Topology of Kač-Moody Groups

by

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B.S. in Mathematics, California Institute of Technology (1993)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1998

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ABSTRACT

The class of complex semisimple Lie algebras can be extended to include infinite dimensional Lie algebras known as Kač-Moody algebras. The correspondence between complex semisimple Lie algebras and complex, connected, simply-connected Lie groups has been extended by Kač-Peterson to a correspondence between Kač-Moody algebras and certain connected, simply-connected topological groups known as Kač-Moody groups. Like Lie groups, they contain a *maximal torus* of finite rank and have an associated *Weyl group*. One can also construct *flag varieties* that are (possibly infinite) projective varieties admitting a *Bruhat decomposition*.

Kač-Moody groups fall into three general types: The *finite type*, the *affine type* and the *indefinite type*. The groups of finite type are the usual simply-connected Lie groups. The groups of affine type are closely related to loop-groups and have been extensively studied in topology and physics. The groups of indefinite type constitute the majority among Kač-Moody groups and very little is known about them.

In this thesis, we explore the topology of Kač-Moody groups. Chapter 1 gives a general overview of the theory of Kač-Moody groups and is a condensed version of [**7**]. Chapter 2 contains some results about the Hopf-algebra structure of the cohomology of Kač-Moody groups. Chapter 3 uses the results of chapter 2 to compute the cohomology of all the Kač-Moody groups of rank 2 which are not of finite type. In this chapter we also compute the cohomology of the classifying spaces of these groups. In chapter 4 we show that the classifying space of a Kač-Moody group which is not of finite type can be realized as a certain homotopy-colimit of the classifying spaces of its proper standard parabolics. Using this we show that the classifying space of the Kač-Moody group is approximated by the classifying space of the normalizer of its maximal torus at all primes not appearing in the torsion of the Weyl group. This extends a well-known classical result to the class of all Kač-Moody groups. Finally, in chapter 5 we construct a fibration using rank 2 Kač-Moody groups and end with a conjecture relating this fibration to certain well-known fibrations known as Anick fibrations [**1**].

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To my parents, Nayana and Chittaranjan, who gave me life and love, and to Haynes for his faith and confidence in me.

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Acknowledgments

The seeds of this thesis were sown during the regular meetings with my advisor Haynes Miller. To Haynes I owe my mathematical training. I feel privileged to be his student and his friend. Without his insight and guidance this work would certainly not have been possible.

I owe much of my knowledge of Kač-Moody groups to V. Kač and D. Peterson without whom my knowledge on this subject would not have been worth a thesis.

I would like to thank my colleagues Ioanid, Gustavo, Gerd, Dan.C, Dan.D and Lior for sharing some of their own knowledge with me. I wish them the best of luck for the future ... clearly we all need as much of it as we can get.

I shared my time at M.I.T with my brothers: Juan, Alok, Ravi, Lior, Victor, Chris, Ioanid and with my friend and companion, Joan, in all of whom I see myself so clearly.

And finally, I would like to thank the wonderful people at Toscanini for making the atmosphere just right for my research.

Contents

CHAPTER 1

Kač-Moody Infinite Dimensional Groups

1. Introduction

This chapter is an attempt to systematically state the properties of the infinite dimensional groups constructed by Kač and Peterson. To benefit from this summary the reader need only know the basic structure theorems of Lie groups and Lie algebras. As a good reference on Lie algebras we cite [**20**]. The author claims originality only for example 1.14.4, all the other material in this chapter can be found in [**7**] and [**8**]. We have tried to be notationally compatible with [**7**]. The reader may assume that the base field is $\mathbb{C}$ unless stated otherwise.

2. Integrable algebras and modules

We begin with some general notions dealing with infinite dimensional Lie algebras. Let V be a (possibly infinite-dimensional) vector space over $\mathbb{C}$ and let A be an endomorphism of V. To motivate the definition of exponential we introduce the following definitions. We say that A is **locally finite** if every $v \in V$ lies in a finite-dimensional A-invariant subspace of V (or equivalently, the vectors $(A^n(v) \mid n = 0, 1, \dots)$ are linearly dependent for every $v \in V$). The endomorphism is called **locally nilpotent** if for every $v \in V$, there is an integer $n > 0$ such that $A^n(v) = 0$. We call A **semisimple** if V admits a basis of eigenvectors for A. Clearly, locally nilpotent and semisimple elements are locally finite.

If A is locally finite we can form the corresponding 1-parameter group of automorphisms of V

$$\exp\, tA = \sum_{n\geq 0} \frac{t^n}{n!} A^n, \qquad t \in \mathbb{C}.$$

Now let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra over $\mathbb{C}$ and let V be a $\mathfrak{g}$-module with action π. An element $x \in \mathfrak{g}$ is called **π - locally finite** if $\pi(x)$ is a locally finite endomorphism of the vector space V. We denote by $F_{\mathfrak{g}}$ the set of all ad-locally finite elements of $\mathfrak{g}$, and by $\mathfrak{g}_{fin}$ the subalgebra of $\mathfrak{g}$ generated by $F_{\mathfrak{g}}$. The Lie algebra $\mathfrak{g}$ is called **integrable** if $\mathfrak{g} = \mathfrak{g}_{fin}$. Denote by $F_{\mathfrak{g},\pi}$ the set of π-locally finite elements of $F_{\mathfrak{g}}$.

Lemma 1.2.1. *The subalgebra of $\mathfrak{g}$ generated by $F_{\mathfrak{g},\pi}$ is the linear span of $F_{\mathfrak{g},\pi}$. In particular, $\mathfrak{g}_{fin}$ is spanned by $F_{\mathfrak{g}}$. If dim $\mathfrak{g} < \infty$ and if $\mathfrak{g}$ is generated by $F_{\mathfrak{g},\pi}$, then V is a locally finite $\mathfrak{g}$-module (i.e. any $v \in V$ is contained in a finite-dimensional $\mathfrak{g}$-submodule).*

We call the $\mathfrak{g}$-module (V, π) **integrable** if $F_{\mathfrak{g},\pi} = F_{\mathfrak{g}}$. Of course the $\mathfrak{g}$-module $(\mathfrak{g}, \text{ad})$ is integrable.

For a special class of Lie algebras known as Kač-Moody algebras (cf. Section 6), the reader may recall that [**8**](§3.6) has a seemingly different definition of an integrable representation. Kač-Peterson has established that these two definitions are indeed the same (cf. [**8**] exercise 3.17).

3. Central extensions

Let R be a complex commutative associative algebra with unity and let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra. Then the Lie algebra $\mathfrak{g}_R = R \otimes_{\mathbb{C}} \mathfrak{g}$ is an integrable Lie algebra over $\mathbb{C}$. This is easily seen by observing that we can choose ad-locally nilpotent generators for $\mathfrak{g}$ (for example the Chevalley generators). For an arbitrary Lie algebra $\mathfrak{g}$, one can also consider the Lie algebra $\mathfrak{g}_R$. We warn the reader that for $\mathfrak{g}_R$ to be integrable it is crucial that the generators of $\mathfrak{g}$ can be chosen to be ad-locally nilpotent; it is not enough for them to be ad-locally finite. For a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, the universal central extension $\tilde{\mathfrak{g}}_R$ of $\mathfrak{g}_R$ is constructed as the following (non-split) exact sequence:

$$0 \to \Omega^1_R/dR \longrightarrow \tilde{\mathfrak{g}}_R = \mathfrak{g}_R \oplus (\Omega^1_R/dR) \xrightarrow{d\tau} \mathfrak{g}_R \to 0 \tag{1.3.1}$$

where Ω^1_R is the space of all formal differentials (i.e. expressions of the form fdg, where f,g $\in R$, with relation d(fg) = fdg + gdf), and the bracket on $\tilde{\mathfrak{g}}_R$ is defined by

$$[r_1 \otimes g_1, r_2 \otimes g_2] = r_1 r_2 \otimes [g_1, g_2] + (g_1 \mid g_2) r_2 dr_1 \mod dR$$

where $(. \mid .)$ is the Killing form on $\mathfrak{g}$. Notice that being central, the bracket is zero on the summand Ω^1_R/dR. One can check that $\tilde{\mathfrak{g}}_R$ is an integrable Lie algebra as well. It is also easy to check that $V_R = R \otimes_{\mathbb{C}} V$ is an integrable $\mathfrak{g}_R$-module for any finite-dimensional $\mathfrak{g}$-module V.

4. Groups associated to integrable algebras

Let $\mathfrak{g}$ be an integrable Lie algebra. We associate to $\mathfrak{g}$ a group G as follows. Let G^* be a free group on the set $F_{\mathfrak{g}}$. Given an integrable $\mathfrak{g}$-module $(V, d\pi)$, we define a G^*-module $(V, \tilde{\pi})$ by

$$\tilde{\pi}(x) = \exp\, d\pi(x) = \sum_{n \geq 0} \frac{d\pi(x)^n}{n!}, \quad x \in F_{\mathfrak{g}}.$$

We put $G = G^*/\cap \operatorname{Ker} \tilde{\pi}$, where the intersection is taken over all integrable $\mathfrak{g}$-modules $d\pi$. Thus, the G^*-module $(V, \tilde{\pi})$ is naturally a G-module (V, π), the integrable $\mathfrak{g}$-module $(V, d\pi)$ being its "differential". We call G the **group associated to the Lie algebra** $\mathfrak{g}$ and (V, π) the G-module associated to the integrable $\mathfrak{g}$-module. Given an element $x \in F_{\mathfrak{g}}$, we denote its image in G under the canonical homomorphism $G^* \to G$ by exp x . Thus, we have by definition

$$\pi(\exp\, x) = \exp\, d\pi(x), \quad x \in F_{\mathfrak{g}}$$

for an integrable $\mathfrak{g}$-module $(V, d\pi)$. Note also that $\{\exp\, tx \mid t \in \mathbb{C}\}$ is a 1-parameter subgroup of G.

Put $F_G = \{\exp\, x \mid x \in F_{\mathfrak{g}}\} \subset G$. A G-module (V, π) is called **differentiable** if all elements of F_G act locally finitely on V and $\exp\, tx$ restricted to any invariant finite-dimensional subspace is analytic in t. Clearly, the G-module (V, π) associated to an integrable $\mathfrak{g}$-module $(V, d\pi)$ is differentiable. In seeking an invertible functor between the categories of integrable $\mathfrak{g}$-modules and differentiable G-modules, we conjecture

CONJECTURE 1.4.1. Let (V, π) be a differentiable G-module. Then there exists a unique action $d\pi$ of $\mathfrak{g}$ on V such that $\pi(\exp x) = \exp d\pi(x)$ for all $x \in F_{\mathfrak{g}}$. $(V, d\pi)$ is then automatically an integrable $\mathfrak{g}$-module.

Notice that uniqueness follows from lemma 1.2.1; it is existence that is hard to show. This conjecture has been established for the class of Kač-Moody algebras discussed in Section 6.

A homomorphism $d\phi : \mathfrak{g}_1 \to \mathfrak{g}$ of integrable Lie algebras is called **integrable** if $d\phi(F_{\mathfrak{g}_1}) \subset F_{\mathfrak{g}}$; then $d\phi(\mathfrak{g}_1)$ is called an **integrable subalgebra** of $\mathfrak{g}$. Given an integrable homomorphism $d\phi$ of Lie algebras, we have a canonically defined homomorphism of the associated groups $\phi : G_1 \to G$, so that $d(\pi \circ \phi)$ exists and is equal to $d\pi \circ d\phi$. The subgroup $\phi(G_1)$ of G is called the subgroup corresponding to the integrable subalgebra $d\phi(\mathfrak{g}_1)$ of $\mathfrak{g}$. It is generated by the exp x with x $\in d\phi(\mathfrak{g}_1) \cap F_{\mathfrak{g}}$. Of course, any isomorphism of integrable Lie algebras is integrable and lifts to an isomorphism of the associated groups.

Now given an ordered set $\bar{x} = (x_1, x_2, \dots, x_n)$ of elements of $F_{\mathfrak{g}}$, one defines $\varphi_{\bar{x}} : \mathbb{C}^n \to G$ by $\varphi_{\bar{x}}(t_1, t_2, \dots, t_n) = (\exp t_1 x_1) \dots (\exp t_n x_n)$. Fix a subset $X \subset F_{\mathfrak{g}}$ such that the set $\{\exp tx \mid x \in X, t \in \mathbb{C}\}$ generates G. We define the topology on G to be the coarsest topology that makes the functions $\varphi_{\bar{x}}$ continuous whenever all the elements of $\bar{x}$ are from X. With this topology G is a connected Hausdorff topological space. The inverse map is continuous but not the multiplication in general. One can show that if X is countable, then G is a topological group.

If G is a (complex) Lie group with Lie algebra $\mathfrak{g}$, then it can be shown that one recovers G by this construction. In fact, the group associated to the Lie algebra $\mathfrak{g}_R$ corresponds to the group of points of G over R in the sense of algebraic geometry.

