Operads, modules and higher Hochschild cohomology

by

Geoffroy Horel

M.Sc., École Polytechnique (2009)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2013

Author . . . .
Department of Mathematics
????????, 2013

Certified by . . . .
Haynes R. Miller
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
X
Chairman, Department Committee on Graduate Theses

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Abstract

To be filled in

Thesis Supervisor: Haynes R. Miller
Title: Professor of Mathematics

Acknowledgments

To be filled in

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Introduction

Chapter 1

Modules over an $\mathcal{O}$-algebra

In this chapter, we give ourselves a one-color operad $\mathcal{O}$ and we construct a family of theories of modules over $\mathcal{O}$-algebras. These module categories are parametrized by associative algebras in the category of right modules over $\mathcal{O}$. Assuming that the symmetric monoidal model category we are working with satisfies certain conditions, these category of modules can be given the structure of a model category. We then show that these categories are organized into an "algebra" over a certain operad.

The reader is invited to refer to the two appendices for background material about operads and model categories.

1.1 Definition of the categories of modules

In this section and the following $(\mathbf{C}, \otimes, \mathbb{I})$ denotes a simplicial symmetric monoidal category. We do not assume any kind of model structure.

Let $\mathcal{O}$ be a one-color operad in $\mathbf{S}$ and A be an object of $\mathbf{C}[\mathcal{O}]$. We want to describe various categories of modules over A. By a module we mean an object M of $\mathbf{C}$ together with operations $A^{\otimes n} \otimes M \to M$.

Definition 1.1.1. Let P be an associative algebra in right modules over $\mathcal{O}$. The operad $P\mathcal{M}od$ of *P-shaped $\mathcal{O}$-modules* has two colors a and m. Its spaces of operations are as follows

$$PMod(a^{\boxplus n}; a) = \mathcal{O}(n)$$
$$PMod(a^{\boxplus n} \boxplus m; m) = P(n)$$

Any other space of operation is empty. The composition is left to the reader.

Any category that can reasonably called a category of modules over an $\mathcal{O}$-algebras arise in the above way as is shown by the following easy proposition:

Proposition 1.1.2. *Let $\mathcal{M}$ be an operad with two colors a and m and satisfying the following properties:*

- *$\mathcal{M}(*; a)$ is empty if $*$ contains the color m.*
- *$\mathcal{M}(a^{\boxplus n}; a) = \mathcal{O}(n)$*
- *$\mathcal{M}(*; m)$ is non empty only if $*$ contains exactly one copy of m.*

Then $\mathcal{M} = P\mathcal{M}od$ for some P in $\mathbf{Mod}_{\mathcal{O}}[\mathcal{A}ss]$.

Proof. We define $P(n) = \mathcal{M}(a^{\boxplus n} \boxplus m; m)$. Using the fact that $\mathcal{M}$ is an operad, it is easy to prove that P is an object of $\mathbf{Mod}_{\mathcal{O}}[\mathcal{A}ss]$ and that $\mathcal{M}$ coincides with $P\mathcal{M}od$. □

We denote by $\mathbf{C}[P\mathcal{M}od]$ the category of algebras over this two-colors operad in the category $\mathbf{C}$. Objects of this category are pairs (A, M) of objects of $\mathbf{C}$. The object A is an $\mathcal{O}$-algebra and the object M has an action of A parametrized by the spaces $P(n)$. Maps in this category are pairs (f, g) preserving all the structure.

Remark 1.1.3. Note that the construction $P \mapsto P\mathcal{M}od$ is a functor from $\mathbf{Mod}_{\mathcal{O}}[\mathcal{A}ss]$ to the category of operads. It preserves weak equivalences between objects of $\mathbf{Mod}_{\mathcal{O}}[\mathcal{A}ss]$. We can in fact improve this homotopy invariance a little bit.

Construction 1.1.4. We construct a category $\mathbf{M}$. Its objects are pairs $(\mathcal{O}, P)$ where $\mathcal{O}$ is a one-color operad and P is an associative algebra in right modules. Its morphisms $(\mathcal{O}, P) \to (\mathcal{O}', P')$ consist of a morphisms of operads $f : \mathcal{O} \to \mathcal{O}'$ together with a morphisms of associative algebras in $\mathcal{O}$-modules $P \to P'$ where P is an seen as an $\mathcal{O}$-module by restriction along f. We say that a map in $\mathbf{M}$ is a *weak equivalence* if it induces a weak equivalence on $\mathcal{O}$ and $\mathcal{P}$.

Proposition 1.1.5. *The functor* $\mathbf{M} \to \mathbf{Oper}$ *sending* $(\mathcal{O}, P)$ *to* $P\mathcal{M}od$ *preserves weak equivalences.* $\square$

Definition 1.1.6. Let A be an $\mathcal{O}$-algebra in $\mathbf{C}$. The *category of P-shaped A-modules* denoted by $P\mathbf{Mod}_A$ is the subcategory of $\mathbf{C}[P\mathcal{M}od]$ on objects of the form (A, M) and of maps of the form (id_A, g).

Note that there is an obvious forgetful functor $P\mathbf{Mod}_A \to \mathbf{C}$. One easily checks that it preserves limits and colimits.

This abstract definition recovers well-known examples. We can try to model left and right modules over associative algebras. Take $\mathcal{O}$ to be $\mathcal{A}ss$ as an operad in the category of sets. The category $\mathbf{Ass}$ is the category of non-commutative sets (it is defined in [Ang09]). Its objects are finite sets and its morphisms are pairs (f, ω) where f is a map of finite sets and ω is the data of a linear ordering of each fiber of f.

Construction 1.1.7. Let $\mathbf{Ass}_-$ (resp. $\mathbf{Ass}_+$) be the category whose objects are based finite sets and whose morphisms are pairs (f, ω) where f is a morphisms of based finite sets and ω is a linear ordering of the fibers of f which is such that the base point is the smallest (resp. largest) element of the fiber over the base point of the target of f.

Let R (resp. L) be the right module over $\mathcal{A}ss$ defined by the formulas:

$$R(n) = \mathbf{Ass}_-(\{*, 1, \ldots, n\}, \{*\})$$
$$L(n) = \mathbf{Ass}_+(\{*, 1, \ldots, n\}, \{*\})$$

Let us construct a pairing:

$$R(n) \times R(m) \to R(n+m)$$

Note that specifying a point in $R(n)$ is equivalent to specifying a linear order of $\{1, \ldots, n\}$. Let f be a point in $R(n)$ and g be a point in $R(m)$. We define their product to be the map whose associated linear order of $\{1, \ldots, n+m\}$ is the linear order induced by n concatenated with the linear order induced by g.

Proposition 1.1.8. *Let A be an associative algebra in $\mathbf{C}$. $L\mathbf{Mod}_A$ (resp. $R\mathbf{Mod}_A$) is isomorphic to the category of left (resp. right) modules over A.*

Proof. Easy. □

Remark 1.1.9. Operadic modules are also a particular case of this construction. Let $\mathcal{O}[1]$ be the shift of the operad $\mathcal{O}$. Explicitely, $\mathcal{O}[1](n) = \mathcal{O}(n+1)$ with action induced by the inclusion $\Sigma_n \to \Sigma_{n+1}$. This is in an obvious way a right module over $\mathcal{O}$. Moreover it has an action of the associative operad:

$$\mathcal{O}[1](n) \times \mathcal{O}[1](m) = \mathcal{O}(n+1) \times \mathcal{O}(m+1) \xrightarrow{\circ_{n+1}} \mathcal{O}(n+m+1) = \mathcal{O}(n+m)[1]$$

It is easy to check that the operad $\mathcal{O}[1]\mathcal{M}od$ is the operad parametrizing operadic $\mathcal{O}$-modules. For instance if $\mathcal{O} = \mathcal{A}ss$, the associative operad, the category $\mathcal{A}ss[1]\mathbf{Mod}_A$ is the category of A-A-bimodules. If $\mathcal{C}om$ is the commutative operad, the category $\mathcal{C}om[1]\mathbf{Mod}_A$ is the category of left modules over A. If $\mathcal{L}ie$ is the operad parametrizing Lie algebra in an additive symmetric monoidal category, the category $\mathcal{L}ie[1]\mathbf{Mod}_{\mathfrak{g}}$ is the category of Lie modules over the Lie algebra $\mathfrak{g}$. That is object M equipped with a map:

$$-.- : \mathfrak{g} \otimes M \to M$$

satisfying the following relation:

$$[X, Y].m = X.(Y.m) - Y.(X.m)$$

1.2 Universal enveloping algebra

In this section, we show that the category $P\mathbf{Mod}_A$ is the category of left modules over a certain associative algebra built out of A and P.

Let $U_A^P = P \circ_{\mathcal{O}} A$. Then by proposition B.2.7, it is an associative algebra in $\mathbf{C}$

Definition 1.2.1. The associative algebra U_A^P is called the *universal enveloping algebra of* $P\mathbf{Mod}_A$.

This name finds its justification in the following proposition.

Proposition 1.2.2. *The category $P\mathbf{Mod}_A$ is equivalent to the category of left modules over the associative algebra U_A^P.*

Proof. Let J be the associative algebra in $\mathbf{Mod}_{\mathcal{O}}$ J which sends 0 to $*$ and everything else to $\emptyset$. J gives rise to a theory of modules. The operad $J\mathcal{M}od$ has the following description:

$$J\mathcal{M}od(a^{\boxplus k}, a) = \mathcal{O}(k)$$
$$J\mathcal{M}od(a^{\boxplus k} \boxplus m, m) = * \text{ if } k = \emptyset,\ \emptyset \text{ otherwise}$$

The theory of modules parametrized by J is the simplest possible. There are no operations $A^{\otimes n} \otimes M \to M$ except the identity map $M \to M$.

There is an obvious operad map $J\mathcal{M}od \to P\mathcal{M}od$ inducing a forgetful functor $\mathbf{C}[P\mathcal{M}od] \to \mathbf{C}[J\mathcal{M}od]$. Let us fix the $\mathcal{O}$-algebra A. One checks easily that $J\mathbf{Mod}_A$ is isomorphic to the category $\mathbf{C}$. We are interested in the left adjoint:

$$\mathbf{C} \cong J\mathbf{Mod}_A \to P\mathbf{Mod}_A$$

Let us first study the left adjoint $F : \mathbf{C}[J\mathcal{M}od] \to \mathbf{C}[P\mathcal{M}od]$. This is an operadic left Kan extension. By B.2.6, we have the equation:

$$F(A, M)(m) \cong P\mathbf{Mod}(-, m) \otimes_{J\mathbf{Mod}} A^{\otimes -} \otimes M^{\otimes -}$$

Note that the only nonempty mapping object in $P\mathbf{Mod}$ with target m are those with source of the form $a^{\boxplus s} \boxplus m$. Hence if we denote $J\mathbf{Mod}_*$ and $P\mathbf{Mod}_*$ the full subcategories with objects of the form $a^{\boxplus s} \boxplus m$, the above coend can be reduced to:

$$F(A, M)(m) \cong P\mathbf{Mod}_*(-, m) \otimes_{J\mathbf{Mod}_*} A^{\otimes -} \otimes M$$

Let us denote by $\mathbf{Fin}_*$ the category whose objects are nonnegative integers n_* and whose morphisms from n_* to m_* are morphisms of finite pointed sets:

$$\{*, 1, \ldots, n\} \to \{*, 1, \ldots, m\}$$

The previous coend is the coequalizer:

$$\bigsqcup_{f \in \mathbf{Fin}_*(s_*, t_*)} P(t) \times \left(\prod_{x \in t} \mathcal{O}(f^{-1}(x)) \right) \times J(f^{-1}(*)) \otimes A^{\otimes s} \otimes M \\ \rightrightarrows \bigsqcup_{s \in \mathbf{Fin}} P(s) \otimes A^{\otimes s} \otimes M$$

Since the right module J takes value $\emptyset$ for any non-empty set, we see that the coproduct on the left does not change if we restrict to maps $s_* \to t_*$ for which the inverse image of the base point of t_* is the base point of s_*. This set of maps is in bijection with the set of unbased maps $s \to t$. Therefore, the coend can be equivalently written as:

$$\bigsqcup_{f \in \mathbf{Fin}(s,t)} P(t) \times \left(\prod_{x \in t} \mathcal{O}(f^{-1}(x)) \right) \otimes A^{\otimes s} \otimes M \\ \rightrightarrows \bigsqcup_{s \in \mathbf{Fin}} P(s) \otimes A^{\otimes s} \otimes M$$

But now we see that M can be pulled out of this coend. Since the tensor product with M commutes with colimits, this is $U_A^P \otimes M$.

One can compute in a similar but easier fashion that $F(A, M)(a) \cong A$.

We have constructed a natural isomorphism:

$$\mathbf{C}[P\mathcal{M}od]((A, U_A^P \otimes M), (A, N)) \cong \mathbf{C}[J\mathcal{M}od]((A, M), (A, N))$$

It is clear that this isomorphisms preserves the subset of maps inducing the identity on A. Hence we have:

$$P\mathbf{Mod}_A(U_A^P \otimes M, N) \cong J\mathbf{Mod}_A(M, N) \cong \mathbf{C}(M, N)$$

This shows that, as functors, the monad associated to the adjunction:

$$\mathbf{C} \leftrightarrows P\mathbf{Mod}_A$$

is isomorphic to the monad associated to the adjunction:

$$\mathbf{C} \leftrightarrows L\mathbf{Mod}_{U_A^P}$$

A little bit of extra-work would show that they are isomorphic as monad. Since both adjunctions are monadic, the result follows. □

The above result is well-known if $P = \mathcal{O}[1]$. See for instance section 4.3. of [Fre09].

Note that there is an involution in the category of associative algebras in right modules over $\mathcal{O}$ sending P to P^{op}. The construction $P \mapsto U_A^P$ sends P^{op} to $(U_A^P)^{\mathrm{op}}$.

Remark 1.2.3. Another source of examples of modules is obtained by the following procedure:

Assume that $\alpha : \mathcal{O} \to \mathcal{Q}$ is a morphism of operads. Let A be an $\mathcal{Q}$ algebra and P be an associative algebra in right modules over $\mathcal{O}$. Then by forgetting along the map $\mathcal{O} \to \mathcal{Q}$, we construct $\alpha^* A$ which is an $\mathcal{O}$-algebra and one may talk about the category $P\mathbf{Mod}_{\alpha^* A}$. The following proposition shows that this category of modules is of the form $Q\mathbf{Mod}_A$ for some Q.

Proposition 1.2.4. *We keep the notation of the previous remark. The object $\alpha_! P = P \circ_{\mathcal{O}} \mathcal{Q}$ is an associative algebra in right modules over $\mathcal{Q}$. Moreover, the category $P\mathbf{Mod}_{\alpha^* A}$ is equivalent to the category $\alpha_! P\mathbf{Mod}_A$.*

Proof. The first part of the claim follows from the fact that $P \circ_{\mathcal{O}} \mathcal{Q}$ is a reflexive coequalizer of associative algebras in right $\mathcal{Q}$-modules and reflexive coequalizers preserve associative algebras.

The second part of the claim follows from a comparison of universal enveloping algebras:

$$\begin{aligned} U_A^{\alpha_! P} &\cong (P \circ_{\mathcal{O}} \mathcal{Q}) \circ_{\mathcal{Q}} A \\ &\cong P \circ_{\mathcal{O}} (\mathcal{Q} \circ_{\mathcal{Q}} A) \\ &\cong P \circ_{\mathcal{O}} \alpha^* A \cong U^P_{\alpha^* A} \end{aligned}$$

□

1.3 Model category structure

We now give a model structure to the category $P\mathbf{Mod}_A$ assuming the category has a good enough model structure. See B.3.4 for the definition of "having a good theory of algebras".

Construction of the model category structure

In the remaining of this chapter, $(\mathbf{C}, \otimes, \mathbb{I}_{\mathbf{C}})$ will denote a cofibrantly generated closed symmetric monoidal simplicial category which either satisfies the monoid axiom (see [SS00]) or is such that $\mathbb{I}_{\mathbf{C}}$ is cofibrant.

Theorem 1.3.1. *Assume that $\mathbf{C}$ has a good theory of algebras (resp. a good theory of algebras over Σ-cofibrant operads). Let $\mathcal{O}$ be an operad (resp. Σ-cofibrant operad) and P be a right $\mathcal{O}$-module (resp. Σ-cofibrant right $\mathcal{O}$-module). Let A a cofibrant $\mathcal{O}$-algebra. There is a model category structure on the category $P\mathbf{Mod}_A$ in which the weak equivalences and fibrations are the weak equivalences and fibrations in $\mathbf{C}$.*

Moreover, this model structure is simplicial and if $\mathbf{C}$ is a $\mathbf{V}$-enriched model category for some monoidal model category $\mathbf{V}$, then so is $P\mathbf{Mod}_A$.

Proof. The category $P\mathbf{Mod}_A$ is isomorphic to $\mathbf{Mod}_{U_A^P}$. The object of $\mathbf{C}$ underlying U_A^P is cofibrant if the unit is cofibrant (B.3.9). The existence of the model structure is then a consequence of [SS00]. If the category satisfies the monoid axiom, then any category of modules can be given a transferred model structure (see [SS00]).

The facts about enrichments come from A.2.7. □

The category $P\mathbf{Mod}_A$ depends on the variables P and A. As expected, there are "base change" Quillen adjunctions.

Proposition 1.3.2. *Let $P \to P'$ be a morphisms of associative algebras in right modules over $\mathcal{O}$ and A be a cofibrant $\mathcal{O}$-algebra, then there is a Quillen adjunction:*

$$P\mathbf{Mod}_A \leftrightarrows P'\mathbf{Mod}_A$$

Similarly, if $A \to A'$ is a morphisms of cofibrant $\mathcal{O}$-algebras then there is a Quillen adjunction:

$$P\mathbf{Mod}_A \leftrightarrows P\mathbf{Mod}_{A'}$$

Proof. In both cases, we get an induced map between the corresponding universal enveloping algebras. The result are then a standard "change of algebras" theorem (see [SS00]). □

In some cases these adjunctions are Quillen equivalences.

Proposition 1.3.3. *Let P be an associative algebra in $\mathbf{Mod}_{\mathcal{O}}$ and A be a cofibrant object of $\mathbf{C}[\mathcal{O}]$. Assume that for any cofibrant object of $P\mathbf{Mod}_A$, N, the functor $- \otimes_{U_A^P} N$ sends weak equivalences of right U_A^P-modules to weak equivalences in $\mathbf{C}$. Then:*

If $P \to P'$ is a weak equivalence of associative algebras in right modules over $\mathcal{O}$, then there is a Quillen equivalence:

$$P\mathbf{Mod}_A \leftrightarrows P'\mathbf{Mod}_A$$

Similarly, if $A \to A'$ is a morphisms of cofibrant $\mathcal{O}$-algebras then there is a Quillen equivalence:

$$P\mathbf{Mod}_A \leftrightarrows P\mathbf{Mod}_{A'}$$

Proof. See [SS00] Theorem 4.3. □

Remark 1.3.4. Having to ask for $-\otimes_{U_A^P} N$ to preserve weak equivalences is a little bit unpleasant but often verified in practice. In particular, it is true for $L_Z p\mathbf{Mod}_E$ and $L_Z a\mathbf{Mod}_E$, $\mathbf{S}$, $\mathbf{Ch}_{\geq 0}(R)$.

Cofibrant replacement in $\mathbf{C}[P\mathcal{M}od]$

The following proposition gives a simple description of the cofibrant objects of $\mathbf{C}[P\mathcal{M}od]$ whose algebra component is cofibrant.

Proposition 1.3.5. *Let A be a cofibrant $\mathcal{O}$-algebra in $\mathbf{C}$. Let M be an object of $P\mathbf{Mod}_A$. The pair (A, M) is a cofibrant object of $\mathbf{C}[P\mathcal{M}od]$ if and only if M is a cofibrant object of $P\mathbf{Mod}_A$.*

Proof. Assume (A, M) is cofibrant in $\mathbf{C}[P\mathcal{M}od]$. For any trivial fibration $N \to N'$ in $P\mathbf{Mod}_A$, the map $(A, N) \to (A, N')$ is a trivial fibration in $\mathbf{C}[P\mathcal{M}od]$. A map of P-shaped A-module $M \to N'$ induces a map of $P\mathcal{M}od$-algebras $(A, M) \to (A, N')$ which can be lifted to a map $(A, M) \to (A, N)$ and this lift has to be the identity on the first component. Thus M is cofibrant.

Conversely, let $(B', N') \to (B, N)$ be a trivial fibration in $\mathbf{C}[P\mathcal{M}od]$. We want to show that any map $(A, M) \to (B, N)$ can be lifted to (B', N'). We do this in two steps. We first lift the first component and then the second component.

Note that if we have a map $A \to B$, any P-shaped module N over B can be seen as a P-shaped module over A by restricting the action along this map. With this in mind, it is clear that any map $(A, M) \to (B, N)$ can be factored as:

$$(A, M) \to (A, N) \to (B, N)$$

where the first map is a map in $P\mathbf{Mod}_A$ and the second map induces the identity on N.

Since the map $(B', N') \to (B, N)$ is a trivial fibration in $\mathbf{C}[P\mathcal{M}od]$, the induced map $B' \to B$ is a trivial fibration in $\mathbf{C}$ which implies that it is a trivial fibration in $\mathbf{C}[\mathcal{O}]$. A is cofibrant as an $\mathcal{O}$-algebra so we can choose a factorization $A \to B' \to B$.

Using this map, we can see N' as an object of $P\mathbf{Mod}_A$ and, we have the following diagram in $\mathbf{C}[P\mathcal{M}od]$:

$$\begin{tikzcd}
 & (A,N') \arrow[r] \arrow[d] & (B',N') \arrow[d] \\
(A,M) \arrow[r] & (A,N) \arrow[r] & (B,N)
\end{tikzcd}$$

We want to construct a map $(A,M) \to (A,N')$ making the diagram to commute. The map $(A,N') \to (A,N)$ is the product of the identity of A and a trivial fibration $N \to N'$ in $\mathbf{C}$. This implies that $(A,N') \to (A,N)$ is a trivial fibration in $P\mathbf{Mod}_A$, hence we can construct a map $(A,M) \to (A,N')$ making the left triangle to commute, this gives us the desired lift $(A,M) \to (B',N')$. $\square$

Pairing between categories of modules

The category of associative algebras in right modules over $\mathcal{O}$ is a symmetric monoidal category. In the end of this section, we want to show that the functor $P \mapsto P\mathbf{Mod}_A$ is symmetric monoidal in a certain sense.

First, notice that if $\mathbf{S}$ is any symmetric monoidal category, the category of associative algebras in $\mathbf{S}$ inherits a symmetric monoidal category structure.

Proposition 1.3.6. *Let A be an object of $\mathbf{C}[\mathcal{O}]$. The functor:*

$$\mathbf{Mod}_{\mathcal{O}}[\mathcal{A}ss] \to \mathbf{C}[\mathcal{A}ss]$$

sending P to U_A^P is monoidal.

Proof. We want to construct an isomorphism:

$$U_A^P \otimes U_A^Q \cong U_A^{P\otimes Q}$$

It is easy to check that for any object X of $\mathbf{C}$ the following identity is satisfied:

$$(P \otimes Q) \circ X \cong (P \circ X) \otimes (Q \circ X)$$

Since the monoidal structure in $\mathbf{C}$ commutes with colimits in each variable, we have:

$$U_A^P \otimes U_A^Q \cong \operatorname{coeq}[(P \circ \mathcal{O} \circ A) \otimes (Q \circ \mathcal{O} \circ A) \rightrightarrows (P \circ A) \otimes (Q \circ A)]$$

Because of the previous observation, this coequalizer can be rewritten as:

$$\operatorname{coeq}[(P \otimes Q) \circ \mathcal{O} \circ A \rightrightarrows (P \otimes Q) \circ A]$$

which is exactly the definition of $U_A^{P\otimes Q}$. □

Proposition 1.3.7. *Let R and S be two associative algebras in $\mathbf{C}$ whose underlying object is cofibrant. The monoidal product of $\mathbf{C}$ extends to a pairing:*

$$\mathbf{Mod}_R \otimes \mathbf{Mod}_S \to \mathbf{Mod}_{R\otimes S}$$

Moreover this pairing is a left Quillen bifunctor.

Proof. The first claim is straightforward.

It suffices to check the pushout-product condition on generating cofibrations and generating trivial cofibrations. If I is a set of generating cofibrations for $\mathbf{C}$ and J is a set of generating trivial cofibration for $\mathbf{C}$, we can take $I \otimes R$ as generating cofibrations in $\mathbf{Mod}_R$ and $J \otimes R$ as generating trivial cofibrations in $\mathbf{Mod}_R$ and similarly for $\mathbf{Mod}_S$ and $\mathbf{Mod}_{R\otimes S}$. With this particular choice, the claim follows directly from the fact that the tensor product of $\mathbf{C}$ itself satisfies the pushout-product axiom. □

Corollary 1.3.8. *Let P and Q be two associative algebras in right modules over $\mathcal{O}$ and A be a cofibrant $\mathcal{O}$-algebra. The monoidal product of $\mathbf{C}$ extends to a pairing:*

$$P\mathbf{Mod}_A \otimes Q\mathbf{Mod}_A \to (P \otimes Q)\mathbf{Mod}_A$$

Moreover, this pairing is a left Quillen bifunctor.

Proof. This follows directly from the previous two propositions together with the identification as model categories:

$$P\mathbf{Mod}_A = \mathbf{Mod}_{U_A^P}, \quad Q\mathbf{Mod}_A = \mathbf{Mod}_{U_A^Q}$$

□

1.4 Functors induced by bimodules

It is well-known that an A-B-bimodule induces a functor from the category of right A-modules to the category of right B-modules. In this section, we study how this functor can be derived in a model category context.

In this section, $\mathbf{V}$ is a cofibrantly generated closed monoidal model category. We make a slight abuse of notation and denote $\mathbf{V}[\mathcal{A}ss]$ the category of associative algebras in $\mathbf{V}$ even though, we have defined the operad $\mathcal{A}ss$ as a symmetric operad.

Proposition 1.4.1. *The category $\mathbf{V}[\mathcal{A}ss]$ of associative algebras in $\mathbf{V}$ with its transferred model structure is such that the forgetful functor:*

$$\mathbf{V}[\mathcal{A}ss] \to \mathbf{V}$$

preserves cofibrations and trivial cofibrations.

Proof. This is a direct application of A.1.3. This is also [SS00] Theorem 4.1. □

Remark 1.4.2. The unit object of $\mathbf{V}$ is the initial associative algebra in $\mathbf{V}$. If it is cofibrant, then this proposition implies that any cofibrant object in $\mathbf{V}[\mathcal{A}ss]$ is cofibrant in $\mathbf{V}$. The reason this is useful is that the category $\mathbf{Mod}_A$ is usually better behaved if the underlying object of A is cofibrant. It is in general not true that any associative algebra is weakly equivalent as an associative algebra to one whose underlying object is cofibrant. However any associative algebra is weakly equivalent to a cofibrant associative algebra.

Proposition 1.4.3. *Let A and B be two associative algebras in $\mathbf{V}$ whose underlying object* *is cofibrant, then the forgetful functor:*

$$ {}_A\mathbf{Mod}_B \to \mathbf{Mod}_B $$

preserves cofibrations.

Proof. This functor is the right adjoint of a Quillen adjunction:

$$ A \otimes - : \mathbf{Mod}_B \rightleftarrows {}_A\mathbf{Mod}_B $$

Moreover the model structure on the right hand side is transferred from the model structure of the left hand-side. The right adjoint preserves filtered colimits and pushouts, therefore by A.1.3, the proposition will be proved if for any generating cofibration g of $\mathbf{V}$, the map $A \otimes g \otimes B$ is a cofibration in $\mathbf{Mod}_B$. But A is cofibrant, therefore, $A \otimes g$ is a cofibration in $\mathbf{V}$ and $A \otimes g \otimes B$ is a cofibration in $\mathbf{Mod}_B$. $\square$

Proposition 1.4.4. *Let A, B and C be three associative algebras in $\mathbf{V}$ whose underlying object is cofibrant. The relative tensor product:*

$$ - \otimes_B - : {}_A\mathbf{Mod}_B \times {}_B\mathbf{Mod}_C \to {}_A\mathbf{Mod}_C $$

is a Quillen bifunctor.

Proof. Let $f : X \to Y$ be a cofibration in ${}_A\mathbf{Mod}$. Then:

$$ X \otimes B \xrightarrow{f \otimes B} Y \otimes B $$

is a cofibration in ${}_A\mathbf{Mod}_B$. Let $g : P \to Q$ be a cofibration in ${}_B\mathbf{Mod}_C$, then the pushout-product of $f \otimes B$ and g is:

$$ X \otimes Q \cup^{X \otimes P} Y \otimes P \to Y \otimes Q $$

It suffices to check that this is a cofibration to prove the proposition. Indeed maps of the for $f \otimes B$ generate all the cofibrations in ${}_A\mathbf{Mod}_B$.

By the previous proposition g is a cofibration in $\mathbf{Mod}_C$. Therefore we have to prove that the pairing:

$$_A\mathbf{Mod} \times \mathbf{Mod}_C \to {}_A\mathbf{Mod}_C$$

satisfies the pushout product axiom which is trivially checked on generators. □

This implies in particular by Ken Brown's lemma that the relative tensor product preserves any weak equivalence between cofibrant objects.

Corollary 1.4.5. *Let M be a cofibrant object of ${}_A\mathbf{Mod}_B$, then:*

$$- \otimes_A M : \mathbf{Mod}_A \to \mathbf{Mod}_B$$

is a left Quillen functor.

Proof. Since $\mathbf{V}$ is closed, this functor is a left adjoint. By the previous proposition, it preserves cofibrations and trivial cofibrations. □

Remark 1.4.6. Functors of the form $- \otimes_A M$ have the property that they preserve colimits. In good cases, *all* colimit preserving functors are of this form up to homotopy. Goodwillie calculus says that any colimit preserving functor from spectra to spectra is the smash product with a given spectrum and we believe the same is true for colimit preserving functors $\mathbf{Mod}_A \to \mathbf{Mod}_B$ with A and B two associative algebras in spectra. See also [Toë07] Corollary 7.6. for a more precise statement in the case of chain complexes.

1.5 Bicategory and A_∞-simplicial categories

1.6 Simplicial operad of algebras and bimodules

The previous section was about constructing a functor from $\mathbf{Mod}_A$ to $\mathbf{Mod}_B$ out of an A-B-bimodule. In this section, we globalize this construction and construct a simplicial category whose objects are associative algebras and whose space of morphisms is the ∞-groupoid of weak equivalences in the ∞-category of A-B-bimodule. Moreover, if we are in a symmetric monoidal category, associative algebras and bimodules can be tensored together and we can extend that category to an operad.

Construction of the category of algebras and bimodules

Let $\mathbf{V}$ be a cofibrantly generated monoidal model category. The category of associative algebras in $\mathbf{V}$ has a transferred model structure (see[SS00]).

Construction 1.6.1. We construct a large bicategory $\mathfrak{BiMod}(\mathbf{V})$.

The object of $\mathfrak{BiMod}(\mathbf{V})$ are associative algebras in $\mathbf{V}$ whose underlying object is cofibrant.

Let A and B be two objects of $\mathfrak{BiMod}(\mathbf{V})$, the category of morphisms $\mathbf{Map}_{\mathfrak{BiMod}(\mathbf{V})}(A, B)$ is the category whose objects are cofibrant objects of ${}_A\mathbf{Mod}_B$ and whose morphisms are weak equivalences. The composition:

$$\mathbf{Map}_{\mathfrak{BiMod}(\mathbf{V})}(A, B) \times \mathbf{Map}_{\mathfrak{BiMod}(\mathbf{V})}(B, C) \to \mathbf{Map}_{\mathfrak{BiMod}(\mathbf{V})}(A, C)$$

is induced by the relative tensor product functor:

$${}_A\mathbf{Mod}_B \times_B \mathbf{Mod}_C \to {}_A\mathbf{Mod}_C$$

Since we restrict to cofibrant bimodules, this map is well-defined (1.4.4). The fact that this data has the structure of a bicategory is checked in [Shu10].

Whenever, we have a bicategory, we can take the nerve of each Hom category. The resulting structure is not simplicial category since the composition is not strictly associative. The structure we get is a $\mathcal{K}$-simplicial category (where $\mathcal{K}$ is the Stasheff operad).

