the inclusion $O(n) \overset{i}{\hookrightarrow} U(n)$ which induces $R(U(n)) \xrightarrow{i^*} R(O(n))$. The real Adams operations come from ~~...~~ additive RO-sequences $\{\psi^k_{\mathbb{R},n}, n \geq 0\}$:

$$\begin{tikzcd} & RO(O(n)) \ni \psi^k_{\mathbb{R},n} = s_k(\Lambda^1, \dots, \Lambda^n) \arrow[d, hook] \\ R(U(n)) \arrow[r, "i^*"] & R(O(n)) \\ \psi^k_n \arrow[r, mapsto] & i^* \psi^k_n \end{tikzcd}$$

Λ^j = jth exterior power of standard representation.

Okay, today we prove Hopf invariant 1.

First, notice this fact about the Adams operations:

<u>Lemma</u> For p prime, $\psi^p(x) \equiv x^p \pmod p$ in $K(X)$ (so think of ψ^p as an improvement of the x^p map that is a ring homomorphism).

<u>Proof</u>: Suppose x is a vector bundle E. Then we saw last time

$$\psi^p(E) = s_p(\Lambda^1 E, \dots, \Lambda^n E).$$

Now $s_p(\sigma_1, \dots, \sigma_n) = \sum_{i=1}^n z_i^p = \left(\sum z_i\right)^p + p \cdot r(\sigma_1, \dots, \sigma_n)$,

that is, $\sum_{i=1}^n z_i^p - \left(\sum z_i\right)^p$ is a polynomial i) all of whose coefficients are divisible by p, and ii) which is certainly symmetric on the z_i's. So it can be written as $p \cdot r(\sigma_1, \dots, \sigma_n)$. Well, this looks pretty good; $\sum_{i=1}^n z_i = \sigma_1$, so

$$\psi^p(E) = s_p(\Lambda^1 E, \dots, \Lambda^n E) =$$

$$\underset{\substack{\shortparallel \\ E^p}}{(\Lambda^1 E)^p} + p \cdot r(\Lambda^1 E, \dots, \Lambda^n E)$$

$$\equiv E^p \pmod p.$$

And $\psi^p(E-F) = \psi^p(E) - \psi^p(F)$

$$= E^p + p \cdot r(E) - F^p - p \cdot r(F)$$

abuse of notation here

$$\equiv E^p - F^p \pmod p$$

$$\equiv (E-F)^p \pmod p.$$

Now we have to talk about products for a while. Spaces will have basepoints. Thinking of $\tilde{K}(X)$ as $\ker\{K(X) \xrightarrow{i^*} K(*)\}$ we get a short exact sequence

$$0 \to \tilde{K}(X) \to K(X) \rightleftarrows \mathbb{Z} \to 0$$

which splits thanks to the map $K(*) \xrightarrow{j^*} K(X)$. So we can think of this as a sequence

$$0 \to \tilde{K}(X) \to \tilde{K}(X) \oplus \mathbb{Z} \to \mathbb{Z} \to 0 .$$

Next tensor the first two terms of this with the same sequence for another space Y; the cokernel is easily computed.

$$0 \to \tilde{K}(X) \otimes \tilde{K}(Y) \to (\tilde{K}(X) \oplus \mathbb{Z}) \otimes (\tilde{K}(Y) \oplus \mathbb{Z}) \xrightarrow{\alpha} \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z} \to 0$$

$$(\tilde{K}(X) \oplus \mathbb{Z}) \otimes (\tilde{K}(Y) \oplus \mathbb{Z}) \cong \tilde{K}(X) \otimes \tilde{K}(Y) \oplus (\tilde{K}(X) \otimes \mathbb{Z}) \oplus (\tilde{K}(Y) \otimes \mathbb{Z}) \oplus \mathbb{Z}$$