For Kač-Moody algebras (cf. Section 6), an alternate construction of the corresponding groups is given in the Appendix at the end of the chapter.

5. Cartan matrices and Coxeter groups

Let $A = (a_{ij})_1^n$ be a generalized Cartan matrix, i.e. $a_{ii} = 2, a_{ij}$ are non-positive integers for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. For a pair of indices i, j such that $i \neq j$ put $m_{ij} = 2, 3, 4$ or 6 if $a_{ij}a_{ji} = 0, 1, 2$ or 3 resp. and put $m_{ij} = 0$ otherwise; put $m_{ii} = 1$. A matrix A is called **decomposable** if it can be made into a non trivial direct sum of matrices after performing the same permutation on its rows and columns. It is called **symmetrizable** if there is an invertible diagonal matrix D and a symmetric matrix B such that $A = DB$. We shall primarily be concerned with indecomposable symmetrizable Cartan matrices.

We associate to A a discrete group $\bar{W}(A)$ on n generators $\bar{r}_1, \dots, \bar{r}_n$ and the following defining relations:

$$(D1) \qquad \bar{r}_j \bar{r}_i^2 \bar{r}_j^{-1} = \bar{r}_i^2 \bar{r}_j^{-2a_{ij}}$$

$$(D2) \quad \bar{r}_i \bar{r}_j \bar{r}_i \dots = \bar{r}_j \bar{r}_i \bar{r}_j \dots (m_{ij} \text{ factors on each side}).$$

We easily see from (D1) that the subgroup $T_{(2)} =< \bar{r}_i^2 \mid i = 1, \dots, n >\subset \bar{W}(A)$ is a normal commutative subgroup. It follows also from (D1) that $\bar{r}_i^4 = 1$.

Let $W(A)$ be the corresponding Coxeter group, i.e. the group on generators $r_1, r_2, \dots, r_n$ and the following defining relations:

$$(r_i r_j)^{m_{ij}} = 1.$$

We then have a homomorphism $\bar{W}(A) \to W(A)$ given by $\bar{r}_i \to r_i$, and we get an exact sequence

$$1 \longrightarrow T_{(2)} \longrightarrow \bar{W}(A) \longrightarrow W(A) \longrightarrow 1. \tag{1.5.1}$$

Let $w = r_{i_1} \dots r_{i_m}$ be a reduced expression of $w \in W$; one defines the length $l(w) = m$. By deleting some of the r_i's from the expression one gets a new element w' and writes $w' \leq w$. The partial ordering $\leq$ on $W(A)$ is called the **Bruhat order**.

One constructs a section of the map $\bar{W}(A) \to W(A)$ by mapping the element $w = r_{i_1} \dots r_{i_m}$ to the element $\bar{w} = \bar{r}_{i_1} \dots \bar{r}_{i_m}$; one can show that $\bar{w}$ is independent of the choice of reduced expression for w. We will construct groups with $\bar{W}(A)$ as a discrete subgroup and $W(A)$ as the "Weyl group".

6. Kač-Moody algebras

Let $(\mathfrak{h}, \Pi, \check{\Pi})$ be a **realization** of the Cartan matrix A of size n, i.e. $\mathfrak{h}$ is a complex vector space of dimension $2n - rankA$, and $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$, $\check{\Pi} = \{h_1, \dots, h_n\} \subset \mathfrak{h}$ are linearly independent sets satisfying $\alpha_j(h_i) = a_{ij}$. One should view the α's as the simple roots and the h's as the corresponding inverse roots. The realization of a Cartan matrix is unique up to isomorphism.

The **Kač-Moody algebra** $\mathfrak{g}(A)$ associated to the generalized Cartan matrix A is the Lie algebra generated by the vector space $\mathfrak{h}$ and symbols (Chevalley generators) e_i and f_i $(i = 1, \dots, n)$, with the following defining relations:

$$(A1) \qquad [h, h] = 0;\ [h, e_i] = \alpha_i(h) e_i;\ [h, f_j] = -\alpha_j(h) f_j \qquad (h \in \mathfrak{h})$$

$$(A2) \qquad [e_i, f_j] = \delta_{ij} h_i;\ (\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0;\ (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0 \qquad (i \neq j).$$

The subalgebra $\mathfrak{g}(A)' \subset \mathfrak{g}(A)$, generated by the subspace $\mathfrak{h}'$ spanned by h_i's and the Chevalley symbols, is sometimes also called the Kač-Moody algebra. If we denote by η_+ (resp. η_-) the subalgebras generated by the e_i's (resp. f_i's) $i = 1, \dots, n$, then we have the **triangular decomposition** $\mathfrak{g}(A) = \eta_- \oplus \mathfrak{h} \oplus \eta_+$ and $\mathfrak{g}(A)' = \eta_- \oplus \mathfrak{h}' \oplus \eta_+$. The center of $\mathfrak{g}(A)$ and $\mathfrak{g}(A)'$ is $\nu = \{h \in \mathfrak{h}' \mid \alpha_i(h) = 0 \text{ for all } i = 1, \dots, n\}$. The center is trivial if and only if $\mathfrak{h} = \mathfrak{h}'$ which happens if and only if $\det A \neq 0$. It can be checked that e_i and f_i are ad-locally nilpotent and elements of $\mathfrak{h}$ are ad-semisimple. Thus $\mathfrak{g}(A)$ and $\mathfrak{g}(A)'$ are both integrable. We remind the reader once again that $\mathfrak{g}(A)_R$ need not be integrable. However, $\mathfrak{g}(A)'_R$ is integrable since it is generated by ad-nilpotent generators.

Notice furthermore that the subalgebras $\mathfrak{g}_i = \mathbb{C} f_i \oplus \mathbb{C} h_i \oplus \mathbb{C} e_i$ and any subspace of $\mathfrak{h}$ are clearly integrable subalgebras of $\mathfrak{g}(A)$. This is also true for the Borel subalgebras $\mathfrak{h}'' \oplus \eta_+$ and $\mathfrak{h}'' \oplus \eta_-$, where $\mathfrak{h}''$ is any subspace of $\mathfrak{h}$.

7. Highest weight modules

The details of this brief section can be found in Chapters 9 and 10 of [**8**].

Given $\Lambda \in \mathfrak{h}'^*$, we extend it in some way to a linear functional $\tilde{\Lambda} \in \mathfrak{h}^*$ and define the **highest weight module** $L(\Lambda)$ over $\mathfrak{g}(A)$ with action $d\pi_\Lambda$ by the properties:

$$(L1) \qquad L(\Lambda) \quad \text{is irreducible}$$

$$(L2) \quad \exists v_\Lambda \neq 0 \in L(\Lambda) \ \text{ with}$$

$$d\pi_\Lambda(e_i)v_\Lambda = 0, i = 1, \ldots, n;\ d\pi_\Lambda(h)v_\Lambda = \tilde{\Lambda}(h)v_\Lambda,\ h \in \mathfrak{h}.$$

The module $L(\Lambda)$ remains irreducible when restricted to $\mathfrak{g}(A)'$ and is independent of the extension of Λ. The reader familiar with Verma modules will recognize $L(\Lambda)$ as the top simple quotient of the Verma module $M(\Lambda)$. Let $P_+ \subset \mathfrak{h}'^*$ be the set of Λ that are integrable. This condition ensures that $\Lambda(h_i)$ are non-negative integers. We put $P_{++} = \{\Lambda \in P_+ \mid \Lambda(h_i) > 0; i = 1, \ldots n\}$. The sets P_+ (resp. P_{++}) are known as integral weights (resp. dominant weights).

8. Structure of $G(A)$

Let $G(A)$ denote the group associated to the Lie algebra $\mathfrak{g}(A)'$. We have the associated $G(A)$ modules $(L(\Lambda), \pi_\Lambda)$, $\Lambda \in P_+$, and the adjoint module $(\mathfrak{g}(A)', \mathrm{Ad})$.

Denote by $G_i, H_i, H, U_+, U_-, B_+$ and B_- the subgroups of $G(A)$ corresponding to the integrable subalgebras $\mathfrak{g}_i, \mathbb{C}h_i, \mathfrak{h}', \eta_+, \eta_-, \mathfrak{h}' \oplus \eta_+$ and $\mathfrak{h}' \oplus \eta_-$ resp. of $\mathfrak{g}(A)$. Let us give a more explicit description of these groups.

We have an integrable homomorphism $d\varphi_i : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}(A)$ defined by

$$d\varphi_i \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = ah_i + be_i + cf_i.$$

Let $\varphi_i : SL_2(\mathbb{C}) \to G(A)$ be the corresponding homomorphism of groups. Put

$$H_i(t) = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

The homomorphisms φ_i are injective and one has

$$G_i = \varphi_i(SL_2(\mathbb{C})); \qquad H_i = \{H_i(t) \mid t \in \mathbb{C}^\times\}$$

$$\exp te_i = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ \exp tf_i = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad t \in \mathbb{C}.$$

Furthermore, H is an abelian group equal to the (finite) direct product of the subgroups H_i. We also have $B_\pm = U_\pm \rtimes H$.

The map

$$\bar{r}_i \mapsto \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\exp\, e_i)(\exp\, -f_i)(\exp\, e_i)$$

extends to an injective homomorphism $\phi : \bar{W}(A) \to G(A)$. We will call the image of this map $\bar{W}$ and denote the image of $\bar{r}_i$ again by $\bar{r}_i \in G(A)$. The image of $T_{(2)}$ is a subgroup of H given by $T_{(2)} = \bar{W} \cap H = \{h \in H \mid h^2 = 1\}$. It follows that $T_{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^n$. The group $\bar{W}$ clearly normalizes H. If we denote by N the subgroup of $G(A)$ generated by H and $\bar{W}$, then N acts on $\mathfrak{h}'$ via the adjoint action, H acting trivially. The map $r_i \mapsto \bar{r}_i H$ extends to an isomorphism $W(A) \longrightarrow W = N/H$. The image of r_i is again denoted by $r_i \in W$. The group W is called the **Weyl group** of

$G(A)$ and the r_i its **fundamental reflections**. Put $S = \{r_1, \dots, r_n\}$. The adjoint action of W on $\mathfrak{h}'$ is

$$r_j.h_i = h_i - a_{ij}h_j \quad (i, j = 1, \dots, n).$$

The following are some general facts about the group $G(A)$:

(A) The group $G(A)$ is generated by the (complex) 1-parameter subgroups $\exp te_i$ and $\exp tf_i$, $i = 1, \dots, n$.

(B) $(G(A), B_+, N, S)$ is a Tits system (cf. [**3**]).

(C) The center $C = \{H_1(t_1)\dots H_n(t_n) \mid t_1^{a_{1j}} \dots t_n^{a_{nj}} = 1\}$ for $j = 1, \dots, n$.

(D) $U_\pm$ is generated by the 1-parameter subgroups $\exp t(\mathrm{Ad}\, w.e_i)$, where $w \in \bar{W}$ is such that $(\mathrm{Ad}\, w)e_i \in \eta_\pm$, $i = 1, \dots, n$.

(E) N is the normalizer of H in $G(A)$.

The proofs of these theorems can be found in [**18**]. Since $\mathbb{C}(\mathrm{Ad}\, w)e_i = \mathfrak{g}_{w\alpha_i}$, fact (D) is saying that $U_\pm$ is generated by $\exp(\mathfrak{g}_\alpha), \alpha \in \Delta_\pm^{re}$ (cf. Appendix for definitions). It is not true in general that $\Delta_\pm^{re}$ can be substituted by any proper subset. A formal consequence of (B) is the **Bruhat decomposition**:

$$G(A) = \coprod_{w \in W} B_+ \bar{w} B_+ \tag{1.8.1}$$

and the **Birkhoff decomposition**:

$$G(A) = \coprod_{w \in W} B_- \bar{w} B_+ \tag{1.8.2}$$

where $\bar{w}$ denotes any preimage of w in $\bar{W}$.

9. The unitary form $K(A)$

If A is a generalized Cartan matrix of finite type, then $K(A)$ is the unitary form of the complex semisimple Lie group $G(A)$. Thus the groups $K(A)$ are infinite dimensional analogs of compact Lie groups.

The Kač-Moody algebra $\mathfrak{g}(A)'$ admits an antilinear involution ω_0 determined by demanding that $\omega_0(e_i) = -f_i$, $i = 1, \dots, n$. We can extend ω_0 to all of $\mathfrak{g}(A)'$ by putting $\omega_0[x, y] = [\omega_0(x), \omega_0(y)]$. Since ω_0 preserves the set of locally finite elements, it can be lifted uniquely to an involution of $G(A)$, which we shall also denote by ω_0. Let $K(A)$ be the fixed point set of this involution on $G(A)$.