Definition 1.6.2. A *$\mathcal{K}$-simplicial category* $\mathbf{X}$ is:

- A set of objects $\mathrm{Ob}(\mathbf{X})$.
- Mapping spaces $\mathrm{Map}_{\mathbf{X}}(X, Y)$ for any pair of objects of $\mathbf{X}$

- Composition morphisms for any n-tuple of objects (including $n = 0$):

$$\mathcal{K}(n) \times \mathrm{Map}_{\mathbf{X}}(X_1, X_2) \times \ldots \times \mathrm{Map}_{\mathbf{X}}(X_{n-1}, X_n) \to \mathrm{Map}_{\mathbf{X}}(X_1, X_n)$$

All this data is required to satisfy the obvious associativity condition compatibly with the operadic composition in $\mathcal{K}$.

Note that a simplicial category is in an obvious way a $\mathcal{K}$-category. If we apply π_0 to each Hom space of a $\mathcal{K}$-simplicial category, we get a honest category that deserves to be called the homotopy category. Now we can say that a functor $f : \mathbf{X} \to \mathbf{Y}$ between $\mathcal{K}$-simplicial categories is a Dwyer-Kan equivalence if the induced map on homotopy categories is an equivalence and the maps

$$\mathrm{Map}_{\mathbf{X}}(x, y) \to \mathrm{Map}_{\mathbf{Y}}(f(x), f(y))$$

are weak equivalences.

The forgetful functors from simplicial categories with Bergner's model structure (see [Ber07]) to $\mathcal{K}$-simplicial categories preserves Dwyer-Kan equivalences. This functor induces an equivalence from the ∞-category of simplicial categories to the ∞-category of $\mathcal{K}$-simplicial category. Although well-known to experts, this theorem does not seem to appear anywhere in the literature.

The following proposition allows one to replace functors from a $\mathcal{K}$-simplicial category to a simplicial category by functors from an equivalent simplicial category.

Proposition 1.6.3. *Let $\mathbf{X}$ be a $\mathcal{K}$-simplicial category. There is a simplicial category X' and an equivalence of $\mathcal{K}$-simplicial categories $X \to X'$ such that any map of $\mathcal{K}$-simplicial categories $\mathbf{X} \to \mathbf{Y}$ with $\mathbf{Y}$ a simplicial category factors through $\mathbf{X}'$.*

Proof. Let S be the set of objects of $\mathbf{X}$. There is an operad in sets $\mathcal{C}_S^{\mathcal{K}}$ whose algebras in $\mathbf{S}$ are $\mathcal{K}$-simplicial categories with set of objects S. Similarly, there is an operad $\mathcal{C}_S$ whose algebras in $\mathbf{S}$ are simplicial algebras with set of objects S. There is a weak equivalence of operads $\rho : \mathcal{C}_S^{\mathcal{K}} \to \mathcal{C}_S$. Moreover, both operads are Σ-cofibrant. Define $X' = \rho^* \rho_! X$.

There is counit map $X \to X'$ which is a weak equivalence since the pair $(\rho_!, \rho^*)$ is a Quillen equivalence (see B.3.4).

Now any map $X \to Y$ factors as $\mathbf{X} \to \mathbf{U} \to \mathbf{Y}$ where $\mathbf{U}$ is a simplicial category with S as set of objects and $\mathbf{U} \to \mathbf{Y}$ is fully faithful. The map $\mathbf{X} \to \mathbf{U}$ is adjoint to a map $\rho_!\mathbf{X} \to \mathbf{U}$ which gives the desired factorization. □

Definition 1.6.4. We denote by $\mathbf{BiMod}(\mathbf{V})$ the $\mathcal{K}$-simplicial category whose objects are $\mathrm{Ob}(\mathfrak{BiMod}(\mathbf{V}))$ and with:

$$\mathrm{Map}_{\mathbf{BiMod}(\mathbf{V})}(A, B) = \mathrm{N}_\bullet(\mathbf{Map}_{\mathfrak{BiMod}(\mathbf{V})}(A, B))$$

Let us recall the definition of the grouplike monoid of homotopy automorphisms of an object P in a model category $\mathbf{X}$.

Construction 1.6.5. If $\mathbf{X}$ is a simplicial model category, the group $\mathrm{Auth}(P)$ has a simple description. First, we take a cofibrant-fibrant replacement P' of P. Then $\mathrm{Auth}(P)$ is the following pullback:

$$\begin{tikzcd}
\mathrm{Auth}(P) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{X}}(P', P') \arrow[d] \\
\pi_0(\mathrm{Map}_{\mathbf{X}}(P', P'))^{\times} \arrow[r] & \pi_0(\mathrm{Map}_{\mathbf{X}}(P', P'))
\end{tikzcd}$$

If $\mathbf{X}$ is not simplicial, it still has a hammock localization as any model category (see [DK80]) denoted $L^H\mathbf{X}$. The space $\mathrm{Map}_{L^H\mathbf{X}}$ can be used instead of $\mathrm{Map}_{\mathbf{X}}$ in the above definition. Note that the two definition coincide when the model category is simplicial.

The space $\mathrm{Map}_{\mathbf{BiMod}(\mathbf{V})}$ has the homotopy type of the moduli space of ${}_A\mathbf{Mod}_B$ (see e.g. [DK84]). More explicitely, it splits as:

$$\mathrm{Map}_{\mathbf{BiMod}(\mathbf{V})}(A, B) \simeq \coprod_{M \in \text{ isom. classes in } \mathbf{Ho}({}_A\mathbf{Mod}_B)} B\mathrm{Auth}(M)$$

Now assume that $G : \mathbf{V} \to \mathbf{W}$ is a monoidal left Quillen functor between monoidal model category.

Proposition 1.6.6. *The functor G induces a functor of bicategories:*

$$\mathfrak{BiMod}(G) : \mathfrak{BiMod}(\mathbf{V}) \to \mathfrak{BiMod}(\mathbf{W})$$

Proof. G is both monoidal and left Quillen. Therefore, it preserves associative algebras whose underlying object is cofibrant. It is then easy to check that G also induces a left Quillen functor between categories of bimodules. The fact that $\mathfrak{BiMod}(G)$ preserves composition is checked in [Shu10]. □

Corollary 1.6.7. *Same notations. G induces a functor of $\mathcal{K}$-simplicial category:*

$$\mathbf{BiMod}(\mathbf{V}) \to \mathbf{BiMod}(\mathbf{W})$$

Construction of the operad of algebras and bimodules

We now want to assume that $\mathbf{V}$ is a *symmetric* monoidal category. In this case, one can prove that $\mathbf{BiMod}(\mathbf{V})$ is a symmetric monoidal bicategory (see [Shu10]). However, for our purposes, we only care about the underlying operad which we now construct.

Definition 1.6.8. Let I be a finite set. For $\{A_i\}_{i\in I}$ an I-indexed family of associative algebras and B an associative algebra, we define:

$$_{\{A_i\}_{i\in I}}\mathbf{Mod}_B$$

to be the category whose objects have a left action by each of the A_i and a right action of B all of these commuting with one another.

Note that $_{\{A_i\}_{i\in I}}\mathbf{Mod}_B$ has a tranferred model structure if each of the A_i and B have a cofibrant underlying object.

Construction 1.6.9 (sketch)**.** There is a $\mathcal{K}$ operad $\mathcal{B}i\mathcal{M}od(\mathbf{V})$ whose colors are associative algebras in $\mathbf{V}$ whose underlying object is cofibrant.

Let I be a finite set. For $\{A_i\}_{i\in I}$ an I-indexed family of associative algebras and B an associative algebra, we define:

$$\mathcal{B}i\mathcal{M}od(\mathbf{V})(\{A_i\}_{i\in I}; B)$$

to be the nerve of the category whose objects are cofibrant objects in ${}_{\{A_i\}_{i\in I}}\mathbf{Mod}_B$ and morphisms are weak equivalences between those.

We did not define what a $\mathcal{K}$ operad is. Let us just say that is is to an operad what a $\mathcal{K}$ simplicial category is to a simplicial category. In fact one could define a notion of bioperad which is the straightforward generalization of a bicategory which allows many inputs. Applying the nerve to the mappings spaces of a bioperad yields a $\mathcal{K}$ operad. The above construction is an example of this procedure.

Proposition 1.6.10. *Let $G : \mathbf{V} \to \mathbf{W}$ be a symmetric monoidal left Quillen functor between symmetric monoidal model category. Then it induces a functor of $\mathcal{K}$ operads:*

$$\mathcal{B}i\mathcal{M}od(G) : \mathcal{B}i\mathcal{M}od(\mathbf{V}) \to \mathcal{B}i\mathcal{M}od(\mathbf{W})$$

Proof. Easy. $\square$

1.7 Simplicial operad of model categories

In this section construct a large category whose objects are model categories and whose space of morphisms can be roughly described as the set of left Quillen functors up to weak equivalences. We then extend this structure into an operad by allowing Quillen functors with several inputs.

The simplicial category of model categories

Definition 1.7.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be two model categories. Let F and G be two left Quillen functors $\mathbf{X} \to \mathbf{Y}$. A *natural weak equivalence* $\alpha : F \to G$ is a natural transformation with the property that $\alpha(x) : F(x) \to G(x)$ is a weak equivalence for any cofibrant $x \in \mathrm{Ob}(\mathbf{X})$.

There is an obvious (vertical) composition between natural weak equivalences but there is also an horizontal composition between natural transformation which preserves natural weak equivalences by the following proposition.

Proposition 1.7.2. *Let* $\mathbf{X}$*,* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be three model categories and let* F*,* G *be two left Quillen functors from* $\mathbf{X}$ *to* $\mathbf{Y}$ *and* K *and* L *be two left Quillen functors from* $\mathbf{Y} \to \mathbf{Z}$*. Let* α *be a natural weak equivalence between* F *and* G *and* β *be a natural weak equivalence between* K *and* L*, then the horizontal composition is again a natural weak equivalence.*

Proof. The horizontal composition evaluated at a cofibrant object x is the composition:

$$KF(x) \xrightarrow{\beta F} LF(x) \xrightarrow{L\alpha} LG(x)$$

Since F is left Quillen, $F(x)$ is cofibrant and the first map is a weak equivalence. The second map is L applied to $\alpha(x) : F(x) \to G(x)$ which is a weak equivalence between cofibrant objects. Since L is left Quillen, this is an equivalence as well. □

Construction 1.7.3. The category **ModCat** is the simplicial category whose objects are model categories and whose space of morphism from $\mathbf{X}$ to $\mathbf{Y}$ is the nerve of the category whose objects are left Quillen functors: $\mathbf{X} \to \mathbf{Y}$ and morphisms are natural weak equivalences between left Quillen functors.

Inspired by [Bar10] we suggest the following definition:

Definition 1.7.4. Let $\mathbf{K}$ be a simplicial category. A *left Quillen diagram* of shape K is a simplicial functor:

$$\mathbf{K} \to \mathbf{ModCat}$$

Note that if $\mathbf{K}$ is an ordinary category a left Quillen presheaf in the sense of [Bar10] is exactly a left Quillen diagram:

$$\mathbf{K}^{\mathrm{op}} \to \mathbf{ModCat}$$

Ths simplicial operad of model categories

Now we want to extend **ModCat** to an operad.

Note that **Cat** is a symmetric monoidal category for the cartesian product; however this structure does not extend well to **ModCat**. For two model categories $\mathbf{X}$ and $\mathbf{Y}$, one can put a product model structure on $\mathbf{X} \times \mathbf{Y}$, but the left Quillen functors from $\mathbf{X} \times \mathbf{Y}$ to $\mathbf{Z}$ are usually not the right thing to consider. The correct notion of "pairing" $\mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ is the notion of a left Quillen bifunctor (see [Hov99], or appendix A).

We need a version of a Quillen multifunctor with more than two inputs. Let us first recall the definition of the cube category.

Definition 1.7.5. The *n-dimensional cube* is the poset of subsets of $\{1, \ldots, n\}$. We use the notation $\mathbf{P}(n)$ to denote that category. Equivalently, $\mathbf{P}(n)$ is the product of n copies of $\mathbf{P}(1)$. The category $\mathbf{P}_1(n)$ is the full subcategory of $\mathbf{P}(n)$ contatining all objects except the maximal element.

Definition 1.7.6. If $(\mathbf{X}_i)_{i \in \{1,\ldots,n\}}$ is a family of categories and f_i is an arrow in $\mathbf{X}_i$ for each i, we denote by $C(f_1, \ldots, f_n)$ the product:

$$\prod_i f_i : \mathbf{P}(n) \to \prod_i \mathbf{X}_i$$

Definition 1.7.7. Let $(\mathbf{X}_i)_{i \in \{1,\ldots,n\}}$ and $\mathbf{Y}$ be model categories. Let $T : \prod_{i=1}^n \mathbf{X}_i \to \mathbf{Y}$ be a functor. We say that T is a *left Quillen n-functor* if it satisfies the following three condition:

- If we fix all variables but one. The induced functor $\mathbf{X}_i \to \mathbf{Y}$ is a left adjoint.
- If $f_i : A_i \to B_i$ is a cofibration in $\mathbf{X}_i$ for $i \in \{1, \ldots, n\}$ then the map:

  $$\mathrm{colim}_{\mathbf{P}_1(n)} T(C(f_1, \ldots, f_n)) \to T(B_1, \ldots, B_n)$$

  is a cofibration in $\mathbf{Y}$
- If further one of the f_i is a trivial cofibration, then the map:

  $$\mathrm{colim}_{\mathbf{P}_1(n)} T(C(f_1, \ldots, f_n)) \to T(B_1, \ldots, B_n)$$

  is a trivial cofibration in $\mathbf{Y}$

Note that the category with one objects and only the identity is the unit of the cartesian product in **Cat**. It is a model category in a unique way. A Quillen 0-functor whose target is $\mathbf{Y}$ is just an object of $\mathbf{Y}$.

Definition 1.7.8. A *natural weak equivalence between left Quillen n-functors* T and S is a natural transformation $T \to T'$ with the property that:

$$T(A_1, \dots, A_n) \to T'(A_1, \dots, A_n)$$

is a weak equivalence whenever A_i is cofibrant for all i.

Construction 1.7.9. We construct a large operad $\mathcal{M}od\mathcal{C}at$ whose colors are model category and whose space of operations $\mathcal{M}od\mathcal{C}at(\{\mathbf{X}_i\}; \mathbf{Y})$ is the nerve of the category of left Quillen n-functors $\prod_i \mathbf{X}_i \to \mathbf{Y}$ and natural weak equivalences.

Now, take $\mathbf{V}$ to be a cofibrantly generated closed monoidal model category.

Proposition 1.7.10. *There is a left Quillen diagram of shape* $\mathbf{BiMod}(\mathbf{V})$ *sending* A *to* $\mathbf{Mod}_A$ *and* M *to:*

$$- \otimes_A M : \mathbf{Mod}_A \to \mathbf{Mod}_B$$

Proof. Both $\mathbf{BiMod}(\mathbf{V})$ and **ModCat** are obtained as nerves of a certain bicategories, therefore it suffices to construct this functor at the bicategorical level. This is then a standard model category argument. □

Now assume that $\mathbf{V}$ is a cofibrantly generated *symmetric* monoidal closed model category.

Proposition 1.7.11. *The functor from* $\mathbf{BiMod}(\mathbf{V})$ *to* **ModCat** *extends to a functor of* $\mathcal{K}$ *operad:*

$$\mathcal{B}i\mathcal{M}od(\mathbf{V}) \to \mathcal{M}od\mathcal{C}at$$

Proof. Again it suffices to do this at the bicategorical level where this is almost tautologous. □

1.8 An algebraic field theory

In this section $\mathbf{C}$ is a symmetric monoidal simplicial cofibrantly generated model category with a good theory of algebras (resp. with a good theory of algebras over Σ-cofibrant operads).

The work of the previous two sections has the following corollary:

Theorem 1.8.1. *Let P be an associative algebra in right modules over some operad (resp. Σ-cofibrant operad) $\mathcal{O}$ whose underlying $\mathcal{O}$-module is cofibrant and A be a cofibrant $\mathcal{O}$-algebra in $\mathbf{C}$. Let $\mathcal{E}nd_P$ be the endomorphism operad of P in the operad $\mathcal{B}i\mathcal{M}od(\mathbf{Mod}_{\mathcal{O}})$. Then,the category $P\mathbf{Mod}_A$ is an $\mathcal{E}nd_P$-algebra in $\mathcal{M}od\mathcal{C}at$.*

More generally, the assignment $P \mapsto P\mathbf{Mod}_A$ defines a $\mathcal{B}i\mathcal{M}od(\mathbf{Mod}_{\mathcal{O}})$-algebra in $\mathcal{M}od\mathcal{C}at$.

Proof. The functor $P \mapsto P \circ_{\mathcal{O}} A$ is left Quillen and symmetric monoidal from $\mathbf{Mod}_{\mathcal{O}}$ to $\mathbf{C}$ therefore by 1.6.10, it induces a morphism of $\mathcal{K}$ operad:

$$\mathcal{B}i\mathcal{M}od(\mathbf{Mod}_{\mathcal{O}}) \to \mathcal{B}i\mathcal{M}od(\mathbf{C})$$

Now we can use 1.7.11 to construct a morphism of $\mathcal{K}$ operad:

$$\mathcal{B}i\mathcal{M}od(\mathbf{C}) \to \mathcal{M}od\mathcal{C}at$$

□

Note that in the above theorem, we have to restrict to categories $P\mathbf{Mod}_A$ defined by an associative algebra in right $\mathcal{O}$-module P whose underlying right $\mathcal{O}$-module is cofibrant. This would not be a problem if any associative algebra in right $\mathcal{O}$-module was equivalent to one of this sort.

If the unit of $\mathbf{C}$ is cofibrant, then the unit of $\mathbf{Mod}_{\mathcal{O}}$ is cofibrant and any cofibrant associative algebra in $\mathbf{Mod}_{\mathcal{O}}$ is cofibrant in $\mathbf{Mod}_{\mathcal{O}}$ (A.1.3). In particular, any associative algebra in $\mathbf{Mod}_{\mathcal{O}}$ is equivalent to one whose underlying object is cofibrant.

There are cases where the unit is not cofibrant like categories of the form $L_{Z}p\mathbf{Mod}_E$ where E is a commutative algebra in symmetric spectra. However, in this case, we can use the absolute model structure. The object underlying U_A^P is cofibrant for the absolute model structure (B.3.9). Therefore the above theorm is true if we give the category $P\mathbf{Mod}_A$ the absolute model structure. This is harmless from the homotopical point of view since the identity map:

$$pP\mathbf{Mod}_A \to aP\mathbf{Mod}_A$$

is a Quillen equivalence.

The title of this section is in reference to the fourth chapter in which we are going to identify a suboperad of $\mathcal{B}i\mathcal{M}od(\mathbf{Mod}_{\mathcal{E}_n})$ as a very close relative of the cobordism category.

Chapter 2

The operad of little disks and its variants

This chapter is mainly technical. We review the traditional definition of the little disk operad. Then we define a topological space of embeddings between framed manifolds possibly with boundary. From these spaces of embeddings we construct a model of the little disk operad and its variants and we show that this model is equivalent to the traditional one.

2.1 Traditional definition

In this section, we give a traditional definition of the *little d-disk operad* $\mathcal{D}_d$ as well as a definition of the *swiss-cheese operad* $\mathcal{SC}_d$ which we denote $\mathcal{D}_d^\partial$. The swiss-cheese operad, originally defined by Voronov (see [Vor99] for a definition when $d = 2$ and [Tho10] for a definition in all dimensions), is a variant of the little d-disk operad which describes the action of an $\mathcal{E}_d$-algebra on an $\mathcal{E}_{d-1}$-algebra.

Space of rectilinear embeddings

Let D denote the open disk of dimension d, $D = \{x \in \mathbb{R}^d, \|x\| < 1\}$.

Definition 2.1.1. Let U and V be connected subsets of $\mathbb{R}^d$, let i_U and i_V denote the inclusion into $\mathbb{R}$. We say that $f : U \to V$ is a *rectilinear embedding* if there is an element L

in the subgroup of $\mathrm{Aut}(\mathbb{R}^d)$ generated by translation and dilations with positive factor such that:

$$i_V \circ f = L \circ i_U$$

We extend this definition to disjoint unions of open subsets of $\mathbb{R}^d$:

Definition 2.1.2. Let $U_1, \ldots, U_n$ and $V_1, \ldots, V_m$ be finite families of connected subsets of $\mathbb{R}^d$. The notation $U_1 \sqcup \ldots \sqcup U_n$ denotes the coproduct of $U_1, \ldots U_n$ in the category of topological spaces. We say that a map from $U_1 \sqcup \ldots \sqcup U_n$ to $V_1 \sqcup \ldots \sqcup V_m$ is a *rectilinear embedding* if it satisfies the following properties:

1. Its restriction to each component can be factored as $U_i \to V_j \to V_1 \sqcup \ldots \sqcup V_m$ where the second map is the obvious inclusion and the first map is a rectilinear embedding $U_i \to V_j$.

2. The underlying map of sets is injective.

We denote by $\mathrm{Emb}_{lin}(U_1 \sqcup \ldots \sqcup U_n, V_1 \sqcup \ldots \sqcup V_m)$ the subspace of $\mathrm{Map}(U_1 \sqcup \ldots \sqcup U_n, V_1 \sqcup \ldots \sqcup V_m)$ whose points are rectilinear embeddings.

Observe that rectilinear embeddings are stable under composition.

The d-disk operad

Definition 2.1.3. The *linear d-disk operad*, denoted $\mathcal{D}_d$, is the operad in topological spaces whose n-th space is $\mathrm{Emb}_{lin}(D^{\sqcup n}, D)$ with the composition induced from the composition of rectilinear embeddings.

There are variants of this definition but they are all equivalent to this one. In the above definition $\mathcal{D}_d$ is an operad in topological spaces. By applying the functor Sing, we get an operad in $\mathbf{S}$. We use the same notation for the topological and the simplicial operad.

The Swiss-cheese operad

As before, we denote by D, the d-dimensional disk and by H the d-dimensional half-disk:

$$H = \{x = (x_1, \ldots, x_d\}), \|x\| < 1, x_d \geq 0\}$$

Definition 2.1.4. The *linear d-dimensional swiss-cheese operad*, denoted $\mathcal{D}_d^\partial$, has two colors z and h and its mapping spaces are:

$$\mathcal{D}_d^\partial(z^{\boxplus n}, z) = \mathrm{Emb}_{lin}(D^{\sqcup n}, D)$$
$$\mathcal{D}_d^\partial(z^{\boxplus n} \boxplus h^{\boxplus m}, h) = \mathrm{Emb}_{lin}^\partial(D^{\sqcup n} \sqcup H^{\sqcup m}, H)$$

where the ∂ superscript means that we restrict to embeddings preserving the boundary.

Proposition 2.1.5. *The full suboperad of $\mathcal{D}_d^\partial$ on the color z is isomorphic to $\mathcal{D}_d$ and the full suboperad on the color h is isomorphic to $\mathcal{D}_{d-1}$.*

Proof. Easy. □

Proposition 2.1.6. *The evaluation at the center of the disks induces a weak equivalence:*

$$\mathcal{D}_d^\partial(z^{\boxplus n} \boxplus h^{\boxplus m}, h) \to \mathrm{Conf}(m, \partial H) \times \mathrm{Conf}(n, H - \partial H)$$

Proof. This map is a Hurewicz fibration whose fibers are contractible. □

2.2 Homotopy pullback in $\mathbf{Top}_W$

The material of this section can be found in [And10]. We have included it mainly for the reader's convenience and also to give a proof of 2.2.4 which is mentioned without proof in [And10].

Homotopy pullback in Top

Let us start by recalling the following well-known proposition:

Proposition 2.2.1. *Let:*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

be a diagram in **Top**. *The homotopy pullback of that diagram can be constructed as the space of triples* (x, p, y) *where* x *is a point in* X, y *is a point in* Y *and* p *is a path from* $f(x)$ *to* $g(y)$ *in* Z. □

Homotopy pullback in $\mathbf{Top}_W$

Let W be a topological space. There is a model structure on $\mathbf{Top}_W$ the category of topological spaces over W in which cofibrations, fibrations and weak equivalences are reflected by the forgetful functor $\mathbf{Top}_W \to \mathbf{Top}$. We want to study homotopy pullbacks in $\mathbf{Top}_W$

We denote a space over W by a single capital letter like X and we write p_X for the structure map $X \to W$.

Let $I = [0, 1]$, for Y an object of $\mathbf{Top}_W$, we denote by Y^I the cotensor in the category $\mathbf{Top}_W$. Concretely, Y^I is the space of paths in Y whose image in W is a constant path.

Definition 2.2.2. Let $f : X \to Y$ be a map in $\mathbf{Top}_W$. We denote by Nf the following pullback in $\mathbf{Top}_W$:

$$\begin{tikzcd} Nf \arrow[r] \arrow[d] & Y^I \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Concretely, Nf is the space of pairs (x, p) where x is a point in X and p is a path in Y whose value at 0 is $f(x)$ and lying over a constant path in W.

Proposition 2.2.3. *Let:*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r] & Z \end{tikzcd}$$

be a diagram in $\mathbf{Top}_W$ *in which* X *and* Z *are fibrant (i.e. the structure maps* p_X *and* p_Z *are fibrations) then the pullback of the following diagram in* $\mathbf{Top}_W$ *is a model for the homotopy pullback:*

$$\begin{tikzcd} & Nf \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Concretely, this proposition is saying that the homotopy pullback is the space of triple (x, p, y) where x is a point in X, y is a point in Y and p is a path in Z between $f(x)$ and $g(y)$ lying over a constant path in W.

Proof of the proposition. The proof is similar to the analogous result in **Top**, it suffices to check that the map $Nf \to Z$ is a fibration in $\mathbf{Top}_W$ which is weakly equivalent to $X \to Z$. Since the category $\mathbf{Top}_W$ is right proper, a pullback along a cofibration is always a homotopy pullback. □

From now on when we talk about a homotopy pullback in the category $\mathbf{Top}_W$, we mean the above specific model. Note that even thoug it looks like the map f plays a special role, this construction is symmetric in X and Y.

Comparison of homotopy pullbacks in Top and in $\mathbf{Top}_W$

For a diagram:

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r] & Z \end{tikzcd}$$

in **Top** (resp. $\mathbf{Top}_W$), we denote by $\mathrm{hpb}(X \to Z \leftarrow Y)$ (resp. $\mathrm{hpb}_W(X \to Z \leftarrow Y)$) the above model of homotopy pullback in **Top** (resp. $\mathbf{Top}_W$).

Note that there is an obvious inclusion:

$$\mathrm{hpb}_W(X \to X \leftarrow Y) \to \mathrm{hpb}(X \to Z \leftarrow Y)$$

which sends a path (which happens to be constant in W) to itself.

Proposition 2.2.4. *Let W be a topological space and $X \to Y \leftarrow Z$ be a diagram in $\mathbf{Top}_W$ in which the structure maps $X \to W$ and $Y \to W$ are fibrations, then the inclusion:*

$$\mathrm{hpb}_W(X \to Y \leftarrow Z) \to \mathrm{hpb}(X \to Y \leftarrow Z)$$

is a weak equivalence.

Proof. [1] Let us consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{hopb}_W(X \to Y \leftarrow Z) \arrow[r] \arrow[d] & \mathrm{hopb}(X \to Y \leftarrow Z) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{hopb}_W(Y \to Y \leftarrow Z) \arrow[r] \arrow[d] & \mathrm{hopb}(Y \to Y \leftarrow Z) \arrow[r] \arrow[d] & Y \\
W \arrow[r] & W^I &
\end{tikzcd}$$

The map $\mathrm{hopb}(Y \to Y \leftarrow Z) \to W^I$ sends a triple (y, p, z) to the image of the path p in W. The map $W \to W^I$ sends a point in W to the constant map at that point. All other maps should be clear.

It is straightforward to check that each square is cartesian.

The category $\mathbf{Top}_W$ is right proper. This implies that a pullback along a fibration is always a homotopy pullback.

Now we make the following three observations:

(1) The map $\mathrm{hopb}(Y \to Y \leftarrow Z) \to W^I$ is a fibration. Indeed it can be identified with the obvious map $Y^I \times_Y Z \to W^I \times_W W$ and $Y^I \to W^I$ and $Z \to W$ are fibrations. This implies that the bottom square is homotopy cartesian.

(2) The map $\mathrm{hopb}(Y \to Y \leftarrow Z) \to Y$ is a fibration. This is almost tautological. We know that fibrations are preserved by pullbacks. In order to construct the homotopy pullback, we replace one of the maps by a fibration and then take the ordinary pullback, so the projection maps from the homotopy pullback to the two factors are fibrations. This implies that the right-hand side square is homotopy cartesian.

(3) The middle line of the diagram $\mathrm{hopb}_W(Y \to Y \leftarrow Z) \to Y$ is a fibration for the same reason. A priori it is a fibration in $\mathbf{Top}_W$ but this is equivalent to being a fibration in $\mathbf{Top}$. This implies that the big horizontal rectangle is homotopy cartesian.

If we combine 2 and 3 we find that the top left-hand side square is homotopy cartesian. If we combine that with 1, we find that the big horizontal rectangle is homotopy cartesian.

[1] The following proof is due to Ricardo Andrade

The map $W \to W^I$ is a weak equivalence. Therefore the map:

$$\mathrm{hopb}_W(X \to Y \leftarrow Z) \to \mathrm{hopb}(X \to Y \leftarrow Z)$$

is a weak equivalence as well. □

2.3 Embeddings between structured manifolds

This section again owes a lot to [And10]. In particular, the definition 2.3.3 can be found in that reference. We then make analogous definitions of embedding spaces for framed manifolds with boundary and S_τ-manifolds which are straightforward generalizations of Andrade's construction.

Topological space of embeddings

There is a topological category whose objects are d-manifolds possibly with boundary and mapping object between M and N is $\mathrm{Emb}(M, N)$, the topological space of smooth embeddings with the weak C^1 topology. The reader should look at [Hir76] for a definition of this topology. We want to emphasize that this topology is metrizable, in particular $\mathrm{Emb}(M, N)$ is paracompact.

Remark 2.3.1. If one is only interested in the homotopy type of this topological space. One could take instead the C^r-topology for any r (even $r = \infty$). The choice of taking the weak (as opposed to strong topology) however is a serious one. The two topologies coincide when the domain is compact. However the strong topology does not have continuous composition maps:

$$\mathrm{Emb}(M, N) \times \mathrm{Emb}(N, P) \to \mathrm{Emb}(M, P)$$

when M is not compact.

Embeddings between framed manifolds

Definition 2.3.2. A *framed d-manifold* is a pair (M, σ_M) where M is a d-manifold and σ_M is a smooth section of the $\mathrm{GL}(d)$-principal bundle $\mathrm{Fr}(TM)$.

If M and N are two framed d-manifolds, we define a space of framed embeddings denoted by $\mathrm{Emb}_f(M, N)$ as in [And10]:

Definition 2.3.3. Let M and N be two framed d-dimensional manifolds. The *topological space of framed embeddings from M to N*, denoted $\mathrm{Emb}_f(M, N)$, is given by the following homotopy pullback in the category of topological spaces over $\mathrm{Map}(M, N)$:

$$\begin{tikzcd}
\mathrm{Emb}_f(M,N) \arrow[r] \arrow[d] & \mathrm{Map}(M,N) \arrow[d] \\
\mathrm{Emb}(M,N) \arrow[r] & \mathrm{Map}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM), \mathrm{Fr}(TN))
\end{tikzcd}$$

The right hand side map is obtained as the composition:

$$\mathrm{Map}(M,N) \to \mathrm{Map}_{\mathrm{GL}(d)}(M \times \mathrm{GL}(d), N \times \mathrm{GL}(d)) \cong \mathrm{Map}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM), \mathrm{Fr}(TN))$$

where the first map is obtained by taking the product with $\mathrm{GL}(d)$ and the second map is induced by the identification $\mathrm{Fr}(TM) \cong M \times \mathrm{GL}(d)$ and $\mathrm{Fr}(TN) \cong N \times \mathrm{GL}(d)$.

It is not hard to show that there are well defined composition maps:

$$\mathrm{Emb}_f(M,N) \times \mathrm{Emb}_f(N,P) \to \mathrm{Emb}_f(M,P)$$

allowing the construction of a topological category $f\mathbf{Man}_d$ (see [And10]).