α takes the degree (i.e. $\mathbb{Z}$) components of $K(X)$ and $K(Y)$ to their product, which is the degree of the tensor product of the elements sitting over the wedge in $K(X \times Y)$. You can convince yourself that α is the map that makes this square commute:

$$\begin{array}{ccc}
K(X) \otimes K(Y) & \xrightarrow{\alpha} & \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z} \\
& & \wr\| \\
\Big\downarrow \times & & \tilde{K}(X \vee Y) \oplus \mathbb{Z} \\
& & \wr\| \\
K(X \times Y) & \longrightarrow & K(X \vee Y)
\end{array}$$

Continuing to look at the bottom row of this square, there is an exact sequence

$$0 \to \tilde{K}(X \wedge Y) \to K(X \times Y) \to K(X \vee Y) \to 0.$$

Why is this exact? Well, the point is that

$$\tilde{K}(X \times Y) \to \tilde{K}(X \vee Y)$$

splits after one suspension:

$$\begin{tikzcd} \tilde{K}(\Sigma(X\times Y)) \arrow[r, shift left] & \tilde{K}(\Sigma X \vee \Sigma Y) \arrow[l, dashed, shift left, "\exists"] \arrow[d, phantom, "\cong?" description] \\ & \Sigma(X \vee Y) \end{tikzcd}$$

and from

$$\begin{tikzcd} \tilde{K}^{+1}(\Sigma(X\vee Y)) \arrow[r] \arrow[d, phantom, "\cong?" description] & \tilde{K}^{+1}(\Sigma(X\times Y)) \arrow[d, phantom, "\cong?" description] \\ \tilde{K}(X \vee Y) \arrow[r] & \tilde{K}(X \times Y) \end{tikzcd}$$

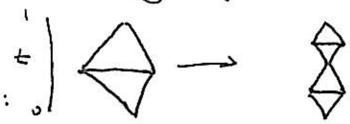
we get

So to do this, we produce a map

$$\begin{tikzcd} \Sigma(X\times Y) \arrow[r] & \Sigma X \vee \Sigma Y \\ \Sigma(X \vee Y) \arrow[u] \arrow[ur, "\simeq"'] \end{tikzcd}$$

so that the composite map is a homotopy equivalence.

The map is



$$\begin{tikzcd} \Sigma(X\times Y) \arrow[r, "\text{pinch map}"] & \Sigma(X\times Y) \vee \Sigma(X\times Y) \arrow[r, "\Sigma pr_1 \vee \Sigma pr_2"] & \Sigma X \vee \Sigma Y \\ \Sigma(X\vee Y) \arrow[u, hook] \end{tikzcd}$$

$$[t,(x,*)] \mapsto \left\{ \begin{array}{ll} 2t\,(x,*) & t \le \frac{1}{2} \\ 2-2t\,(x,*) & t \ge \frac{1}{2} \end{array} \right\} \to \begin{array}{ll} (2t,x) & t \le \frac{1}{2} \quad \subset \Sigma X \\ * & t \ge \frac{1}{2} \end{array}$$

$$\uparrow$$

$$(t,x)$$

& similarly for a point (t,y). But the top half $(t \ge \frac{1}{2})$ is just a cone, which is contractible, so we get the splitting we wanted.

All this goes to show that $\tilde{K}(X\times Y) \rightleftarrows \tilde{K}(X\vee Y)$ splits, so the sequence $\tilde{K}(X\wedge Y) \to \tilde{K}(X\times Y) \to \tilde{K}(X\vee Y)$ is short exact; augmenting the last two terms doesn't harm this, and we get

$$0 \to \tilde{K}(X\wedge Y) \longrightarrow K(X\times Y) \rightleftarrows K(X\vee Y) \to 0.$$

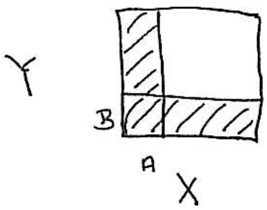
Now we can put these two short exact sequences together using the square at the bottom of two pages ago, to get

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\tilde{K}(X)\otimes\tilde{K}(Y) \arrow[r, dashed, "\wedge"] \arrow[d] & \tilde{K}(X\wedge Y) \arrow[d] \\
K(X)\otimes K(Y) \arrow[r, "\times"] \arrow[d, "\alpha"'] & K(X\times Y) \arrow[d] \\
K(X)\vee K(Y) \arrow[r, equal] \arrow[d] & K(X\vee Y) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The induced map we get $\tilde{K}(X)\otimes\tilde{K}(Y) \longrightarrow \tilde{K}(X\wedge Y)$ is the smash product map we defined (sort of) and that appeared in the statement of Bott periodicity.