Provided that A is symmetrizable and indecomposable, the Kač-Moody algebra $\mathfrak{g}(A)'$ carries a (unique up to a constant factor) symmetric , invariant, bilinear form $(.\mid.)$ (cf. [**8**] Ch. 2). The reader should consider this the analog of the Killing form. We remind the reader that a form is called invariant if $([x, y] \mid z) = (x \mid [y, z])$. This form has the property that $(e_i \mid f_i) > 0$. Put $(x \mid y)_0 = -(x \mid \omega_0(y))$. The form $(.\mid.)_0$ is a Hermitian form and the triangular decomposition is orthogonal with respect to it. It can be shown that $(.\mid.)_0$ is positive definite on η_+ and η_-.

The involution ω_0 preserves the subgroups G_i, H_i and H (cf. Section 8); we denote by K_i, T_i and T respectively the corresponding fixed point subgroups. It follows that

$$K_i = \varphi_i(SU_2), \quad \text{and}$$

$$T_i = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : \| t \| = 1.$$

T_i is a maximal torus of K_i and $T = \prod_i T_i$ is a **maximal torus** of $K(A)$. The maximal torus T is a maximal connected abelian subgroup of $K(A)$ and all such subgroups are indeed conjugate (cf. [**13**]). Put

$$H_i^+ = \varphi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R}, \quad t > 0.$$

If we denote by $H_+ = \prod_i H_i^+$; then $H = T \times H^+$.

10. The structure of $K(A)$

Let D (resp. $\overset{\circ}{D}$) $= \{u \in \mathbb{C} : \| u \| \leq 1$ (resp. $\| u \| < 1)\}$ be the unit disc (resp. its interior) and let $S^1 = D - \overset{\circ}{D}$ be the unit circle. Given $u \in D$, put

$$z(u) = \begin{pmatrix} u & (1 - \| u \|^2)^{\frac{1}{2}} \\ -(1 - \| u \|^2)^{\frac{1}{2}} & \bar{u} \end{pmatrix} \in SU_2$$

and put $z_i(u) = \varphi_i(z(u))$. We have $\bar{r}_i = z_i(0) \in K_i$, hence $\bar{W} \subset K(A) \subset G(A)$. Put

$$Y_i = \{z_i(u) \mid u \in \overset{\circ}{D}\} \subset K_i.$$

It can be shown (cf. [**19**] Lemma 43(b)) that

$$B_+ \bar{r}_i B_+ = Y_i B_+ \text{ (uniquely)} \tag{1.10.1}$$

where by uniquely we mean that every element on the right hand side can be expressed uniquely as a product of elements from Y_i and B_+.

Let $w = r_{i_1} \dots r_{i_m}$ be a reduced expression of $w \in W$ and let $\bar{w}$ be its preimage in $\bar{W}$. Using 1.10.1, and a similar argument one shows

$$B_+ \bar{w} B_+ = Y_{i_1} \dots Y_{i_m} B_+ \text{ (uniquely).} \tag{1.10.2}$$

Put $K_w = K(A) \cap B_+ \bar{w} B_+$. Put $Y_w = Y_{i_1} \dots Y_{i_m}$; this is independent of the choice of the reduced expression for w. We have by 1.10.2

$$K_w = Y_w T \text{ (uniquely).} \tag{1.10.3}$$

Put $\bar{K}_w = K_{i_1} \dots K_{i_m} T$; this expression is also independent of the reduced expression of w, as follows from

$$\bar{K}_w = \coprod_{w' \leq w} K_{w'}. \tag{1.10.4}$$

By the Bruhat decomposition, we have

$$K(A) = \coprod_{w \in W} K_w. \tag{1.10.5}$$

In particular, $K(A)$ is generated by K_i, $i = 1, \dots, n$. Finally, we also have the **Iwasawa decomposition**:

$$G(A) = K(A) H_+ U_+ \text{ (uniquely).} \tag{1.10.6}$$

11. A Presentation of the group $K(A)$

$K(A)$ should be thought of as a "real analytic continuation" of the presentation of the group $\bar{W}(A)$. We have the following relations coming from SU_2:

$$(R1) \quad (i)\ z_i(u_1)z_i(u_2) = z_i(u_1u_2) \ \text{ if } \ u_1, u_2 \in S^1,$$

$$(ii)\ z_i(u)z_i(-\bar{u}) = z_i(-1) \ \text{ if } \ u \in \overset{\circ}{D},$$

$$(iii)\ z_i(u_1)z_i(u_2) = z_i(u_1')z_i(u_2') \ \text{ if } \ u_1, u_2 \in \overset{\circ}{D}, u_1 \neq u_2$$

and $u_1 \neq u_2$, for some unique $u_1' \in \overset{\circ}{D}$ and $u_2' \in S^1$.

Furthermore, T_i normalizes K_j and the conjugation is given by

$$(R2) \quad z_i(u_1)z_j(u_2)z_i(u_1)^{-1} = z_j(u_1^{a_{ij}}u_2)z_j(u_1^{-a_{ij}}) \ \text{ if } \ u_1 \in S^1, u_2 \in D.$$

Finally, if $m_{ij} \neq 0$, then $r_ir_jr_i \ldots = r_jr_ir_j \ldots$ (m_{ij} factors on both sides). Hence $Y_iY_jY_i \ldots = Y_jY_iY_j \ldots$ (uniquely). In other words, we have

$$(R3) \quad z_i(u_1)z_j(u_2)z_i(u_3) \ldots = z_j(u_1')z_i(u_2')z_j(u_3') \ldots$$

with m_{ij} factors on each side, if $u_1, u_2, \ldots \in \overset{\circ}{D}$, for some unique $u_1', u_2', \ldots \in \overset{\circ}{D}$.

THEOREM 1.11.1. *The group $K(A)$ is a group on generators $z_i(u)$ defined above, $i = 1, \ldots, n; u \in D$, with defining relations (R1), (R2) and (R3).*

12. The topology of $K(A)$

All the subgroups that we have seen so far are closed. The bijection provided by by the Iwasawa decomposition 1.10.6 is a homeomorphism. Furthermore, H_+ and U_+ are contractible. Thus as in the finite-dimensional case, $G(A)$ is homotopically equivalent to $K(A)$.

The topology on $K(A)$ can be described as follows. Given $w \in W$ with reduced expression $w = r_{i_1} \ldots r_{i_m}$, one can define a map

$$(SU_2)^m \times T \longrightarrow K(A) \qquad (k_1, \ldots, k_m, t) \mapsto \varphi_{i_1}(k_1) \ldots \varphi_{i_m}(k_m)t.$$

The image of this map is $\bar{K}_w$, and we take the quotient topology on it. This topology is independent of the choice of the reduced expression and makes $\bar{K}_w$ a connected Hausdorff compact topological space. Then a subset F of $K(A)$ is closed if and only if $F \cap \bar{K}_w$ is closed in $\bar{K}_w$ for all $w \in W$. It follows that $\bar{K}_w$ is the closure of K_w and that $\bar{K}_{w'} \leq \bar{K}_w$ if and only if $w' \leq w$. Thus, as a topological space, $K(A)$ is a colimit with respect to the Bruhat order of the compact spaces $\bar{K}_w$. Of course, $K(A)$ is also a linear colimit of compact spaces with respect to the length function

$$K(A) = \varinjlim_k \bigcup_{l(w)=k} \bar{K}_w.$$

13. The Flag Variety and CW-Decompositions

The most natural way to study the topology of $K(A)$ is to consider the fibration

$$\pi : K(A) \longrightarrow K(A)/T.$$

The topological space $\mathfrak{F}(A) = K(A)/T$ is called the **flag variety** of the group $K(A)$ and of $G(A)$. Put $C_w = \pi(Y_w)$. Then by 1.10.3 and 1.10.5 we get a cellular decomposition

$$\mathfrak{F}(A) = \coprod_{w\in W} C_w.$$

To show that this is a CW-complex, one has to only construct attaching maps. For that, given $w \in W$, choose a reduced expression $w = r_{i_1} \dots r_{i_m}$ and define a map from the disc $D^s = D \times D \dots \times D$

$$\alpha_w : D^s \longrightarrow \mathfrak{F}(A) \quad \alpha_w(u_1, \dots , u_s) = z_{i_1}(u_1) \dots z_{i_s}(u_s) \mod T.$$

This gives a homeomorphism of $\overset{\circ}{D}{}^s$ onto Y_w by 1.10.2. Since $\bar{K}_w$ is the closure of K_w, by 1.10.4 we have

$$\bar{C}_w = \coprod_{z'\leq w} C_{w'} \tag{1.13.1}$$

where $\bar{C}_w$ is the closure of C_w. It is clear from the defining relations of $K(A)$ that

$$\alpha_w(D^{k-1} \times S^1 \times D^{s-k}) \subset \bar{C}_{w'}$$

where w' is obtained from w by dropping r_{i_k}. Thus we get a CW-decomposition for $\mathfrak{F}(A)$.

Since the dimensions of the cell $C_w = 2l(w)$, there are no cells of odd dimensions. Thus $H_*(\mathfrak{F}(A); \mathbb{Z})$ and $H^*(\mathfrak{F}(A); \mathbb{Z})$ are free $\mathbb{Z}$-modules on generators of degree $2l(w), w \in W$. Putting $W(q) = \sum_{w\in W} q^{l(w)}$, we notice that the Poincaré series for homology and cohomology of $\mathfrak{F}(A)$ over any ring is $W(q^2)$.

As in the finite-dimensional case, $\mathfrak{F}(A)$ can be given a natural structure of a complex projective variety. For this note that, by the Iwasawa decomposition, we have a homeomorphism $\mathfrak{F}(A) \xrightarrow{\cong} G(A)/B_+$. But $G(A)/B_+$ can be identified with the orbit space $G.v_\Lambda$ (cf. Section 7) in the projective space $\mathbb{P}L(\Lambda)$ for $\Lambda \in P_{++}$. This is a closed subvariety of $\mathbb{P}L(\Lambda)$. One can give a definition that is independent of the choice of Λ.

As a result, the $\bar{C}_w$ become finite-dimensional projective varieties, called **Schubert varieties**, and $\mathfrak{F}(A)$ is their inductive limit with respect to the Bruhat order (or with respect to the length function).

REMARK 1.13.2. If we take a standard cellular decomposition of T, then 1.10.3 and 1.10.5 together give us a cellular decomposition of $K(A)$. This may not be a CW-complex since the cells may be attached in the wrong order. This, however, tells us that $K(A)$ does have the homotopy type of a CW-complex since we can always move the cells up to homotopy to get them in the right order.

14. Cohomology

We would like to calculate the ring structure in cohomology for these spaces. In order to do that we need to introduce some terminology.

Let $\check{\mathbb{Q}} = \sum_i \mathbb{Z}h_i$ and let $P = \{\lambda \in \mathfrak{h}'^* \mid \lambda(h_i) \in \mathbb{Z}, i = 1, \dots , n\}$ be the dual lattice. Let $S(P) = \oplus_{j\geq 0} S^j(P)$ be the symmetric algebra over the lattice P, and let $S(P)^+ = \oplus_{j>0} S^j(P)$ denote the augmentation ideal. Given a field $\mathbb{F}$, we denote

$S(P)_{\mathbb{F}} = \mathbb{F} \otimes_{\mathbb{Z}} S(P)$. We now proceed to define the **characteristic homomorphism** $\psi : S(P) \longrightarrow H^*(\mathfrak{F}(A);\mathbb{Z})$ as follows. Given $\lambda \in P$, we have the corresponding character of T and the associated line bundle ζ_λ on $\mathfrak{F}(A)$. Put $\psi(\lambda) \in H^2(\mathfrak{F}(A);\mathbb{Z})$ equal to the first Chern class of ζ_λ and extend ψ by multiplicativity to the whole of $S(P)$. Denote by $\psi_{\mathbb{F}}$ the extension of ψ by linearity to $S(P)_{\mathbb{F}}$. Let us denote the image of $\psi_{\mathbb{F}}$ by $R_{\mathbb{F}}$. Similarly, let $I_{\mathbb{F}}$ denote the kernel of $\psi_{\mathbb{F}}$.

PROPOSITION 1.14.1. *Let $\mathbb{F}$ be a field. Then $H^*(\mathfrak{F}(A);\mathbb{F})$ is a free module over $R_{\mathbb{F}}$. Furthermore, any minimal system of homogeneous generators of the ideal $I_{\mathbb{F}}$ is a regular sequence.*

Let $CH(G(A),\mathbb{F})$ denote the quotient (graded) algebra of $H^*(\mathfrak{F}(A);\mathbb{F})$ by the ideal generated by $R_{\mathbb{F}}$; this is called the **Chow algebra** of $G(A)$ over $\mathbb{F}$. It shall follow from the next theorem that $CH(G(A),\mathbb{F}) = \pi^*(H^*(\mathfrak{F}(A);\mathbb{F}))$. Denote the degrees of the elements of a minimal system of homogeneous generators of the ideal $I_{\mathbb{F}}$ by $d_1,\dots,d_s (s \le n)$. These degrees are well-defined; we will call them the **degrees of basic generators of** $I_{\mathbb{F}}$.