Taking a homotopy pullback in the category of spaces over $\mathrm{Map}(M, N)$ is not strictly necessary. Taking the homotopy pullback of the underlying diagram of spaces would have given the same homotopy type by 2.2.4. However, this definition has the psychological advantage that any point in the space $\mathrm{Emb}_f(M, N)$ lies over a point in $\mathrm{Map}(M, N)$ in a canonical way. If we had taken the homotopy pullback in the category of spaces, the resulting object would have had two distinct maps to $\mathrm{Map}(M, N)$, one given by the upper horizontal arrow and the other given as the composition $\mathrm{Emb}_f(M, N) \to \mathrm{Emb}(M, N) \to \mathrm{Map}(M, N)$.

Embeddings between framed manifolds with boundary

If N is a manifold with boundary, n a point of the boundary, and v is a vector in $TN_n - T(\partial N)_n$, we say that v is pointing inward if it can be represented as the tangent vector at 0 of a curve $\gamma : [0,1) \to N$ with $\gamma(0) = n$.

Definition 2.3.4. A *d-manifold with boundary* is a pair (N, ϕ) where N is a d-manifold with boundary in the traditional sense and ϕ is an isomorphism of d-dimensional vector bundles over ∂N:

$$\phi : T(\partial N) \oplus \mathbb{R} \to TN_{|\partial N}$$

which is required to restrict to the canonical inclusion $T(\partial N) \to TN_{|\partial N}$, and which is such that for any n on the boundary, the point $1 \in \mathbb{R}$ is sent to an inward pointing vector through the composition:

$$\mathbb{R} \to T_n(\partial N) \oplus \mathbb{R} \xrightarrow{\phi_n} T_n N$$

Definition 2.3.5. Let (M, ϕ) and (N, ψ) be two d-manifolds with boundary, we define $\mathrm{Emb}(M, N)$ to be the topological space of smooth embeddings from M into N sending ∂M to ∂N, preserving the splitting of the tangent bundles along the boundary $T(\partial M) \oplus \mathbb{R} \to T(\partial N) \oplus \mathbb{R}$. The topology on this space is the weak C^1-topology.

We now introduce framings on manifolds with boundary. We require a framing to interact well eith the boundary.

Definition 2.3.6. Let (N, ϕ) be a d-manifold with boundary. We say that a section σ_N of $\mathrm{Fr}(TN)$ is compatible with the boundary if for each point n on the boundary of N there is a splitting preserving isomorphism:

$$T_n(\partial N) \oplus \mathbb{R} \xrightarrow{\phi_n} T_n N \xrightarrow{\sigma_N} \mathbb{R}^{d-1} \oplus \mathbb{R}$$

A framed d-manifold with boundary is a d-manifold with boundary together with the datum of a compatible framing.

In particular, if ∂M is empty, $\mathrm{Emb}(M, N) = \mathrm{Emb}(M, N - \partial N)$. If ∂N is empty and ∂M is not empty, $\mathrm{Emb}(M, N) = \emptyset$.

Definition 2.3.7. Let M and N be two framed d-manifolds with boundary. We denote $\mathrm{Map}^{\partial}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM), \mathrm{Fr}(TN))$ as the topological space of $\mathrm{GL}(d)$-equivariant maps sending $\mathrm{Fr}(TM_{|\partial M})$ to $\mathrm{Fr}(TN_{|\partial N})$ and preserving the $\mathrm{GL}(d-1)$-subbundle consisting of framings that are compatible with the boundary.

Definition 2.3.8. Let M and N be two framed d-manifolds with boundary. The topological space of framed embeddings from M to N, denoted $\mathrm{Emb}_f(M,N)$, is the following homotopy pullback in the category of topological spaces over $\mathrm{Map}((M,\partial M),(N,\partial N))$:

$$\begin{tikzcd}
\mathrm{Emb}_f(M,N) \arrow[r] \arrow[d] & \mathrm{Map}((M,\partial M),(N,\partial N)) \arrow[d] \\
\mathrm{Emb}(M,N) \arrow[r] & \mathrm{Map}^{\partial}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM),\mathrm{Fr}(TN))
\end{tikzcd}$$

Concretely, a point in $\mathrm{Emb}_f(M,N)$ is a pair (ϕ, p) where $\phi : M \to N$ is an embedding of manifolds with boundary and p is the data at each point m of M of a path between the two trivializations of T_mM (the one given by the framing of M and the one induced by ϕ). These paths are required to vary smoothly with m. Moreover if m is a point on the boundary, the path between the two trivializations of T_mM must be such that at any time, the first $d-1$-vectors are in $T_m\partial M \subset T_mM$.

The simplicial category $\mathbf{Man}^{\partial}_d$ is the category whose objects are manifolds with boundary and whose space of morphism from M to N is the space $\mathrm{Emb}(M,N)$. Similarly, the simplicial category $f\mathbf{Man}^{\partial}_d$ is the category whose objects are framed manifolds with boundary and whose space of morphism from M to N is $\mathrm{Emb}^{\partial}_f(M,N)$. Note that $\mathbf{Man}^{\partial}_d$ contains $\mathbf{Man}_d$ as a full subcategory and similarly $f\mathbf{Man}^{\partial}_d$ contains $f\mathbf{Man}_d$ as a full subcategory.

Manifolds with fixed boundary

In this subsection S is a compact $(d-1)$-manifold.

Definition 2.3.9. An S-*manifold* is a triple (M,ϕ,f) where (M,ϕ) is a d-manifold with boundary and $f : S \to \partial M$ is a diffeomorphism.

Definition 2.3.10. A *collared S-manifold* is a triple (M, ϕ, f) where (M, ϕ) is a d-manifold with boundary and $f : S \times [0, 1) \to M$ is an embedding whose restriction to the boundary induces a diffeomorphism $S \cong \partial M$

If we restrict the collar to the boundary, a collared S-manifolds is an S-manifold. Moreover, it is a standard fact that the space of collars for a given S-manifold is non-empty and contractible. Therefore up to homotopy the two notions are the same.

Definition 2.3.11. A *d-framing* of a $(d-1)$-manifold S is a trivialization of the d-dimensional bundle $TS \oplus \mathbb{R}$ where $\mathbb{R}$ is a trivial line bundle.

Definition 2.3.12. Let τ be a d-framing of S. A *framed S_τ-manifold* is an S-manifolds (M, ϕ, f) with the datum of a framing of TM such that the following composition:

$$TS \oplus \mathbb{R} \xrightarrow{Tf \oplus \mathbb{R}} T(\partial M) \oplus \mathbb{R} \xrightarrow{\phi} TM_{|\partial M}$$

sends τ to the given framing on the right-hand side.

Definition 2.3.13. A framed collared S_τ-manifold is a collared S-manifold (M, ϕ, f) with the datum of a framing of TM such that for some real number ϵ in $(0, 1)$, the following composition of embeddings:

$$S \times [0, \epsilon) \to S \times [0, 1) \xrightarrow{f} M$$

preserves the framing when we give $S \times [0, \epsilon)$ the framing τ.

Remark 2.3.14. We want to emphasize that a framed S_τ-manifold is not necessarily a framed manifold with boundary. It is a manifold with boundary as well as a framed manifold but the two structures are not required to be compatible.

Definition 2.3.15. Let (M, ϕ, f) and (M, ψ, g) be two framed S_τ-manifolds. The *topological space of framed embeddings from M to N*, denoted $\mathrm{Emb}_f^{S_\tau}(M, N)$, is the following homotopy pullback taken in the category of topological spaces over $\mathrm{Map}^S(M,N)$:

$$\begin{tikzcd}
\mathrm{Emb}_f^{S_\tau}(M,N) \arrow[r] \arrow[d] & \mathrm{Map}^S(M,N) \arrow[d] \\
\mathrm{Emb}^S(M,N) \arrow[r] & \mathrm{Map}^{S_\tau}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM),\mathrm{Fr}(TN))
\end{tikzcd}$$

Any time we use the S superscript, we mean that we are considering the subspace of maps commuting with the given map from S. The topological space in the lower right corner is the space of morphisms of $\mathrm{GL}(d)$-bundles inducing the identity $\tau \to \tau$ over the boundary.

Definition 2.3.16. Let (M,ϕ,f) and (M,ψ,g) be two collared framed S_τ-manifolds.

We define $\mathrm{Map}^{cS}(M,N)$ to be the subspace of $\mathrm{Map}^S(M,N)$ consisting of maps inducing the identity on $S\times[0,\epsilon]$ for some ϵ. We define $\mathrm{Emb}^{cS}(M,N)$ and $\mathrm{Map}^{cS_\tau}(\mathrm{Fr}(TM),\mathrm{Fr}(TN))$ in a similar fashion.

The *topological space of framed embeddings from M to N*, denoted $\mathrm{Emb}_f^{cS_\tau}(M,N)$, is the following homotopy pullback taken in the category of topological spaces over $\mathrm{Map}^{cS}(M,N)$:

$$\begin{tikzcd}
\mathrm{Emb}_f^{cS_\tau}(M,N) \arrow[r] \arrow[d] & \mathrm{Map}^{cS}(M,N) \arrow[d] \\
\mathrm{Emb}^{cS}(M,N) \arrow[r] & \mathrm{Map}^{cS_\tau}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM),\mathrm{Fr}(TN))
\end{tikzcd}$$

We can extend the notation $\mathrm{Emb}^S(-,-)$ or $\mathrm{Emb}^{cS}(-,-)$ to manifolds without boundary:

- $\mathrm{Emb}^S(M,N)=\mathrm{Emb}(M,N)$ if M is a manifold without boundary and N is either an S-manifold or a manifold without boundary.

- $\emptyset$ if M is an S-manifold and N is a manifold without boundary.

Using these as spaces of morphisms, there is a simplicical category $\mathbf{Man}_d^S$ (resp. $\mathbf{Man}_d^{cS}$) whose objects are S-manifolds (resp. collared S-manifolds). Similarly, we can extend the notation $\mathrm{Emb}_f^{S_\tau}(-,-)$ and Emb^{cS_τ} to framed manifolds without boundary as above and construct a simplicical category $f\mathbf{Man}_d^{S_\tau}$ (resp. $f\mathbf{Man}^{cS_\tau}$) whose objects are framed S_τ-manifolds (resp. collared framed S_τ-manifolds).

2.4 Homotopy type of spaces of embeddings

We want to analyse the homotopy type of spaces of embeddings described in the previous section. None of the result presented here are surprising. Some of them are proved in greater generality in [Cer61]. However the author of [Cer61] is working with the strong topology on spaces of embeddings and for our purposes, we needed to use the weak topology.

As usual, D denotes the d-dimensional open disk of radius 1 and H is the upper half-disk of radius 1

We will make use of the following two lemmas.

Lemma 2.4.1. *Let X be a topological space with an increasing filtration by open subsets $X = \bigcup_n U_n$. Let Y be another space and $f : X \to Y$ be a continuous map such that for all n, the restriction of f to U_n is a weak equivalence. Then f is a weak equivalence.*

Proof. It suffices to show that the induced map $f_* : [K, X] \to [K, Y]$ is an isomorphism for all finite CW-complexes.

Since $f_{|U_1}$ is a weak equivalence, the composition $[K, U_1] \to [K, X] \to [K, Y]$ is surjective this forces $[K, X] \to [K, Y]$ to be surjective.

Let a, b be two points in $[K, X]$ whose image in $[K, Y]$ are equal, let α, β be continuous maps $K \to X$ representing a and b and such that $f \circ \alpha$ is homotopical to $f \circ \beta$. Since the topological space K is compact, α and β are maps $K \to U_n$ for some n. The composite $U_n \to X \xrightarrow{f} Y$ is a weak equivalence, thus α is homotopical to β in U_n. This implies that α is homotopical to β in X or equivalently that $a = b$. □

Lemma 2.4.2. *(Cerf) Let G be a topological group and let $p : E \to B$ be a map of G-topological spaces. Assume that for any $x \in B$, there is a neighborhood of x on which there is a section of the map:*

$$G \to B$$

$$g \mapsto g.x$$

Then if we forget the action, the map p is a locally trivial fibration. In particular, if B is paracompact, it is a Hurewicz fibration.

Proof. See [Cer]. □

Let $\mathrm{Emb}^*(D,D)$ (resp. $\mathrm{Emb}^{\partial,*}(H,H)$) be the topological space of self embeddings of D (resp. H) mapping 0 to 0.

Proposition 2.4.3. *The "derivative at the origin" map from* $\mathrm{Emb}^*(D,D)$ *to* $\mathrm{GL}(d)$ *is a Hurewicz fibration and a weak equivalence. The analogous result for the map* $\mathrm{Emb}^*(H,H) \to \mathrm{GL}(d-1)$ *also holds.*

Proof. Let us first show that the derivative map:

$$\mathrm{Emb}^*(D,D) \to \mathrm{GL}(d)$$

is a Hurewicz fibration.

The group $\mathrm{GL}(d)$ acts on the source and the target and the derivative map commutes with this action. We use lemma 2.4.2, it suffices to show that for any $u \in \mathrm{GL}(d)$, we can define a section of the multiplication by u map:

$$\mathrm{GL}(d) \to \mathrm{GL}(d)$$

which is trivial.

Now we show that the fibers are contractible. Let $u \in \mathrm{GL}(d)$ and let $\mathrm{Emb}^u(D,D)$ be the space of embedding whose derivative at 0 is u, we want to prove that $\mathrm{Emb}^u(D,D)$ is contractible. It is equivalent but more convenient to work with $\mathbb{R}^d$ instead of D. Let us consider the following homotopy:

$$\begin{aligned} \mathrm{Emb}^u(\mathbb{R}^d,\mathbb{R}^d) \times (0,1] &\to \mathrm{Emb}^u(\mathbb{R}^d,\mathbb{R}^d) \\ (f,t) &\mapsto \left(x \mapsto \frac{f(tx)}{t}\right) \end{aligned}$$

At $t=1$ this is the identity of $\mathrm{Emb}^u(D,D)$. We can extend this homotopy by declaring that its value at 0 is constant with value the linear map u. Therefore, the inclusion $\{u\} \to \mathrm{Emb}^u(D,D)$ is a deformation retract.

The proof for H is similar. □

Proposition 2.4.4. *Let M be a manifold (possibly with boundary). The map:*

$$\mathrm{Emb}(D, M) \to \mathrm{Fr}(TM)$$

is a weak equivalence and a Hurewicz fibrations. Similarly the map:

$$\mathrm{Emb}(H, M) \to \mathrm{Fr}(T\partial M)$$

is a weak equivalence and a Hurewicz fibration.

Proof. The fact that these maps are Hurewicz fibrations will follow again from lemma 2.4.2. We will assume that M has a framing because this will make the proof easier and and we will only apply this result with framed manifolds. However the result remains true in general.

Let's do the proof for D. The derivative map:

$$\mathrm{Emb}(D, M) \to \mathrm{Fr}(TM) \cong M \times \mathrm{GL}(d)$$

is equivariant with respect to the action of the group $\mathrm{Diff}(M) \times \mathrm{GL}(d)$. It suffices to show that for any $x \in \mathrm{Fr}(TM)$, the "action on x" map:

$$\mathrm{Diff}(M) \times \mathrm{GL}(d) \to M \times \mathrm{GL}(d)$$

has a section in a neighborhood of x. Clearly it is enough to show that for any x in M, the "action on x" map:

$$\mathrm{Diff}(M) \to M$$

has a section in a neighborhood of x

We can restrict to neighborhoods U such that $U \subset \bar{U} \subset V \subset M$ in which U and V are diffeomorphic to $\mathbb{R}^d$.

Let us consider the group $\mathrm{Diff}^c(V)$ of diffeomorphisms of V that are the identity outside a compact subset of V. Clearly we can prolong one of these diffeomorphism by the identity and there is a well define inclusion of topological groups:

$$\mathrm{Diff}^c(V) \to \mathrm{Diff}(M)$$

Now we have made the situation local. It is equivalent to construct a map:

$$\phi : D \to \mathrm{Diff}^c(\mathbb{R}^d)$$

with the property that $\phi(x)(0) = x$.

Let f be a smooth function from $\mathbb{R}^d$ to $\mathbb{R}$ which is such that:

- $f(0) = 1$
- $\|\nabla f\| \leq \frac{1}{2}$
- f is compactly supported

We claim that $\phi(x)(u) = f(u)x+u$ satisfies the requirement which proves that $\mathrm{Emb}(D, M) \to \mathrm{Fr}(TM)$ is a Hurewicz fibration. The case of H is similar.

Now let us prove that this derivative maps are weak equivalences.

We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Emb}(D, M) \arrow[r] \arrow[d] & \mathrm{Fr}(TM) \arrow[d] \\
M \arrow[r, "="] & M
\end{tikzcd}$$

Each of the vertical map is a Hurewicz fibration, therefore it suffices to check that the induced map on fibers is a weak equivalence. We denote by $\mathrm{Emb}^m(D, M)$ the subspace consisting of those embeddings sending 0 to m. Hence all we have to do is prove that for any point $m \in M$ the derivative map $\mathrm{Emb}^m(D, M) \to \mathrm{Fr}T_m M$ is a weak equivalence. If M is D, this is the previous proposition. In general, we pick an embedding $f : D \to M$ centered at m. Let $U_n \subset \mathrm{Emb}^m(D, M)$ be the subspace of embeddings mapping D_n to the image of f (where $D_n \subset D$ is the subspace of points of norm at most $1/n$). Clearly U_n is open in $\mathrm{Emb}^m(D, M)$ and $\bigcup_n U_n = \mathrm{Emb}^m(D, M)$, by 2.4.1 it suffices to show that the map $U_n \to \mathrm{Fr}(T_m M)$ is a weak equivalence for all n.

Clearly the inclusion $U_1 \to U_n$ is a deformation retract for all n, therefore, it suffices to check that $U_1 \to \mathrm{Fr}(T_m M)$ is a weak equivalence. Equivalently, it suffices to prove that $\mathrm{Emb}^0(D, D) \to \mathrm{GL}(d)$ is a weak equivalence and this is the previous proposition. □

This result extends to disjoint union of copies of H and D:

Proposition 2.4.5. *The derivative map:*

$$\mathrm{Emb}(D^{\sqcup p} \sqcup H^{\sqcup q}, M) \to \mathrm{Fr}(T\mathrm{Conf}(p, M - \partial M)) \times \mathrm{Fr}(T\mathrm{Conf}(q, \partial M))$$

is a weak equivalence and a Hurewicz fibration.

Proposition 2.4.6. *The evaluation at the center of the disks induces a weak equivalence:*

$$\mathrm{Emb}_f(D^{\sqcup p} \sqcup H^{\sqcup q}, M) \to \mathrm{Conf}(p, M - \partial M) \times \mathrm{Conf}(q, \partial M)$$

Proof. To simplify notations, we restrict to studying $\mathrm{Emb}_f(H, M)$, the general case is similar. By definition 2.3.8 and proposition 2.2.4, we need to study the following homotopy pullback:

$$\begin{tikzcd} & \mathrm{Map}((H, \partial H), (M, \partial M)) \arrow[d] \\ \mathrm{Emb}(H, M) \arrow[r] & \mathrm{Map}^{\partial}_{\mathrm{GL}(d-1)}(\mathrm{Fr}(TH), \mathrm{Fr}(TM)) \end{tikzcd}$$

This diagram is weakly equivalent to:

$$\begin{tikzcd} & \partial M \arrow[d] \\ \mathrm{Fr}(T(\partial M)) \arrow[r] & \mathrm{Fr}(T(\partial M)) \end{tikzcd}$$

where the bottom map is the identity. Therefore, $\mathrm{Emb}_f(H, M) \simeq \partial M$. □

Now we want to study the spaces $\mathrm{Emb}^S(M, N)$ and $\mathrm{Emb}_f^{S_\tau}(M, N)$. Note that the manifold $S \times [0, 1)$ is canonically an S-manifold and even a collared S-manifolds whose collar is the identity.

The splitting of $TS \oplus \mathbb{R}$ on the boundary comes from the identification:

$$T(S \times [0,1)) \cong TS \oplus T([0,1)) \cong TS \oplus \mathbb{R}$$

If τ is a framing of $TS \oplus \mathbb{R}$, the above identification makes $S \times [0,1)$ into a framed S_τ-manifold and a collared S_τ-manifold.

Lemma 2.4.7. *Let M be an S-manifold with S compact. The space $\mathrm{Emb}^S(S \times [0,1), M)$ is weakly contractible. Similarly, the space $\mathrm{Emb}^{cS}(S \times [0,1), M)$ is weakly contractible.*

Proof. We do the proof for Emb^S. The case of Emb^{cS} is easier.

Let us choose one of these embeddings $\phi : S \times [0,1) \to M$ and let's denote its image by C. For $n > 0$, let U_n be the subset of $\mathrm{Emb}^S(S \times [0,1), M)$ consisting of embeddings f with the property that $f(S \times [0, \frac{1}{n}]) \subset C$. By definition of the weak C^1-topology, U_n is open in $\mathrm{Emb}^S(S \times [0,1), M)$, moreover $\mathrm{Emb}^S(S \times [0,1), M) = \bigcup_n U_n$, therefore by 2.4.1, it is enough to prove that U_n is contractible for all n.

Let us consider the following homotopy:

$$H : \left[0, 1 - \frac{1}{n}\right] \times U_n \to U_n$$

$$(t, f) \mapsto ((s,u) \mapsto f(s, (1-t)u))$$

It is a homotopy between the identity of U_n and the inclusion $U_1 \subset U_n$. Therefore U_1 is a deformation retract of each of the U_n and all we have to prove is that U_1 is contractible. But each element of U_1 factors through $C = \mathrm{Im}\phi$, hence all we need to do is prove the lemma when $M = S \times [0,1)$. It is equivalent and notationally simpler to do it for $S \times \mathbb{R}_{\geq 0}$[2].

For $t \in (0,1]$, let $h_t : S \times \mathbb{R}_{\geq 0} \to S \times \mathbb{R}_{\geq 0}$ be the diffeomorphism sending (s,u) to (s, tu)

Let us consider the following homotopy

$$(0,1] \times \mathrm{Emb}^S(S \times \mathbb{R}_{\geq 0}, S \times \mathbb{R}_{\geq 0}) \to \mathrm{Emb}^S(S \times \mathbb{R}_{\geq 0}, S \times \mathbb{R}_{\geq 0})$$

$$(t, f) \mapsto h_{1/t} \circ f \circ h_t$$

[2] The following was suggested to us by Søren Galatius

At time 1, this is the identity of $\mathrm{Emb}^S(S \times [0,+\infty), S \times [0,+\infty))$. At time 0 it has as limit the map:

$$(s,u) \mapsto \left(s, u\frac{\partial f}{\partial u}(s,0)\right)$$

that lies in the subspace of $\mathrm{Emb}^S(S \times [0,+\infty), S \times [0,+\infty))$ consisting of element which are of the form $(s,u) \mapsto (s, a(s)u)$ for some smooth function $a : S \to \mathbb{R}_{>0}$. This space is obviously contractible and we have shown that it is deformation retract of $\mathrm{Emb}^S(S \times [0,+\infty), S \times [0,+\infty))$. □

A similar proof yields the following proposition:

Proposition 2.4.8. *Let M be a d-manifold with compact boundary. The "restriction on the boundary" map:*

$$\mathrm{Emb}^\partial(S \times [0,1), M) \to \mathrm{Emb}(S, \partial M)$$

is a weak equivalence. □

Proposition 2.4.9. *Let M be a framed d-manifold with compact boundary. The "restriction to the boundary" map:*

$$\mathrm{Emb}_f^\partial(S \times [0,1), M) \to \mathrm{Emb}_f(S, \partial M)$$

is a weak equivalence.

Proof. There is a restriction map comparing the pullback diagram defining $\mathrm{Emb}_f(S \times [0,1), M)$ to the pullback diagram defining $\mathrm{Emb}_f(S, \partial M)$. Each of the three maps is a weak equivalence (one of them because of the previous proposition) therefore, the homotopy pullbacks are equivalent. □

Lemma 2.4.10. *Let N be a framed S_τ-manifold. The space $\mathrm{Emb}_f^{S_\tau}(S \times [0,1), N)$ is contractible. Similarly if N is collared, the space $\mathrm{Emb}_f^{cS_\tau}(S \times [0,1), N)$ is contractible.*

Proof. Again we do the proof for $\mathrm{Emb}_f^{S_\tau}(S \times [0,1), N)$, the case of $\mathrm{Emb}_f^{cS_\tau}(S \times [0,1), N)$ being similar.

This space is homotopy equivalent to the following homotopy pullback by 2.2.4:

$$\begin{tikzcd} & \mathrm{Map}^S(S\times[0,1),N) \arrow[d] \\ \mathrm{Emb}^S(S\times[0,1),N) \arrow[r] & \mathrm{Map}^{S_\tau}_{\mathrm{GL}(d)}(\mathrm{Fr}(T(S\times[0,1))),\mathrm{Fr}(TN)) \end{tikzcd}$$

The upper right corner is obviously contractible and by the previous lemma, the lower left corner is contractible. The bottom right corner is equal to:

$$\mathrm{Map}^S(S\times[0,1),N\times\mathrm{GL}(d))$$

where $S\to N\times\mathrm{GL}(d)$ is the product of the map $f:S\to N$ and a constant map $S\to\mathrm{GL}(d)$. This space is clearly contractible. Therefore, the pullback has to be contractible. □

We are now ready to define the operads $\mathcal{E}_d$, $\mathcal{E}_d^\partial$.

Definition 2.4.11. The operad $\mathcal{E}_d$ is the simplicial operad whose n-th space is $\mathrm{Emb}_f(D^{\sqcup n},D)$. Equivalently, $\mathcal{E}_d$ is the endomorphism operad of D in $f\mathcal{M}an_d$.

Note that here is an inclusion of operads:

$$\mathcal{D}_d\to\mathcal{E}_d$$

Proposition 2.4.12. *This map is a weak equivalence of operads.*

Proof. It is enough to check it degreewise. The map:

$$\mathcal{D}_d\to\mathrm{Conf}(n,D)$$

is a weak equivalence which factors through $\mathcal{E}_d(n)$ by 2.4.6, the map $\mathcal{E}_d(n)\to\mathrm{Conf}(n,D)$ is a weak equivalence. □

Recall that H is the subspace of $\mathbb{R}^d$:

$$H=\{(x_1,\dots,x_d)\in\mathbb{R}^d,\|x\|<1,x_d\geq 0\}$$

Definition 2.4.13. We define the operad $\mathcal{E}_d^\partial$ to be the full suboperad of $f\mathcal{M}an_d^\partial$ on the colors D and H.

There is an obvious inclusion of operads:

$$\mathcal{D}_d^\partial \to \mathcal{E}_d^\partial$$

Proposition 2.4.14. *This map is a weak equivalence of operads.*

Proof. Similar to 2.4.12. □

Chapter 3

Factorization homology

Factorization homology is a family of pairings between geometric objects and algebraic objects. The general idea is to start with a (simplicial) category with coproducts $\mathbf{M}$ and a full subcategory $\mathbf{E}$ with typically a small number of objects. The objects of $\mathbf{E}$ are the "basic" objects of $\mathbf{M}$ in the sense that each object of $\mathbf{M}$ is obtained by "glueing" of objects of $\mathbf{E}$. Then, one can consider the suboperad of $(\mathbf{C}, \sqcup)$ on the objects of $\mathbf{E}$. Any algebra over that operad can be pushed forward to the operad $(\mathbf{C}, \sqcup)$ and evaluated at a particular object. This process is called factorization homology.

If we try to do that for the category of d-manifolds and embeddings, the reasonable set of basic objects is the singleton consisting of the manifold $\mathbb{R}^d$. The endomorphism operad of $\mathbb{R}^d$ is the (framed) little d-disk operad. Factorization homology is then a pairing between manifolds and algebras over the framed little disk operad. We could also work with framed d-manifolds. In that case factorization homology would be a pairing between framed d-manifolds and $\mathcal{E}_d$-algebras. One should refer to [Fra12] for a good overview of the subject. There are lots of variants of this idea. One could change the tangential structure on the manifolds or allow manifolds with certain singularities (like boundary, corners, base point, etc.). A very general theory of factorization homology for manifolds is developed in [AFT12].

We can also define factorization homology for spaces. Any space can be obtained from contractible spaces through the process of forming homotopy pushout. In this sense it is reasonable to take the point as our unique basic objects. The endomorphism operad of the point in $\mathbf{S}$ is the commutative operad. Hence in this case, factorization homology is a pairing between spaces and commutative algebras. We show that factorization homology of a commutative algebra over a space which happens to have a d-manifold structure coincides with the factorization homology of the underlying $\mathcal{E}_d$-algebra. A similar construction can be found in [GTZ10].

We give a definition of factorization homology for framed manifolds possibly with boundary and for S_τ-manifolds for S a $(d-1)$-manifold with a d-framing. The difference between this chapter and [AFT12] is that we use model categories instead of quasi-category. Note that a model category version of factorization homology for ordinary d-manifolds can be found in [And10]. The definition of [And10] is slightly different from ours since it is defined as an ordinary left Kan extension instead of an operadic left Kan extension. The two definitions coincide as is explained in B.3.10 but we found our definition easier to work with.

3.1 Preliminaries

Let $\mathfrak{M}$ be the set of framed d manifolds whose underlying manifold is a submanifold of $\mathbb{R}^\infty$. Note that $\mathfrak{M}$ contains at least an element of each diffeomorphism class of framed d-manifold.

Definition 3.1.1. We denote by $f\mathcal{M}an_d$ an operad whose set of colors is $\mathfrak{M}$ and with mapping objects:

$$f\mathcal{M}an_d(\{M_1, \ldots, M_n\}, M) = \mathrm{Emb}_f(M_1 \sqcup \ldots \sqcup M_n, M)$$

As usual, we denote by $f\mathbf{Man}_d$ the free symmetric monoidal category on the operad $\mathcal{M}an_f$.

We can see $D \subset \mathbb{R}^d \subset \mathbb{R}^\infty$ as an element of $\mathfrak{M}$. We denote by $\mathcal{E}_d$ the full suboperad of $f\mathcal{M}an_d$ on the color D. The category $\mathbf{E}_d$ is the full subcategory of $f\mathbf{Man}_d$ on objects of the form $D^{\sqcup n}$ with n a nonnegative integer.

Similarly, we define $\mathfrak{M}^\partial$ to be the set of submanifold of $\mathbb{R}^\infty$ possibly with boundary. $\mathfrak{M}^\partial$ contains at least an element of each diffeomorphism class of framed d-manifold with boundary.

Definition 3.1.2. We denote by $f\mathcal{M}an_d^{\partial}$ the operad whose set of colors is $\mathfrak{M}^{\partial}$ and with mapping objects:

$$f\mathcal{M}an_d^{\partial}(\{M_1, \ldots, M_n\}, M) = \mathrm{Emb}_f(M_1 \sqcup \ldots \sqcup M_n, M)$$

We denote by $f\mathbf{Man}_d^{\partial}$ the free symmetric monoidal category on the operad $f\mathcal{M}an_d^{\partial}$. We define the suboperad $\mathcal{E}_d^{\partial}$ as the full suboperad of $f\mathbf{Man}_d^{\partial}$ on the colors D and H.

Let S be a compact $(d-1)$-manifold and τ be a d-framing on S. Let $\mathfrak{M}^{S_\tau}$ be the set of S_τ-manifolds whose underlying manifold is a submanifold of $\mathbb{R}^\infty$.

Definition 3.1.3. The operad $f\mathcal{M}an_d^{S_\tau}$ has the set $\mathfrak{M} \sqcup \mathfrak{M}^{S_\tau}$ as set of colors. Its spaces of operations are given by:

$$\begin{aligned} f\mathcal{M}an_d^{S_\tau}(\{M_i\}_{i\in I}; N) &= \emptyset, \text{ if } \{M_i\}_{i\in I} \text{ contains more than 1 element of } \mathfrak{M}^{S_\tau} \\ &= \mathrm{Emb}_f^{S_\tau}(\sqcup_i M_i, N) \text{ otherwise} \end{aligned}$$

One can consider the full suboperad on the colors D and $S \times [0, 1)$ and check that it is isomorphic to $S_\tau\mathcal{M}od$ (see 4.1.1).

3.2 Definition of factorization homology

In this section and the following, we assume that $\mathbf{C}$ is a cofibrantly generated symmetric monoidal simplicial category with a good theory of algebras over Σ-cofibrant operads.

Definition 3.2.1. Let A be an object of $\mathbf{C}[\mathcal{E}_d]$. We define factorization homology with coefficients in A to be the derived operadic left Kan extension of A along the map of operads $\mathcal{E}_d \to f\mathcal{M}an_d$.

We denote by $M \mapsto \int_M A$ the symmetric monoidal functor $f\mathbf{Man}_d \to \mathbf{C}$ induced by that pushforward.