A few words on relative K-theory. Suppose $* \in A \subseteq X$ and the inclusion is nice, i.e. a cofibration. Then $K(X,A) \overset{\text{def}}{=} \tilde{K}(X/A)$. There is a product

$$\begin{array}{ccc} K(X,A) \otimes K(Y,B) & & \\ \| & & \\ \tilde{K}(X/A) \otimes \tilde{K}(Y/B) & \xrightarrow{\ \wedge\ } & \tilde{K}(X/A \wedge Y/B) \\ & & \| \\ & & \tilde{K}(X \times Y, X \times B \cup A \times Y) \end{array}$$



$\leftarrow$ this picture helps to understand this $\Rightarrow \|$

Now suppose $X = Y$; then we have a diagonal map

$$(X, A \cup B) \xrightarrow{\ \Delta\ } (X \times X, X \times B \cup A \times X)$$

which induces a cup-product:

$$\begin{array}{ccc} K(X,A) \otimes K(X,B) & \xrightarrow{\ \cup\ } & K(X, A \cup B) \\ \| & & \uparrow \Delta^* \\ \tilde{K}(X/A) \otimes \tilde{K}(X/B) & \xrightarrow{\ \wedge\ } & K(X \times X, X \times B \cup A \times X) \end{array}$$

Note when $A \simeq \{*\} \simeq B$ we get

$$\begin{array}{ccc} K(X,A) \otimes K(X,B) & \xrightarrow{\ \cup\ } & K(X, A \cup B) \\ \downarrow \cong & & \downarrow \cong \\ \tilde{K}(X) \otimes \tilde{K}(X) & \xrightarrow{\ \wedge\ } & \tilde{K}(X). \end{array}$$

If in addition $A \cup B = X$, we get $K(X, A \cup B) = 0$, so the smash product map is trivial. We have shown

Lemma If $X = A \cup B$ and $A \simeq * \simeq B$, then $\tilde{K}(X)^2 = 0$.

For example, this is true of any suspension and in particular of spheres. So, we know that $\tilde{K}(S^2) = \mathbb{Z}\langle \underbrace{L - 1}_{x} \rangle$ (by Bott periodicity), and we have proved the important relation $(L-1)^2 = 0$.

Hopf

So $K(S^2) = \mathbb{Z}[x]/(x^2)$.

Why not compute the Adams operations as well:

$$\psi^k(x) = \psi^k(L-1) = L^k - 1.$$

$$L^k = (1+x)^k \equiv 1 + kx - 1 \mod (x^2) = kx.$$

So $\psi^k(x) = kx$.

For other spheres, well, by Bott periodicity

$$\begin{array}{ccc} \tilde{K}(S^2)^{\otimes n} & \xrightarrow{\cong} & \tilde{K}(S^{2n}) \\ x^{\otimes n} & \longmapsto & x_n \end{array}$$

Once again $x_n^2 = 0$, so $K(S^{2n}) = \mathbb{Z}[x_n]/(x_n^2)$. Now on $X \times Y$,

$\psi^k(x \times y) = \psi^k x \times \psi^k y$; from the ladder two pages ago it follows that

$\psi^k(pr_1^* x \cdot pr_2^* y)$ //

$$\psi^k(x^{\otimes n}) = \psi^k(x)^n = (kx)^n = k^n x^n$$

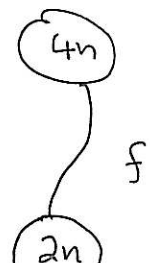
So $\psi^k(x_n) = k^n x_n$;

the Adams operations detect the dimension of the sphere!