THEOREM 1.14.2. *Let $\mathbb{F}$ be a field. Then:*

(A) $CH(G(A),\mathbb{Q})$ is a polynomial algebra on (in general an infinite number of) homogeneous generators. The Poincaré series of the algebra $CH(G(A),\mathbb{F})$ can be written as $W(q^2)(1-q^2)^n/\prod_1^s(1-q^{2d_i})$. The (graded) algebra $H^*(K/T;\mathbb{Q})$ is (non-canonically) isomorphic to the tensor product of $R_{\mathbb{Q}}$ and $CH(G(A),\mathbb{Q})$.

(B) The Serre spectral sequence in cohomology $E_r(K(A);\mathbb{F})$ for the principal fibration $\pi : K(A) \to \mathfrak{F}(A)$ collapses at $r = 3$.

(C) π^* induces an injective homomorphism of $CH(G(A),\mathbb{F})$ into $H^*(K(A);\mathbb{F})$ and into $E_\infty(K(A);\mathbb{F})$, the image being a Hopf subalgebra of $H^*(K(A);\mathbb{F})$.

(D) The algebra $E_\infty(K(A);\mathbb{F})$ is isomorphic to a tensor product of $CH(G(A),\mathbb{F})$ and the cohomology algebra of the Kozul complex $(\Lambda(P)\otimes R_{\mathbb{F}}, d)$ where the differentials are given by $d(\lambda\otimes u) = \psi(\lambda) \smile u$. The latter algebra is an exterior algebra on homogeneous generators of degree $2d_1-1,\dots,2d_s-1$. The Poincaré series of $H^*(K(A);\mathbb{F})$ is equal to the product of the Poincaré series of $CH(G(A),\mathbb{F})$ and the polynomial $\prod_1^s(1+q^{2d_i-1})$.

Notice that (B) does not say that the spectral sequence $E_r(K(A);\mathbb{Z})$ collapses at $r = 3$. However, we shall see an example in a moment where this does indeed happen. The classical theorems on the cohomology of compact Lie groups and their flag manifolds can be seen as corollaries of this theorem (cf. [**7**]). We also state another corollary for arbitrary $K(A)$:

COROLLARY 1.14.3. *K(A) is a connected simply connected topological group and $H^2(K(A);\mathbb{Z}) = 0$.*

EXAMPLE 1.14.4. *Consider the rank 2 case where the Cartan matrix is given by $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ with $ab \ge 4$. Since $n = 2$, the spectral sequence $E_r(K(A);\mathbb{Z})$ collapses at $r = 3$ for trivial reasons and one can compute the cohomology explicitly to get the additive structure of $H^*(K(A);\mathbb{Z})$. This computation is done in detail in chapter 3 and the main results are as follows:*

Define sequences of integers c_j, d_j for $j \in \mathbb{Z}$ by the following recurrence formula:

$$c_0 = d_0 = 0,\ c_1 = d_1 = 1,\ c_{j+2} = ad_{j+1} - c_j,\ d_{j+2} = bc_{j+1} - d_j.$$

Let g_k be the greatest common divisor of c_k and d_k. Then

$$H^{2j}(K(A); \mathbb{Z}) \cong H^{2j+3}(K(A); \mathbb{Z}) \cong \mathbb{Z}/g_j\mathbb{Z}.$$

Notice that $g_j = j$ if $a = b = 2$. It can be shown using fact (C) in section 8 that the center of $K(A)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times 2\mathbb{Z}/(ab-4)\mathbb{Z}$ if $a = b = 0 \bmod 2$ and $\mathbb{Z}/(ab-4)\mathbb{Z}$ otherwise. Note that if $ab = 4$, then we define $\mathbb{Z}/(ab-4)\mathbb{Z}$ to be the circle S^1.

15. The Affine Case

A Kač-Moody algebra $\mathfrak{g}(A)$ is finite dimensional if and only if A is of finite type (i.e. all principal minors of A are positive). The class of these algebras coincides with the class of finite-dimensional semisimple Lie algebras. The associated group $G(A)$ is the Lie group of $\mathbb{C}$-points of the connected simply-connected algebraic group whose Lie algebra is $\mathfrak{g}(A)$. The group $K(A)$ is the compact form of $G(A)$. H is the Cartan subgroup of $G(A)$, B_+ and B_- are "opposite" Borel subgroups, etc. In this case most results are well known.

Let us discuss in more detail the case when the matrix A is of affine type, i.e. all proper principal minors of A are positive, but $\det A = 0$ (A is then automatically indecomposable and symmetrizable). An example of such a matrix is the extended Cartan matrix of a simple finite-dimensional Lie algebra (which we describe in a moment). All affine Cartan matrices can essentially be obtained in this fashion and have rank one less than their size. Let us now describe the construction of the extended Cartan matrix.

Let $\mathfrak{g}(\mathring{A})$ be a simple finite-dimensional Lie algebra corresponding to the Cartan matrix $\mathring{A}$ of finite type. Then by the **dual algebra** $\mathfrak{g}({}^t\mathring{A})$ we shall mean the simple finite-dimensional Lie algebra corresponding to the Cartan matrix ${}^t\mathring{A}$ of finite type given by the transpose of $\mathring{A}$. Let $(\mathfrak{h}, \Pi, \check{\Pi})$ be a realization for $\mathring{A}$, then it is easy to see that $(\mathfrak{h}^*, \check{\Pi}, \Pi)$ is a realization for ${}^t\mathring{A}$. Let $\theta = a_1\alpha_1 + \ldots + a_n\alpha_n$ be a highest root of the simple finite-dimensional Lie algebra $\mathfrak{g}(\mathring{A})$, and $\check{\theta} = \check{a}_1\check{\alpha}_1 + \ldots + \check{a}_n\check{\alpha}_n$ be the highest short root of the Lie algebra $\mathfrak{g}({}^t\mathring{A})$. Put $a_{00} = 2$, $a_{i0} = -(a_n a_{i1} + \ldots + a_n a_{in})$, $a_{0j} = -(\check{a}_1 a_{1j} + \ldots + \check{a}_n a_{nj})$. Then $A = (a_{ij})_0^n$ is called the **extended Cartan matrix**.

The Kač-Moody algebra corresponding to the extended Cartan matrix is nothing other than the central extension $\tilde{\mathfrak{g}}(\mathring{A})_{\mathbb{L}}$ (cf. Section 3), where we abbreviate the ring of Laurent series $\mathbb{C}[z, z^{-1}]$ by $\mathbb{L}$. The isomorphism $\mathfrak{g}(A)' \to \tilde{\mathfrak{g}}(\mathring{A})_{\mathbb{L}}$ is given by

$$e_i \mapsto 1 \otimes e_i^\circ, \quad f_i \mapsto 1 \otimes f_i^\circ,\ i = 1, \ldots, n$$

$$e_0 \mapsto z \otimes e_{-\theta}, \quad f_0 \mapsto z^{-1} \otimes e_\theta$$

where θ is the highest root of $\mathfrak{g}(\mathring{A})$, e_θ and $e_{-\theta}$ are the corresponding normalized root vectors (normalized so that $(e_\theta \mid e_{-\theta}) = \frac{2}{(\theta|\theta)}$) and finally e_i° and f_i°, $i = 1, \ldots, n$ are the Chevalley generators of $\mathfrak{g}(\mathring{A})$.

Now recalling the definitions from Section 3, we observe that $\Omega^1_{\mathbb{L}} = \mathbb{C}\frac{dz}{z} + d\mathbb{L}$, hence $\frac{dz}{z}$ is a basis of $\Omega^1_{\mathbb{L}}/d\mathbb{L}$. We have

$$[e_0, f_0] = \frac{2}{(\theta \mid \theta)}\frac{dz}{z} + 1 \otimes [e_{-\theta}, e_\theta]$$

where the bracket on the left is in $\tilde{\mathfrak{g}}(\mathring{A})_{\mathbb{L}}$ while that on the right is in $\mathfrak{g}(\mathring{A})$. It is a general fact from Lie theory that for any root α and a normalized root vector e_α, $[e_\alpha, e_{-\alpha}] = \check{\alpha}$. Consequently, we find that

$$h_0 = [e_0, f_0] = \frac{2}{(\theta \mid \theta)}\frac{dz}{z} - 1 \otimes \check{\theta}.$$

16. Central extension of the free loop group

If G° is the (complex) simple Lie group associated to the Lie algebra $\mathfrak{g}(\mathring{A})$, then it is not difficult to check that the group associated to the integrable Lie algebra $\mathfrak{g}(\mathring{A})_{\mathbb{L}}$ is the group of polynomial maps $\mathbb{C}^\times \to G^\circ$, denoted by $G^\circ_{\mathbb{L}}$, which is known to be homotopy equivalent to the free loop group ΛG°. Corresponding to the exact sequence 1.3.1, we have an exact sequence

$$1 \longrightarrow \mathbb{C}^\times \xrightarrow{\mu} G(A) \xrightarrow{\tau} G^\circ_{\mathbb{L}} \longrightarrow 1. \tag{1.16.1}$$

This sequence is a sequence of topological groups where $\mathbb{C}^\times$ has the metric topology and $G^\circ_{\mathbb{L}}$ is topologized as the colimit of the finite-dimensional subspaces given by $G^{\circ,m}_{\mathbb{L}} = \{f : f(z) = \sum_{-m}^m A_i z^i\}$ where we have fixed an embedding $G^\circ \hookrightarrow GL_n(\mathbb{C})$. This topology is called the **box topology**. Notice that there is a canonical embedding of $G^\circ \hookrightarrow G^\circ_{\mathbb{L}}$ via constant loops. So we can pull back the above exact sequence to G°. Since there is a canonical map of Lie algebras between $\mathfrak{g}(\mathring{A})$ and $\mathfrak{g}(A)'$, we have a section of this pullback sequence and it splits. Let $\psi : G^\circ \hookrightarrow G(A)$ be this section. It is easy to see that the maps $\varphi_i : SL_2(\mathbb{C}) \to G(A)$ and $\psi \circ \varphi_i^\circ : SL_2(\mathbb{C}) \to G^\circ \hookrightarrow G(A)$ are equal for $i = 1, \ldots, n$ (cf. Section 8). Consequently, via the embedding ψ we have

$$G_i = G_i^\circ,\ H_i = H_i^\circ,\ \bar{r}_i = \bar{r}_i^\circ,\ \ i = 1, \ldots, n$$

where the subgroups H_i(resp. H_i°), G_i(resp. G_i°) and elements $\bar{r}_i$ (resp. $\bar{r}_i^\circ$) were defined in Section 8. Furthermore, we also have the map $\varphi_0 : SL_2(\mathbb{C}) \to G(A)$. Since $\check{\theta} = \check{a}_1 h_1 + \ldots + \check{a}_n h_n$, it follows that $\check{a}_0 h_0 + \ldots + \check{a}_n h_n$ spans the center of $\mathfrak{g}(A)'$ where $\check{a}_0 = 1$. Consequently, the map μ is given by

$$\mu(t) = \prod_0^n H_i(t^{\check{a}_i})$$

and the center of $G(A)$ is $C = \mu(\mathbb{C}^\times) \times C^\circ$, where C° is the center of G°.