We have $\int_M A = \mathrm{Emb}_f(-, M) \otimes_{\mathbf{E}_d} QA$ where $QA \to A$ is a cofibrant replacement in the category $\mathbf{C}[\mathcal{E}_d]$. We use the fact that the operad $\mathcal{E}_d$ is Σ-cofibrant and that the right module $\mathrm{Emb}_f(-, M)$ is Σ-cofibrant.

We can define factorization homology of an object of $f\mathbf{Man}_d^\partial$ with coefficients in an algebra over $\mathcal{E}_d^\partial$.

Definition 3.2.2. Let (B, A) be an algebra over $\mathcal{E}_d^\partial$ in $\mathbf{C}$. Factorization homology with coefficients in (B, A) is the derived operadic left Kan extension of (B, A) along the obvious inclusion of operads $\mathcal{E}_d^\partial \to f\mathcal{M}an_d^\partial$. We write $\int_M(B, A)$ to denote the value at $M \in f\mathbf{Man}_d^\partial$ of the induced functor.

Again, we have $\int_M(B, A) = \mathrm{Emb}_f^\partial(-, M) \otimes_{\mathbf{E}_d^\partial} Q(B, A)$ where $Q(B, A) \to (B, A)$ is a cofibrant replacement in the category $\mathbf{C}[\mathcal{E}_d^\partial]$. We use the fact that $\mathcal{E}_d^\partial$ is Σ-cofibrant and that $\mathrm{Emb}_f^\partial(-, M)$ is Σ-cofibrant as a right module over $\mathcal{E}_d^\partial$.

We can define, in a similar fashion, factorization homology on an S_τ-manifold. This gives a pairing between S_τ-manifolds and $S_\tau\mathcal{M}od$-algebras (see 4.1.1 for a definition of the operad $S_\tau\mathcal{M}od$).

Definition 3.2.3. Let (A, M) be an $S_\tau\mathcal{M}od$-algebra in $\mathbf{C}$. Factorization homology with coefficients in (A, M) is the left derived functor of the pushforward of (A, M) along the map of operad:

$$S_\tau\mathcal{M}od \to f\mathcal{M}an_d^{S_\tau}$$

3.3 Factorization homology as a homotopy colimit

In this section, we show that factorization homology can be expressed as the homotopy colimit of a certain functor on the poset of open sets of M that are diffeomorphic to a disjoint union of disks. Note that this result in the case of manifolds without boundary is proved in [Lur11].

We will rely heavily on the following theorem:

Theorem 3.3.1. *Let X be a topological space and $\mathbf{U}(X)$ be the poset of open subsets of X. Let $\chi : \mathbf{A} \to \mathbf{U}(X)$ be a functor from a small discrete category $\mathbf{A}$. For a point $x \in X$, denote by $\mathbf{A}_x$ the full subcategory of A whose objects are those that are mapped by χ to open sets containing x. Assume that for all x, the nerve of $\mathbf{A}_x$ is contractible. Then the obvious* *map:*

$$\mathrm{hocolim}\chi \to X$$

is a weak equivalence.

Proof. See [Lur11] Theorem A.3.1. p. 971. □

Let M be an object of $f\mathbf{Man}_d$. Let $\mathbf{D}(M)$ the poset of subset of M that are diffeomorphic to a disjoint union of disks. Let us choose for each object V of $\mathbf{D}(M)$ a framed diffeomorphism $V \cong D^{\sqcup n}$ for some uniquely determined n. Each inclusion $V \subset V'$ in $\mathbf{D}(M)$ induces a morphism $D^{\sqcup n} \to D^{\sqcup n'}$ in $\mathbf{E}_d$ by composing with the chosen parametrization. Therefore each choice of parametrization induces a functor $\mathbf{D}(M) \to \mathbf{E}_d$. Up to homotopy this choice is unique since the space of automorphisms of D in $\mathbf{E}_d$ is contractible.

In the following we assume that we have one of these functors $\delta : \mathbf{D}(M) \to \mathbf{E}_d$. We fix a cofibrant algebra $A : \mathbf{E}_d \to \mathbf{C}$.

Lemma 3.3.2. *The obvious map:*

$$\mathrm{hocolim}_{V\in\mathbf{D}(M)}\mathrm{Emb}_f(-,V) \to \mathrm{Emb}_f(-,M)$$

is a weak equivalence in $\mathrm{Fun}(\mathbf{E}_d, \mathbf{S})$*.*

Proof. It suffices to prove that for each n, there is a weak equivalence in spaces:

$$\mathrm{hocolim}_{V\in\mathbf{D}(M)}\mathrm{Emb}_f(D^{\sqcup n},V) \simeq \mathrm{Emb}_f(D^{\sqcup n},M)$$

We can apply theorem 3.3.1 to the functor:

$$\mathbf{D}(M) \to \mathbf{U}(\mathrm{Emb}_f(D^{\sqcup n},M))$$

sending V to $\mathrm{Emb}_f(D^{\sqcup n},V) \subset \mathrm{Emb}_f(D^{\sqcup n},M)$. For a given point ϕ in $\mathrm{Emb}_f(D^{\sqcup n},M)$, we have to show that the poset of open sets $V \in \mathbf{D}(M)$ such that $\mathrm{im}(\phi) \subset V$ is contractible. But this poset is filtered, thus its nerve is contractible. □

Corollary 3.3.3. *We have:*

$$\int_M A \simeq \mathrm{hocolim}_{V\in\mathbf{D}(M)} \int_{\delta(V)} A$$

Proof. By B.3.10, we know that $\int_M A$ is weakly equivalent to the Bar construction $\mathrm{B}(\mathrm{Emb}_f(-,M),\mathbf{E}_d,A)$. Therefore we have:

$$\int_M A \simeq \mathrm{B}(*,\mathbf{D}(M),\mathrm{B}(\mathrm{Emb}_f(-,-),\mathbf{E}_d,A))$$

The right hand side is the realization of a bisimplicial object and its value is independant of the order in which we do the realization. □

Corollary 3.3.4. *There is a weak equivalence:*

$$\int_M A \simeq \mathrm{hocolim}_{V\in\mathbf{D}(M)} A(\delta(V))$$

Proof. By 3.3.3 the left-hand side is weakly equivalent to:

$$\mathrm{hocolim}_{V\in\mathbf{D}(M)} \int_{\delta(V)} A$$

Let U be an object of $\mathbf{E}_d$. The object $\int_U A$ is the coend :

$$\mathrm{Emb}_f(-,U)\otimes_{\mathbf{E}_d} A$$

Yoneda's lemma implies that this coend is isomorphic to $A(U)$. Moreover, this isomorphism is functorial in U. Therefore we have the desired identity. □

We want to use a similar approach for manifolds with boundaries. Let M be an object of $f\mathbf{Man}_d$ and let $M\times[0,1)$ be the object of $f\mathbf{Man}_d^\partial$ whose framing is the direct sum of the framing of M and the obvious framing of $[0,1)$. We identify $\mathbf{D}(M)$ with the poset of open sets of $M\times[0,1)$ of the form $V\times[0,1)$ with V an open set of M that is diffeomorphic to a disjoint union of disks. As before we can pick a functor $\delta:\mathbf{D}(M)\to\mathbf{E}_d^\partial$.

Lemma 3.3.5. *The obvious map:*

$$\mathrm{hocolim}_{V\in\mathbf{D}(M)}\mathrm{Emb}_f(-,V\times[0,1))\to\mathrm{Emb}_f(-,M\times[0,1))$$

is a weak equivalence in $\mathrm{Fun}((\mathbf{E}_d^{\partial})^{\mathrm{op}},\mathbf{S})$.

Proof. It suffices to prove that for each p,q, there is a weak equivalence in spaces:

$$\mathrm{hocolim}_{V\in\mathbf{D}(M)}\mathrm{Emb}_f(D^{\sqcup p}\sqcup H^{\sqcup q},V\times[0,1))\simeq\mathrm{Emb}_f(D^{\sqcup p}\sqcup H^{\sqcup q},M\times[0,1))$$

It suffices to show, by 3.3.1, that for any $\phi\in\mathrm{Emb}(D^{\sqcup p}\sqcup H^{\sqcup q},M\times[0,1))$, the poset $\mathbf{D}(M)_\phi$ (which is the subposet of $\mathbf{D}(M)$ on open sets V that are such that $V\times[0,1)\subset M\times[0,1)$ contains the image of ϕ) is contractible. But it is easy to see that $\mathbf{D}(M)_\phi$ is filtered. Thus it is contractible. □

Proposition 3.3.6. *Let* $(B,A):\mathbf{E}_d^{\partial}\to\mathbf{C}$ *be a cofibrant* $\mathcal{E}_d^{\partial}$*-algebra, then we have:*

$$\int_{M\times[0,1)}(B,A)\simeq\mathrm{hocolim}_{V\in\mathbf{D}(M)}(B,A)(\delta(V))$$

Proof. The proof is a straightforward modification of 3.3.4. □

There is a morphism of operad $\mathcal{E}_{d-1}\to\mathcal{E}_d^{\partial}$ sending the unique color of $\mathcal{E}_{d-1}$ to H. Indeed H is diffeomorphic to the product of the $(d-1)$-dimensional disk with $[0,1)$.

Corollary 3.3.7. *Let* (B,A) *be an* $\mathcal{E}_d^{\partial}$*-algebra, then we have a weak equivalence:*

$$\int_{M\times[0,1)}(B,A)\simeq\int_M A$$

Proof. Because of the previous proposition, the left hand side is weakly equivalent to $\mathrm{hocolim}_{V\in\mathbf{D}(M)}A(\delta(V))$ which by 3.3.4 is weakly equivalent to $\int_M A$ □

3.4 Factorization homology of spaces

We define a version of factorization homology which allows to work over a general simplicial set, on the other hand, we need to restrict to commutative algebras as coefficients. The definition is a straightforward variant of factorization homology. Such a construction was made by Pirashvili (see [Pir00]) in the category of chain complexes over a field of characteristic zero. See also [GTZ10].

In this section and the following $(\mathbf{C}, \otimes, \mathbb{I}_{\mathbf{C}})$ denotes a symmetric monoidal simplicial cofibrantly generated model category with a good theory of algebras.

Let $\mathfrak{S}$ be a set of connected simplicial sets containing the point, we denote $\mathbf{S}^{\mathfrak{S}}$ the operad with colors $\mathfrak{S}$ and with spaces of operations:

$$\mathcal{S}pace^{\mathfrak{S}}(\{s_i\}_{i\in I}; t) := \mathrm{Map}(\sqcup_I s_i, t)$$

Note that the full suboperad on the point is precisely the operad $\mathcal{C}om$, therefore, we have a morphism of operads:

$$\mathcal{C}om \to \mathcal{S}pace^{\mathfrak{S}}$$

We assume that $\mathbf{C}$ is a symmetric monoidal model category in which the commutative algebras have a transferred model structure. Note that this is quite restrictive. For instance it does not work for $\mathbf{S}$. It does work for $\mathbf{Spec}$ and $\mathbf{Ch}_{\geq 0}(R)$ with R a $\mathbb{Q}$-algebra.

Definition 3.4.1. Let A be a commutative algebra in $\mathbf{C}$, let X be an object of the symmetric monoidal category $\mathbf{Space}^{\mathfrak{S}}$, we define $\int_X A$ to be the operadic left Kan extension of A along the map $\mathcal{C}om \to \mathcal{S}pace^{\mathfrak{S}}$.

Note that the value of $\int_X A$ is:

$$\mathrm{Map}(-, X) \otimes_{\mathbf{Fin}} QA$$

where $QA \to A$ is a cofibrant replacement of A as a commutative algebra. In particular, it is independant of the set $\mathfrak{S}$. In the following we will write $\int_X A$ for any simplicial set X without mentioning the set $\mathfrak{S}$.

Proposition 3.4.2. *The functor $X \mapsto \int_X A$ preserves weak equivalences.*

Proof. The functor $X \mapsto \mathrm{Map}(-, X)$ sends any weak equivalence in $\mathbf{S}$ to a weak equivalence in $\mathrm{Fun}(\mathbf{Fin}^{\mathrm{op}}, \mathbf{S})$. The result then follows from B.3.9. □

We now want to compare $\int_X A$ with $\int_M A$ where M is a framed manifold.

Lemma 3.4.3. *There is a weak equivalence:*

$$\mathrm{hocolim}_{\mathbf{D}(M)}\mathbf{Fin}(S,\pi_0(-)) \simeq \mathrm{Map}(S,M)$$

Proof. Note that for $U \in \mathbf{D}(M)$, we have $\mathbf{Fin}(S,\pi_0(U)) \simeq \mathrm{Map}(S,U)$, thus, we are reduced to showing:

$$\mathrm{hocolim}_{U\in\mathbf{D}(M)}\mathrm{Map}(S,U) \simeq \mathrm{Map}(S,M)$$

We use 3.3.1 again, there is a functor $\mathbf{D}(M) \to \mathbf{U}(\mathrm{Map}(S,M))$ sending U to the open set of maps whose image is contained in U. For $f \in \mathrm{Map}(S,M)$, the subcategory of $U \in \mathbf{D}(M)$ containing the image of f is filtered, therefore, it is contractible. $\square$

Let F be any functor $\mathbf{Fin} \to \mathbf{C}$. We have the following diagram:

$$\mathbf{D}(M) \overset{\alpha}{\to} \mathbf{Fin} \overset{F}{\to} \mathbf{C}$$

Proposition 3.4.4. *There is a weak equivalence:*

$$\mathrm{hocolim}_{\mathbf{D}(M)}\alpha^*F \simeq \mathrm{Map}(-,M) \otimes^{\mathbb{L}}_{\mathbf{Fin}} F$$

Proof. The hocolim can be written as a coend:

$$* \otimes^{\mathbb{L}}_{\mathbf{D}(M)} \alpha^*F$$

We use the adjuction induced by α, and find:

$$\mathrm{hocolim}_{\mathbf{D}(M)}\alpha^*F \simeq \mathbb{L}\alpha_!(*) \otimes_{\mathbf{Fin}} F$$

Now $\mathbb{L}\alpha_!(*)$ is the functor whose value at S is:

$$\mathbf{Fin}^{\mathrm{op}}(\pi_0(-),S) \otimes^{\mathbb{L}}_{\mathbf{D}(M)^{\mathrm{op}}} * \simeq \mathrm{hocolim}_{\mathbf{D}(M)}\mathbf{Fin}(S,\pi_0(-))$$

The results then follows from the previous lemma. $\square$

Corollary 3.4.5. *Let M be a framed manifold and A a commutative algebra in $\mathbf{C}$, then $\int_{\mathrm{Sing}(M)} A$ is weakly equivalent to $\int_M A$*

Proof. We have by 3.3.4:

$$\int_M A \simeq \mathrm{hocolim}_{\mathbf{D}(M)} \alpha^* A$$

By B.3.10:

$$\int_{\mathrm{Sing}(M)} A \simeq \mathrm{Map}(-, \mathrm{Sing}(M)) \otimes^{\mathbb{L}}_{\mathbf{Fin}} A$$

Hence the result is a trivial corollary of the previous proposition. $\square$

Comparison with McClure, Schwanzl and Vogt description of THH.

In [MSV97], the authors show that THH of a commutative ring spectrum R coincides with the tensor $S^1 \otimes R$ in the simplicial category of commutative ring spectra. We want to generalize this result and show that for a commutative algebra A, there is a natural weak equivalence of commutative algebras:

$$\int_X A \simeq X \otimes A$$

Let X be a simplicial set. There is a category Δ/X called the category of simplices of X whose objects are pairs $([n], x)$ where x is a point of X_n and whose morphisms from $([n], x)$ to $([m], y)$ are the datum of a map $d : [n] \to [m]$ in Δ such that $d^* y = x$. Note that there is a functor:

$$F_X : \Delta/X \to \mathbf{S}$$

sending $([n], x)$ to $\Delta[n]$. The colimit of that functor is obviously X again.

Theorem 3.4.6. *The map:*

$$\mathrm{hocolim}_{\Delta/X} F_X \to \mathrm{colim}_{\Delta/X} F_X \cong X$$

is a weak equivalence.

Proof. see [Lur09], proposition 4.2.3.14. □

Corollary 3.4.7. *Let U be a functor from $\mathbf{S}$ to a model category $\mathbf{Y}$. Assume that U preserves weak equivalences and homotopy colimits. Then U is weakly equivalent to:*

$$X \mapsto \mathrm{hocolim}_{\Delta/X} U(*)$$

In particular, if U and V are two such functors, and $U(*) \simeq V(*)$, then $U(X) \simeq V(X)$ for any simplicial set X.

Proof. Since U preserves weak equivalences and homotopy colimits, we have a weak equivalence:

$$\mathrm{hocolim}_{\Delta/X} U(*) \simeq U(\mathrm{hocolim}_{\Delta/X} *) \simeq U(X)$$

□

We now have the following theorem:

Theorem 3.4.8. *Let A be a cofibrant commutative algebra in $\mathbf{C}$. The functor $X \mapsto \int_X A$ and the functor $X \mapsto X \otimes A$ are weakly equivalent as functors from $\mathbf{S}$ to $\mathbf{C}[\mathcal{C}om]$.*

Proof. The two functors obviously coincide on the point. In order to apply 3.4.7, we need to check that both functors preserve weak equivalences and homotopy colimits.

Since A is cofibrant and $\mathbf{C}$ is simplicial, $X \mapsto X \otimes A$ preserves weak equivalences between cofibrant $\mathbf{S}$. Since all simplicial sets are cofibrant it preserves all weak equivalences. The functor $X \mapsto \int_X A$ also preserves weak equivalences by 3.4.2, the result then follows from B.3.9.

Now assume $Y \simeq \mathrm{hocolim}_{\mathbf{A}} F$ where F is some functor from a small category $\mathbf{A}$ to $\mathbf{S}$, then $Y \simeq \mathrm{B}(*, \mathbf{A}, F)$. Tensoring with A preserves colimits since it is a left adjoint, therefore, we have:

$$\begin{aligned} Y \otimes A &\simeq |\mathrm{B}_\bullet(*, \mathbf{A}, F(-))| \otimes A \\ &\simeq (\Delta[-] \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(*, \mathbf{A}, F(-))) \otimes A \\ &\simeq \Delta[-] \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(*, \mathbf{A}, F(-) \otimes A) \\ &\simeq \mathrm{hocolim}_{\mathbf{A}} F(-) \otimes A \end{aligned}$$

Therefore $X \mapsto X \otimes A$ preserves homotopy colimit. Similarly, one can prove that $P \mapsto P \otimes_{\mathbf{Fin}} A$ preserves homotopy colimits in the variable $P \in \mathbf{Mod}_{\mathcal{C}om}$. Moreover, $Y \simeq \mathrm{hocolim}_{\mathbf{A}} F$ implies the identity $\mathrm{Map}(-, Y) \simeq \mathrm{hocolim}_{\mathbf{A}} \mathrm{Map}(-, F)$ in $\mathbf{Mod}_{\mathcal{C}om}$. This concludes the proof. □

3.5 The commutative field theory

This section is a toy-example of what we are going to consider in the fourth chapter. Let us define first the category $\mathbf{Cospan}(\mathbf{S})$.

If X is a space, we denote by $\mathbf{S}^X$, the category of simplicial sets under X with the model structure whose cofibrations, fibrations and weak equivalences are reflected by the forgetful functor $\mathbf{S}^X \to \mathbf{S}$.

The objects of $\mathbf{Cospan}(\mathbf{S})$ are $\mathbf{S}$.

The morphisms space $\mathrm{Map}_{\mathbf{Cospan}(\mathbf{S})}(X, Y)$ is the nerve of the category of weak equivalences between cofibrant objects in $\mathbf{S}^{X \sqcup Y}$. More concretely, it is the nerve of the category whose objects are diagrams of cofibrations:

$$X \to U \leftarrow Y$$

and whose morphisms are commutative diagrams:

$$\begin{tikzcd} & U \arrow[dd, "\simeq"] & \\ X \arrow[ur] \arrow[dr] & & Y \arrow[ul] \arrow[dl] \\ & V & \end{tikzcd}$$

whose middle arrow is a weak equivalence.

The composition:

$$\mathrm{Map}_{\mathbf{Cospan}(\mathbf{S})}(X,Y) \times \mathrm{Map}_{\mathbf{Cospan}(\mathbf{S})}(Y,Z) \to \mathrm{Map}_{\mathbf{Cospan}(\mathbf{S})}(X,Z)$$

is deduced from the Quillen bifunctor:

$$\mathbf{S}^{X \sqcup Y} \times \mathbf{S}^{Y \sqcup Z} \to \mathbf{S}^{X \sqcup Z}$$

taking $(X \to A \leftarrow Y, Y \to B \leftarrow Z)$ to $X \to A \sqcup^{Y} B \leftarrow Z$.

The category $\mathbf{Cospan}(\mathbf{S})$ is the underlying category of an operad $\mathcal{C}\mathrm{ospan}(\mathbf{S})$.

A multi-cospan from $\{X_i\}_{i \in I}$ to Y is a diagram:

$$\begin{tikzcd} X_i \arrow[ddr] & & \\ X_j \arrow[dr] & & \\ \cdots & A & Y \arrow[l] \\ X_k \arrow[ur] & & \end{tikzcd}$$

where all the objects X_i for $i \in I$ appear on the left of the diagram.

There is a model category on multi-cospans from $\{X_i\}_{i \in I}$ to Y.

The space of multi-morphisms from $\{X_i\}_{i \in I}$ to Y in the operad $\mathcal{C}\mathrm{ospan}(\mathbf{S})$ is the nerve of the category of weak equivalences between cofibrant multi-cospans from $\{X_i\}_{i \in I}$ to Y.

Theorem 3.5.1. *Let A be a cofibrant commutative algebra in $\mathbf{C}$. There is a morphism of operad $\mathcal{C}\mathrm{ospan}(\mathbf{S}) \to \mathcal{M}od\mathcal{C}\mathrm{at}$ sending X to $\mathbf{Mod}_{\int_X A}$.*

Proof. (SKETCH) Let us first construct a morphism of operad $\mathcal{C}\mathrm{ospan}(\mathbf{S}) \to \mathcal{B}i\mathcal{M}od(\mathbf{Mod}_{\mathcal{C}om})$. We do this by sending the color X to the right $\mathcal{C}om$-module $\mathrm{Map}(-,X)$. $\mathrm{Map}(-,X)$ is a commutative algebra in $\mathbf{Mod}_{\mathcal{C}om}$ and any map of simplicial sets $X \to Y$ induces a commutative algebra map $\mathrm{Map}(-,X) \to \mathrm{Map}(-,Y)$ making $\mathrm{Map}(-,Y)$ into a left module over $\mathrm{Map}(-,X)$. This observation implies that any multicospan from $\{X_i\}_{i\in I}$ to Y represents an object of ${}_{\{X_i\}_{i\in I}}\mathbf{Mod}_Y$.

Moreover observe that if $X \leftarrow U \to Y$ is a diagram in $\mathbf{S}$ in which both maps are cofibrations, then the functor on finite sets $\mathrm{Map}(-, X \sqcup^U Y)$ is isomorphic (not just weakly equivalent) to the functor $\mathrm{Map}(-,X) \otimes_{\mathrm{Map}(-,U)} \mathrm{Map}(-,Y)$. Indeed, both functors can be identified with the following functor:

$$S \mapsto \bigsqcup_{S=A\cup B} \mathrm{Map}((A, A\cap B),(X,U)) \times_{\mathrm{Map}(A\cap B,U)} \mathrm{Map}((B,A\cap B),(Y,U))$$

This proves that the assignment $X \mapsto \mathrm{Map}(-,X)$ is a morphism of operads from $\mathcal{C}\mathrm{ospan}(\mathbf{S})$ to $\mathcal{B}i\mathcal{M}od(\mathbf{Mod}_{\mathcal{C}om})$.

We have already constructed a morphism of operad from $\mathcal{B}i\mathcal{M}od(\mathbf{Mod}_{\mathcal{C}om})$ to $\mathcal{M}od\mathcal{C}\mathrm{at}$ in the first chapter. We can compose it with the map we have just constructed. $\square$

Chapter 4

Modules over algebras over the little disks operad

In this chapter $(\mathbf{C}, \otimes, \mathbb{I}_{\mathbf{C}})$ is a symmetric monoid simplicial cofibrantly generated model category with a good theory of algebras over Σ-cofibrant operads.

4.1 Definition

Let S be a compact $(d-1)$-manifold and let τ be a d-framing of S.

Definition 4.1.1. The $\mathcal{E}_d$-right module S_τ is given by:

$$S_\tau(n) = \mathrm{Emb}_f^{S_\tau}(D^{\sqcup n} \sqcup S \times [0,1), S \times [0,1))$$

It is clearly a right modules over $\mathcal{E}_d$. Moreover, we have a composition:

$$-\square- : S_\tau(n) \times S_\tau(m) \to S_\tau(n+m)$$

which makes S_τ into an associative algebra in right S_τ-modules. The composition is as follows:

Let ϕ be an element of $S_\tau(n)$ and ψ be an element of $S_\tau(m)$. Let ψ^S be the restriction of ψ to $S \times [0,1)$. We define $\phi\square\psi$ to be the element of $S_\tau(n+m)$ whose restriction to

$S \times [0,1) \sqcup D^{\sqcup n}$ is $\psi^S \circ \phi$ and whose restriction to $D^{\sqcup n}$ is $\psi_{|D^{\sqcup m}}$.

To remember how the composition works, note that if we represents $[0,1)$ on a horizontal axis with 0 to the left of 1 and we represent the disks of $\phi\Box\psi$ on $S \times [0,1)$, the disks of ϕ lie on the left of the disks of ψ $\phi\Box\psi$ in accordance with way we write $\phi\Box\psi$.

The general theory of the first chapter gives rise to an operad $S_\tau\mathcal{M}od$ and for any $\mathcal{E}_d$-algebra A in $\mathbf{C}$, a category $S_\tau\mathbf{Mod}_A$.

Example 4.1.2. The unit sphere inclusion $S^{d-1} \to \mathbb{R}^d$ has a trivial normal bundle. Thus, there is a d-framing on S^{d-1} that we denote κ. Using 4.1.1, we can construct an operad $S^{d-1}_\kappa\mathcal{M}od$. We will show in 4.3.1 that the theory of modules defined by this operad is equivalent to the theory of operadic modules over $\mathcal{E}_d$.

Let $V \cong Q \oplus \mathbb{R}$ be a d-dimensional vector space equipped with a decomposition into the direct sum of a hyperplane and a line. There is a unique endomorphism of V whose restriction to Q is the identity and whose restriction to $\mathbb{R}$ is $-\mathrm{id}$. This endomorphism is an involution and hence induces an involution of $\mathrm{Fr}(V)$ that we denote by a minus sign:

$$\tau \mapsto -\tau$$

We can extend this operation to framings of vector bundles which split as the direct sum of a codimension 1 factor and a line.

Proposition 4.1.3. *There is an isomorphism of associative algebras in right modules over $\mathcal{E}_d$:*

$$S_\tau^{\mathrm{op}} \cong S_{-\tau}$$

Proof. Easy. $\square$

4.2 Linearization of embeddings

In this section, we construct a smaller model of the right module S_τ. We use this model to compare the universal enveloping algebra of S_τ-shaped modules to the factorization homology over a certain manifold.

We will need the following technical result which insures that certain maps are fibrations.

Proposition 4.2.1. *Let N be an S_τ-manifold and let M be an object of $S_\tau\mathbf{Mod}$ which can be expressed as a disjoint union:*

$$M = P \sqcup Q$$

in which one of the factor is an S_τ-manifold and the other is a manifold without boundary. Then the restriction maps:

$$\mathrm{Emb}_f^{S_\tau}(M, N) \to \mathrm{Emb}_f^{S_\tau}(P, N)$$

is a fibration.

Recall that we have extended the definition of Emb^{S_τ} to manifolds without boundary. The above theorem can be applied in the case where P and Q are both manifolds without boundary.

Proof. By the enriched Yoneda's lemma, the space $\mathrm{Emb}_f^{S_\tau}(M, N)$ can be identified with the space of natural transformations:

$$\mathrm{Map}_{\mathrm{Fun}(S_\tau\mathbf{Mod}^{\mathrm{op}}, \mathbf{S})}(\mathrm{Emb}_f^{S_\tau}(-, M), \mathrm{Emb}_f^{S_\tau}(-, N))$$

and similarly for $\mathrm{Emb}_f^{S_\tau}(P, N)$ and $\mathrm{Emb}_f^{S_\tau}(Q, N)$. The category $\mathrm{Fun}(S_\tau\mathbf{Mod}^{\mathrm{op}}, \mathbf{S})$ is a symmetric monoidal model category in which fibrations and weak equivalences are objectwise.

Indeed more generally, if $\mathbf{A}$ is a small simplicial symmetric monoidal category, the category of simplicial functors to simplicial sets $\mathrm{Fun}(\mathbf{A}, \mathbf{S})$ with the projective model structure and the Day tensor product is a symmetric monoidal model category (this is proved in [Isa09] proposition 2.2.15). It is easy to check that in this model structure, a representable functor is automatically cofibrant (this comes from the characterization in terms of lifting against trivial fibrations together with the fact that trivial fibration in $\mathbf{S}$ are epimorphisms). Moreover, we have the identity:

$$\mathrm{Emb}_f^{S_\tau}(-, M) = \mathrm{Emb}_f^{S_\tau}(-, P) \otimes \mathrm{Emb}_f^{S_\tau}(-, Q)$$

This immediatly implies that $\mathrm{Emb}_f^{S_\tau}(-, P) \to \mathrm{Emb}_f^{S_\tau}(-, M)$ is a cofibration in $\mathrm{Fun}(S_\tau\mathbf{Mod}^{\mathrm{op}}, \mathbf{S})$.

But the category $\mathrm{Fun}(S_\tau\mathbf{Mod}^{\mathrm{op}}, \mathbf{S})$ is also a model category enriched in $\mathbf{S}$, therefore, the induced map:

$$\mathrm{Map}_{\mathrm{Fun}(S_\tau\mathbf{Mod}^{\mathrm{op}},\mathbf{S})}(\mathrm{Emb}_f^{S_\tau}(-,M), \mathrm{Emb}_f^{S_\tau}(-,N))$$

$$\to \mathrm{Map}_{\mathrm{Fun}(S_\tau\mathbf{Mod}^{\mathrm{op}},\mathbf{S})}(\mathrm{Emb}_f^{S_\tau}(-,P), \mathrm{Emb}_f^{S_\tau}(-,N))$$

is a fibration. □

Definition 4.2.2. Let S be a $(d-1)$-manifold, we define the space $l\mathrm{Emb}^S(S \times [0,1), S \times [0,1))$ to be the space of embedding whose underlying map is of the form:

$$(s,t) \mapsto (s, at)$$

for some fixed number $a \in (0,1]$.

If τ is a d-framing of S, there is an obvious map $l\mathrm{Emb}^S(S\times[0,1), S\times[0,1)) \to \mathrm{Emb}_f^{S_\tau}(S\times [0,1), S\times[0,1))$, we denote its image by $l\mathrm{Emb}_f^{S_\tau}(S\times[0,1), S\times[0,1))$.

More generally, we denote by $l\mathrm{Emb}^S(S\times[0,1) \sqcup D^{\sqcup n}, S\times[0,1))$ the space of embeddings whose restriction to $S\times[0,1)$ is a point of $l\mathrm{Emb}^S(S\times[0,1), S\times[0,1))$. We define $l\mathrm{Emb}_f^{S_\tau}(S\times [0,1) \sqcup D^{\sqcup n}, S\times[0,1))$ in a similar fashion.

Definition 4.2.3. For any d-framing τ of S, we define an associative algebra in right module over $\mathcal{E}_d$ denoted lS_τ:

$$lS_\tau(n) = l\mathrm{Emb}_f^{S_\tau}(S\times[0,1) \sqcup D^{\sqcup n}, S\times[0,1))$$

Theorem 4.2.4. *The inclusion of right modules $lS_\tau \to S_\tau$ is a weak equivalences of associative algebra in right modules over $\mathcal{E}_d$.*

Proof. The map is obviously a map of $\mathcal{A}ss$-algebras in right $\mathcal{E}_d$-modules. All we have to do is check that they are objectwise weak equivalences.