OK, so now we can use all this equipment to prove the Hopf-invariant 1 problem. Remember the set-up:

this is cofiber; continue the Barratt-Puppe seq. ?

$$\begin{array}{ccccccc} S^{4n-1} & \xrightarrow{f} & S^{2n} & \xrightarrow{j} & X & \xrightarrow{k} & S^{4n} \\ \uparrow & & 0 & \longleftarrow & y & \longleftarrow & x_{2n} \\ 0 \text{ on} & & x_n & \longleftarrow & x & & \\ \text{K-theory!} & & & & & & \end{array}$$



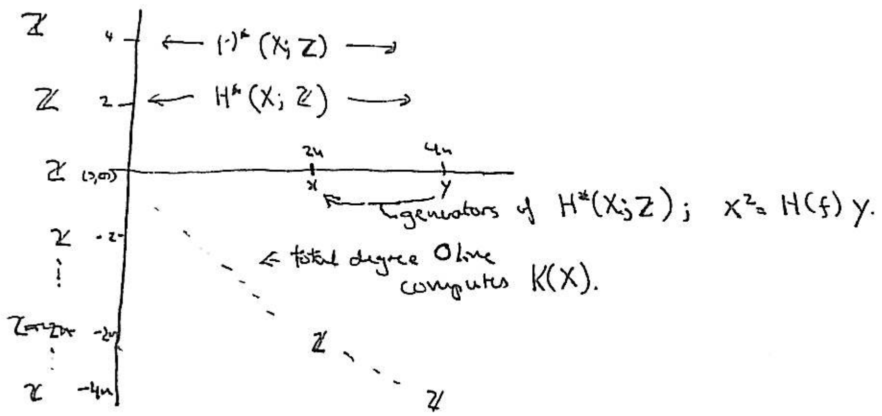
Since f is 0 on K-theory, the long exact sequence degenerates to a short exact sequence. So there is an $x \in \tilde{K}(X)$ such that it pulls back to x_n. In addition, $\tilde{K}(X) \xleftarrow{k^*} \tilde{K}(S^{4n})$ is injective, so $y^2 = k^* x_{2n}^2 = 0$. Thus $K(X)$ has three generators $\langle 1, x, y \rangle$, and relations $y^2 = 0$, $x^2 = ay$.
($x^2 = ay + bx$, but $b = 0$, since $0 = x_n^2 = j^*(x^2) = b x_n$)

<u>Claim</u> a is the Hopf invariant.

There are a variety of ways to justify this claim; for example you could use the Chern character, which provides a ring homomorphism $K(X) \longrightarrow H^*(X;\mathbb{Q})$. Also the Atiyah-Hirzebruch spectral sequence works nicely:

$$E_2^{**} = H^*(X; K^*(*)) \Rightarrow K^*(X).$$

X is very simple, so we can write this down easily.
Here's the E_2-term.



But everything in the E_2-term happens in even dimensions, so there are no differentials. And since the SS is nice with respect to all the algebraic structure in sight, clearly the x and y represent the generators of $\tilde{K}(X)$, and $a = H(f)$.

Claim $a = \text{odd} \Rightarrow n = 1, 2, \text{ or } 4.$

Step 1 Calculate ψ^k.

$$\psi^k(x_{2n}) = k^{2n} x_{2n}, \text{ so}$$

$$\psi^k(y) = k^{2n} y.$$

$$j^* \psi^k(x) = \psi^k j^*(x) = k^n x_n, \text{ so}$$

$$\psi^k(x) = k^n + b_k y, \quad b_k \in \mathbb{Z}.$$

Recalling the Lemma from the beginning of today's discussion,

$$ay = x^2 \underset{(\text{mod } 2)}{\equiv} \psi^2(x) = 2^n x + b_2 y \underset{(\text{mod } 2)}{\equiv} b_2 y.$$

So the new claim is

Claim $b_2 = \text{odd} \Rightarrow n = 1, 2, \text{ or } 4.$

Need one more ψ^k, namely ψ^3; recall $\psi^3\psi^2 = \psi^3\psi^2$ ($= \psi^6$, but don't need that)

$$\psi^3\psi^2(x) = \psi^3(2^n x + b_2 y) = 2^n(3^n x + b_3 y) + b_2 3^{2n} y$$

$$\psi^2\psi^3(x) = \psi^2(3^n x + b_3 y) = 3^n(2^n x + b_2 y) + b_3 2^{2n} y$$

hence (collecting terms in y):

$$\underbrace{b_3(2^{2n} - 2^n)} = b_2(3^{2n} - 3^n).$$

2^n divides this, so $2 | b_2$ as long as 2^n fails to divide $3^{2n} - 3^n$;

equivalently, 2^n fails to divide $3^n - 1$.