Let $\tilde{G}^\circ_{\mathbb{L}} = \{f \in G^\circ_{\mathbb{L}} : f(1) = e\}$ (where $e \in G^\circ$ is the identity) be the based loops. Then G° acts on $\tilde{G}^\circ_{\mathbb{L}}$ via conjugation and we have an isomorphism

$$\tilde{G}^\circ_{\mathbb{L}} \rtimes G^\circ \xrightarrow{\cong} G^\circ_{\mathbb{L}} \qquad (f, g) \mapsto fg.$$

Using the obvious notation, this equation restricts to an isomorphism

$$\tilde{H}^\circ_{\mathbb{L}} \rtimes N^\circ \xrightarrow{\cong} N^\circ_{\mathbb{L}}.$$

Now if we define $M = \sum \mathbb{Z} h_i^\circ$ to be the inverse root lattice of $\mathfrak{g}(\mathring{A})$, we have an isomorphism $M \to \tilde{H}_{\mathbb{L}}^\circ$ given by $h_i^\circ \mapsto H_i^\circ(z)$, $i = 1, \dots, n$. Thus we get a subgroup $M \rtimes \bar{W}^\circ$ of $G_{\mathbb{L}}^\circ$. Over this subgroup, equation 1.16.1 restricts to

$$(1.16.2) \qquad 1 \longrightarrow \{\pm 1\} \longrightarrow \bar{W} \longrightarrow M \rtimes \bar{W}^\circ \longrightarrow 1$$

which splits over $\bar{W}^\circ$ but over M gives the non-split exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow L \longrightarrow M \longrightarrow 1.$$

Of course, $W(A) = M \rtimes W(\mathring{A})$. Consider the triangular decomposition of $\mathfrak{g}(A)'$ given by $\mathfrak{g}(A)' = \eta_- \oplus \mathfrak{h} \oplus \eta_+$, where

$$\eta_- = (z^{-1}\mathbb{C}[z^{-1}] \otimes (\eta_+^\circ + \mathfrak{h}^\circ)) + \mathbb{C}[z^{-1}] \otimes \eta_-^\circ$$

$$\eta_+ = (z\mathbb{C}[z] \otimes (\eta_-^\circ + \mathfrak{h}^\circ)) + \mathbb{C}[z] \otimes \eta_+^\circ.$$

Clearly, the triangular decomposition of $\mathfrak{g}(\mathring{A})_{\mathbb{L}} = \eta_- \oplus \mathfrak{h}^\circ \oplus \eta_+$ differs from that of $\mathfrak{g}(A)'$ only in its Cartan subalgebra. Let $\tilde{U}_\pm$ be the subgroups of $G_{\mathbb{L}}^\circ$ associated to the integrable subalgebras $\eta_\pm$ respectively. Then

$$\tilde{U}_+ = \{a(z) \in G_{\mathbb{C}[z]}^\circ \mid a(0) \in U_+^\circ\}$$

$$\tilde{U}_- = \{a(z^{-1}) \in G_{\mathbb{C}[z^{-1}]}^\circ \mid a(\infty) \in U_-^\circ\}.$$

The exact sequence 1.16.1 splits over $\tilde{U}_\pm$ canonically but not uniquely. Let us define in a similar fashion

$$\tilde{B}_+ = \{a(z) \in G_{\mathbb{C}[z]}^\circ \mid a(0) \in B_+^\circ\}$$

$$\tilde{B}_- = \{a(z^{-1}) \in G_{\mathbb{C}[z^{-1}]}^\circ \mid a(\infty) \in B_-^\circ\}.$$

Notice that the image of the Borel subgroups B_+ and B_- of $G(A)$, under the map τ, are the subgroups $\tilde{B}_+$ and $\tilde{B}_-$ respectively. Once we make the observation that $M \cap G_{\mathbb{C}[z^\pm]}^\circ = e$ and $\bar{W}^\circ \subset G^\circ$, the Bruhat and Birkhoff decompositions 1.8.1 and 1.8.2 along with equation 1.16.2 descend to give us the classical results:

$$G_{\mathbb{L}}^\circ = G_{\mathbb{C}[z]}^\circ M G_{\mathbb{C}[z]}^\circ$$

$$G_{\mathbb{L}}^\circ = G_{\mathbb{C}[z^{-1}]}^\circ M G_{\mathbb{C}[z]}^\circ.$$

17. Connected covers of loop groups

Let us take a look at the unitary form $K(A)$ of $G(A)$. Let ω_0° be the antilinear involution of $\mathfrak{g}(\mathring{A})$ introduced earlier. This involution induces an involution of G° whose fixed point set is the the compact form of G° denoted by K°. Now ω_0° extends to an obvious involution $\omega_{0,\mathbb{L}}$ of $\mathfrak{g}(\mathring{A})_{\mathbb{L}}$ which maps z to z^{-1}. The involution ω_0 of $\mathfrak{g}(A)'$ is compatible with $\omega_{0,\mathbb{L}}$ via the map $d\tau$. The fixed point set of the (unique) lift of $\omega_{0,\mathbb{L}}$ to $G_{\mathbb{L}}^\circ$ is the group of polynomial maps $\mathbb{C}^\times \to G^\circ$ for which the image of the unit circle is contained in K°, these are called polynomial loops on K° and denoted by $K_{\mathbb{L}}^\circ$. It is a well-known result that $K_{\mathbb{L}}^\circ \hookrightarrow \Lambda K^\circ$ is a homotopy equivalence, where ΛK° denotes the space of free loops on K°. If we denote $\tilde{K}_{\mathbb{L}}^\circ = \{f \in K_{\mathbb{L}}^\circ : f(1) = e\}$, then it follows that $\tilde{K}_{\mathbb{L}}^\circ \hookrightarrow \Omega K^\circ$ is a homotopy equivalence, where ΩK° is the space of based loops on K°. The exact sequence 1.16.1 restricts to the following exact sequence:

$$1 \longrightarrow S^1 \xrightarrow{\mu} K(A) \xrightarrow{\tau} K^\circ_{\mathbb{L}} \longrightarrow 1. \tag{1.17.1}$$

As before, we have $\tilde{K}^\circ_{\mathbb{L}} \rtimes K^\circ \xrightarrow{\cong} K^\circ_{\mathbb{L}}$ and the above sequence splits canonically over K°. Let $\psi : K^\circ \to K(A)$ be the section over K°. Denote by $\Omega(K^\circ)\langle 2\rangle$ the group $\tau^{-1}(\tilde{K}^\circ_{\mathbb{L}})$ (the reason for this notation will soon be apparent). Now K° acts via ψ by conjugation on $\Omega(K^\circ)\langle 2\rangle$ and we have

$$\Omega(K^\circ)\langle 2\rangle \rtimes K^\circ \xrightarrow{\cong} K(A).$$

Since $K(A)$ is simply-connected, it follows easily from the fibration sequence

$$S^1 \longrightarrow \Omega(K^\circ)\langle 2\rangle \longrightarrow \tilde{K}^\circ_{\mathbb{L}}$$

that $\Omega(K^\circ)\langle 2\rangle$ is homotopy equivalent to the two-connected cover of $\Omega(K^\circ)$ (and hence the notation).

We have the following pullback diagram of principal S^1-bundles:

$$\begin{array}{ccccc}
S^1 & \xrightarrow{=} & S^1 & \xrightarrow{=} & S^1 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega(K^\circ)\langle 2\rangle & \longrightarrow & K(A) & \longrightarrow & ES^1 \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{K}^\circ_{\mathbb{L}} & \longrightarrow & K^\circ_{\mathbb{L}} & \xrightarrow{\zeta} & \mathbb{C}P^\infty
\end{array}$$

where ζ is the classifying map. ζ generates $H^2(K^\circ_{\mathbb{L}};\mathbb{Z}) \cong \mathbb{Z}$ and hence $K(A)$ is called the **universal central extension** of ΛK°.

Notice that the map τ in equation 1.17.1 maps the maximal torus of $K(A)$ to the maximal torus $T \subset K^\circ \subset K^\circ_{\mathbb{L}}$. Hence

$$\mathfrak{F}(A) \xrightarrow{\cong} \frac{\tilde{K}^\circ_{\mathbb{L}} \rtimes K^\circ}{1 \rtimes T} \xrightarrow{\kappa} \Omega(K^\circ) \times (K^\circ/T)$$

where the map κ is a homotopy equivalence. Thus $\Omega(K^\circ)$ can be viewed as a partial flag variety. Results of Section 14 provide important information on the cohomology of the spaces $\Omega(K^\circ)$ and $\Omega(K^\circ)\langle 2\rangle$.

Now consider the multiplication map

$$\tilde{K}^\circ_{\mathbb{L}} \times G^\circ_{\mathbb{C}[z]} \longrightarrow G^\circ_{\mathbb{L}}.$$

It is a classical result that this map is a homeomorphism. This can easily be seen from the Iwasawa decomposition of $G(A)$, 1.10.6, using the exact sequences 1.16.1 and 1.17.1.

18. The Indefinite Case

There is a third and final type of Cartan matrix. It is characterized by the property that there is a vector α with positive integer entries such that $A\alpha$ has negative entries. While the Cartan matrices of finite and affine types have been classified and are symmetrizable, the matrices of indefinite type remain unclassified. However, one of its subclasses, the class of hyperbolic type was classified just recently. The groups associated to Cartan matrices of Indefinite type remain a mystery and no example of such a group is known to have a natural description.

19. Appendix

In this section we describe an alternate construction of the groups associated to the Kač-Moody algebras. Let us build up some background before we begin.

Let $\mathfrak{g}(A)$ be a Kač-Moody algebra associated to the generalized Cartan matrix A. We have the **root space decomposition** $\mathfrak{g}(A) = \oplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha$, where the individual root spaces are defined as $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. Put $Q = \sum \mathbb{Z}\alpha_i$, $Q_+ = \sum \mathbb{Z}_+\alpha_i$ (where $\mathbb{Z}_+ = \{0, 1, \dots\}$), and define a partial order on $\mathfrak{h}^*$ by : $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$. A **root** (resp. **positive root**) is an element of $\Delta = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$ (resp. $\Delta_+ = \Delta \cap Q_+$). We have

$$\mathfrak{h} = \mathfrak{g}_0 \ , \qquad \eta_\pm = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{\pm\alpha}.$$

For $\alpha = \sum k_i\alpha_i \in \Delta$, we write the **height** $\operatorname{ht}\alpha = \sum k_i$. The Weyl group W preserves the root system Δ. A **real root** is an element of $\Delta^{re} = \{w(\alpha) \mid w \in W, \alpha \in \Pi\}$. If $\alpha \in \Delta^{re}$, then $\dim\mathfrak{g}_\alpha = 1$. A root which is not a real root is called an **imaginary root**. The set of imaginary roots is denoted by Δ^{im}. If the Cartan matrix is symmetrizable then a root α is imaginary if and only if $(\alpha \mid \alpha) \leq 0$. For $\alpha \in \Delta^{re}$, write $\alpha = w(\alpha_i)$ for some $w \in W$; then we define $r_\alpha = wr_iw^{-1}$ which depends only on α.

Let G^* be the free product of the additive groups $\mathfrak{g}_\alpha, \alpha \in \Delta^{re}$, with canonical inclusions $\iota_\alpha : \mathfrak{g}_\alpha \to G^*$. For any integrable $\mathfrak{g}(A)'$-module (V, π), define a homomorphism $\pi^* : G^* \to \operatorname{Aut}(V)$ by $\pi^*(\iota_\alpha(y)) = \exp\pi(y)$. Let N^* be the intersections of all $\operatorname{Ker}(\pi^*)$, put $G = G^*/N^*$, and let $q : G^* \to G$ be the canonical homomorphism. For $y \in \mathfrak{g}_\alpha(\alpha \in \Delta^{re})$, put $\exp(y) = q(\iota_\alpha(y))$, so that the group $U_\alpha = \exp(\mathfrak{g}_\alpha)$ is an additive 1-parameter subgroup of G. The topology on G is defined exactly as in Section 4 by taking X to be the set Δ^{re}. This construction has been taken from [**11**].

CHAPTER 2

Cohomology of Kač-Moody Groups

1. Introduction

Let K be the unitary form of a Kač-Moody group G. One has the principal fibration

$$T \longrightarrow K \xrightarrow{\pi} K/T \tag{2.1.1}$$

where $T \subset K$ is a maximal torus. In [**9**] a proof was outlined to show that for coefficients in any field $\mathbb{F}$, the Serre spectral sequence for this fibration collapses at E_3. The details of this proof were to appear in [**10**] but that paper never made it in print. From the collapse of the spectral sequence one obtains an extension of graded algebras

$$1 \longrightarrow \pi^* H(K/T;\mathbb{F}) \longrightarrow H(K;\mathbb{F}) \longrightarrow E \longrightarrow 1$$

where E is an exterior algebra on finitely many generators of odd homogeneous degree. It was also stated in [**9**] that the above extension is in fact an extension of Hopf-algebras.

In this chapter we provide proofs of the above claims as suggested in [**9**]. The proofs given here are by no means original and have been directly influenced by the lectures given by D. Peterson at M.I.T in the Spring of '97. The author would like to take this opportunity to thank D. Peterson and V. Kač for their help during the numerous conversations he had with them.

Our proofs make essential use of operators A_i that act on the cohomology of the flag variety K/T. These operators were introduced by Kač-Peterson in [**7**] and Kostant-Kumar in [**14**]. Following D. Peterson, we interpret them as integration along the fiber for suitable fibrations. Using this definition we derive an inductive formula for the action of the Steenrod algebra on $H^*(K/T;\mathbb{F}_p)$. We show as a consequence that $H^*(K/T;\mathbb{F}_p)$ is locally finite as a module over the mod p Steenrod algebra. The same holds for $H^*(K;\mathbb{F}_p)$.

2. Background

Let $\mathfrak{g}$ be the Kač-Moody algebra associated to a generalized Cartan matrix A of size n. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots. For each root α_i one has a simple reflection r_i of $\mathfrak{h}^*$ and the elements r_i generate the Weyl group W. The Weyl group admits a partial order known as the Bruhat order. Let G be the Kač-Moody group associated to the Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Let B denote its Borel subgroup, and $H \subset B$ its maximal torus. Define P_i to be the minimal parabolic subgroup associated to the root α_i. Denote by K the unitary form of G. Then $T = K \cap H = K \cap B$ is the maximal torus of K and has rank n.

Let us now recall some properties of the homogeneous space $K/T = G/B$. The space G/B admits a decomposition

$$G/B = \coprod_{w \in W} BwB/B$$

known as the Bruhat decomposition of 1.8.1. The quotients BwB/B have the natural structure of finite-dimensional affine spaces $\mathbb{C}^{l(w)}$. This decomposition is indeed a CW-decomposition. For any $w \in W$, the subcomplex

$$X_w = \coprod_{w' \leq w} Bw'B/B$$

is called the Schubert variety corresponding to w, and it has the natural structure of a projective variety. G/B is a direct limit of such Schubert varieties.