For a given n, we want to show that the inclusion $lS_\tau(n) \to S_\tau(n)$ is a weak equivalence. The restriction map $S_\tau(n) \to \mathrm{Emb}_f(D^{\sqcup n}, S\times[0,1))$ is a fibration and similarly for the restriction map $lS_\tau(n) \to \mathrm{Emb}_f(D^{\sqcup n}, S\times[0,1))$. We have the following pullback diagram where the right vertical map is a fibration by 4.2.1:

$$\begin{tikzcd} lS_\tau(n) \arrow[r] \arrow[d] & S_\tau(n) \arrow[d] \\ (0,1] \arrow[r] & \mathrm{Emb}_f^{S_\tau}(S\times[0,1), S\times[0,1)) \end{tikzcd}$$

The bottom map sends a number a to the product of the identity of S with $t \mapsto at$. Since the category of spaces is right proper and the bottom map is a weak equivalence by 2.4.10, the top map is a weak equivalence. $\square$

Let S be a $(d-1)$-manifold and let τ be a d-framing of S. Let A be an $\mathcal{E}_d$-algebra, the factorization homology $\int_{S\times(0,1)} A$ is an $\mathcal{E}_1$ algebra. Indeed there is a morphism of operad:

$$\mathrm{Emb}_f((0,1)^{\sqcup n}, (0,1)) \to \mathrm{Emb}_f(S\times(0,1)^{\sqcup n}, S\times(0,1))$$

obtained by taking the product with the identity of S.

Proposition 4.2.5. *The map $lS_\tau \to \mathrm{Emb}_f(-, S\times(0,1))$ is a weak equivalence of right $\mathcal{E}_d$-modules*

Proof. This is clear. $\square$

Corollary 4.2.6. *For a cofibrant $\mathcal{E}_d$-algebra A, there is a zig-zag of weak equivalences:*

$$U_A^{S_\tau} \xleftarrow{\simeq} U_A^{lS_\tau} \xrightarrow{\simeq} \int_{S\times(0,1)} A$$

Proof. By the previous proposition and 4.2.4, there is a zig-zag of weak equivalences of right $\mathcal{E}_d$-modules:

$$S_\tau \leftarrow lS_\tau \to \mathrm{Emb}_f(-, S\times(0,1))$$

Then it suffices to apply B.3.9 to this zig-zag. $\square$

4.3 Equivalence with operadic modules

In this section, we prove the following theorem (see [Fra] for a similar result):

Theorem 4.3.1. *S^{d-1}_κ and $\mathcal{E}_d[1]$ are weakly equivalent as associative algebras in right modules over $\mathcal{E}_d$. In particular, for a cofibrant $\mathcal{E}_d$-algebra A, the category $S^{d-1}_\kappa\mathbf{Mod}_A$ is related to $\mathcal{E}_d[1]\mathbf{Mod}_A$ through a zig-zag of Quillen equivalences.*

Proof. We have a chain of weak equivalences:

$$\mathcal{E}_d[1] \leftarrow \mathcal{E}_d^* \leftarrow l\mathcal{E}_d^* \rightarrow lS^{d-1}_\kappa \rightarrow S^{d-1}_\kappa$$

The definition of the intermediate terms and the proof of the weak equivalences is done in the remaining of the section. □

Definition 4.3.2. Let $\mathcal{E}_d^*$ be the right $\mathcal{E}_d$-module:

$$\mathcal{E}_d^*(n) = \mathrm{Emb}_f^*(D^{\sqcup n} \sqcup D^*, D^*)$$

where D^* is the manifold D pointed at 0 and Emb_f^* denotes the space of framed embeddings preserving the base point.

There is clearly a map of right $\mathcal{E}_d$-modules $\mathcal{E}_d^* \to \mathcal{E}_d[1]$.

Proposition 4.3.3. *This map is a weak equivalence.*

Proof. It suffices to check it for any n. We have a commutative diagram where the right vertical map is a fibration by 4.2.1:

$$\begin{tikzcd}
\mathcal{E}_d^*(n) \arrow[r] \arrow[d] & \mathcal{E}_d[1](n) \arrow[d] \\
\mathrm{Emb}_f^*(D^*, D^*) \arrow[r, "\simeq"] & \mathrm{Emb}_f(D, D)
\end{tikzcd}$$

Moreover, this diagram is by definition a pullback square. Since the category of spaces is right proper, the top map is a weak equivalence. □

Definition 4.3.4. Let $l\mathcal{E}_d^*$ be the right module over $\mathcal{E}_d$ whose value at n is the following pullback:

$$\begin{tikzcd} l\mathcal{E}_d^*(n) \arrow[r] \arrow[d] & \mathcal{E}_d^*(n) \arrow[d] \\ (0,1] \arrow[r] & \mathrm{Emb}_f^*(D^*, D^*) \end{tikzcd}$$

where the bottom horizontal map sends a to the multiplicaiton by a and the right vertical map is the restriction on the D^*-component. In other words, $l\mathcal{E}_d^*(n)$ is the subspace of $\mathcal{E}_d^*(n)$ whose points are the embeddings whose restriction to D^* is linear.

Proposition 4.3.5. *The obvious inclusion of right $\mathcal{E}_d$-modules $l\mathcal{E}_d^* \to \mathcal{E}_d^*$ is a weak equivalence.*

Proof. The fact that this is a map of right module is easy. Therefore, it suffices to check that it is a degreewise weak equivalence. The right vertical map in the pullback diagram of the previous definition is a fibration by 4.2.1, moreover the bottom map is a weak equivalence since both sides are contractible. Since the category of spaces is right proper, the top horizontal map is a weak equivalence. □

We now want to compare lS_κ^{d-1} to $l\mathcal{E}_d^*$.

Let n be a nonnegative integer. We construct a map $lS_\kappa^{d-1}(n) \to l\mathcal{E}_d^*(n)$. A point in the left-hand-side is a pair (a, f) where a is a point in $(0,1]$ and f is an embedding of $D^{\sqcup n}$ in the complement of $S^{d-1} \times [0,a]$, a point in the right hand side is a pair (b, g) where b is a point in $(0,1]$ and g is an embedding of $D^{\sqcup n}$ in the complement of the disk of center 0 and radius b in D. There is an obvious diffeomorphism ϕ_a from the complement of $[0,a] \times S^{d-1}$ in $[0,1) \times S^{d-1}$ to the complement of the disk of radius a in D obtained by passing to polar coordinate. Moreover this diffeomorphism preserves the framing on the nose if $[0,1) \times S^{d-1}$ is given the framing κ. We thus define the image of (a, f) to be $(a, \phi_a \circ f)$.

Proposition 4.3.6. *The above maps are weak equivalences for any n. Moreover they assemble into a morphism of associative algebras in right $\mathcal{E}_d$-modules.*

Proof. There is a commutative diagram:

$$\begin{tikzcd} l\mathcal{E}_d^*(n) \arrow[r] \arrow[d] & lS_\kappa^{d-1}(n) \arrow[d] \\ (0,1] \arrow[r, "="] & (0,1] \end{tikzcd}$$

in which the vertical maps are fibrations. The construction of the top horizontal map makes it clear that it is a fiberwise weak equivalence (even a homeomorphism) therefore it is a weak equivalence.

It is clear that the map $l\mathcal{E}_d^* \to lS_\kappa^{d-1}$ is a morphism of right $\mathcal{E}_d$-modules. A straightforward computation shows that it preserves the associative algebra structure. □

Remark 4.3.7. The category $S_\kappa^{d-1}\mathbf{Mod}_A$ is

4.4 $\mathcal{E}_1$-modules and their tensor product

We denote by L the right module over $\mathcal{E}_1$ induced by the one-point manifold and the negative framing. More precisely, this is the framing on $T(*) \oplus \mathbb{R} \cong \mathbb{R}$ given by the real number -1. Similarly, we define R to be the right-module over $\mathcal{E}_1$ induced by the one-point manifold and the positive framing. Finally we denote by B the right module induced by the manifold S^0 and the framing κ.

We denote by cL cR and cB the collared versions of L, R and B.

In this section, A is a cofibrant $\mathcal{E}_1$-algebra in $\mathbf{C}$.

Proposition 4.4.1. *Let M be an object of $L\mathbf{Mod}_A$ and N be an object of $R\mathbf{Mod}_A$, then $M \otimes N$ is an object of $B\mathbf{Mod}_A$. Moreover, if A is cofibrant, the pairing:*

$$L\mathbf{Mod}_A \times R\mathbf{Mod}_A \to B\mathbf{Mod}_A$$

is a left Quillen bifunctor.

We have a similar result for collared left and right modules.

Proof. As right modules over $\mathcal{E}_1$, $L \otimes R$ is isomorphic to S_κ^0. The result then follows from 1.3.8 □

The manifold $[0,1]$ with positive orientation is in an obvious way a (collared) S^0_κ-manifold.

In more concrete terms, let us denote by $\mathcal{G}(M,A,N)$, the functor on $\mathbf{E}_1^{S^0_\kappa}$ which sends $L^{\sqcup i} \sqcup D^{\sqcup n} \sqcup R^{\sqcup j}$ to $M^{\otimes i} \otimes A^{\otimes n} \otimes N^{\otimes j}$ where i and j are in $\{0,1\}$ and n is any nonnegative integer.

Definition 4.4.2. We define $M \otimes_A^{[0,1]} N$ as the coend:

$$M \otimes_A N = \mathrm{Emb}_f^{S^0_\kappa}(-,[0,1]) \otimes_{\mathbf{E}_1^{S^0_\kappa}} \mathcal{G}(M,A,N)$$

Definition 4.4.3. We denote by $M \otimes_A^{\mathbb{L}[0,1]} N$ the value of $QM \otimes_A^{[0,1]} QN$ where $QM \to M$ is a cofibrant replacement in $L\mathbf{Mod}_A$ and $QN \to N$ is a cofibrant replacement in $R\mathbf{Mod}_A$.

Equivalently, this is the factorization homology of the $S^0_\kappa\mathcal{M}od$ algebra in $\mathbf{C}$ $(A, M \otimes N)$ over the S^0_κ-manifold $[0,1]$.

The following proposition shows that this is indeed the derived tensor product:

Proposition 4.4.4. *If $M \to M'$ and $N \to N'$ are weak equivalences between cofibrant objects in $L\mathbf{Mod}_A$ and $R\mathbf{Mod}_A$, then the induced map $M \otimes_A^{[0,1]} N \to M' \otimes_A^{[0,1]} N'$ is a weak equivalence.*

Proof. The condition on A and M, N, M' and N' implies that the algebra $(A, M \otimes N)$ and $(A, M' \otimes N')$ over $B\mathcal{M}od$ are cofibrant and weakly equivalent. □

Comparison with the tensor product of modules over an associative algebra

Definition 4.4.5. $\mathbf{Ass}^{-+}$ is the category whose objects are finite sets with two distinguished elements $-$ and $+$ and whose morphisms are maps of finite sets f preserving $-$ and $+$ together with the extra data of a linear ordering of each fiber which is such that $-$ (resp. $+$) is the smallest (resp. largest) element in the fiber over $-$ (resp $+$).

The functor π_0 which sends a disjoint union of intervals to the set of connected components is an equivalence of simplicial categories from $B\mathbf{Mod}$ to $\mathbf{Ass}^{-+}$.

Let A be an associative algebra and M (resp. N) be a left (resp. right) module over it. We define $G(M,A,N)$ to be the obvious functor $\mathbf{Ass}^{-+} \to \mathbf{C}$ sending $\{-,1,\ldots,n,+\}$ to $M \otimes A^{\otimes n} N$.

There is an obvious functor $\Delta^{\mathrm{op}} \to \mathbf{Ass}^{-+}$ which sends a totally ordered set with minimal element $-$ and maximal element $+$ to the underlying finite set and an order preserving map to the underlying map with the data of the linear ordering of each fiber.

Proposition 4.4.6. *Let A be an associative algebra and M (resp. N) be a left (resp. right) module over it. The precomposition of $G(M, A, N)$ along the above functor $\Delta^{\mathrm{op}} \to \mathbf{Ass}^{-+}$ is the bar construction $B_\bullet(M, A, N)$*

Proof. Trivial. □

Let us denote by $P : (\mathbf{Ass}^{-+})^{\mathrm{op}} \to \mathbf{S}$ the left Kan extension of the constant cosimplicial set $[n] \to *$ along this map. Concretely P sends a finite set with $-$ and $+$ to the set of linear oredring of that set whose smallest element is $-$ and largest element is $+$. Note that a linear ordering of each fiber of a map is exactly the data we need to pullback such an ordering along that map.

Corollary 4.4.7. *Let A be a cofibrant associative algebra and M (resp. N) be a left (resp. right) modules over it. Then:*

$$M \otimes^{\mathbb{L}}_A N \simeq P \otimes^{\mathbb{L}}_{\mathbf{Ass}^{-+}} G(M, A, N)$$

Proof. Assume that M and N are cofibrant as left and right modules. If they are not , we take a cofibrant replacement. The left hand side is:

$$|[n] \to \mathrm{B}_n(M, A, N) = M \otimes A^{\otimes n} \otimes N|$$

According to the cofibrancy assumption, this simplicial functor is Reedy cofibrant, therefore the realization coincides with the homotopy colimit. Hence we have:

$$M \otimes^{\mathbb{L}}_A N \simeq * \otimes^{\mathbb{L}}_{\Delta^{\mathrm{op}}} \mathrm{B}(M, A, N) \simeq P \otimes^{\mathbb{L}}_{\mathbf{Ass}^{-+}} G(M, A, N)$$

□

Proposition 4.4.8. *Let (M, A, N) be a triple consisting of a cofibrant associative algebra A, a left A-module M and a right A-module N, then there is a weak equivalence:*

$$M \otimes_A^{\mathbb{L}} N \simeq M \otimes_A^{\mathbb{L}[0,1]} N$$

Proof. First notice that if A is cofibrant as an associative algebra, then the underlying $\mathcal{E}_1$-algebra is cofibrant. Let us assume that M and N are already cofibrant (otherwise take a cofibrant replacement).

The left hand-side is the derived coend:

$$P \otimes^{\mathbb{L}}_{\mathbf{Ass}^{-+}} G(M, A, N)$$

which can be computed as the realization of the Reedy cofibrant simplicial object:

$$\mathrm{B}_\bullet(P, \mathbf{Ass}^{-+}, G(M, A, N))$$

The right hand side is the realization of the Reedy fibrant simplicial object:

$$\mathrm{B}_\bullet(\mathrm{Emb}^{S^0}(-, [0,1]), B\mathbf{Mod}, \mathcal{G}(M, A, N))$$

It is clear that both simplicial object are degreewise weakly equivalent which concludes the proof. □

Note that a similar construction of the bar construction for a left and right module over an algebra over the Stasheff operad is made in [Ang09].

4.5 Tensor product of S_τ-shaped modules

In this section, A is a cofibrant $\mathcal{E}_d$-algebra.

Let M be an S_τ-shaped module and N be an $S_{-\tau}$-shaped module. Abstractly, we can form the relative tensor product:

$$M \otimes_{U_A^{S_\tau^{\mathrm{op}}}} N$$

However, $U_A^{S_\tau}$ is a complicated colimit and we would like a more geometric construction involving only A, M and N.

Notice that the triple (M, A, N) forms an algebra over the operad $(S_\tau \sqcup S_{-\tau})\mathcal{M}od$. It makes sense to take factorization homology of such a structure on an $S_\tau \sqcup S_{-\tau}$-manifold. One particularly simple example of such a manifold is the product $S \times [0,1]$ with the framing induced by τ.

Proposition 4.5.1. *The factorization homology $\int_{S\times[0,1]}(M, A, N)$ is naturally weakly equivalent to $M \otimes^{\mathbb{L}}_{U_A^{S_\tau}} N$.*

Proof. If we fix A and a cofibrant S_τ-shaped module N, both functors are Quillen left functors from $S_{-\tau}\mathbf{Mod}_A$ to $\mathbf{C}$. It suffices to check that both functors coincide on free right $U_A^{S_\tau}$-modules. Let X be a cofibrant object of $\mathbf{C}$ and $M = X \otimes U_A^{S_\tau}$ be the free module on X. Then $M \otimes_{U_A^{S_\tau}} N \cong X \otimes N$.

Now let us compute $\int_{S\times[0,1]}(M, A, N)$. (TO BE FILLED IN) $\square$

Let us give a slightly different interpretation of the above construction:

Let M be an S_τ-shaped module over an $\mathcal{E}_d$-algebra A. We can give M the structure of a right module over the $\mathcal{E}_1$-algebra $\int_{S\times(0,1)} A$. Let:

$$[0,1) \sqcup (0,1)^{\sqcup n} \to [0,1)$$

be a framed embedding. We can take the product with S and get an embedding in $f\mathbf{Man}_d^{S_\tau}$:

$$S \times [0,1) \sqcup (S \times (0,1))^{\sqcup n} \to S \times [0,1)$$

Evaluating $\int_{-}(M, A)$ over this embedding, we find a map:

$$M \otimes \left(\int_{S\times(0,1)} A\right)^{\otimes n} \to M$$

All these maps assemble exactly into a structure of a right $\int_{S\times(0,1)} A$-module on M.

Proposition 4.5.2. *Let M be an S_τ-shaped module over A and N be an $S_{-\tau}$-shaped module* *over A. Then there is a weak equivalence:*

$$M \otimes^{\mathbb{L}[0,1]}_{\int_{S\times(0,1)} A} N \xrightarrow{\simeq} \int_{S\times[0,1]} (M, A, N)$$

Proof. There is an obvious functor from $B\mathbf{Mod}$ to $\mathrm{Fun}((S_\tau \sqcup S_{-\tau})\mathbf{Mod}, \mathbf{S})$:

$$U \mapsto \mathrm{Emb}_f^{S_\tau \sqcup S_{-\tau}}(-, U \times S)$$

We claim that:

$$\mathrm{Emb}_f^{S^0_\kappa}(-, [0,1]) \otimes^{\mathbb{L}}_{B\mathbf{Mod}} \mathrm{Emb}_f^{S_\tau \sqcup S_{-\tau}}(-, - \times S) \simeq \mathrm{Emb}_f^{S_\tau \sqcup S_{-\tau}}(-, S \times [0,1])$$

The proof of that claim is entirely analogous to 3.3.2.

The result follows by associativity of double coends. □

4.6 Hom between modules over an $\mathcal{E}_1$-algebra.

In this section $\mathbf{C}$ is a closed symmetric monoidal category whose inner Hom is denoted $\underline{\mathrm{Hom}}$. Let A be a cofibrant $\mathcal{E}_1$-algebra. We have the model category $L\mathbf{Mod}_A$ of left A-modules. We want to define a functor:

$$L\mathbf{Mod}_A^{\mathrm{op}} \times L\mathbf{Mod}_A \to \mathbf{C}$$

which can be called a Hom object in the category of left modules over A.

Construction 4.6.1. Let M and N be two left modules over an $\mathcal{E}_1$-algebra. We define a functor:

$$\mathcal{F}(M, A, N)(-) : B\mathbf{Mod}^{\mathrm{op}} \to \mathbf{C}$$

Its value on $[0,1) \sqcup I^{\sqcup n} \sqcup (0,1]$ is $\underline{\mathrm{Hom}}(M \otimes A^{\otimes n}, N)$. In order to explain the functoriality, notice that any map in $B\mathbf{Mod}$ can be decomposed as a disjoint union of any of the following three types:

- An embedding $[0,1) \sqcup I^{\sqcup k} \to [0,1)$.

- An embedding $I^{\sqcup l} \to I$.

- An embedding $I^{\sqcup l} \sqcup (0,1] \to (0,1]$.

Let ϕ be an embedding $[0,1) \sqcup I^{\sqcup n} \sqcup (0,1] \to [0,1) \sqcup I^{\sqcup m} \sqcup (0,1]$ and let:

$$\phi = \phi_- \sqcup \psi_1 \sqcup \ldots \sqcup \psi_r \sqcup \phi_+$$

be its decomposition with ϕ_- of the first type, ϕ_+ of the third type and ψ_i of the second type for each i. We need to extract from this data a map:

$$\underline{\mathrm{Hom}}(M \otimes A^{\otimes m}, N) \to \underline{\mathrm{Hom}}(M \otimes A^{\otimes n}, N)$$

The action of ϕ_- and of the ψ_i comes in an obvious way from the $\mathcal{E}_1$-structure of A and the left module structure on M. The only non trivial part is the action of ϕ_+. We can hence assume that $\phi = \mathrm{id}_{L \sqcup I^{\sqcup p}} \sqcup \phi_+$ where ϕ_+ is an embedding $I^{\sqcup n} \sqcup R \to R$. We want to construct:

$$\underline{\mathrm{Hom}}(M \otimes A^{\otimes p}, N) \to \underline{\mathrm{Hom}}(M \otimes A^{\otimes p} \otimes A^{\otimes n}, N)$$

To do that, notice that $\underline{\mathrm{Hom}}(M \otimes A^{\otimes p}, N)$ has the structure of a left A module induced from N. The map ϕ_+ therefore induces a map:

$$\underline{\mathrm{Hom}}(M \otimes A^{\otimes p}, N) \otimes A^{\otimes n} \to \underline{\mathrm{Hom}}(M \otimes A^{\otimes p}, N)$$

This maps is adjoint to a map:

$$\underline{\mathrm{Hom}}(M \otimes A^{\otimes p}, N) \to \underline{\mathrm{Hom}}(M \otimes A^{\otimes p} \otimes A^{\otimes n}, N)$$

which we define to be the action of ϕ.

Definition 4.6.2. Let $(\mathbf{C}, \underline{\mathrm{Hom}}_{\mathbf{C}}(-,-))$ be a category enriched over $\mathbf{V}$. Let $\underline{\mathrm{hom}} : \mathbf{V}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$ be the cotensor. Let $\mathbf{A}$ be a small category, F a functor from $\mathbf{A}$ to $\mathbf{C}$ and G a functor from $\mathbf{A}$ to $\mathbf{C}$. We denote by $\underline{\hom}_{\mathbf{A}}(F,G)$ the end:

$$\int_{\mathbf{A}} \underline{\hom}(F(-),G(-))$$

Definition 4.6.3. We define $\mathbb{R}\underline{\mathrm{Hom}}_A^{[0,1]}(M,N)$ to be the homotopy end:

$$\mathbb{R}\underline{\hom}_{B\mathbf{Mod}}(\mathrm{Emb}^{S^0}(-,[0,1]),\mathcal{F}(QM,A,RN))$$

where $QM \to M$ is a cofibrant replacement as a left module over A and $N \to RN$ is a fibrant replacement.

Proposition 4.6.4. *Let A be an associative algebra and M and N be two left A-modules in $\mathbf{C}$. Then:*

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{[0,1]}(M,N) \simeq \mathbb{R}\underline{\mathrm{Hom}}_A(M,N)$$

Proof. Similar to 4.4.8. □

4.7 Hom of S_τ-modules.

In this short section, we dualize the results about tensor products of S_τ-shaped modules.

Let A be an $\mathcal{E}_d$-algebra which we assume to be cofibrant and M and N be two S_τ-shaped modules. We define a functor

$$\mathcal{F}(M,A,N) : (S_\tau \sqcup S_{-\tau})\mathbf{Mod}^{\mathrm{op}} \to \mathbf{C}$$

its value on $S\times[0,1)^{\sqcup\epsilon}\sqcup D^{\sqcup n}\sqcup S\times(-1,0]^{\sqcup\epsilon'}$ is $\underline{\mathrm{Hom}}(M^{\otimes\epsilon}\otimes A^{\otimes n}, N^{\otimes\epsilon'})$. The functoriality is analogous to 4.6.1.

Definition 4.7.1. We define $\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times[0,1]}(M,N)$ to be the homotopy end:

$$\mathbb{R}\underline{\hom}_{(S_\tau\sqcup S_{-\tau})\mathbf{Mod}^{\mathrm{op}}}(\mathrm{Emb}_f^{S_\tau\sqcup S_{-\tau}}(-,S\times[0,1]),\mathcal{F}(QM,A,RN))$$

where $QM \to M$ is a cofibrant replacement as an S_τ-shaped module over A and $N \to RN$ is a fibrant replacement.

Proposition 4.7.2. *There is a weak equivalence:*

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times[0,1]}(M,N) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{U_A^{S_\tau}}(M,N)$$

Proof. Similar to 4.5.1. □

4.8 Functor induced by a bordism

Let S_σ and T_τ be two $(d-1)$-manifold with a d-framing.

Definition 4.8.1. A bordism from S_σ to T_τ is a collared $S_\sigma \sqcup T_{-\tau}$-manifold.

Definition 4.8.2. Let W be a bordism from S to T. We define $\widetilde{W}$ to be the manifold:

$$(S\times(-1,0]) \cup_S W \cup_T (T\times[0,1))$$

Note that $\widetilde{W}$ is a manifold without boundary which contains W as a deformation retract. The collar is needed to give $\widetilde{W}$ a smooth structure. It is clear that the framing on each piece assemble and induce a framing on $\widetilde{W}$.

Proposition 4.8.3. *The functor represented by $\widetilde{W}$ is an cS_σ-cT_τ-bimodule in the category* $\mathbf{Mod}_{\mathcal{E}_d}$.

Proof. Let us describe the right cT_τ-module structure.

Let ϕ be a collared embedding of $T_{-\tau}\times[0,1)\sqcup D^{\sqcup n}$ into $T_{-\tau}\times[0,1)$. It is obvious that ϕ induces an embedding of $\widetilde{W}\sqcup D^{\sqcup n}$ into $\tilde{W}$. This makes $\widetilde{W}$ into a cT_τ-module over D in $f\mathbf{Man}_d$. The Yoneda embedding from $f\mathbf{Man}_d$ to $\mathrm{Fun}(\mathbf{E}_d^{\mathrm{op}},\mathbf{S})$ is simplicial and monoidal. Therefore, it preserves any operadically defined algebraic structure.

The left cS_σ-module structure is similar. □

Therefore, according to the first chapter. For any cofibrant $\mathcal{E}_d$-algebras in $\mathbf{C}$, a bordism W generates a functor:

$$cS_\sigma\mathbf{Mod}_A \to cT_\tau\mathbf{Mod}_A$$

Definition 4.8.4. Let V be a bordism from S_σ to T_τ and W be a bordism from T_τ to U_υ. We define $W \circ W'$ to be the manifold:

$$V \cup_T (T \times [0,1]) \cup_T W$$

Note that, with its obvious framing, $W \circ W'$ is a bordism from S_σ to U_υ.

Theorem 4.8.5. *There is a weak equivalence:*

$$\int_{T\times[0,1]} (\int_{\widetilde{V}} A, A, \int_{\widetilde{W}} A) \simeq \int_{\widetilde{V\circ W}} A$$

Proof. There is an obvious functor from $(cT_\tau \sqcup cT_{-\tau})\mathbf{Mod}$ to $\mathbf{Mod}_{\mathcal{E}_d}$ sending $cT_\tau \sqcup D^{\sqcup n} \sqcup cT_{-\tau}$ to $\mathrm{Emb}_f^{cS_\tau \sqcup cS_{-\tau}}(-, \widetilde{V} \sqcup D^{\sqcup n} \sqcup \widetilde{W})$. The homotopy coend of that functor with $\mathrm{Emb}_f(-, T\times[0,1])$ is weakly equivalent to $\mathrm{Emb}_f(-, \widetilde{V \circ W})$ (the proof is again similar to 3.3.2). The result then follows from associativity of double coends. $\square$

We can generalize the definition 4.7.1.

Let W be bordism from S_σ to T_τ. Let M be an S_σ-shaped module over A and N be a $T_{-\tau}$-shaped module. We can construct a functor $\mathcal{F}(M, A, N)$ from $(S_\sigma \sqcup T_{-\tau})\mathbf{Mod}^{\mathrm{op}}$ to $\mathbf{C}$ which sends $S_\sigma^{\sqcup\epsilon} \sqcup D^{\sqcup n} \sqcup T_\tau^{\sqcup\epsilon'}$ to $\underline{\mathrm{Hom}}_{\mathbf{C}}(M^{\otimes\epsilon} \otimes A^{\otimes n}, N^{\otimes\epsilon'})$. We define $\mathbb{R}\underline{\mathrm{Hom}}_A^W(M, N)$ to be the homotopy end:

$$\mathbb{R}\underline{\mathrm{Hom}}_A^W(M, N) = \mathbb{R}\underline{\mathrm{hom}}_{(S_\sigma \sqcup T_{-\tau})\mathbf{Mod}^{\mathrm{op}}}(\mathrm{Emb}_f^{S_\sigma \sqcup T_{-\tau}}(-, W), \mathcal{F}(M, A, N))$$

Theorem 4.8.6. *There is a weak equivalence:*

$$\mathbb{R}\underline{\mathrm{Hom}}_A^W(M, N) \simeq \mathbb{R}\underline{\mathrm{Hom}}_A^{T\times[0,1]}(M \otimes_{U_A^{S_{-\sigma}}} \int_{\widetilde{W}} A, N)$$

Corollary 4.8.7. *Let $\bar{D}$ be the closed unit ball in $\mathbb{R}^d$ seen as a bordism from the empty manifold to S_κ^{d-1}. There is a weak equivalence:*

$$\mathrm{HH}_{\mathcal{E}_d}(A, M) := \mathbb{R}\underline{\mathrm{Hom}}_{S_\kappa^{d-1}\mathbf{Mod}_A}(A, M) \simeq \mathbb{R}\underline{\mathrm{Hom}}_A^D(\mathbb{I}_{\mathbf{C}}, M)$$

Proof. It suffices to apply the previous theorem. $\mathbb{I}_{\mathbf{C}}$ is an object of $\emptyset\mathbf{Mod}_A$ and its push-forward along the bordism D is equivalent to A. $\square$

This has the following surprising consequence. Observe that $\mathrm{Emb}_f^{S^{d-1}}(D, D)$ is isomorphic to $\mathrm{Diff}_f^{S^{d-1}}(D)$.

Corollary 4.8.8. *The group* $\mathrm{Diff}_f^{S^{d-1}}(D)$ *acts on* $\mathrm{HH}_{\mathcal{E}_d}(A, M)$*.*

Note that there is a fiber sequence:

$$\mathrm{Diff}_f^{S^{d-1}}(D) \to \mathrm{Diff}^{S^{d-1}}(D) \to \Omega^n \mathrm{O}(n)$$

4.9 Cobordism category

Let S_σ and T_τ be two $(d-1)$-manifold with a d-framing. For a bordism W between S and T, we define $\mathrm{Diff}^{\sqcup T}(W)$ to be the group of diffeomorphisms of W as an $S_{-\sigma} \sqcup T_\tau$-manifold. Note that any embedding from a compact manifold to itself is surjective. Therefore $\mathrm{Diff}^{\sqcup T}(W) \cong \mathrm{Emb}_f^{\sqcup T}(W, W)$.

We define $f\mathbf{Cob}_d(S, T)$ as the disjoint union over all diffeomorphism classes of framed bordisms W from S_σ to T_τ of the space:

$$B\mathrm{Diff}_f^{\sqcup T}(W)$$

The cobordism category $f\mathbf{Cob}_d$ is a simplicial category whose objects are diffeomorphism classes of $(d-1)$-manifolds and whose space of morphism from S to T is equivalent to $f\mathbf{Cob}_d(S, T)$ and whose composition is given by glueing of bordisms. See [GMTW09] for a precise definition.

An embedding calculus version of the cobordism category

Embedding calculus replaces framed manifold by the functor they represent on $\mathbf{E}_d$. In that sends, we can see the category $\mathbf{Mod}_{\mathcal{E}_d}$ as a category of "generalized manifolds". The functor $f\mathbf{Man}_d \to \mathbf{Mod}_{\mathcal{E}_d}$ is symmetric monoidal. The cobordism category $f\mathbf{Cob}_d$ has an embedding calculus "shadow" in the world of right modules over $\mathcal{E}_d$ that we now describe.

We construct a subcategory $f\widehat{\mathbf{Cob}}_d$ of $\mathbf{NBiMod}(\mathbf{Mod}_{\mathcal{E}_d})$.

The objects of $f\widehat{\mathbf{Cob}}_d$ are associative algebras in right $\mathcal{E}_d$-module of the form cS_σ where S is a $(d-1)$-manifold and σ is a d-framing. The space of morphisms $S_\sigma \to T_\tau$ is the nerve of the full subcategory of ${}_{S_\sigma}\mathbf{Mod}_{T_\tau}$ on objects that are cofibrant and weakly equivalent to $\mathrm{Emb}_f(-, \widetilde{W})$ for some framed bordism W from S_σ to T_τ. We have shown in 4.8.5 that the relative tensor product of two bimodule represented by bordisms is a bimodule represented by a bordism. Therefore, the composition is well defined in $f\widehat{\mathbf{Cob}}_d$.