Finally, we have

<u>Claim</u>

$$\nu_2(3^n-1) = \begin{cases} 1 & n \text{ odd} \\ \nu_2(n)+2 & n \text{ even.} \end{cases}$$

Using this, we see

n	1	2	3	4	5	6	7	8
$\nu_2(3^n-1)$	1	3	1	4	1	3	1	5

so $\nu_2(3^n-1) \geq n$ only when $n = 1, 2,$ or 4.

<u>Proof</u> of the last claim (following David Anick).

Write $k = \nu_2(n)$, and $n = 2^k d$ where d is odd

If $k=0$, write $n = 2j+1$, and

$$3^{2j+1} - 1 = 9^j \cdot 3 - 1 \equiv 2 \bmod 8, \text{ so}$$

2 divides $3^d - 1$ once. Note also

$$3^{2j+1} + 1 = 9^j \cdot 3 + 1 \equiv 4 \bmod 8, \text{ so } 2 \text{ divides this twice.}$$

For $k > 0$, note $(3^{2^k d} - 1) = (3^{2^{k-1}d} + 1)(3^{2^{k-1}d} - 1)$; continuing we get

$$(3^{2^k d} - 1) = (3^d + 1)(3^d - 1)(3^{2d} + 1) \cdots (3^{2^{k-1}d} + 1)$$

Now for $k > 0$,

$$3^{2^k d} + 1 = 9^{2^{k-1}d} + 1 \equiv 2 \bmod 8, \text{ so } 2 \text{ divides } 3^{2^k d} + 1 \text{ once.}$$

Counting up terms, we find

$$\nu_2(3^n-1) = \begin{cases} 1 & n \text{ odd} \\ \nu_2(n)+2 & n \text{ even} \end{cases} \text{ as advertised.}$$

Lemma If $X = A \cup B$ and $A \simeq * \simeq B$, then $\tilde{K}(X)^2 = 0$;
e.g. any suspension has this property.

For example, $\tilde{K}(S^2) = \mathbb{Z}\langle L - \underline{1} \rangle$ is subject to the relation $(L-\underline{1})^2 = 0$.
(Bott) $\quad\uparrow$ Hopf bundle

So $K(S^2) = \mathbb{Z}[x]/(x^2)$. While we're at it, let's compute the Adams operations. L is a line bundle, so $\psi^k(L) = L^k$. And

$$\psi^k(x) = \psi^k(L - \underline{1}) = L^k - \underline{1}.$$

$$L = \underline{1} + x; \quad \psi^k(x) = (\underline{1}+x)^k - \underline{1} = kx.$$

By Bott, again, we have

$$\begin{array}{ccc} \tilde{K}(S^{2n}) & \xleftarrow{\cong} & \tilde{K}(S^2)^{\otimes n} \\ \underset{\text{generator}}{x_n} & \longleftarrow & x^{\otimes n} \end{array}$$

So $K(S^{2n}) \cong \mathbb{Z}[x_n]/(x_n^2)$.

As for $\psi^k(x_n)$, well, from the ladder two pages ago, the map $\tilde{K}(X) \otimes \tilde{K}(S^2) \longrightarrow \tilde{K}(S^2 \wedge X)$ of Bott periodicity comes from the cross-product on K-theory; that is to say

$$\begin{aligned} \psi^k(x \times y) &= \psi^k(pr_1^* x \cdot pr_2^* y) \\ &= pr_1^* \psi^k(x) \cdot pr_2^* \psi^k(y). \end{aligned}$$

So $\psi^k(x_n) = \psi^k(x^{\otimes n}) = \psi^k(x)^n = (kx)^n = k^n x^n$.

The Adams operations detect the dimension of an even-dimensional sphere!