The above discussion also applies to the spaces G/P_i in which case the cells, BwP_i/P_i, are indexed by the cosets W/W_i where $W_i = \{1, r_i\}$. Let $w = r_{i_1} \dots r_{i_k}$ be some (not necessarily reduced) expression of $w \in W$. Consider the variety

$$P_{i_1,\dots,i_k} = P_{i_1} \times_B P_{i_2} \times_B \dots \times_B P_{i_k}/B.$$

We have a morphism of varieties

$$j : P_{i_1,\dots,i_k} \hookrightarrow G \times_B G \dots \times_B G/B \xrightarrow{\mu} G/B$$

where μ is the map induced by group multiplication. The image of j under this morphism is the Schubert variety X_w. Notice that $P_{i_1,\dots,i_k}$ is a smooth projective variety over $\mathbb{C}$ and hence has a canonical orientation class. In the case when the expression $w = r_{i_1} \dots r_{i_k}$ is reduced, the map j provides a desingularization of X_w and the pair $(P_{i_1,\dots,i_k}, j)$ is known as the Bott-Samelson resolution. These resolutions are the key ingredient in showing that the Bruhat decomposition is a CW-decomposition. In particular, for a reduced expression $w = r_{i_1} \dots r_{i_k}$, the map

$$j : P_{i_1,\dots,i_k} \longrightarrow X_w$$

is an isomorphism on the top-dimensional integral cohomology group $H^{2k}(\quad; \mathbb{Z})$.

From the CW-decomposition it is clear that $H^*(G/B; \mathbb{Z})$ is a free $\mathbb{Z}$-module on canonical generators δ^w indexed by the Weyl group. These will henceforth be referred to as the Schubert basis. Under the identification

$$H^*(G/B; R) = H^*(G/B; \mathbb{Z}) \otimes R,$$

the elements $\delta^w_R = \delta^w \otimes 1$ form a basis of $H^*(G/B; R)$ for any coefficient ring R.

For a Kač-Moody group of rank n we have

Theorem 2.2.1. *For any sequence $i_1, \dots, i_k$ with $i_j \leq n$, consider the map*

$$j : P_{i_1,\dots,i_k} \longrightarrow G/B.$$

If $w \in W$ is any element of length k, then

$$j^*(\delta^w) = 0 \qquad \text{if} \qquad w \neq r_{i_1} \dots r_{i_k}$$

$$j^*(\delta^w) = \sigma \qquad \text{if} \qquad w = r_{i_1} \dots r_{i_k}$$

where σ refers to the canonical element of $H^{2k}(P_{i_1,\dots,i_k}; \mathbb{Z})$ dual to the orientation class. We shall refer to σ as the top class of $P_{i_1,\dots,i_k}$.

Proof. If $r_{i_1} \dots r_{i_k}$ is not in reduced form then the map j factors through a lower skeleton and so in this case the theorem is obvious. If $w = r_{i_1} \dots r_{i_k}$ is in reduced form then j factors as the Bott-Samelson resolution of X_w followed by the inclusion of X_w into G/B and so the result follows from the comments made earlier. $\square$

3. The operators A_i^R and D_i^R

Before constructing these operators, we define what we mean by the term integration along the fiber. Let

$$F \longrightarrow E \xrightarrow{\pi} B$$

be a fibration with the property that $H^i(F; R) = 0$ for $i > n$. Assume that the action of $\pi_1(B)$ on $H^n(F; R)$ is trivial. Let

$$\tau : H^n(F; R) \longrightarrow R$$

be a fixed homomorphism of R-modules. We shall refer to such data as an oriented fibration. We define a homomorphism of R-modules

$$\int_{\pi,\tau} : H^*(E; R) \longrightarrow H^{*-n}(B; R)$$

as follows. Consider the Serre spectral sequence for the above fibration. By assumption $E_\infty^{p,q} = 0$ if $q > n$. Thus we have a map

$$H^*(E; R) \twoheadrightarrow E_\infty^{*-n,n} \subset E_2^{*-n,n} = H^{*-n}(B; H^n(F; R)).$$

Now $\int_{\pi,\tau}$ is defined as the composite

$$H^*(E; R) \longrightarrow H^{*-n}(B; H^n(F; R)) \xrightarrow{\tau_*} H^{*-n}(B; R).$$

We leave it to the reader to verify the following properties of this map:

(A) Consider a map of fibrations

$$\begin{array}{ccc} F_1 & \xrightarrow{f} & F_2 \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{g} & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{h} & B_2. \end{array}$$

Assume that $H^i(F_1; R) = H^i(F_2; R) = 0$ for $i > n$. Let

$$\tau_2 : H^n(F_2; R) \longrightarrow R$$

be a fixed homomorphism of R-modules and define

$$\tau_1 : H^n(F_1; R) \longrightarrow R$$

to be the homomorphism $f^*(\tau_2)$. Then we have the following equality:

$$h^* \int_{\pi_2,\tau_2} (z) = \int_{\pi_1,\tau_1} g^*(z). \tag{2.3.1}$$

Using 2.3.1 we also have

$$g^*\pi_2^* \int_{\pi_2,\tau_2} (z) = \pi_1^* \int_{\pi_1,\tau_1} g^*(z). \tag{2.3.2}$$

We shall refer to property (A) as naturality with respect to maps of fibrations.

(B) $\int_{\pi,\tau}$ is a $H^*(B;R)$-module homomorphism, where $H^*(E;R)$ is seen as a $H^*(B;R)$-module via π.

(C) For an oriented spherical fibration with fiber S^n and

$$\tau \in \mathrm{Hom}(H^n(S^n;R),R) = H_n(S^n;R)$$

being the fundamental class, the homomorphism $\int_{\pi,\tau}$ is given by the composite

$$H^*(E;R) \xrightarrow{\delta} H^{*+1}(D,E;R) \xrightarrow{t} H^{*-n}(B;R)$$

where $D \longrightarrow B$ is the disc-fibration corresponding to the spherical fibration E (the space D is defined as the cofiber of the map π), δ is the boundary homomorphism and t stands for the Thom isomorphism.

We now proceed to construct operators A_i^R and D_i^R. Consider the oriented $\mathbb{C}P^1$-bundle

$$P_i/B \longrightarrow G/B \xrightarrow{\pi} G/P_i \tag{2.3.3}$$

where $\tau \in \mathrm{Hom}(H^2(P_i/B;R),R) = H_2(P_i/B;\mathbb{Z}) \otimes R$ is the standard orientation class of $\mathbb{C}P^1$. Define

$$A_i^R : H^*(G/B;R) \longrightarrow H^{*-2}(G/B;R)$$

by

$$A_i^R(z) = \pi^* \int_{\pi,\tau} (z). \tag{2.3.4}$$

We shall denote $A_i^{\mathbb{Z}}$ by A_i. By naturality it is clear that A_i^R corresponds to $A_i \otimes id$ under the identification $H^*(G/B;R) = H^*(G/B;\mathbb{Z}) \otimes R$.

Now let $H_i \subset P_i$ be the Levi subgroup $H_i = P_i \cap K$. Note that $H_i/T = P_i/B$ and that the fibration

$$H_i/T \longrightarrow K/T \xrightarrow{\pi} K/H_i$$

is equivalent to the fibration 2.3.3. One also has the oriented fibration

$$H_i/T \longrightarrow BT \xrightarrow{p} BH_i.$$

Hence we can define operators

$$D_i^R : H^*(BT;R) \longrightarrow H^{*-2}(BT;R)$$

by

$$D_i^R(z) = p^* \int_{p,\tau} (z). \tag{2.3.5}$$

Denote $D_i^{\mathbb{Z}}$ by D_i. As before, D_i^R corresponds to $D_i \otimes id$ under the obvious identification. The operators D_i^R are known as Bernstein-Gelfand-Gelfand (BGG) operators since they were first introduced in the compact Lie group setting in [**5**]. These operators have a rather nice expression:

$$D_i^R(z) = (z - r_i(z))\,/\alpha_i.$$

This equality can be verified using [**4**](example VI.5) and [**2**](thm 4.3) but we shall not be needing it for the purposes of the present document. In the above equality $r_i(z)$ denotes the action of W on $H^*(BT;R)$ induced by the action of W on T. The element $\alpha_i \in H^2(BT;R)$ refers to the Euler class of the complex line bundle

$$ET \times_{\rho_i} \mathbb{C} \longrightarrow BT$$

where

$$\rho_i : T \longrightarrow U(1) \tag{2.3.6}$$

is the representation obtained by exponentiating the root α_i.

Now consider the following pullback diagram of oriented $\mathbb{C}P^1$-bundles:

$$\begin{array}{ccc} H_i/T & \xrightarrow{=} & H_i/T \\ \downarrow & & \downarrow \\ K/T & \xrightarrow{\psi} & BT \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ K/H_i & \longrightarrow & BH_i \end{array}$$

where ψ is the map classifying the principal T-fibration

$$K \longrightarrow K/T.$$

Using the definitions 2.3.4 and 2.3.5 along with property 2.3.2, we have

$$\psi^* D_i^R(z) = A_i^R(\psi^* z). \tag{2.3.7}$$

Henceforth we shall refer to ψ^* as the characteristic homomorphism.

4. Properties of the operators A_i^R and D_i^R

Perhaps the most useful property of the operators A_i^R is their diagonal form with respect to the Schubert basis.

THEOREM 2.4.1. *Let δ_R^w be an element in the Schubert basis. Then*

$$A_i^R(\delta_R^w) = \delta_R^{wr_i} \qquad \textit{if} \qquad l(wr_i) < l(w)$$

$$A_i^R(\delta_R^w) = 0 \qquad \textit{if} \qquad l(wr_i) > l(w).$$

Proof. Since $\delta^w_R = \delta^w \otimes 1$ and $A^R_i = A_i \otimes id$ under the identification

$$H^*(G/B;R) = H^*(G/B;\mathbb{Z}) \otimes R,$$

it is sufficient to prove the theorem for the case $R = \mathbb{Z}$. Assume that $l(w) = n+1$. For $v \in W$ and $l(v) = n$, let $v = r_{i_1} \dots r_{i_n}$ be a reduced expression. Recall the map

$$j : P_{i_1,\dots,i_n} \longrightarrow G/B.$$

By theorem 2.2.1, we need to show that

$$j^*A_i(\delta^w) = 0 \qquad \text{if} \qquad v \neq wr_i$$

$$j^*A_i(\delta^w) = \sigma \qquad \text{if} \qquad v = wr_i$$

where σ denotes the top class of $P_{i_1,\dots,i_n}$. Now we have the pullback diagram of oriented $\mathbb{C}P^1$-bundles

$$\begin{array}{ccccc}
 & & P_i/B & \xrightarrow{=} & P_i/B \\
 & & \downarrow & & \downarrow \\
P_{i_1,\dots,i_n} & \xrightarrow{s} & P_{i_1,\dots,i_n,i} & \xrightarrow{j'} & G/B \\
 & & \downarrow \rho & & \downarrow \pi \\
 & & P_{i_1,\dots,i_n,i}/P_i & \longrightarrow & G/P_i
\end{array}$$

where s is the morphism

$$P_{i_1,\dots,i_n} = P_{i_1} \times_B \dots \times_B P_{i_n} \times_B B/B \longrightarrow P_{i_1,\dots,i_n,i}$$

induced by the inclusion $B \hookrightarrow P_i$ in the last factor. It is clear that $j's = j$, thus

$$j^*A_i(\delta^w) = s^*j'^*A_i(\delta^w) = s^*\rho^* \int_{\rho,\tau} j'^*(\delta^w) \tag{2.4.2}$$

the second equality following from property 2.3.2 applied to the above pullback diagram. Let σ_1 be the top class of $P_{i_1,\dots,i_n,i}$. By theorem 2.2.1, we know that

$$j'^*(\delta^w) = 0 \qquad \text{if} \qquad vr_i \neq w$$

$$j'^*(\delta^w) = \sigma_1 \qquad \text{if} \qquad vr_i = w.$$

This fact along with 2.4.2 will complete the proof once we have verified that

$$s^*\rho^* \int_{\rho,\tau} \sigma_1 = \sigma. \tag{2.4.3}$$

Now notice that the morphism $\rho \circ s$ is an equivalence. Thus we can identify $P_{i_1,\dots,i_n,i}/P_i$ with $P_{i_1,\dots,i_n}$ via $\rho \circ s$. Under this identification 2.4.3 reduces to showing

$$\int_{\rho,\tau} \sigma_1 = \sigma$$

which follows from an easy spectral sequence argument and is left to the reader. $\square$