Now let us compare $f\mathbf{Cob}_d$ and $f\widehat{\mathbf{Cob}}_d$. In the two categories the objects are the same, namely $(d-1)$-manifolds with a d-framing. In $\mathbf{Cob}_d$, the space of maps from S_σ to T_τ is equivalent to:

$$\bigsqcup_W B\mathrm{Emb}_f^{\sqcup T}(W, W)$$

where the disjoint union is taken over all diffeomorphism classes of bordisms.

In $f\widehat{Cob}_d$, the space of maps from S_σ to T_τ is equivalent to:

$$\bigsqcup_W B\widehat{\mathrm{Emb}}_f^{\sqcup T}(W, W)$$

where the disjoint union is taken over the same set and $\widehat{\mathrm{Emb}}_f^{\sqcup T}(W)$ denotes the inverse limit of the embedding calculus tower converging to $\mathrm{Emb}_f^{\sqcup T}(W, W)$. Indeed, we have showed in the first chapter that in $\mathbf{NBiMod}_{\mathbf{E}_d}$, the space of morphisms between S_σ and T_τ splits as a disjoint union over all equivalence classes of S_σ-T_τ-bimodules M of $B\mathrm{Auth}(M)$.

The grouplike monoid $\mathrm{Auth}(\mathrm{Emb}_f(-, W))$ is by definition the invertible components in the monoid $\widehat{\mathrm{Emb}}_f^{\sqcup T}(W, W)$

Chapter 5

Chromatic homotopy computations

We give two methods for computing factorization homology. The first one is based on embedding calculus, the second one is just a traditional homotopy colimit spectral sequence. We then shows how to compute higher Hochschild homology and cohomology when the algebra is étale in a sense that we make precise. As an application, we compute higher Hochschild cohomology of the Lubin-Tate ring spectrum.

5.1 Embedding calculus spectral sequence

Embedding calculus is a tool for studying presheaves on the category of manifolds and embeddings (see [BdBW12] for an account of embedding calculus close to what we are describing in this section)

Let $\mathcal{F}$ be a functor from $\mathbf{E}_d$ to $\mathbf{Mod}_E$. Ultimately we are interested in taking $\mathcal{F}$ to be a monoidal functor (i.e. a functor coming from an $\mathcal{E}_d$-algebra) but this will not be necessary in this section.

Definition 5.1.1. Let M be an object of $f\mathbf{Man}_d$. We define the *factorization homology* of M with coefficients in $\mathcal{F}$to be the value at M of the homotopy left Kan extension of $\mathcal{F}$ to the category $f\mathbf{Man}_d$. We use the notation $\int_M \mathcal{F}$ to denote that object.

If $\mathcal{F}$ comes from an $\mathcal{E}_d$-algebra A, this coincides up to homotopy with the factorization homology of M with coefficients in A according to B.3.10.

We define a filtration of the category $\mathbf{E}_d$:

Definition 5.1.2. The category $\mathbf{E}_d^{\leq n}$ is the full subcategory of $\mathbf{E}_d$ on objects with at most n connected components. We denote by i_n the inclusion functor:

$$\mathbf{E}_d^{\leq n} \to \mathbf{E}_d$$

Construction 5.1.3. We construct a filtered object converging to $\mathcal{F}$. Assume that $\mathcal{F}$ is a cofibrant object of $\mathrm{Fun}(\mathbf{E}_d, \mathbf{S})$ equipped with the projective model structure.

For any natural number n, there is a map in $\mathrm{Fun}(\mathbf{E}_d, \mathbf{S})$:

$$(i_n)_! i_n^* \mathcal{F} \to (i_{n+1})_! i_{n+1}^* \mathcal{F}$$

To define this map we can start from the identity map:

$$i_n^* \mathcal{F} \to i_n^* \mathcal{F}$$

Let us denote by j_n the fully faithful inclusion $j_n : \mathbf{E}_d^{\leq n} \to \mathbf{E}_d^{\leq n+1}$. We have $i_n = i_{n+1} \circ j_n$ therefore the identity map can be rewritten:

$$i_n^* \mathcal{F} \to j_n^* i_{n+1}^* \mathcal{F}$$

This is adjoint to:

$$(j_n)_! i_n^* \mathcal{F} \to i_{n+1}^* \mathcal{F}$$

Then we apply $(i_{n+1})_!$ on both sides and we obtain the desired map.

Note that the cofibrancy assumption insures that $i_n^* \mathcal{F}$ is cofibrant for all n (this is because i_n^* preserves cofibrations), therefore $(i_n)_! i_n^* \mathcal{F}$ has the right homotopy type.

Definition 5.1.4. We denote by $P_n \mathcal{F}$ the functor $(i_n)_! i_n^* \mathcal{F}$. We call it the n-th *polynomial approximation* of $\mathcal{F}$. If $\mathcal{F}$ is not cofibrant, $P_n \mathcal{F}$ is defined as $(i_n)_! i_n^* Q\mathcal{F}$ where $Q\mathcal{F}$ is a cofibrant replacement of $\mathcal{F}$ in $\mathrm{Fun}(\mathbf{E}_d, \mathbf{S})$.

Definition 5.1.5. The filtered object:

$$\mathcal{F}(\emptyset) \cong P_0 \mathcal{F} \to \ldots \to P_n \mathcal{F} \to P_{n+1} \mathcal{F} \to \ldots \to \mathcal{F}$$

is called the *embedding calculus filtered object* associated to $\mathcal{F}$.

Traditional embeddings calculus deals with *contravariant functors* on $\mathbf{E}_d$ (or **Man**) our construction is formally dual to the embedding calculus tower.

Proposition 5.1.6. *This diagram exhibits $\mathcal{F}$ as the homotopy colimits of the $P_n\mathcal{F}$.*

Proof. It suffices to check that the map from the homotopy colimit of the $P_n\mathcal{F}$ to $\mathcal{F}$ is an objectwise weak equivalence.

Let U be an object of $\mathbf{E}_d$, then U is in $\mathbf{E}_d^{\leq n}$ for all sufficiently big n. Therefore, if we evaluate this diagram on U, we find that $P_n\mathcal{F}(U) \cong \mathcal{F}(U)$ for all sufficiently big n. For a constant diagram, the ordinary colimit is necessarily the homotopy colimit. To conclude, notice that evaluating at U preserves colimits (since in presheaves categories, colimits are computed objectwise). $\square$

Let j be the fully faithful inclusion $\mathbf{E}_d \to f\mathbf{Man}_d$. We can apply $j_!$ to each term of the filtered object of the $P_n\mathcal{F}$'s. Applying $j_!$ is equivalent to applying its left derived functor since everything is cofibrant. As a left adjoint, $j_!$ preserves colimits, the diagram:

$$j_! P_0\mathcal{F} \to \ldots \to j_! P_n\mathcal{F} \to \ldots \to j_!\mathcal{F}$$

exhibits $j_!\mathcal{F}$ as the homotopy colimits of the $j_! P_n\mathcal{F}$.

Evaluating at M we obtain the following:

Proposition 5.1.7. *The diagram:*

$$\int_M P_0\mathcal{F} \to \ldots \to \int_M P_n\mathcal{F} \to \ldots \to \int_M \mathcal{F}$$

exhibits $\int_M \mathcal{F}$ as the homotopy colimit of the $\int_M P_n\mathcal{F}$.

Proof. As we have mentioned before, evaluation at an object preserves homotopy colimits in a functor category. $\square$

As is the case with any filtered object, there is a spectral sequence converging to the homotopy groups of the colimit whose E^1-page involves the associated graded pieces:

Definition 5.1.8. We define $D_n\mathcal{F}$, the n-th *homogeneous piece* of $\mathcal{F}$ by the cofiber sequence:

$$P_{n-1}\mathcal{F} \to P_n\mathcal{F} \to D_n\mathcal{F}$$

Proposition 5.1.9. *There is a spectral sequence:*

$$\mathrm{E}^1_{s,t} = \pi_s\left(\int_M D_t\mathcal{F}\right) \implies \pi_{s+t}\left(\int_M \mathcal{F}\right)$$

Proof. Taking cofiber is a special kind of homotopy colimits, in particular it commutes with factorization homology. Therefore, we have a cofiber sequence:

$$\int_M P_{n-1}\mathcal{F} \to \int_M P_n\mathcal{F} \to \int_M D_n\mathcal{F}$$

The existence of the spectral sequence is then a standard fact (see for instance [Lur11] 1.2.2.). □

Note that we do not have to worry about convergence in this case contrary to the case of embedding calculus. The reason is that contrary to what is ordinarily the case in embedding calculus, the functor on $f\mathbf{Man}_d$ that we are interested in is by definition the left Kan extension of a functor on $\mathbf{E}_d$.

We can be quite explicit about the homogeneous pieces $D_n\mathcal{F}$ in a way that is analogous to [Wei99]. In the case where $\mathcal{F} = A$ comes from an $\mathcal{E}_d$-algebra, and the manifold M is compact, we have:

$$D_n\mathcal{F} = \mathrm{Conf}(n, M)^* \otimes_{\Sigma n} A^{\otimes n}$$

where $\mathrm{Conf}(n, M)^*$ is the one point compactification of $\mathrm{Conf}(n, M)^*$. (see [Fra12]).

5.2 Pirashvili's higher Hochschild homology

We will need a version of $\int_X A$ for commutative algebras in $\mathbf{Ch}_{\geq 0}(R)$ (the category of non-negatively graded chain complexes over a commutative ring R) where R is not necessarily a $\mathbb{Q}$-algebra. In this case there is not necessarily a model structure on commutative algebras in $\mathbf{Ch}_{\geq 0}(R)$. However, we have the projective model category structure on functors

$\mathbf{Fin} \to \mathbf{Ch}_{\geq 0}(R)$, in which weak equivalences are objectwise and fibrations are objectwise epimorphisms. The following definition was made by Pirashvilli at least in the characteristic 0 case (see [Pir00], [GTZ10])

Definition 5.2.1. Let A be a degreewise projective commutative algebra in $\mathbf{Ch}_{\geq 0}(R)$ where R is any commutative ring and let X be a simplicial set. We denote by $\mathrm{HH}^X(A|R)$ the homotopy coend:

$$\mathrm{Map}(-, X) \otimes^{\mathbb{L}}_{\mathbf{Fin}} A$$

By B.3.10, if R is a $\mathbb{Q}$-algebra, then $\mathrm{HH}^X(A)$ is quasi-isomorphic to $\int_X A$. The advantage of this construction is that it is defined for any R. In practice, we can take $\mathrm{HH}^X(A)$ to be the realization of the simplicial object:

$$\mathrm{B}_\bullet(\mathrm{Map}(-, X), \mathbf{Fin}, A^{\otimes -})$$

This construction preserves quasi-isomorphism between degreewise projective commutative algebras.

Proposition 5.2.2. *Let A be a degreewise projective commutative algebra in $\mathbf{Ch}_{\geq 0}(R)$, then $\mathrm{HH}^X(A|R)$ is a commutative algebra in $\mathbf{Ch}_{\geq 0}(R)$ naturally in the variable X.*

Proof. The category $\mathrm{Fun}(\mathbf{Fin}^{\mathrm{op}}, \mathbf{S})$ equipped with the convolution tensor product is a symmetric monoidal model category. It is easy to check that there is an isomorphism:

$$\mathrm{Map}(-, X) \otimes \mathrm{Map}(-, Y) \cong \mathrm{Map}(-, X \sqcup Y)$$

Moreover $A : \mathbf{Fin} \to \mathbf{Ch}_{\geq 0}(R)$ is a commutative algebra for the convolution tensor product, this makes $\mathrm{HH}^X(A|R)$ into a symmetric monoidal in the X variable. To conclude, it suffices to observe that any simplicial set is a commutative monoid with respect to the disjoint union in a unique way and natural way. Therefore, $\mathrm{HH}^X(A|R)$ is a commutative algebra. □

Proposition 5.2.3. *Let A be a degreewise projective commutative algebra in $\mathbf{Ch}_{\geq 0}(R)$.*

Let:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & P \end{tikzcd}$$

be a homotopy pushout of Kan complexes. Then there is a weak equivalence:

$$\mathrm{HH}^P(A|R) \simeq |\mathrm{B}_\bullet(\mathrm{HH}^Y(A|R), \mathrm{HH}^X(A|R), \mathrm{HH}^Z(A|R))|$$

Proof. First, notice that the maps $X \to Z$ and $X \to Y$ induce commutative algebra maps $\mathrm{HH}^X(A|R) \to \mathrm{HH}^Y(A|R)$ and $\mathrm{HH}^X(A|R) \to \mathrm{HH}^Z(A|R)$. In particular $\mathrm{HH}^Z(A|R)$ and $\mathrm{HH}^Y(A|R)$ are modules over $\mathrm{HH}^X(A|R)$. This explains the bar construction in the proof.

We can explicitely construct P as the realization of the following simplicial space:

$$[p] \mapsto Y \sqcup X^{\sqcup p} \sqcup Z$$

For a finite set S, and any simplicial space $U_\bullet$, there is an isomorphism:

$$|U_\bullet^S| \cong |U_\bullet|^S$$

Therefore, there is a weak equivalence of functors on $\mathbf{Fin}$:

$$\mathrm{Map}(-, P) \simeq |\mathrm{B}_\bullet(\mathrm{Map}(-, Y), \mathrm{Map}(-, X), \mathrm{Map}(-, Z))|$$

where the bar construction on the right hand side is in the category $\mathrm{Fun}(\mathbf{Fin}, \mathbf{S})$ with the convolution tensor product.

Now, we have the bisimplicial object:

$$\mathrm{B}_\bullet(\mathrm{B}_\bullet(\mathrm{Map}(-, Y), \mathrm{Map}(-, X), \mathrm{Map}(-, Z)), \mathbf{Fin}, A)$$

By the previous observation, if we first realize with respect to the inner simplicial variable and then the outer one, we find something equivalent to $\mathrm{HH}^P(A|R)$. If we first realize with respect to the outer variable, we find:

$$\mathrm{B}_\bullet(\mathrm{HH}^Y(A|E), \mathrm{HH}^X(A|E), \mathrm{HH}^Z(A|E))$$

The two realizations are equivalent which concludes the proof. $\square$

Corollary 5.2.4. *Let A be a degreewise projective commutative algebra in $\mathbf{Ch}_{\geq 0}(R)$, then $\mathrm{HH}^{S^1}(A)$ is quasi-isomorphic to $\mathrm{HH}(A)$.*

Proof. We can write S^1 as the homotopy pushout of:

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & \mathrm{pt} \\ \mathrm{pt} & \end{tikzcd}$$

If S is a finite set $\mathrm{HH}^S(A) = A^{\otimes S}$ with the obvious commutative algebra structure. In particular, the previous theorem gives:

$$\mathrm{HH}^{S^1}(A) \simeq |\mathrm{B}_\bullet(A, A \otimes A, A)|$$

Since $A = A^{\mathrm{op}}$, the right hand side is quasi-isomorphic to $A \otimes^{\mathbb{L}}_{A \otimes A^{\mathrm{op}}} A$ $\square$

5.3 Another spectral sequence

We construct another spectral sequence converging to factorization homology with Pirashvili's higher Hochschild homology as the E^2-page.

Definition 5.3.1. Let $\mathbf{I}$ be a small discrete category and $F : \mathbf{I} \to gr\mathbf{Mod}_R$ be a functor landing in the category of graded modules over a (possibly graded) associative ring. We define the homology of $\mathbf{I}$ with coefficients in F to be the homology groups of the homotopy colimit of F seen as a functor from $\mathbf{I}$ to $\mathbf{Ch}_{\geq 0}(R)$, the category of chain complexes with values in $\mathbf{A}$.

We write $\mathrm{H}^R_*(\mathbf{I}, F)$ for the homology of $\mathbf{I}$ with coefficients in F.

Note that since we consider graded modules, the chain complexes are gaded objects in chain complexes and the homology groups are bigraded.

There is an explicit model for this homology. We construct the simplicial object of $gr\mathbf{Mod}_R$ whose p simplices are:

$$\mathrm{B}_b(R, \mathbf{I}, F) = \prod_{i_0 \to \ldots \to i_p} F(i_p)$$

The realization of this simplicial object is an object of $\mathbf{Ch}_{\geq 0}(R)$ which models the homotopy colimit of F. In particular, its homology groups are the homology groups of $\mathbf{I}$ with coefficients in F.

Proposition 5.3.2. *Let $F : \mathbf{I} \to \mathbf{Mod}_E$ be a functor from a discrete category to the category of right modules over a cofibrant associative algebra in symmetric spectra E. There is a spectral sequence of E_*-modules:*

$$\mathrm{E}^2_{s,t} \cong \mathrm{H}^{E_*}_s(\mathbf{I}, \pi_*(F)[t]) \implies \pi_{s+t}(\mathrm{hocolim}_{\mathbf{I}} F)$$

Proof. The homotopy colimit can be computed by taking an objectwise cofibrant replacement of F and then take the realization of the Bar construction:

$$\mathrm{hocolim}_{\mathbf{I}} F \simeq |\mathrm{B}_\bullet(*, \mathbf{I}, QF(-))|$$

We can then use the standard spectral sequence associated to a simplicial object $\square$

Let A be an $\mathcal{E}_d$-algebra in $\mathbf{Mod}_E$. Let M be a framed manifold and let $\mathbf{D}(M)$ be the poset of open sets of M that are diffeomorphic to a disjoint union of copies of D. Up to a choice of framed diffeomorphism $U \to D^{\sqcup k}$ there is a functor $\mathbf{D}(M) \to \mathbf{E}_d$. We proved in 3.3.4 that the factorization homology of A over M can be computed as the homotopy colimit of the composition:

$$\mathbf{D}(M) \to \mathbf{E}_d \xrightarrow{A} \mathbf{Mod}_E$$

We are in a situation where we can apply the previous proposition:

Proposition 5.3.3. *There is a spectral sequence of E_*-modules:*

$$\mathrm{H}_*^{E_*}(\mathbf{D}(M), \pi_*(A)) \implies \pi_*(\int_M A)$$

We want to exploit the fact that A is a monoidal functor to obtain a more explicit model for the left hand side in some cases. Let K be an associative algebra in ring spectra with a $\mathbb{Z}/2$-equivariant Künneth isomorphism.

Example of such spectra are the Eilenberg-MacLane spectra Hk for any field k or $K(n)$ the Morava K-theory of height n at odd primes. The previous proposition can be rewritten as:

Proposition 5.3.4. *There is a spectral sequence of $K_*(E)$-modules:*

$$\mathrm{H}_*^{K_*E}(\mathbf{D}(M), K_*(A)) \implies K_*(\int_M A)$$

Proof. We just smash the functor A with K objectwise and apply the previous proposition to $K \otimes A$ seen as a functor with value in $K \otimes E$-modules. □

Now we want to identify $K_*(A)$ as a functor on $\mathbf{D}(M)$.

Proposition 5.3.5. *Let $\mathcal{O}$ be an operad. Let R be a homotopy commutative ring spectrum. Let A be an $\mathcal{O}$-algebra in $\mathbf{Mod}_E$, then R_*A is an $\pi_0(\mathcal{O})$-algebra in R_*E-modules.*

Proof. The functor R_* is lax monoidal. Hence it is easy to see that R_*A is an $R_*(\Sigma_+^\infty\mathcal{O})$-algebra. But the unit map $S \to R$ induces a morphism of operad:

$$\pi_*^s(\mathcal{O}_+) \to R_*(\Sigma_+^\infty\mathcal{O})$$

There are obvious maps of operad:

$$\pi_0(\mathcal{O}) \to \pi_0^s(\mathcal{O}_+) \to \pi_*^s(\mathcal{O}_+)$$

Therefore, the $R_*(\Sigma_+^\infty\mathcal{O})$-algebra structure induces a $\pi_0(\mathcal{O})$-algebra structure. □

Corollary 5.3.6. *If $d > 1$, $K_*(A)$ is a commutative algebra in the category of K_*E-modules. If $d = 1$, $K_*(A)$ is an associative algebra in K_*E-modules.*

Proof. This follows from the fact that $\pi_0(\mathcal{E}_1) \cong \mathcal{A}ss$ and $\pi_0(\mathcal{E}_d) \cong \mathcal{C}om$ if $d \geq 2$. □

Remark 5.3.7. One can show that for any n, the spectrum $\Sigma^\infty_+\mathcal{E}_d(n)$ splits as a wedge of spheres. The homology of $\mathcal{E}_d$ is the operad of $d-1$-Gerstenhaber algebras (i.e. Gerstenhaber algebras with a degree $d-1$ Lie bracket). This computation has the consequence that for any ring spectrum R, the operad $R_*(\mathcal{E}_d)_+$ is the operad of $d-1$-Gerstenhaber algebras in R_*-modules. In particular, $K_*(A)$ is not only a commutative algebra. It also has a degree $(d-1)$ Lie bracket which is a derivation in both variables. It would be interesting to understand how this structure interacts with the spectral sequence.

Proposition 5.3.8. *The functor $K_*(A) : \mathbf{D}(M) \to \mathbf{Mod}_{K_*E}$ is induced by the $\mathcal{E}_d$-algebra structure on $K_*(A)$ coming from $\mathcal{C}om$-algebra structure.*

Proof. The category **Fin** is the free symmetric monoidal category on the operad $\mathcal{C}om$, therefore the commutative algebra K_*A gives rise to a monoidal functor:

$$\mathbf{Fin} \to \mathbf{Mod}_{K_*E}$$

It is easy to check that the functor $K_*A : \mathbf{D}(M) \to \mathbf{Mod}_{K_*E}$ factors as:

$$\mathbf{D}(M) \to \mathbf{E}_d \to \mathbf{Fin} \to \mathbf{Mod}_{K_*E}$$

where the functor $\mathbf{E}_d \to \mathbf{Fin}$ comes from the map of operads $\mathcal{E}_d \to \mathcal{C}om$. □

Corollary 5.3.9. *There is an isomorphism:*

$$\mathrm{H}_*^{K_*E}(\mathbf{D}(M), K_*A) \cong \mathrm{HH}_*^{\mathrm{Sing}(M)}(K_*A|K_*E)$$

Multiplicative structure

Let us start with the general homotopy colimit spectral sequence:

Proposition 5.3.10. *Let $F : \mathbf{I} \to \mathbf{Mod}_E$ and $G : \mathbf{I} \to \mathbf{Mod}_E$ be functors.*

$$\mathrm{hocolim}_{\mathbf{I}\times\mathbf{J}} F \otimes_E G \simeq (\mathrm{hocolim}_{\mathbf{I}} F) \otimes_E (\mathrm{hocolim}_{\mathbf{J}} G)$$

We denote by $\mathrm{E}^r_{**}(\mathbf{I}, F)$ the spectral sequence computing the homotopy colimit of F.

Proposition 5.3.11. *We keep the notations of the previous proposition. There is a pairing of spectral sequences of E_*-modules:*

$$\mathrm{E}^r_{**}(\mathbf{I}, F) \otimes_{E_*} \mathrm{E}^r_{**}(\mathbf{J}, G) \to \mathrm{E}^r_{**}(\mathbf{I} \times \mathbf{J}, F \otimes_E G)$$

Proof. It suffices to write the simplicial object computing the hocolim over $\mathbf{I} \times \mathbf{J}$ as the objectwise tensor product of the simplicial object computing the hocolim over $\mathbf{I}$ with the simplicial object computing the hocolim over $\mathbf{J}$. The result is then a standard fact about pairing of spectral sequences associated to simplicial objects (see for instance [Pal07]). □

Let us specialize to the case of factorization homology. We consider an $\mathcal{E}_d$-algebra A in $\mathbf{Mod}_E$ a homology theory with $\mathbb{Z}/2$-equivariant Künneth isomorphism K and a framed manifold of dimension d M. We denote by $\mathrm{E}^r_{**}(M, A, K)$ the spectral sequence of the previous section.

Proposition 5.3.12. *Let M and N be two framed d-manifolds. There is a pairing of spectral sequences:*

$$\mathrm{E}^r_{**}(M, A, K) \otimes_{K_*E} \mathrm{E}^r_{**}(N, A, K) \to \mathrm{E}^r_{**}(M \sqcup N, A, K)$$

Proof. This follows from the previous proposition as well as the observation that $\mathbf{D}(M \sqcup N) \cong \mathbf{D}(M) \times \mathbf{D}(N)$ and the fact that $A \otimes_E A$ as a functor on $\mathbf{D}(M) \times \mathbf{D}(N)$ is equivalent to A as a functor on $\mathbf{D}(M \sqcup N)$. □

In other words, we have proved that the spectral sequence $\mathrm{E}^r_{**}(M, A, K)$ is a lax monoidal functor of the variable M. In particular it preserves associative algebras.

Assume now that M is an associative algebra up to isotopy in $f\mathbf{Man}_d$. One possible example is to take $M = N \times \mathbb{R}$ with a framing induced from a framing of $TN \oplus \mathbb{R}$. In that case, M is an $\mathcal{E}_1$-algebra in $f\mathbf{Man}_d$.

Proposition 5.3.13. *Let M be an associative algebra up to isotopy. The spectral sequence $\mathrm{E}^r_{**}(M, A, K)$ has a commutative multiplicative structure converging to the associative algebra structure on $K_* \int_M A$.*

On the E^2-page, this multiplication is induced by the unique commutative algebra structure on $\mathrm{Sing}(M)$ in the category $(\mathbf{S}, \sqcup)$.

Moreover this structure is functorial with respect to embeddings of d-manifolds $M \to M'$ preserving the multiplication up to isotopy.

Proof. According to the previous proposition there is a multiplication operation on the spectral sequence converging to the associative algebra structure on $K_* \int_M^E A$.

It is easy to see that the multiplication on the E^2-page is what is stated. Since $\mathrm{Sing}(M)$ is commutative, the multiplication on the E^2-page is commutative. The homology of a commutative differential graded algebra is a commutative algebra, therefore the multiplication is commutative on each page.

The functoriality is clear. □

Now we want to construct an edge homomorphism

Let S be a $(d-1)$-manifold with a d-framing τ. Let ϕ be a framed embedding of $\mathbb{R}^{d-1} \times \mathbb{R}$ into $S \times \mathbb{R}$ commuting with the projection to $\mathbb{R}$. Applying factorization homology we get a map of $\mathcal{E}_1$-algebras:

$$u_\phi : A \cong \int_{\mathbb{R}^{d-1} \times \mathbb{R}} A \to \int_{S \times \mathbb{R}} A$$

On the other hand for any point x of $S \times \mathbb{R}$, we get a morphism of commutative algebra over K_*E:

$$u_x : K_*(A) \cong \mathrm{HH}^{\mathrm{pt}}(K_*A|K_*E) \to \mathrm{HH}^{\mathrm{Sing}(S)}(K_*A|K_*E)$$

Proposition 5.3.14. *For any framed embedding $\phi : \mathbb{R}^{d-1} \times \mathbb{R} \to S \times \mathbb{R}$, there is a edge homomorphism:*

$$K_*A \to E^r_{0,*}(S \times \mathbb{R}, A, K)$$

On the E^2*-page it is identified with the* K_*E*-algebra homomorphism:*

$$u_{\phi(0,0)} : K_*(A) \to \mathrm{HH}^{\mathrm{pt}}(K_*A|K_*E) \to \mathrm{HH}^{\mathrm{Sing}(S)}(K_*A|K_*E)$$

and it converges to the K_*E*-algebra homomorphism:*

$$K_*(u_\phi) : K_*A \to K_*\left(\int_{N\times\mathbb{R}} A\right)$$

Proof. The spectral sequence computing $K_* \int_{\mathbb{R}^{d-1}\times\mathbb{R}} A$ has its E^2-page K_*A concentrated on the 0-th column. For degree reason, it is degenerate.

Then the result follows directly from the functoriality of the spectral sequence applied to the map ϕ. □

Note that the edge homomorphism only depends on the connected component of the image of ϕ.

In the case of the sphere $S^{d-1}\times\mathbb{R}$ with the framing κ, we can say more:

Lemma 5.3.15. *For any framed embedding* $\phi : \mathbb{R}^{d-1}\times\mathbb{R} \to (S^{d-1}\times\mathbb{R})_\kappa$ *commuting with the projection to* $\mathbb{R}$*, the map:*

$$u_\phi : A \to \int_{S^{d-1}\times\mathbb{R}} A$$

has a splitting in the homotopy category of $\mathbf{Mod}_E$

Proof. There is an embedding:

$$S^{d-1}\times\mathbb{R} \to \mathbb{R}^d$$

sending (θ, x) to $e^x\theta$. This embedding preserves the framing. Moreover, the composition:

$$\mathbb{R}^d \xrightarrow{\phi} S^{d-1}\times\mathbb{R} \to \mathbb{R}^d$$

is isotopic to the identity (because $\mathrm{Emb}_f(\mathbb{R}^d,\mathbb{R}^d)$ is contractible). We can apply $\int_{\_} A$ to this sequence of morphisms of framed manifolds and we obtain the desired splitting. □

Although we will not need it, this has the following corollary:

Corollary 5.3.16. *The image of the edge homomorphism in $E^r_{**}((S^{d-1}\times\mathbb{R})_\kappa, A, K)$ consists of permanent cycles.*

5.4 Computations

Proposition 5.4.1. *Let A_* be a degreewise projective commutative graded algebra over a commutative graded ring R_*. Assume that A_* is a sequential colimit of étale algebras over R_*. Then, for all $d \geq 1$, the unit map:*

$$A_* \to \mathrm{HH}^{S^d}(A_*|R_*)$$

is a quasi isomorphism of commutative R_*-algebras.

Proof. We proceed by induction on d. For $d = 1$, $\mathrm{HH}^{S^1}(A_*|R_*)$ is quasi-isomorphic to the ordinary Hochschild homology $\mathrm{HH}(A_*|R_*)$ (5.2.4). If A_* is étale, the result is well-known (see for instance [WG91]). If A_* is a sequential colimit of étale algebras, the result follows from the fact that Hochschild homology commutes with sequential colimits.

Now assume that $A_* \to \mathrm{HH}^{S^{d-1}}(A_*|R_*)$ is a quasi-isomorphism of commutative algebras. The sphere S^d is part of the following homotopy pushout diagram:

$$\begin{tikzcd} S^{d-1} \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & S^d \end{tikzcd}$$

Applying 5.2.3, we find:

$$\mathrm{HH}^{S^d}(A_*|R) \simeq |\mathrm{B}_\bullet(A_*, \mathrm{HH}^{S^{d-1}}(A_*|R_*), A_*)|$$

The quasi-isomorphism $A_* \to \mathrm{HH}^{S^{d-1}}(A_*|R_*)$ induces a degreewise quasi-isomorphism between Reedy cofibrant simplicial objects:

$$\mathrm{B}_\bullet(A_*, A_*, A_*) \to \mathrm{B}_\bullet(A_*, \mathrm{HH}^{S^{d-1}}(A_*|R_*), A_*)$$

This induces a quasi-isomorphism between their realization:

$$A_* \simeq \mathrm{HH}^{S^d}(A_*|R_*)$$

□

Corollary 5.4.2. *Let A be an $\mathcal{E}_d$-algebra in* **Spec** *such that $K_*(A)$ is a directed colimits of étale algebras over K_*, then the unit map:*

$$A \to \int_{S^{d-1}\times\mathbb{R}} A$$

is a K-local equivalence.

Proof. It suffices to check that the K-homology of this map is an isomorphism. This can be computed as the edge homomorphism of the spectral sequence $\mathrm{E}^2(S^{d-1}\times\mathbb{R}, A, K)$. By the previous proposition, the edge homomorphism is an isomorphism on the E^2-page. Therefore, the spectral sequence collapses at the E^2-page for degree reasons. □

Let us fix a prime p. We denote by E_n, the Lubin-Tate ring spectrum of height n at p and $K(n)$ the Morava K-theory of height n. Recall that:

$$\begin{aligned}(E_n)_* &\cong \mathbb{W}(\mathbb{F}_{p^n})[[u_0,\dots,u_{n-1}]][u^{\pm}],\ |u_i| = 0\ |u| = 2\\ K(n)_* &\cong \mathbb{F}_p[v_n^{\pm}],\ |v_n| = 2(p^n-1)\end{aligned}$$

The spectrum E_n is known to have a unique $\mathcal{E}_1$-structure inducing the correct multiplication on homotopy groups (this is a theorem of Hopkins and Miller, see [Rez98]) and a unique $\mathcal{C}om$-structure (see [GH04]). As far as we know, there is no published proof that the space of $\mathcal{E}_d$-structure for $d \geq 2$ is contractible although evidence suggests that it is the case.

Recall also that $K(n)$ has a $\mathbb{Z}/2$-equivariant Künneth isomorphism if p is odd. If $p = 2$, the equivariance is not satisfied. However, this is true if we restrict $K(n)_*$ to spectra whose $K(n)$-homology is concentrated in even degree like E_n and our argument works modulo this minor modification.

Corollary 5.4.3. *For any positive integer n, and any $\mathcal{E}_d$-algebra structure on E_n inducing the unique $\mathcal{E}_1$-structure, the unit map*

$$E_n \to \int_{S^{d-1}\times\mathbb{R}} E_n$$

is an equivalence in the $K(n)$-local category.

Proof. For any such $\mathcal{E}_d$-structure on E, $K(n)_*(E_n) \cong K_*[t_1, t_2, \cdots]/(v_n^{p^k-1}t_k - t_k^{p^n}, k \geq 1)$ (see [Rav92], Theorem B.7.4) which is obviously a directed colimit of étale algebras over K_*. □

Corollary 5.4.4. *Same notations, the action map $\mathrm{HH}_{\mathcal{E}_d}(E_n) \to E_n$ is an equivalence.*

Proof. We have

$$\mathrm{HH}_{\mathcal{E}_d}(E_n) := \mathbb{R}\underline{\mathrm{Hom}}_{U_{E_n}^{\mathcal{E}_d[1]}}(E_n, E_n)$$

This can be computed as the totalization of the cosimplicial object

$$\underline{\mathrm{Hom}}(E_n \otimes (U_{E_n}^{\mathcal{E}_d[1]})^{\otimes \bullet}, E_n)$$

The spectrum E_n is $K(n)$-local, therefore, $\underline{\mathrm{Hom}}(-, E_n)$ sends $K(n)$-equivalences to equivalences. We know that as an $\mathcal{E}_1$-algebra, $U_{E_n}^{\mathcal{E}_d[1]} \simeq \int_{S^{d-1}\times\mathbb{R}} E_n$.

We can apply the previous corollary. The cosimplicial object is degreewise weakly equivalent to:

$$\underline{\mathrm{Hom}}(E_n \otimes (E_n)^{\otimes \bullet}, E_n)$$

This cosimplicial object computes $\mathbb{R}\underline{\mathrm{Hom}}_{E_n}(E_n, E_n)$ which is weakly equivalent to E_n and it is easy to check that up to homotopy this is the action map

$$\mathrm{HH}_{\mathcal{E}_d}(E_n) \to E_n$$

□

Let $E(n) = BP/(v_{n+1}, v_{n+2}, \ldots)[v_n^{-1}]$ be the Johnson-Wilson spectrum. Let $\widehat{E}(n)$ be $L_{K(n)}E(n)$. An anologous proof yields the following result:

Proposition 5.4.5. *For any $\mathcal{E}_d$-algebra structure on $\widehat{E}(n)$ inducing the unique $\mathcal{E}_1$-structure, the action map*

$$\mathrm{HH}_{\mathcal{E}_d}(\widehat{E}(n)) \to \widehat{E}(n)$$

is a weak equivalence.

5.5 Étale base change for Hochschild cohomology

In this section we assume that $(\mathbf{C}, \otimes)$ is the category $\mathbf{Mod}_E$ of modules over some commutative symmetric spectrum. We want to put the previous result in the wider context of derived algebraic geometry over $\mathcal{E}_d$-algebra see ([Fra]).

Let $\alpha : \mathcal{E}_1 \to \mathcal{E}_d$ be the morphism of operad sending $(0,1)$ to $(0,1) \times \mathbb{R}^{d-1}$.

Let L be the associative algebra in $\mathbf{Mod}_{\mathcal{E}_1}$ parametrizing left modules over an $\mathcal{E}_1$-algebra. One can form the pushforward $\alpha_!(L) = L \circ_{\mathcal{E}_1} \mathcal{E}_d$. This is an associative algebra in $\mathbf{Mod}_{\mathcal{E}_d}$. If A is an $\mathcal{E}_d$-algebra, the category $\alpha_! L\mathbf{Mod}_A$ can be identified with the category of left modules over the undelying (induced by α) $\mathcal{E}_1$-algebra of A.

Construction 5.5.1. We construct a morphism of associative algebra in $\mathbf{Mod}_{\mathcal{E}_d}$ from $\alpha_! L$ to S^{d-1}_κ which encodes the fact that an S^{d-1}_κ-module over A has the structure of a left module over the undelying $\mathcal{E}_1$-algebra of A.

(TO BE FILLED IN)

Using the above construction, there is an adjunction

$$F : L\mathbf{Mod}_{\alpha^* A} \leftrightarrows S^{d-1}_\kappa \mathbf{Mod}_A : G$$

Proposition 5.5.2. *Let A be an $\mathcal{E}_d$-algebra in spectra. Recall that A is S^{d-1}_κ-module over itself. The comonad FG applied to A is equivalent to $U_A^{S^{d-1}_\kappa}$.*

Proof. See Francis ([Fra]). $\square$

Following Francis ([Fra]) we make the following definition:

Definition 5.5.3. The cotangent complex L_A of A over E is defined to be the n-fold desuspension of the cofiber of the counit map

$$U_A^{S^{d-1}} \to A$$

This is not a very good definition of the cotangent complex. Francis actually defines the cotangent complex as the object representing the derivations:

$$\mathbb{R}\underline{\mathrm{Hom}}_{S^{d-1}\mathbf{Mod}_A}(L_A, M) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{Mod}_E[\mathcal{E}_d]/A}(A, A \oplus M) := \mathrm{Der}(A, M)$$

The above definition is then a theorem of Francis.

Proposition 5.5.4. *The map $U_A^{S^{d-1}} \to A$ used in the definition of the cotangent complex coincides with the map*

$$\int_{S^{d-1}\times(0,1)} A \to A \simeq \int_{\mathbb{R}^d} A$$

induced by the "polar coordinate" embedding $S^{d-1} \times (0,1) \to \mathbb{R}^d$.

Proof. Both sides of the map commutes with colimits of $\mathcal{E}_d$-algebras, therefore it suffices to check it for free $\mathcal{E}_d$-algebras. Francis in [Fra] computes $U_A^{S^{d-1}}$ for a free $\mathcal{E}_d$-algebra. The proposition follows rather easily from his explicit computation. □

Definition 5.5.5. We say that an $\mathcal{E}_d$-algebra A is étale over E if L_A is contractible.

Equivalently A is étale if the unit map $A \to U_A^{S^{d-1}}$ is an equivalence. Indeed we have shown in 5.3.15 that the unit map is a section of $U_A^{S^{d-1}} \to A$.

Proposition 5.5.6. *If A is a commutative algebra and is étale as an $\mathcal{E}_d$-algebra, then it is étale as an $\mathcal{E}_{d+1}$-algebra.*

Proof. This is very similar to 5.4.1. □

Remark 5.5.7. It does not seem that being étale as an $\mathcal{E}_1$-algebra is a reasonable thing to require from an $\mathcal{E}_1$-algebra. This amounts to checking that the multiplication map

$$A \otimes_E A \to A$$

is a weak equivalence and we do not know any interesting example where this is the case.

Remark 5.5.8. If A is a commutative algebra, then A is étale as an $\mathcal{E}_2$-algebra if and only if it is THH-étale (see [Rog08]). Indeed, for commutative algebras (and in fact for an $\mathcal{E}_3$-algebras), $\mathrm{THH}(A)$ coincides with $\int_{S^1\times\mathbb{R}}$. Note that is is *not* true for $\mathcal{E}_2$-algebras as the product framing on $S^1\times\mathbb{R}$ is not connected to the κ-framing in the space of framings of $S^1\times\mathbb{R}$.

Remark 5.5.9. If A is a commutative algebra, $U_A^{S^d}\simeq S^d\otimes A$. Therefore, A is étale as an $\mathcal{E}_{d+1}$-algebra if and only if the space $\mathrm{Map}_{\mathbf{Mod}_E[\mathcal{C}om]}(A,B)$ is d-truncated for any B.

The main theorem of this section is the following:

Theorem 5.5.10. *Let T be a is a commutative algebra in $\mathbf{C}=\mathbf{Mod}_E$ that is étale as an $\mathcal{E}_d$-algebras, then for any $\mathcal{E}_d$-algebra A over T, the base-change map*

$$\mathrm{HH}_{\mathcal{E}_d}(A|E)\overset{\simeq}{\to}\mathrm{HH}_{\mathcal{E}_d}(A|T)$$

is an equivalence

Proof. We write $A|T$ whenever we want to emphasize the fact that we are seeing A as an $\mathcal{E}_d$-algebra over T. We write U_A instead of $U_A^{S^{d-1}_\kappa}$.

By Francis ([Fra]), there is cofiber sequence

$$u_!L_T\to L_A\to L_{A|T}$$

where $u:T\to A$ is the unit map and $u_!$ is the corresponding functor

$$u_!:S^{d-1}\mathbf{Mod}_T\to S^{d-1}\mathbf{Mod}_A$$

By hypothesis L_T is contractible, therefore $L_A\to L_{A|T}$ is an equivalence.

We have a base-change map of cofiber sequences

$$\begin{tikzcd}
\Sigma^{d-1}L_A \arrow[r] \arrow[d] & U_A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & \Sigma^d L_A \arrow[d] \\
\Sigma^{d-1}L_{A|T} \arrow[r] & U_{A|T} \arrow[r] & A \arrow[r] & \Sigma^d L_{A|T}
\end{tikzcd}$$

This implies that $U_A \to U_{A|T}$ is a weak equivalence of associative algebras. Therefore, the category $S^{d-1}\mathbf{Mod}_{A|T}$ is equivalent to $S^{d-1}\mathbf{Mod}_A$. The theorem is a particular case of this fact. □

The computation of the previous section shows that $L_{K(n)}S \to E_n$ is an étale morphism of $\mathcal{E}_d$-algebras for all d in the $K(n)$-local category. Therefore, given a $K(n)$-local E_n-algebra A, one can compute its (higher) Hochschild cohomology over E_n or over S without affecting the result. This fact is used by Angeltveit (see [Ang08]) in the case of ordinary Hochschild cohomology.

5.6 A rational computation

We end up this chapter with a rational computation. Let $K = K(1)$ and $E = \widehat{E}(1)$.

Angeltveit (see [Ang08]) computes the homotopy groups of $\mathrm{HH}_{\mathcal{E}_1}(K)$ for p odd

$$\pi_*\mathrm{HH}_{\mathcal{E}_1}(K|E) = \mathbb{Z}_p[v_1^{\pm}, q]/(q^{p-1} - pv_1)$$

where q is some class of degree 2.

This is an isomorphism of E_*-algebra. From the homotopy group we see that $\mathrm{HH}_{\mathcal{E}_1}(K|E)$ is a wedge of copies of E. Therefore if we write H for $H\mathbb{Q}$, we find

$$H(\mathrm{HH}_{\mathcal{E}_1}(K|E)) = \mathbb{Q}_p[v_1^{\pm}, q]/(q^{p-1} - pv_1)$$

Again this is true as an $H(E)$-algebra.

Proposition 5.6.1. *The graded algebra $H(\mathrm{HH}_{\mathcal{E}_1}(K|E))$ is an étale algebra over $H(E) = \mathbb{Q}_p[v_1^{\pm}]$*

Proof. We can apply the Jacobian criterion. Let $f(q) = q^{p-1} - pv_1$. We have $H(\mathrm{HH}_{\mathcal{E}_1}(K|E)) = H(E)[q]/(f(q))$ we need to prove that $f'(q) = (p-1)q^{p-2}$ is invertible in $H(\mathrm{HH}_1(K|E))$. It suffices to prove that it is prime to $f(q)$. We have

$$qf'(q) - (p-1)f(q) = pv_1$$

Since pv_1 is a unit we are done. $\square$

Unfortunately $K(1)_*(\mathrm{HH}_{\mathcal{E}_1}(K|E))$ is not étale over $K(1)_*(E)$ which makes a $K(1)$-local computation a lot more complicated.

By Deligne's conjecture, $\mathrm{HH}_{\mathcal{E}_1}(K|E)$ has an $\mathcal{E}_2$-structure. We can compute the unit map $\mathrm{HH}_{\mathcal{E}_1}(K|E) \to \int_{S^1\times\mathbb{R}} \mathrm{HH}_{\mathcal{E}_1}(K|E)$. By 5.4.2, this unit map is a rational equivalence. This implies a rational equivalence

$$\mathrm{HH}_{\mathcal{E}_2}(\mathrm{HH}_{\mathcal{E}_1}(K|E)) \to \mathrm{HH}_{\mathcal{E}_1}(K|E)$$

The same argument can be iterated to give a proof of the following:

Proposition 5.6.2. *For all n the rational homology of the iterated centers $\mathrm{HH}_{\mathcal{E}_d} \circ \mathrm{HH}_{\mathcal{E}_{d-1}} \circ \cdots \circ \mathrm{HH}_{\mathcal{E}_1}(K|E)$ is isomorphic to $H(\mathrm{HH}_{\mathcal{E}_1}(K|E))$.*

Chapter 6

Calculus à la Kontsevich Soibelman

Let A be an associative algebra over a field k. The Hochschild Kostant Rosenberg theorem (see [HKR09]) suggests that the Hochschild homology of A should be interpreted as the graded vector space of differential forms on the non commutative space "Spec$A$". Similarly, the Hochschild cohomology of A should be interpreted as the space of polyvector fields on Spec$A$.

If M is a smooth manifold, let $\Omega_*(M)$ be the (homologicallly graded) vector space of de Rham differential forms and $V^*(M)$ be the vector space of polyvector fields (i.e. global sections of the exterior algebra on TM). This pair of graded vector spaces supports the following structure:

- The de Rham differential : $d : \Omega_*(M) \to \Omega_{*-1}(M)$.
- The cup product of vector fields : $-.- : V^i(M) \otimes V^j(M) \to V^{i+j}(M)$.
- The Schouten-Nijenhuis bracket : $[-,-] : V^i \otimes V^j \to V^{i+j-1}$.
- The cap product : $\Omega_i \otimes V^j \to \Omega_{i-j}$ denoted by $\omega \otimes X \mapsto i_X\omega$.
- The Lie derivative : $\Omega_i \otimes V^j \to \Omega_{i-j+1}$ denoted by $\omega \otimes X \mapsto L_X\omega$.

This structure satisfies some properties:

- The de Rham differential is a differential, i.e. $d \circ d = 0$.

- The cup product and the Schouten-Nijenhuis bracket make $V^*(M)$ into a Gerstenhaber algebra. More precisely, the cup product is graded commutative and the bracket is a derivation in each variable.

- The cap product and the Lie derivative make $\Omega_*(M)$ into a Gerstenhaber $V^*(M)$-module.

The Gerstenhaber module structure means that the following formulas are satisfied:

$$\begin{aligned} L_{[X,Y]} &= [L_X, L_Y] \\ i_{[X,Y]} &= [i_X, L_Y] \\ i_{X.Y} &= i_X i_Y \\ L_{X.Y} &= L_X i_Y + (-1)^{|X|} i_X L_Y \end{aligned}$$

Finally there is the following formula relating the Lie derivative, the exterior product and the de Rham differential:

$$L_X = [d, i_X]$$

Note that there is even more structure available in this situation. For example, the de Rham differential forms are equipped with a commutative DGA structure. However we will ignore this additional structure since it is not available in the non commutative case.

There is an operad $\mathcal{C}$alc in graded vector spaces such that a $\mathcal{C}$alc-algebra is a pair (V^*, Ω_*) together with all the structure we have just mentioned.

It turns out that any associative algebra gives rise to a $\mathcal{C}$alc-algebra pair:

Theorem 6.0.3. *Let A be an associative algebra over a field k, let $\mathrm{HH}_*(A)$ (resp. $\mathrm{HH}^*(A)$) denote the Hochschild homology (resp. cohomology) of A, then the pair $(\mathrm{HH}^*(A), \mathrm{HH}_*(A))$ is an algebra over $\mathcal{C}$alc.*

A natural question is to lift this action to an action at the level of chains inducing the $\mathcal{C}$alc-action in homology in the same way that there is an $\mathcal{D}_2$-action on Hocshild cochains inducing the Gerstenhaber structure on Hochschild cohomology.

Kontsevich and Soibelman in [KS09] have constructed a topological operad denoted $\mathcal{KS}$ whose homology is $\mathcal{C}$alc. The purpose of this chapter is to construct an action of $\mathcal{KS}$ on the pair consisting of topological Hochschild cohomology and topological Hochschild homology. We also construct obvious higher dimensional analogues of the operad $\mathcal{KS}$ and show that they describe the action of higher Hochschild cohomology on chiral homology.

6.1 $\mathcal{KS}$ and its higher versions.

In this section, we recall the definition of the operad $\mathcal{KS}$ defined in [KS09]. We construct an equivalent version of that operad as well as higher dimensional analogues of it.

Definition 6.1.1. Let D be the 2-dimensional disk. An injective continuous map $D \to S^1 \times (0,1)$ is said to be rectilinear if it can be factored as

$$D \xrightarrow{l} \mathbb{R} \times (0,1) \to \mathbb{R} \times (0,1)/\mathbb{Z} = S^1 \times (0,1)$$

where the map l is rectilinear and the second map is the quotient by the $\mathbb{Z}$-action.

We say that an embedding $S^1 \times [0,1) \to S^1 \times [0,1)$ is rectilinear if it is of the form $(z,t) \mapsto (z+z_0, at)$ for some fixed $z_0 \in S^1$ and $a \in (0,1)$.

We denote by $\mathrm{Emb}^{\partial}_{lin}(S^1 \times [0,1) \sqcup D^{\sqcup n}, S^1 \times [0,1)$ the topological space of injective maps whose restriction to each disk and to $S^1 \times [0,1)$ is rectilinear.

Definition 6.1.2. We define Q, an associative algebra in right modules over $\mathcal{D}_2$ by

$$Q(n) = \mathrm{Emb}^{\partial}_{lin}(S^1 \times [0,1) \sqcup E^{\sqcup n}, S^1 \times [0,1))$$

We define the Kontsevich-Soibelman's operad $\mathcal{KS}$ by

$$\mathcal{KS} = Q\mathcal{M}od$$

Now we define generalizations of $\mathcal{KS}$.

Definition 6.1.3. Let S be a $(d-1)$-manifold with framing τ. We define $S^{\circlearrowleft}_{\tau}$ to be the associative algebra in right module defined by

$$S_\tau^{\circlearrowright}(n) = \mathrm{Emb}_f^{\partial}(S \times [0,1) \sqcup D^{\sqcup n}, S \times [0,1))$$

Note that a linear embedding preserves the framing on the nose. Therefore, there is a well defined inclusion

$$\mathcal{KS} \to (S^1)_\tau^{\circlearrowright}$$

Proposition 6.1.4. *This map is a weak equivalence.*

Proof. There is an obvious restriction map

$$S_\tau^{\circlearrowright}(n) \to \mathrm{Emb}_f(D^{\sqcup n}, S \times [0,1))$$

This map is a fibration by an argument similar to 4.2.1. Its fiber over a particular configuration of disks is the space of embeddings of $S \times [0,1)$ into the complement of that configuration. By 2.4.9, this space is weakly equivalent to $\mathrm{Emb}_f(S,S)$ through the obvious map.

We have a diagram

$$\begin{tikzcd}
\mathrm{Emb}_{lin}^{\partial}(S^1 \times [0,1) \sqcup D^n, S^1 \times [0,1) \arrow[r] \arrow[d] & \mathrm{Emb}_f^{\partial}(S^1 \times [0,1) \sqcup D^{\sqcup n}, S^1 \times [0,1)) \arrow[d] \\
\mathrm{Emb}_{lin}(D^{\sqcup n}, S^1 \times (0,1)) \arrow[r] & \mathrm{Emb}_f(D^{\sqcup n}, S^1 \times (0,1))
\end{tikzcd}$$

Both vertical maps are fibrations. The bottom map is a weak equivalence since both sides are weakly equivalent to $\mathrm{Conf}(n, S^1 \times (0,1))$. The map induced on fibers is weakly equivalent to the inclusion

$$S^1 \to \mathrm{Emb}_f(S^1, S^1)$$

Showing that this map is an equivalence is a standard exercise. $\square$

6.2 Action of the higher version of $\mathcal{KS}$

Let (B, A) be an algebra over the operad $\mathcal{E}_d^\partial$ in the category $\mathbf{C}$. Let M be a framed $(d-1)$-manifold and τ be the product framing on $TM \oplus \mathbb{R}$. This is the main theorem of this chapter:

Theorem 6.2.1. *The pair $(B, \int_M A)$ is weakly equivalent to an algebra over the operad $M_\tau^\circlearrowleft$.*

Proof. The construction $\int_{\_}(B, A)$ is a simplicial functor $f\mathbf{Man}_d^\partial \to \mathbf{C}$. Hence, $\int_{\_}(B, A)$ is a functor from the full subcategory of $f\mathbf{Man}_d^\partial$ spanned by disjoint unions of copies of D and $M \times [0, 1)$ to $\mathbf{C}$. Moreover this functor is symmetric monoidal. The operad $M_\tau^\circlearrowleft$ has a map to the endomorphism operad of the pair $D, M \times [0, 1)$ in the symmetric monoidal category $f\mathbf{Man}_d^\partial$, therefore $(\int_D(B, A), \int_{M\times[0,1)}(B, A))$ is an algebra over $M_\tau^\circlearrowleft$. To conclude, we use the fact that $\int_D(B, A) \cong B$ by Yoneda's lemma and $\int_{M\times[0,1)}(B, A) \simeq \int_M A$ by 3.3.7. □

This theorem is mainly interesting because of the following theorem due to Thomas (see [Tho10]):

Theorem 6.2.2. *Let A be an $\mathcal{E}_d$-algebra in $\mathbf{C}$, then there is an algebra (B', A') over $\mathcal{E}_d^\partial$ such that B' is weakly equivalent to $\mathrm{HH}_{\mathcal{E}_d}(A)$ and A' is weakly equivalent to A.*

This has the following immediate corollary:

Corollary 6.2.3. *We keep the notations of 6.2.1. The pair $(\mathrm{HH}_{\mathcal{E}_d}(A), \int_M A)$ is weakly equivalent to an algebra over the operad $M_\tau\mathcal{M}od_G$.*

Appendix A

A few facts about model categories

A.1 Cofibrantly generated model categories

Definition A.1.1. A *cofibrantly generated model category* is a model category $\mathbf{X}$ with the extra data of two sets I and J of arrows of $\mathbf{X}$. Such that

- The set I and J permit the small object argument.
- The fibrations are the map with the right lifting property with respect to the maps of J.
- The trivial fibrations are the map with the right lifting property with respect to the maps of I.

We will not spell out what is meant by "permit the small object argument". If the domain of the elements of I and J are compact, then they permit the small object argument. A cofibrantly generated model category has functorial factorization of maps as a cofibration followed by a trivial fibration or as a trivial cofibration followed by a fibration. In particular there is a fibrant replacement functor and a cofibrant replacement functor. See [Hov99] for more details.

Let $\mathbf{X}$ be a cofibrantly generated model category and

$$F : \mathbf{X} \leftrightarrows \mathbf{Y} : U$$

be an adjunction.

Definition A.1.2. The *transferred model category structure* on $\mathbf{Y}$ is the model category structure satisfying on of the following equivalent conditions:

- The fibrations (resp. weak equivalences) are the maps whose image through U are fibrations resp. weak equivalences
- It is the cofibrantly generated model category whose generating cofibrations (resp. generating trivial cofibrations) are FI (resp. FJ).

Note that this model structure does not necessarily exist but if it does, it is unique. Moreover notice that if the transferred model category structure exists, the adjunction is a Quillen adjunction.

In practice, one often uses the following lemma to prove that the transferred model structure exists.

Lemma A.1.3. *Let*

$$F : \mathbf{X} \rightleftarrows \mathbf{Y} : U$$

be an adjunction in which $\mathbf{X}$ *is cofibrantly generated. Assume that*

- U *preserves colimit indexed over ordinals.*
- *For any (trivial) cofibration* i *in* $\mathbf{X}$ *and any pushout diagram*

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d, "F(i)"'] & Y \\ F(X') & \end{tikzcd}$$

  the functor U *sends the pushout of* $F(i)$ *to a (trivial) cofibration in* $\mathbf{X}$.

Then the transferred model structure exists on $\mathbf{Y}$ *and* U *preserves cofibrations and trivial cofibrations.*

Proof. See [Fre09], 11.1.14 $\square$

A.2 Monoidal and enriched model categories

Definition A.2.1. Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be three model categories. A pairing $T : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ is said to satisfies the *pushout-product axiom* if for each pair of cofibrations $f : A \to B$ of $\mathbf{X}$ and $g : K \to L$ of $\mathbf{Y}$, the induced map

$$T(B, K) \sqcup^{T(A,K)} T(A, L) \to T(B, L)$$

is a cofibration which is trivial if one of f and g is.

We say that T is a *left Quillen bifunctor* if it satisfies the pushout-product axiom and if it is a left adjoint if one variable is fixed.

One useful consequence of the pushout-product axiom is that if A is cofibrant $T(A, -)$ preserves trivial cofibrations between cofibrant objects. Then by Ken Brown's lemma (see [Hov99]) it preserves all weak equivalences between cofibrant objects.

Definition A.2.2. A *(closed) monoidal model* category is a model category structure on a (closed) monoidal category $(\mathbf{V}, \otimes, \mathbb{I})$ which is such that

- The functor $- \otimes - : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ satisfies the pushout-product axiom.
- The map $Q\mathbb{I} \to \mathbb{I}$ induces a weak equivalence $Q\mathbb{I} \otimes V \to V$ for each V.

A symmetric monoidal model category is a model category structure on a symmetric monoidal category which makes the underlying monoidal category into a monoidal model category.

Recall that if $\mathbf{X}$ is a model category $\mathbf{X}^{\text{op}}$ has a canonical model structure in which (trivial) fibrations are opposite of (trivial) cofibrations.

Definition A.2.3. Let $\mathbf{V}$ be a monoidal model category. Let $(\mathbf{X}, \underline{\mathrm{Hom}}_{\mathbf{X}}(-, -))$ be a $\mathbf{V}$-enriched category. A $\mathbf{V}$-*enriched model structure* on $\mathbf{X}$ is a model category structure on the underlying category of $\mathbf{X}$ that is such that the functor

$$\underline{\mathrm{Hom}}_{\mathbf{X}}^{\text{op}} : \mathbf{X} \times \mathbf{X}^{\text{op}} \to \mathbf{V}^{\text{op}}$$

satisfies the pushout-product axiom.

Definition A.2.4. Let $(\mathbf{X}, \underline{\mathrm{Hom}}_{\mathbf{X}})$ be a $\mathbf{V}$-enriched category. Let T be a monad on $\mathbf{X}$. Let us define the following equalizer:

$$\underline{\mathrm{Hom}}_{\mathbf{X}[T]}(X,Y) \to \underline{\mathrm{Hom}}_{\mathbf{X}}(X,Y) \rightrightarrows \underline{\mathrm{Hom}}_{\mathbf{X}}(TX,Y)$$

where the top map is obtained by precomposition with the structure map $TX \to X$ and the bottom map is the composition

$$\underline{\mathrm{Hom}}_{\mathbf{X}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathbf{X}}(TX,TY) \to \underline{\mathrm{Hom}}_{\mathbf{X}}(TX,Y)$$

Definition A.2.5. A *simplicial model category* is a model structure on the category underlying a *closed* simplicial category.

This is more than just requiring the category to be enriched over $\mathbf{S}$. We also ask for a tensor and a cotensor:

$$\mathbf{S} \times \mathbf{C} \to \mathbf{C} \quad \mathbf{S}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$$

Definition A.2.6. A *symmetric monoidal simplicial model category* is a category with a simplicial enrichment, a symmetric monoidal structure and a model category structure such that

- It is a symmetric monoidal model category.
- It is a simplicial model category.
- The simplicial and monoidal structure are compatible in the sense that the functor $K \mapsto K \otimes \mathbb{I}$ from $\mathbf{S}$ to $\mathbf{C}$ is symmetric monoidal.

Let $\mathbf{X}$ be a category enriched over a complete monoidal category $\mathbf{V}$ and let T be a monad on $\mathbf{X}$. It is classical that we can give an enrichment to the category $\mathbf{X}[T]$ of T-algebras. For A and B two T-algebras, we define $\underline{\mathrm{Hom}}_{X[T]}(A,B)$ as a certain equalizer.

$$\underline{\mathrm{Hom}}_{\mathbf{X}[T]}(A,B) \to \underline{\mathrm{Hom}}_{\mathbf{X}}(A,B) \rightrightarrows \underline{\mathrm{Hom}}_{\mathbf{X}}(TA,B)$$

Proposition A.2.7. *Let* $\mathbf{X}$ *be a cofibrantly generated model category. If the category* $\mathbf{X}[T]$ *can be given the transferred model structure. Then* $\mathbf{X}[T]$ *equipped with* $\underline{\mathrm{Hom}}_{\mathbf{X}[T]}$ *is a* $\mathbf{V}$*-enriched model category.*

Proof. Let $f : U \to V$ be a (trivial) cofibration and $p : X \to Y$ be a fibration in $\mathbf{C}[T]$. We want to show that the obvious map

$$\underline{\mathrm{Hom}}_{\mathbf{X}[T]}(V, X) \to \underline{\mathrm{Hom}}_{\mathbf{X}[T]}(U, X) \times_{\underline{\mathrm{Hom}}_{\mathbf{X}[T]}(U,Y)} \underline{\mathrm{Hom}}_{\mathbf{X}[T]}(V, Y)$$

is a (trivial) fibration in $\mathbf{V}$. It suffices to do it for all generating (trivial) cofibration f. Hence it suffices to do this for a free map $f = Tm : TA \to TB$ where m is a (trivial) cofibration in $\mathbf{X}$. But then the statement reduces to proving that

$$\underline{\mathrm{Hom}}_{\mathbf{C}}(B, X) \to \underline{\mathrm{Hom}}_{\mathbf{C}}(A, X) \times_{\underline{\mathrm{Hom}}_{\mathbf{C}}(A,Y)} \underline{\mathrm{Hom}}_{\mathbf{C}}(B, Y)$$

is a (trivial) fibration which is true because $\mathbf{C}$ is a $\mathbf{V}$-enriched model category. □

A.3 Homotopy colimits and bar construction

See [DHKS05] or [Shu06] for a general definition of derived functors. We will use the following:

Proposition A.3.1. *Let* $\mathbf{X}$ *be a model category tensored over* $\mathbf{S}$ *and* $s\mathbf{X}$ *be the category of simplicial objects in* $\mathbf{X}$ *with the Reedy model structure. Then the geomeotric realization functor*

$$|-| : s\mathbf{X} \to \mathbf{X}$$

is left Quillen

Proof. See [GJ09] VII.3.6. □

Proposition A.3.2. *Let* $\mathbf{X}$ *be a simplicial model category, let* $\mathbf{K}$ *be a simplicial category and let* $F : \mathbf{K} \to \mathbf{X}$ *and* $W : \mathbf{K}^{\mathrm{op}} \to \mathbf{S}$ *be simplicial functors. Then the Bar construction*

$$\mathrm{B}_\bullet(W, \mathbf{K}, F)$$

is Reedy cofibrant if F is objectwise cofibrant.

Proof. See [Shu06]. □

Definition A.3.3. Same notation as in the previous proposition. Assume that $\mathbf{X}$ has a simplicial cofibrant replacement functor Q. We denote by $W \otimes^{\mathbb{L}}_{\mathbf{K}} F$ the realization of the simplicial object

$$\mathrm{B}_\bullet(W, \mathbf{K}, Q \circ F)$$

This object is homotopy invariant in the following strong sense:

Proposition A.3.4. *Let $(W, \mathbf{K}, F)$ and $(W', \mathbf{K}', F)$ be two triple whose middle term is a simplicial category whose left term is a contravariant functor from that simplicial category to $\mathbf{S}$ and whose right term is a covariant functor from that simplicial category to $\mathbf{X}$. Let $\alpha : \mathbf{K} \to \mathbf{K}'$ be a simplicial functor which is weakly fully faithful and an isomorphism on objects and $F \to \alpha^* F'$ and $W \to \alpha^* W'$ be two objectwise weak equivalences. Then the obvious map*

$$W \otimes^{\mathbb{L}}_{\mathbf{K}} F \to W' \otimes^{\mathbb{L}}_{\mathbf{K}'} F'$$

is a weak equivalence.