Our next objective is to describe how the operators $A_i^{\mathbb{F}_p}$ relate to the mod p Steenrod algebra. First we introduce some terminology. Recall the homomorphism of 2.3.6 obtained by exponentiating the root α_i. By $\alpha_i \in H^2(K/T;R) = H^2(G/B;R)$ we shall mean the Euler class of the complex line bundle

$$K \times_{\rho_i} \mathbb{C} \longrightarrow K/T. \tag{2.4.4}$$

Now fix a prime p and let $\mathbb{F}_p$ denote the field of p elements. Let $\mathcal{P}^i$ denote the Steenrod power operation ($\mathcal{P}^i = Sq^i$ if $p = 2$). Let

$$\mathcal{P} = \sum_i \mathcal{P}^i$$

denote the total Steenrod operation. We have

THEOREM 2.4.5.

$$A_i^{\mathbb{F}_p}\mathcal{P}(z) = (1 + {\alpha_i}^{p-1})\mathcal{P}A_i^{\mathbb{F}_p}(z).$$

PROOF. Let $E \xrightarrow{\pi} B$ denote the fibration

$$K/T \xrightarrow{\pi} K/H_i$$

where H_i was defined to be the Levi subgroup of the parabolic P_i. Let $D \longrightarrow B$ be the disc-bundle associated to the spherical fibration π. For $z \in H^*(E; \mathbb{F}_p)$, we are interested in showing

$$A_i^{\mathbb{F}_p}\mathcal{P}(z) = (1 + {\alpha_i}^{p-1})\mathcal{P}A_i^{\mathbb{F}_p}(z).$$

Using 2.3.4 this is equivalent to

$$\pi^* \int_{\pi,\tau} \mathcal{P}(z) = (1 + {\alpha_i}^{p-1})\mathcal{P}\pi^* \int_{\pi,\tau} (z).$$

Using property (C) for integration along the fiber, we need to show

$$\pi^* t\mathcal{P}\delta(z) = (1 + {\alpha_i}^{p-1})\mathcal{P}\pi^* t\delta(z)$$

where δ was the boundary homomorphism and t the Thom isomorphism between the appropriate cohomology groups. Hence it is sufficient to show that

$$\pi^* t\mathcal{P}(x) = (1 + {\alpha_i}^{p-1})\mathcal{P}\pi^* t(x) \tag{2.4.6}$$

where $x \in H^*(D, E; \mathbb{F}_p)$ is any element. Using the Cartan formula, 2.4.6 reduces to showing

$$\pi^* t\mathcal{P}(u) = (1 + {\alpha_i}^{p-1}) \tag{2.4.7}$$

where $u \in H^3(D, E; \mathbb{F}_p)$ is the Thom class. Now consider the pullback diagram of oriented $\mathbb{C}P^1$-bundles

$$\begin{array}{ccc} K \times_T (H_i/T) & \xrightarrow{\mu} & K/T \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} \\ K/T & \xrightarrow{\pi} & K/H_i \end{array}$$

where $\mu[k, hT] = khT$. Let $\bar{E} \xrightarrow{\bar{\pi}} \bar{B}$ denote the fibration

$$K \times_T (H_i/T) \xrightarrow{\bar{\pi}} K/T$$

and let $\bar{D} \longrightarrow \bar{B}$ denote the associated disc-bundle. The above pullback diagram along with the naturality of the Thom isomorphism with respect to pullbacks reduces

2.4.7 to

$$\mathcal{P}(\bar{u}) = (1 + {\alpha_i}^{p-1})\bar{u}$$

where $\bar{u} \in H^3(\bar{D}, \bar{E}; \mathbb{F}_p)$ is the Thom class. We now proceed to establish the above equality.

It is easy to verify that the left action of T on H_i/T is the compactification of the action of T on $\mathbb{C}$ via the homomorphism ρ_i. It follows that the Thom space $\bar{D}/\bar{E}$ is equivalent to the Thom space of the vector bundle $L_i \oplus \mathbf{1}$, where L_i is the complex line bundle 2.4.4 and $\mathbf{1}$ denotes the trivial 1-dimensional real line bundle. Hence $\bar{u} = \Sigma v$, where v is a Thom class for L_i. Thus

$$\mathcal{P}(\bar{u}) = \Sigma\mathcal{P}(v) = \Sigma(v + v^p) = \Sigma(v + {\alpha_i}^{p-1}v) = (1 + {\alpha_i}^{p-1})\bar{u}$$

which is what we wanted to establish. □

Before we proceed with the properties of the operators A_i^R and D_i^R, we make a few comments relating the operators A_i to those introduced by Kostant-Kumar. In [**14**] Kostant-Kumar introduce operators acting on $H^*(K/T; \mathbb{C})$ and satisfying the formulas described in theorem 2.4.1 for the case $R = \mathbb{C}$. In particular, their operators preserve the subgroup $H^*(K/T; \mathbb{Z}) \subset H^*(K/T; \mathbb{C})$ and the restriction of these operators to $H^*(K/T; \mathbb{Z})$ agrees with the operators A_i. We are therefore justified in using results from [**14**] to prove the next property:

Proposition 2.4.8.

$$A_i^R(u \smile v) = A_i^R(u) \smile r_i(v) + u \smile A_i^R(v)$$

where $r_i(v)$ refers to the action of W on $H^*(K/T; R)$ induced from its action on K/T.

Proof. As usual it is sufficient to prove the proposition for the case $R = \mathbb{Z}$. But this case is exactly lemma (4.26) in [**14**]. □

Proposition 2.4.9. *Let $u \in H^*(K/T; R)$ be any element such that $A_i^R(u) = 0$ for all i, then $u \in H^0(K/T; R)$.*

Proof. The proof is an easy consequence of theorem 2.4.1 and is left to the reader. □

5. The cohomology of K

We are now ready to examine $H^*(K; \mathbb{F})$, where $\mathbb{F}$ stands for any field. The program has been outlined in [**9**] and we merely have to fill in the details. We begin with a definition

Definition 2.5.1. *Let R be a ring. Let*

$$I_R = \{u \in H^*(BT; R)^+ \mid D_{i_1}^R \dots D_{i_k}^R(u) \in H^*(BT; R)^+ \ \ \forall \ \ i_1, \dots, i_k\}$$

where $H^*(BT; R)^+$ refers to the homogeneous elements of positive degree.

Let ψ denote the map

$$\psi : K/T \longrightarrow BT$$

classifying the principal T-bundle given by 2.1.1. The induced homomorphism ψ^* is called the characteristic homomorphism. The next few theorems are generalization to the infinite case of the corresponding theorems in [**9**].

THEOREM 2.5.2.

$$I_R = \mathrm{Ker}\{\psi^* : H^*(BT; R) \longrightarrow H^*(K/T; R)\}.$$

PROOF. By proposition 2.4.9 it follows that $z \in H^*(K/T; R)^+$ is nonzero if and only if there exists a sequence $i_1, \dots, i_k$ with the property

$$0 \neq A^R_{i_1} \dots A^R_{i_k}(z) \in H^0(K/T; R).$$

Now observe that ψ^* is an isomorphism in degree 0, hence the statement

$$D^R_{i_1} \dots D^R_{i_k}(u) \in H^*(BT; R)^+ \ \ \forall \ \ i_1, \dots, i_k$$

is equivalent to

$$\psi^* D^R_{i_1} \dots D^R_{i_k}(u) \in H^*(K/T; R)^+ \ \ \forall \ \ i_1, \dots, i_k.$$

By 2.3.7, the latter statement is equivalent to

$$A^R_{i_1} \dots A^R_{i_k} \psi^*(u) \in H^*(K/T; R)^+ \ \ \forall \ \ i_1, \dots, i_k$$

which, as we observed, happens if and only if $\psi^*(u) = 0$. □

The ideal I_R is better know as the 'ideal of generalized invariants' (cf. [**12**] and [**17**]).

THEOREM 2.5.3. *Let $\mathbb{F}$ be a field. Then the ideal $I_{\mathbb{F}}$ is generated by a regular sequence of length $\leq n$.*

The proof of this theorem can be taken word-for-word from [**12**] (see also [**17**] and [**9**]) and we omit it for the sake of brevity.

NOTE 2.5.4. If $\mathrm{char}(\mathbb{F}) > 0$, then the subgroup of $GL_n(\mathbb{F})$ generated by the reflections r_i is finite and hence the proof of the above theorem given in [**17**] shows that the length of the regular sequence generating $I_{\mathbb{F}}$ is exactly n.

Now let $R_{\mathbb{F}}$ denote the image of the characteristic homomorphism

$$R_{\mathbb{F}} = \mathrm{Im}\{\psi^* : H^*(BT; \mathbb{F}) \longrightarrow H^*(K/T; \mathbb{F})\}.$$

It follows from 2.5.4 that if $\mathrm{char}(\mathbb{F}) > 0$ then $R_{\mathbb{F}}$ is a finite-dimensional vector space.

THEOREM 2.5.5. *Let $\mathbb{F}$ be any field, then $H^*(K/T; \mathbb{F})$ is a free $R_{\mathbb{F}}$-module.*

PROOF. The proof proceeds exactly as in [**9**]. Choose a homogeneous basis $\{\bar{y}_\alpha\}$ for the $\mathbb{F}$-vector space $\mathbb{F} \otimes_{R_{\mathbb{F}}} H^*(K/T; \mathbb{F})$. Lift the elements $\bar{y}_\alpha$ to homogeneous elements $y_\alpha \in H^*(K/T; \mathbb{F})$. It is clear that the elements y_α generate $H^*(K/T; \mathbb{F})$ as a $R_{\mathbb{F}}$-module. Now assume we have a homogeneous relation of minimal degree

$$\sum_i a_i y_i = 0 \qquad \text{with} \qquad 0 \neq a_i \in R_{\mathbb{F}}.$$

If the above relation has homogeneous degree m, then we can order the elements y_i appearing above so that

$$m \geq deg(y_1) \geq deg(y_2) \geq \dots \qquad .$$

By the choice of the basis y_α we have

$$0 < deg(a_1) \leq deg(a_2) \leq \dots \qquad .$$

From proposition 2.4.9 we can find a j such that $A_j^{\mathbb{F}}(a_1) \neq 0$. Applying $A_j^{\mathbb{F}}$ to the above relation and using proposition 2.4.8 we have

$$\sum_i A_j^{\mathbb{F}}(a_i) y_i + \sum_i r_j(a_i) A_j^{\mathbb{F}}(y_i) = 0. \tag{2.5.6}$$

Now we can write

$$A_j^{\mathbb{F}}(y_i) = \sum_k c_{ik} y_k$$

where c_{ik} are homogeneous elements of $R_{\mathbb{F}}$ with $c_{i1} = 0$ for dimensional reasons. Substituting this back into 2.5.6 we obtain another nontrivial homogeneous relation of lesser degree, which is a contradiction by the assumption of minimality of m. □

With theorems 2.5.2,2.5.3 and 2.5.5 we have all the details in place to prove the following theorem claimed by V. Kač in [**9**]:

THEOREM 2.5.7. *Let K be the unitary form of a Kač-Moody group of rank n with maximal torus T. Then the Serre spectral sequence for the fibration*

$$T \longrightarrow K \xrightarrow{\pi} K/T$$

converging to $H^*(K;\mathbb{F})$ collapses at E_3 with

$$E_3 = E_\infty = \mathbb{F} \otimes_{R_{\mathbb{F}}} H^*(K/T;\mathbb{F}) \otimes_{\mathbb{F}} E(x_1, \dots, x_l)$$

where $E(x_1, \dots, x_l)$ stands for an exterior algebra on generators x_i of degree $2d_i - 1$. Here $d_1, \dots, d_l$ are the degrees of the regular sequence generating $I_{\mathbb{F}}$ and so $l \leq n$ with $l = n$ if $char(\mathbb{F}) > 0$.

PROOF. The proof of theorem 2.5.7 is identical to the one given for compact Lie groups in [**9**]. Consider the Kozul resolution for the field $\mathbb{F}$ over the algebra $H^*(BT;\mathbb{F})$ given by

$$H^*(T;\mathbb{F}) \otimes H^*(BT;\mathbb{F}) \longrightarrow \mathbb{F}$$

with the differentials behaving as in the universal fibration over BT. From this it is easy to see that the E_3 term of the Serre spectral sequence for the fibration in the statement of the theorem is computing

$$\mathrm{Tor}_{H^*(BT;\mathbb{F})}(\mathbb{F}; H^*(K/T;\mathbb{F})).$$

Now notice that

$$\mathrm{Tor}^0_{H^*(BT;\mathbb{F})}(\mathbb{F}; H^*(K/T;\mathbb{F})) = \mathbb{F} \otimes_{R_{\mathbb{F}}} H^*(K/T;\mathbb{F}).$$

Using 2.5.3, it is easy to establish the equality

$$\mathrm{Tor}_{H^*(BT;\mathbb{F})}(\mathbb{F}; R_{\mathbb{F}}) = E(x_1, \dots, x_l)$$

where $E(x_1, \dots, x_l)$ stands for an exterior algebra on generators x_i of total homogeneous degree $2d_i - 1$. Here $d_1, \dots, d_l$ are the degrees of the regular sequence generating $I_{\mathbb{F}}$. It follows from above that we have a map of algebras

$$\mathbb{F} \otimes_{R_{\mathbb{F}}} H^*(K/T;\mathbb{F}) \otimes_{\mathbb{F}} E(x_1, \dots, x_l) \longrightarrow \mathrm{Tor}_{H^*(BT;\mathbb{F})}(\mathbb{F}; H^*(K/T;\mathbb{F}).$$

By 2.5.5, it is clear that this map is an isomorphism of algebras. For dimensional reasons, the generators of this algebra are permanent cycles in the Serre spectral sequence and hence the spectral sequence collapses at E_3.