Proof. This map is the realization of a weak equivalence between simplicial objects of $\mathbf{X}$ which are both Reedy cofibrant. □

Note that this proposition is already useful when $\alpha = \mathrm{id}$. Finally let us mention the following proposition which insures that having a simplicial cofibrant replacement diagram is not a strong restriction:

Proposition A.3.5. *Let $\mathbf{X}$ be a cofibrantly generated simplicial model category. Then $\mathbf{X}$ has a simplicial cofibrant replacement functor.*

Proof. See [BR12] theorem 6.1. □

The bar construction is often useful because of the following result:

Proposition A.3.6. *Let $\mathbf{X}$ be a simplicial model category. Let $\alpha : \mathbf{K} \to \mathbf{L}$ be a simplicial functor. Let $F : \mathbf{K} \to \mathbf{X}$ be a simplicial functor. The functor*

$$l \mapsto \mathbf{L}(\alpha(-), l) \otimes^{\mathbb{L}}_{\mathbf{K}} F$$

is the homotopy left Kan extension of F along α

A.4 Model structure on symmetric spectra

Let E be a an associative algebra in symmetric spectra. Then $\mathbf{Mod}_E$ has (at least) two simplicial cofibrantly generated model category structures in which the weak equivalences are the stable equivalences of the underlying symmetric spectrum:

- The positive model structure $p\mathbf{Mod}_E$.
- The absolute model structure $a\mathbf{Mod}_E$.

Moreover if E is commutative, these model structure are closed symmetric monoidal model categories.

The identity functor induces a Quillen equivalence

$$p\mathbf{Mod}_E \leftrightarrows a\mathbf{Mod}_E$$

Both model structures have their advantages. The absolute model structure has more cofibrant objects (for instance E itself is cofibrant which is often convenient). On the other hand the positive model structure has few cofibrant objects but a very well-bahaved monoidal structure. A very pleasant property of this monoidal model structure is described in proposition A.4.5.

Proposition A.4.1. *A morphism $f : E \to F$ of algebra in symmetric spectra induces a Quillen adjunction:*

$$f_! : \mathbf{Mod}_E \leftrightarrows \mathbf{Mod}_F : f^*$$

in the positive or absolute model structure. Moreover, this is a Quillen equivalence if f is a weak equivalence of the underlying symmetric spectra.

Proof. See [Sch07]. ☐

Now let Z be a positively cofibrant symmetric spectrum. We say that a map f of symmetric spectra is a Z-equivalence if $Z \otimes f$ is a weak equivalence.

Proposition A.4.2. *For any algebra in symmetric spectra E, there is a simplicial model category structure on $\mathbf{Mod}_E$ denoted $L_Z\mathbf{Mod}_E$ whose cofibrations are positive (resp. absolute) cofibrations in $\mathbf{Mod}_E$ and whose weak equivalences are Z-equivalences. Moreover if E is commutative, both these model categories are closed symmetric monoidal categories for the relative tensor product $- \otimes_E -$.*

Proof. See [Bar10]. ☐

Proposition A.4.3. *A morphism $f : E \to F$ of algebras in symmetric spectra induces a Quillen adjunction:*

$$f_! : L_Z\mathbf{Mod}_E \leftrightarrows L_Z\mathbf{Mod}_F : f^*$$

in the positive or absolute Z-local model structure. Moreover, this is a Quillen equivalence if f is a Z equivalence of the underlying symmetric spectra.

Proof. The following proof works indifferently for the positive and absolute model structure.

The functor $f_!$ preserves cofibrations since they are the same in $\mathbf{Mod}_E$ and $L_Z\mathbf{Mod}_E$.

Notice that the fibrant objects in $L_Z\mathbf{Mod}_E$ or $L_Z\mathbf{Mod}_F$ are exactly the objects that are Z-local and fibrant as spectra. Let $M \to N$ be a Z-equivalence and a cofibration in $\mathbf{Mod}_E$. Let L be a Z-local fibrant F-module, then we want to show that the map

$$\mathrm{Map}_{\mathbf{Mod}_F}(N \otimes_E F, L) \to \mathrm{Map}_{\mathbf{Mod}_F}(M \otimes_E F, L)$$

is an equivalence in $\mathbf{S}$. But by adjunction, this map is

$$\mathrm{Map}_{\mathbf{Mod}_E}(N, L) \to \mathrm{Map}_{\mathbf{Mod}_E}(M, L)$$

which is an equivalence since L is Z-local and fibrant in $\mathbf{Mod}_E$. ☐

See 1.7.6 for the definition of C. Note that if f is a map in a model category $\mathbf{C}$, the map $C(f,\dots,f)$ with n copies of f is naturally a map in the category $\mathbf{C}^{\Sigma_n}$ of objects of $\mathbf{C}$ with a Σ_n-action.

The following definition is due to Lurie (see [Lur11]):

Definition A.4.4. Let $\mathbf{C}$ be a cofibrantly generated symmetric monoidal model category. A map $f : X \to Y$ is said to be a power cofibration if, for each n, the map $C(f,\dots,f)$ is a cofibration in $\mathbf{C}^{\Sigma_n}$ with the projective model structure.

Proposition A.4.5. *In the category $\mathbf{Mod}_E$ with the positive model structure, any cofibration is a power cofibration. The same is true for the positive model structure of $L_Z\mathbf{Mod}_E$ for any Z.*

Proof. The paper [EM06] prove that it is the case if E is the sphere spectrum. To prove the result for $\mathbf{Mod}_E$, it suffices to check it for generating cofibrations. Generating cofibrations in $\mathbf{Mod}_E$ can be chosen of the form $f \otimes E$ where E is a cofibration in **Spec**, therefore, the result follows from the case of **Spec**.

To take care of the Z local case, it suffices to notice that, for any finite group G, we have the identity as model categories:

$$(L_Z\mathbf{Mod}_E)^G = L_Z(\mathbf{Mod}_E^G)$$

indeed in both cases the weak equivalences are the Z-equivalences and the generating cofibrations are the maps $G \otimes f$ where f is a generating cofibration of $\mathbf{Mod}_E$. $\square$

In particular, this property is saying that if X is cofibrant in $\mathbf{Mod}_E$, then $X^{\otimes_E n}$ is cofibrant in $\mathbf{Mod}_E^{\Sigma_n}$. This situation is very specific to symmetric monoidal model structures on spectra. It fails in $\mathbf{S}$, **Top** or $\mathbf{Ch}_{\geq 0}(R)$.

Appendix B

Operads and modules

B.1 Colored operad

We recall the definition of a colored operad (also called a multicategory). In this paper we will restrict ourselves to the case of operads in **S** but the same definitions could be made in any symmetric monoidal category. Note that we use the word "operad" even when the operad has several colors. When we want to specifically talk about operads with only one color, we say "one-color operad".

Definition B.1.1. An operad in the category of simplicial sets consists of:

- a set of colors $\mathrm{Col}(\mathcal{M})$
- for any finite sequence $\{a_i\}_{i\in I}$ in $\mathrm{Col}(\mathcal{M})$ indexed by a finite set I, and any color b, a simplicial set:

$$\mathcal{M}(\{a_i\}_I; b)$$

- a base point $* \to \mathcal{M}(a; a)$ for any color a
- for any map of finite sets $f : I \to J$, whose fiber over $j \in J$ is denoted I_j, we have compositions operations

$$\left(\prod_{j\in J} \mathcal{M}(\{a_i\}_{i\in I_j}; b_j)\right) \times \mathcal{M}(\{b_j\}_{j\in J}; c) \to \mathcal{M}(\{a_i\}_{i\in I}; c)$$

All these data are required to satisfy unitality and associativity conditions (see for instance [Lur11] Definition 2.1.1.1.).

A map of operads $\mathcal{M} \to \mathcal{N}$ is a map $f : \mathrm{Col}(\mathcal{M}) \to \mathrm{Col}(\mathcal{N})$ together with the data of maps

$$\mathcal{M}(\{a_i\}_I; b) \to \mathcal{N}(\{f(a_i)\}_I; f(b))$$

compatible with the compositions and units.

With the above definition, it is not clear that there is a category of operads since there is no set of finite sets. However it is easy to fix this by checking that the only data needed is the value $\mathcal{M}(\{a_i\}_{i\in I}; b)$ on sets I of the form $\{1, \ldots, n\}$. The above definition has the advantage of avoiding unnatural identification between finite sets.

Note that the last point of the definition can be used with an automorphism $\sigma : I \to I$. Using the unitality and associativity of the composition structure, it is not hard to see that $\mathcal{M}(\{a_i\}_{i\in I}; b)$ supports an action of the group $\mathrm{Aut}(I)$. This is another advantage of this definition. We do not need to include this action as extra structure.

Definition B.1.2. Let $\mathcal{M}$ be an operad. The underlying simplicial category of $\mathcal{M}$ denoted $\mathbf{M}$ is the simplicial category whose objects are the colors of $\mathcal{M}$ and with

$$\mathrm{Map}_{\mathbf{M}}(m, n) = \mathcal{M}(\{m\}; n)$$

We define the following notation which is useful to write operads explicitely:

Let $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ be two sequences of colors of $\mathcal{M}$. We denote by $\{a_i\}_{i\in I} \boxplus \{b_j\}_{j\in J}$ the sequence indexed over $I \sqcup J$ whose restriction to I (resp. to J) is $\{a_i\}_{i\in I}$ (resp. $\{b_j\}_{j\in J}$).

For instance if we have two colors a and b, we can write $a^{\boxplus n} \boxplus b^{\boxplus m}$ to denote the sequence $\{a, \ldots, a, b, \ldots, b\}_{\{1,\ldots,n+m\}}$ with n a's and m b's.

Any symmetric monoidal category can be seen as an operad:

Definition B.1.3. Let $(\mathbf{A}, \otimes, \mathbb{I}_{\mathbf{A}})$ be a small symmetric monoidal category enriched in $\mathbf{S}$. Then $\mathbf{A}$ has an underlying operad $\mathcal{U}\mathbf{A}$ whose colors are the objects of A and whose spaces of operations are given by

$$\mathcal{U}\mathbf{A}(\{a_i\}_{i\in I}; b) = \mathrm{Map}_{\mathbf{A}}(\bigotimes_{i\in I} a_i, b)$$

Definition B.1.4. We denote by **Fin** the category whose objects are nonnegative integers n and whose morphisms $n \to m$ are maps of finite sets

$$\{1, \dots, n\} \to \{1, \dots, m\}$$

We allow ourselves to write $i \in n$ when we mean $i \in \{1, \dots, n\}$.

The construction $\mathbf{A} \mapsto \mathcal{U}\mathbf{A}$ sending a symmetric monoidal category to an operad has a left adjoint that we define now. The underlying category of the left adjoint applied to $\mathcal{M}$ is $\mathbf{M}$. For this reason, we can safely use the letter $\mathbf{M}$ to denote that symmetric monoidal category.

Definition B.1.5. Let $\mathcal{M}$ be an operad, the objects of the free symmetric monoidal category $\mathbf{M}$ are given by

$$\mathrm{Ob}(\mathbf{M}) = \coprod_{n\in\mathrm{Ob}(\mathbf{Fin})} \mathrm{Col}(\mathbf{M})^n$$

Morphisms are given by

$$\mathbf{M}(\{a_i\}_{i\in n}, \{b_j\}_{j\in m}) = \coprod_{f:n\to m} \prod_{i\in m} \mathcal{M}(\{a_j\}_{j\in f^{-1}(i)}; b_i)$$

It is easy to check that there is a functor $\mathbf{M}^2 \to \mathbf{M}$ which on objects is

$$(\{a_i\}_{i\in n}, \{b_j\}_{j\in m}) \mapsto \{a_1 \dots, a_n, b_1, \dots, b_m\}$$

Proposition B.1.6. *This functor can be extended to a symmetric monoidal structure on* $\mathbf{M}$. $\square$

We define an algebra over an operad with value in a symmetric monoidal category $(\mathbf{C}, \otimes, \mathbb{I}_{\mathbf{C}})$:

Definition B.1.7. Let S be a set, and let $A : S \to \mathrm{Ob}(\mathbf{C})$ be a map. We define the endomorphism operad $\mathcal{E}nd_A$ of A to be the operad with set of colors S and with

$$\mathcal{E}nd_A(\{a_i\}_{i\in I}; b) = \mathbf{C}(\otimes_{i\in I} A(a_i), A(b))$$

Definition B.1.8. Let $\mathcal{M}$ be an operad. We define the category of $\mathcal{M}$-algebras in $\mathbf{C}$.

Its objects are functions $A : \mathrm{Col}(\mathcal{M}) \to \mathrm{Ob}(\mathbf{C})$ together with maps of operads inducing the identity on colors:

$$\mathcal{M} \to \mathcal{E}nd_A$$

A morphism $f : A \to B$ is the data of a map $f_c : A(c) \to B(c)$ for each color c of $\mathcal{M}$ such that the following triangle of operads commutes:

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[dr] & \mathcal{E}nd_A \arrow[d, "f"] \\ & \mathcal{E}nd_B \end{tikzcd}$$

We denote by $\mathbf{C}[\mathcal{M}]$ the category of $\mathcal{M}$-algebras in $\mathbf{C}$.

Equivalently, an $\mathcal{M}$-algebra in $\mathbf{C}$ is a map of operads $\mathcal{M} \to \mathcal{U}C$. With this definition, it is tautologous that an algebra over $\mathcal{M}$ induces a (symmetric monoidal) functor $\mathbf{M} \to \mathbf{C}$. We will use the same notation for the two objects and allow oursleves to switch between them without mentioning it.

B.2 Right modules over operads

Definition B.2.1. Let $\mathcal{M}$ be an operad. A *right $\mathcal{M}$-module* is a simplicial functor

$$R : \mathbf{M}^{\mathrm{op}} \to \mathbf{S}$$

When $\mathcal{O}$ is a single-color operad, we denote by $\mathbf{Mod}_{\mathcal{O}}$ the category of right modules over $\mathcal{O}$.

Remark B.2.2. If $\mathcal{O}$ is a single-color operad, it is easy to verify that the category of right modules over $\mathcal{O}$ in the above sense is isomorphic to the category of right modules over $\mathcal{O}$ in the usual sense (i.e. a right module over the monoid $\mathcal{O}$ with respect to the monoidal structure on symmetric sequences given by the composition product).

Note that the analogous result for left modules over $\mathcal{O}$ is false.

Let Σ be the category whose objects are the finite sets $\{1, \dots, n\}$ with $n \in \mathbb{Z}_{\geq 0}$ and morphisms are bijections. Σ is a symmetric monoidal category for the disjoint union operation.

Let $\mathcal{I}$ be the initial one-color operad (i.e. $\mathcal{I}(1) = *$ and $\mathcal{I}(k) = \emptyset$ for $k \neq 1$). It is clear that the free symmetric monoidal category associated to $\mathcal{I}$ is the category Σ. Let $\mathcal{O}$ be an operad and $\mathbf{O}$ be the free symmetric monoidal category associated to $\mathcal{O}$. By functoriality of the free symmetric monoidal category construction, there is a symmetric monoidal functor $\Sigma \to \mathbf{O}$ which induces a functor

$$\mathrm{Fun}(\mathbf{O}^{\mathrm{op}}, \mathbf{S}) \to \mathrm{Fun}(\Sigma^{\mathrm{op}}, \mathbf{S})$$

Recall the definition of the Day tensor product:

Definition B.2.3. Let $(\mathbf{A}, \square, \mathbb{I}_{\mathbf{A}})$ be a small symmetric monoidal category, then the category $\mathrm{Fun}(\mathbf{A}, \mathbf{S})$ is a symmetric monoidal category for the operation $\otimes$ defined as the following coend:

$$F \otimes G(a) = \mathbf{A}(-\square-, a) \otimes_{\mathbf{A}\times\mathbf{A}} F(-) \times G(-)$$

Now we can make the following proposition:

Proposition B.2.4. *Let $\mathcal{O}$ be a single-color operad. The category of right $\mathcal{O}$-modules has a symmetric monoidal structure such that the restriction functor*

$$\mathrm{Fun}(\mathbf{O}^{\mathrm{op}}, \mathbf{S}) \to \mathrm{Fun}(\Sigma^{\mathrm{op}}, \mathbf{S})$$

is symmetric monoidal when the target is equipped with the Day tensor product.

Proof. We have the following identity for three symmetric sequences in $\mathbf{S}$ (see [Fre09] 2.2.3.):

$$(M \otimes N) \circ P \cong (M \otimes P) \circ (N \otimes P)$$

If P is an operad, this identity gives a right P-module structure on the tensor product $M \otimes N$. $\square$

The category $\mathbf{Mod}_{\mathcal{O}}$ is a symmetric monoidal category tensored over $\mathbf{S}$. Therefore if $\mathcal{P}$ is another operad, we can talk about the category $\mathbf{Mod}_{\mathcal{O}}[\mathcal{P}]$.

It is easy to check that the category $\mathbf{Mod}_{\mathcal{O}}[\mathcal{P}]$ is isomorphic to the category of $\mathcal{P}$-$\mathcal{O}$-bimodules in the category of symmetric sequences in $\mathbf{S}$.

Any right module R over a single-color operad $\mathcal{O}$ gives rise to a functor $\mathbf{C}[\mathcal{O}] \to \mathbf{C}$

$$A \mapsto R \circ_{\mathcal{O}} A = \operatorname{coeq}(R \circ \mathcal{O}(A) \rightrightarrows R(A))$$

Here the first map of the coequalizer is given by the $\mathcal{O}$-algebra structure on A and the second one by the right $\mathcal{O}$-action on R.

It is sometimes psychologically easier to describe $R \circ_{\mathcal{O}} X$ as an enriched coend. The next proposition does this:

Proposition B.2.5. *There is an isomorphism*

$$R \circ_{\mathcal{O}} A \cong R \otimes_{\mathbf{O}} A$$

This kind of coend often occurs because of the following proposition:

Proposition B.2.6. *Let $\alpha : \mathcal{M} \to \mathcal{N}$ a map of operads, the forgetful functor $\mathbf{C}[\mathcal{N}] \to \mathbf{C}[\mathcal{M}]$ has a left adjoint $\alpha_!$.*

For $A \in \mathbf{C}[\mathcal{M}]$, the value at the color n of $\mathrm{Col}(\mathcal{N})$ of $\alpha_! A$ is given by

$$\alpha_! A(n) = \mathbf{N}(\alpha(-), n) \otimes_{\mathbf{M}} A(-)$$

Proof. Easy. MAYBE I SHOULD PROVIDE A REFERENCE. $\square$

Proposition B.2.7. *Let R be a $\mathcal{P}$-algebra in $\mathbf{Mod}_{\mathcal{O}}$. The functor $A \mapsto R \circ_{\mathcal{O}} A$ factors through the forgetful functor $\mathbf{C}[\mathcal{P}] \to \mathbf{C}$.*

Proof. This functor is defined as a reflexive coequalizer. The forgetful functor $\mathbf{C}[\mathcal{P}] \to \mathbf{C}$ preserves reflexive coequalizer (this is because the category defining reflexive coequalizers is sifted).Each term in this reflexive coequalizer is a $\mathcal{P}$-algebra. Therefore, the coequalizer has a $\mathcal{P}$-algebra structure. □

B.3 Homotopy theory of operads and modules

Definition B.3.1. an operad $\mathcal{M}$ is said to be Σ-cofibrant if for any sequence of colors $\{a_i\}_{i\in n}$ and any color b, the space $\mathcal{M}(\{a_i\};b)$ is a cofibrant object in $\mathbf{S}^{\Sigma_n}$ with its projective model structure for the Σ_n-action described in B.1.

Similarly, a right module P over $\mathcal{M}$ is Σ-cofibrant if for any sequence of colors $\{a_i\}_{i\in n}$, the Σ_n-simplicial set $P(\{a_i\})$ is cofibrant in $\mathbf{S}^{\Sigma_n}$.

Remark B.3.2. A G-simplicial set is cofibrant if the G-action is free. In this work, anytime, we claim that a simplicial set is G-cofibrant, we use this fact.

Definition B.3.3. A weak equivalence between operads is a morphism of operad $f : \mathcal{M} \to \mathcal{N}$ which satisfies:

- (Homotopical fully faithfulness) For each $\{m_i\}_{i\in I}$ a finite set of colors of $\mathcal{M}$ and each m a color of $\mathcal{M}$, the map

$$\mathcal{M}(\{m_i\};m) \to \mathcal{N}(\{f(m_i)\};f(m))$$

  is a weak equivalence.

- (Essential surjectivity) The underlying map of simplicial categories $\mathbf{M} \to \mathbf{N}$ is essantially surjective (i.e. it is such when we apply π_0 to each space of maps).

The homotopy theory of simplicial operads with respect to the above definition of weak equivalences can be structured into a model category (see [**?**] or [Rob11]) but we will not need this fact in this work.

Definition B.3.4. A cofibrantly generated symmetric monoidal model category $(\mathbf{C}, \otimes, \mathbb{I})$ has a good theory of algebras (resp. a good theory of algebras over Σ-cofibrant operads) if:

- For any operad $\mathcal{M}$ (resp. Σ-cofibrant operad) in $\mathbf{S}$, the category $\mathbf{C}[\mathcal{M}]$ of $\mathcal{M}$-algebras in $\mathbf{C}$ has a model category structure where weak equivalences and fibrations are created by the forgetful functors $\mathbf{C}[\mathcal{M}] \to \mathbf{C}[\mathrm{Col}(\mathcal{M})]$.

- If $\alpha : \mathcal{M} \to \mathcal{N}$ is a morphism of operad (resp. Σ-cofibrant operads), the adjunction

$$\alpha_! : \mathbf{C}[\mathcal{M}] \leftrightarrows \mathbf{C}[\mathcal{N}] : \alpha^*$$

  is a Quillen adjunction which is a Quillen equivalence if α is a weak equivalence (i.e. induces an isomorphism on the set of colors and weak equivalences between corresponding spaces of operations).

- For any operad $\mathcal{M}$ (resp. Σ-cofibrant operad) in $\mathbf{S}$, the right adjoint $\mathbf{C}[\mathrm{Col}(\mathcal{M})] \leftrightarrows \mathbf{C}[\mathcal{M}]$ preserves cofibrations.

Remark B.3.5. In practice, one proves the first point of this definition by using the lemma A.1.3. In that case, the third point is automatically satisfied.

Remark B.3.6. In $\mathbf{S}$, the category $\mathbf{S}[\mathcal{C}om]$ has a tranferred model structure as is proved in [BM03]. However, this model structure does not encode the homotopy theory of $\mathcal{E}_\infty$-spaces. The second axiom of the above definition is here to insure that the homotopy theory underlying these model structure is homotopically correct.

Let us mention two families of examples where these conditions are satisfied:

Theorem B.3.7. *Let $\mathbf{C}$ be a symmetric monoidal simplicial cofibrantly generated model category. Assume that $\mathbf{C}$ has a monoidal fibrant replacement functor and a cofibrant unit. Then $\mathbf{C}$ has a good theory of algebras over Σ-cofibrant operads.*

Proof. The proof is essentially done in [BM05]. The idea is that $H = \mathrm{Sing}([0,1])$ is a cocommutative monoid in $\mathbf{S}$, therefore for any $\mathcal{M}$-algebra A, the object A^H furnishes a path object in $\mathbf{C}[\mathcal{M}]$. □

For instance $\mathbf{S}$ and $\mathbf{Top}$ obviously satisfy the conditions. If R is a $\mathbb{Q}$-algebra, the category $\mathbf{Ch}_{\geq 0}(R)$ with its projective model structure (i.e., the model structure for which weak equivalences are quasi-isomorphisms and fibrations are epimorphisms) satisfies the condition. One can take $C_*([0,1])$ as interval object.

If $\mathbf{C}$ satisfies the conditions of the theorem, and $\mathbf{I}$ is any small simplicial category. Then $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ with the objectwise tensor product and projective model structure also satisfies the conditions.

Proposition B.3.8. *Let E be a commutative symmetric ring spectrum and Z be any symmetric spectrum. Then the positive model structure on $\mathbf{Mod}_E$ has a good theory of algebras. Similarly, the Bousfield localization $L_Z\mathbf{Mod}_E$ with the positive model structure has a good theory of algebras.*

Proof. The paper [EM06] only deals with modules over the sphere spectrum but it is easy to check that their proof can be adapted to this more general situtation. The main ingredient being A.4.5. □

Proposition B.3.9. *Let $\mathbf{C}$ be a symmetric monoidal model category with a good theory of algebras (resp. with a good theory of algebras over Σ-cofibrant operads). Let $\mathcal{M}$ be an operad (resp. Σ-cofibrant operad) and let $\mathbf{M}$ be the free symetric monoidal category on $\mathcal{M}$. Let $A : \mathbf{M} \to \mathbf{C}$ be an algebra. Then*

1. *Let $P : \mathbf{M}^{\mathrm{op}} \to \mathbf{S}$ be a right module (resp. Σ-cofibrant right module). Then $P \otimes_{\mathbf{M}} -$ preserves weak equivalences between cofibrant $\mathcal{M}$-algebras.*

2. *Let $P : \mathbf{M}^{\mathrm{op}} \to \mathbf{S}$ be a right module (resp. Σ-cofibrant right module). Then $P \otimes_{\mathbf{M}} -$ sends cofibrant $\mathcal{M}$-algebras to cofibrant objects of $\mathbf{C}$.*

3. *If A is a cofibrant algebra and $\mathbb{I}_{\mathbf{C}}$ is cofibrant, the functor $- \otimes_{\mathbf{M}} A$ is a left Quillen functor from right modules over $\mathcal{M}$ to $\mathbf{C}$.*

4. *Moreover the functor $- \otimes_{\mathbf{M}} A$ preserves all weak equivalences between right modules (resp. Σ-cofibrant right modules).*

Proof. For P any simplicial functor $\mathbf{M}^{\mathrm{op}} \to \mathbf{C}$, we denote by $\mathcal{M}_P$ the operad whose colors are $\mathrm{Col}(\mathcal{M})\sqcup\infty$ and whose spaces of operations are the following:

$$\mathcal{M}_P(\{m_1,\ldots,m_k\},n)=\mathcal{M}(\{m_1,\ldots,m_k\},n)\ \text{ if } \infty\notin\{m_1,\ldots,m_k\}$$
$$\mathcal{M}_P(\{m_1,\ldots,m_k\},n)=\emptyset\ \text{ if } \infty\in\{m_1,\ldots,m_k\}$$
$$\mathcal{M}_P(\{m_1,\ldots,m_k\};\infty)=P(\{m_1,\ldots,m_k\})$$

It is easy to see that there is an operad map $\alpha_P:\mathcal{M}\to\mathcal{M}_P$. Moreover by B.2.6 we have

$$\mathrm{ev}_\infty(\alpha_P)_!A=P\otimes_{\mathbf{M}}A$$

Proof of the first claim. If $A\to B$ is a weak equivalence between cofibrant $\mathcal{M}$-algebras, then $(\alpha_P)_!A$ is weakly equivalent to $(\alpha_P)_!B$ by the previous theorem. Since ev_∞ preserves all weak equivalences, we are done.

Proof of the second claim. Since $(\alpha_P)_!$ is a left Quillen functor, $(\alpha_P)_!A$ is a cofibrant $\mathcal{M}_P$-algebra and by B.3.4, $\mathrm{ev}_\infty(\alpha_P)_!A$ is cofibrant in $\mathbf{C}$.

Proof of the third claim. To show that $P\mapsto P\otimes_{\mathbf{M}}A$ is left Quillen it suffices to check that it sends generating (trivial) cofibrations to (trivial) cofibrations.

For $m\in\mathrm{Ob}(\mathbf{M})$, denote by ι_m the functor $\mathbf{S}\to\mathrm{Fun}(\mathrm{Ob}(\mathbf{M}),\mathbf{S})$ sending X to the functor sending m to X and everything else to $\emptyset$. Denote by $F_{\mathbf{M}}$ the left Kan extension functor

$$F_{\mathbf{M}}:\mathrm{Fun}(\mathrm{Ob}(\mathbf{M})^{\mathrm{op}},\mathbf{S})\to\mathrm{Fun}(\mathbf{M}^{\mathrm{op}},\mathbf{S})$$

We can take as generating (trivial) cofibrations are the maps of the form $F_{\mathbf{M}}\iota_mI$ ($F_{\mathbf{M}}\iota_mJ$). We have:

$$F_{\mathbf{M}}\iota_mI\otimes_{\mathbf{M}}A\cong I\otimes A(m)$$

Since A is cofibrant as an algebra its value at each object of $\mathbf{M}$ is either cofibrant or the unit of $\mathbf{C}$ which is assumed to be cofibrant. Moreover, the left tensoring $\mathbf{S}\times\mathbf{C}$ is a Quillen bifunctor by hypothesis, therefore $F_{\mathbf{M}}\iota_mI\otimes_{\mathbf{M}}A$ consists of cofibrations. Similarly, $F_{\mathbf{M}}\iota_mJ\otimes_{\mathbf{M}}A$ consists of trivial cofibrations.

Proof of the fourth claim. Let $P\to Q$ be a weak equivalence between functors $\mathbf{M}^{\mathrm{op}}\to\mathbf{S}$.

This induces a weak equivalence between operads $\beta : \mathcal{M}_P \to \mathcal{M}_Q$. It is clear that $\alpha_Q = \beta \circ \alpha_P$, therefore $(\alpha_Q)_! A = \beta_! (\alpha_P)_! A$. We apply β^* to both side and get

$$\beta^* \beta_! (\alpha_P)_! A = \beta^* (\alpha_Q)_! A$$

Since $(\alpha_P)_! A$ is cofibrant, the adjunction map $(\alpha_P)_! A \to \beta^* \beta_! (\alpha_P)_! A$ is a weak equivalence by definition of a Quillen equivalence. Therefore the obvious map

$$(\alpha_P)_! A \to \beta^* (\alpha_Q)_! A$$

is a weak equivalence.

If we evaluate this at the color ∞, we find a weak equivalence

$$P \otimes_{\mathbf{M}} A \to Q \otimes_{\mathbf{M}} A$$

□

Operadic vs categorical homotopy left Kan extension

Proposition B.3.10. *Assume* $\mathbf{C}$ *has a good theory of algebras (resp. a good theory of algebras over* Σ*-cofibrant operads) and assume that* $\mathbf{C}$ *has a cofibrant unit. Let* $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$ *be a morphism of simplicial operads (resp.* Σ*-cofibrant operads). Let* A *be an algebra over* $\mathcal{M}$*. The derived operadic pushforward* $\alpha_!(A)$ *is weakly equivalent to the homotopy left Kan extension of* $A : \mathbf{M} \to \mathbf{C}$ *along the induced map* $\mathbf{M} \to \mathbf{N}$*.*

Proof. Let $QA \to A$ be a cofibrant replacement of A as an $\mathcal{M}$-algebra. The value at n of the homotopy left Kan extension of A can be computed as the geometric realization of the Bar construction

$$\mathrm{B}_\bullet(\mathbf{N}(\alpha -, n), \mathbf{M}, QA)$$

By B.3.4, QA is objectwise cofibrant (we use the fact that a tensor product of cofibrant objects is cofibrant) or the unit $\mathbb{I}_{\mathbf{C}}$. Therefore, the bar construction is Reedy-cofibrant (A.3.2) if $\mathbb{I}_{\mathbf{C}}$ is cofibrant.

We can rewrite this simplicial object as

$$\mathrm{B}_\bullet(\mathbf{N}(\alpha-,n),\mathbf{M},\mathbf{M})\otimes_{\mathbf{M}} A$$

The geometric realization is

$$|\mathrm{B}_\bullet(\mathbf{N}(\alpha-,n),\mathbf{M},\mathbf{M})|\otimes_{\mathbf{M}} A$$

It is a classical fact that the map

$$|\mathrm{B}_\bullet(\mathbf{N}(\alpha-,n),\mathbf{M},\mathbf{M})|\to\mathbf{N}(\alpha-,n)$$

is a weak equivalence of functors on $\mathbf{E}_d^{\mathrm{op}}$. Therefore by B.3.9, the Bar construction is weakly equivalent to $\alpha_! A$. □

This result is also true in $L_Z p\mathbf{Mod}_E$:

Proposition B.3.11. *Let A be an object of $L_Z p\mathbf{Mod}_E[\mathbf{C}]$. The derived operadic left Kan extension $\alpha_!(A)$ is weakly equivalent to the homotopy left Kan extension of $A:\mathbf{M}\to\mathbf{C}$ along the induced map $\mathbf{M}\to\mathbf{N}$.*

Proof. We can consider the bar construction as an object of $L_Z a\mathbf{Mod}_E$. In that case, it is Reedy cofibrant and the rest of the argument of the previous proposition works. □

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