See [**16**] for an alternate proof using the Eilenberg-Moore spectral sequence. □

Remark 2.5.8. The horizontal edge homomorphism of the spectral sequence gives us an exact sequence of graded algebras

$$H^*(BT;\mathbb{F}) \xrightarrow{\psi^*} H^*(K/T;\mathbb{F}) \xrightarrow{\pi^*} H^*(K;\mathbb{F}) \longrightarrow E(x_1,\dots,x_l) \longrightarrow 1.$$

Remark 2.5.9. For a field $\mathbb{F}$ of characteristic 0 and a Kač-Moody group with an indecomposable nonsingular Cartan matrix which is not of finite type, it can be shown that the exterior algebra in 2.5.8 is an exterior algebra on a single generator in dimension 3.

6. The Hopf-algebra structure of $H^*(K;\mathbb{F})$

By remark 2.5.8, we have an extension of graded algebras

$$1 \longrightarrow \pi^*H(K/T;\mathbb{F}) \longrightarrow H^*(K;\mathbb{F}) \longrightarrow E(x_1,\dots,x_l) \longrightarrow 1.$$

The purpose of this section is to show that this is an extension of Hopf-algebras. Clearly, it is sufficient to show that $\pi^*H(K/T;\mathbb{F})$ is a sub Hopf-algebra of $H^*(K;\mathbb{F})$. Since the quotient algebra, $E(x_1,\dots,x_l)$, is an exterior algebra on generators of odd homogeneous degree, the general theory of Hopf-algebras allows us to assume that these generators are primitive.

The basic idea in this section is to introduce a co-algebra structure on $H^*(K/T;\mathbb{F})$ and show that the map

$$\pi^* : H^*(K/T;\mathbb{F}) \longrightarrow H^*(K;\mathbb{F})$$

is a map of co-algebras. Here and for the rest of this section $\mathbb{F}$ denotes a field. We shall use R to denote a ring.

Define a co-algebra structure on $H^*(K/T;R)$ by

$$\Delta(\delta^w_R) = \sum_{uv=w,l(u)+l(v)=l(w)} \delta^u_R \otimes \delta^v_R. \tag{2.6.1}$$

With this co-algebra structure on $H^*(K/T;\mathbb{F})$, we have the following crucial theorem of D. Peterson:

Theorem 2.6.2.

$$\pi^* : H^*(K/T;\mathbb{F}) \longrightarrow H^*(K;\mathbb{F})$$

is a map of co-algebras.

The proof of 2.6.2 proceeds in various steps. The first step is to understand the cohomology of the space $K \times_T (K/T)$ considered as a module over the cohomology of K/T via the map

$$\rho : K \times_T (K/T) \longrightarrow K/T, \qquad [k,k'T] \mapsto kT.$$

Denote the space $K \times_T (K/T)$ by E and consider the fibration

$$K/T \longrightarrow E \xrightarrow{\rho} K/T.$$

Given $w \in W$, we have the Schubert variety X_w. Since X_w is a left T-space, we have a commutative triangle

$$\begin{array}{ccc} K \times_T (X_w) & \xrightarrow{j_w} & K \times_T (K/T) \\ \downarrow{\scriptstyle \rho_w} & \swarrow{\scriptstyle \rho} & \\ K/T & & \end{array}$$

where j_w is induced by the inclusion of X_w into K/T. Denote the space $K \times_T (X_w)$ by E_w and let

$$\tau_w : H^{2l(w)}(X_w; R) \longrightarrow R$$

be the unique homomorphism with the property $\tau_w(\delta_R^w) = 1$. We define operators

$$\psi_w : H^*(E; R) \longrightarrow H^{*-2l(w)}(K/T; R)$$

as the composite

$$H^*(E; R) \xrightarrow{j_w^*} H^*(E_w; R) \xrightarrow{\int_{\rho_w, \tau_w}} H^{*-2l(w)}(K/T; R).$$

It is immediate from property (B) for the integration along the fiber that ψ_w are $H^*(K/T; R)$-module homomorphisms. We have

Proposition 2.6.3. *There is a unique basis* $\{\sigma_R^w\}_{w \in W}$ *of* $H^*(E; R)$ *as an* $H^*(K/T; R)$*-module that satisfies*

$$\psi_w(\sigma_R^v) = 1 \quad \text{if} \quad v = w$$

$$\psi_w(\sigma_R^v) = 0 \quad \text{if} \quad v \neq w.$$

Proof. Consider the Serre spectral sequence for the fibration

$$K/T \longrightarrow E \longrightarrow K/T.$$

The E_2 term for this spectral sequence is a free module over $H^*(K/T; R) = E_2^{*,0}$. Since $E_2 = E_\infty$, one readily verifies that $H^*(E; R)$ is isomorphic to the E_2 term as $H^*(K/T; R)$-modules. Hence it suffices to prove the analogous statement for the E_2 term. It is straight forward to see that in this case the elements

$$1 \otimes \delta_R^w \in E_2^{0,2l(w)}$$

provide this unique basis. ∎

One can similarly consider the cohomology of the space $Y = ET \times_T (K/T)$ as a module over the cohomology of BT. As before we have operators

$$\phi_w : H^*(Y; R) \longrightarrow H^{*-2l(w)}(BT; R).$$

The following proposition is proved in exactly the same style as proposition 2.6.3.

Proposition 2.6.4. *There is a unique basis* $\{\zeta_R^w\}_{w \in W}$ *of* $H^*(Y; R)$ *as an* $H^*(BT; R)$*-module that satisfies*

$$\phi_w(\zeta_R^v) = 1 \quad \text{if} \quad v = w$$

$$\phi_w(\zeta_R^v) = 0 \quad \text{if} \quad v \neq w.$$

Now one has a pullback diagram

$$\begin{array}{ccc} K \times_T (K/T) & \xrightarrow{\bar{\psi}} & ET \times_T (K/T) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ K/T & \xrightarrow{\psi} & BT \end{array}$$

where we take EK as a model for ET and the map $\bar{\psi}$ is induced by the inclusion $K \hookrightarrow EK$ as the fiber of the universal fibration

$$K \hookrightarrow EK \longrightarrow BK.$$

¿From the naturality of the integration along the fiber with respect to maps of fibrations, we derive

$$\bar{\psi}^*(\zeta_R^w) = \sigma_R^w. \tag{2.6.5}$$

Now assume that we have two reduced expressions $r_{i_1} \dots r_{i_k}$ and $r_{i_{k+1}} \dots r_{i_n}$. Let $K_{i_1,\dots,i_k}$ denote the right T-space $H_{i_1} \times_T \dots \times_T H_{i_k}$. Consider the map

$$f : K_{i_1,\dots,i_k} \times_T (P_{i_{k+1},\dots,i_n}) \longrightarrow K \times_T (K/T)$$

induced by group multiplication. For the element $\delta_R^u \smile \sigma_R^v \in H^*(K \times_T (K/T); R)$ with $l(u) + l(v) = n$, we have

THEOREM 2.6.6.

$$f^*(\delta_R^u \smile \sigma_R^v) = \sigma \quad \textit{if} \quad u = r_{i_1} \dots r_{i_k} \quad \textit{and} \quad v = r_{i_{k+1}} \dots r_{i_n}$$

$$f^*(\delta_R^u \smile \sigma_R^v) = 0 \quad \textit{if} \quad u \neq r_{i_1} \dots r_{i_k} \quad \textit{or} \quad v \neq r_{i_{k+1}} \dots r_{i_n}$$

where σ refers to the top class of $P_{i_1,\dots,i_n} = K_{i_1,\dots,i_k} \times_T (P_{i_{k+1},\dots,i_n})$.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} K_{i_1,\dots,i_k} \times_T (P_{i_{k+1},\dots,i_n}) & \xrightarrow{f} & K \times_T (K/T) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \rho} \\ P_{i_1,\dots,i_k} & \xrightarrow{j} & K/T \end{array}$$

where p is the projection onto the first factor. Now let τ_p be the top class of $P_{i_{k+1},\dots,i_n}$ and let w be the element $r_{i_{k+1}} \dots r_{i_n}$. Then it follows from the naturality of integration along the fiber with respect to maps of fibrations that

$$j^*\psi_w(x) = \int_{p,\tau_p} f^*(x) \tag{2.6.7}$$

for all $x \in H^*(E; R)$. It is easy to verify that

$$\int_{p,\tau_p} \sigma = \sigma_1$$

where σ is the top class of $P_{i_1,\dots,i_n}$ and σ_1 is the top class of $P_{i_1,\dots,i_k}$. Applying 2.6.7 to the element $x = \delta_R^u \smile \sigma_R^v$ we have

$$\int_{p,\tau_p} f^*(\delta_R^u \smile \sigma_R^v) = \sigma_1 \quad \text{if} \quad u = r_{i_1}\dots r_{i_k} \quad \text{and} \quad v = r_{i_{k+1}}\dots r_{i_n}$$

$$\int_{p,\tau_p} f^*(\delta_R^u \smile \sigma_R^v) = 0 \quad \text{if} \quad u \neq r_{i_1}\dots r_{i_k} \quad \text{or} \quad v \neq r_{i_{k+1}}\dots r_{i_n}.$$

The result follows. □

THEOREM 2.6.8. *Let*

$$\mu : K \times_T (K/T) \longrightarrow K/T, \quad \mu[k, k'T] = kk'T$$

be the action map. Then

$$\mu^*\delta_R^w = \sum_{uv=w, l(u)+l(v)=l(w)} \delta_R^u \smile \sigma_R^v$$

PROOF. Using the terminology of theorem 2.6.6, we need to show that

$$(\mu \circ f)^*\delta_R^w = \sigma.$$

But notice that the map $\mu \circ f$ is none other than the map

$$j : P_{i_1,\dots,i_n} \longrightarrow K/T$$

and so the result follows by theorem 2.2.1. □

Let

$$\pi' : K \times K \longrightarrow K \times_T (K/T)$$

be the map induced by the projection π and let

$$\pi_1 : K \times K \longrightarrow K,$$

$$\pi_2 : K \times K \longrightarrow K$$

denote the two projection maps onto the left and right factor respectively.

THEOREM 2.6.9. *Let δ_R^w be any element of the Schubert basis. Then*

$$\pi'^*(\delta_R^w) = \pi_1^*\pi^*(\delta_R^w),$$

$$\pi'^*(\sigma_R^w) = \pi_2^*\pi^*(\delta_R^w).$$

PROOF. The first statement follows clearly from the definition of the elements $\delta_R^w \in H^*(K \times_T (K/T); R)$ and the commutativity of the diagram

$$\begin{array}{ccc} K \times K & \xrightarrow{\pi'} & K \times_T (K/T) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \rho} \\ K & \xrightarrow{\pi} & K/T. \end{array}$$

So it remains to show the second statement. Consider the composite

$$\lambda : K \times K \xrightarrow{\pi'} K \times_T (K/T) \xrightarrow{\bar{\psi}} ET \times_T (K/T)$$

where $\bar{\psi}$ was the map from equation 2.6.5. Recall that we take EK as our model for ET and the map $\bar{\psi}$ is induced by the inclusion

$$\iota : K \hookrightarrow EK$$

of the fiber in the universal principal K-fibration. The composite map λ can also be written as the composite

$$\lambda : K \times K \xrightarrow{\iota \times \pi} ET \times (K/T) \longrightarrow ET \times_T (K/T).$$

By 2.6.5 we have

$$\pi'^*(\sigma_R^w) = \pi'^* \bar{\psi}^*(\zeta_R^w) = \lambda^*(\zeta_R^w).$$

Now since ET is contractible, the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} K \times K & \xrightarrow{\iota \times \pi} & ET \times (K/T) & \longrightarrow & ET \times_T (K/T) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} & \nearrow{\scriptstyle \eta} & \\ K & \xrightarrow{\pi} & K/T & & \end{array}$$

where η includes K/T into $ET \times_T (K/T)$ by choosing an arbitrary basepoint in ET. It is clear from the definition of the elements ζ_R^w that $\eta^*(\zeta_R^w) = \delta_R^w$. The result now follows by chasing the diagram. □

We now come to the proof of theorem 2.6.2. Consider the commutative diagram

$$\begin{array}{ccc} K \times K & \xrightarrow{\pi'} & K \times_T (K/T) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ K & \xrightarrow{\pi} & K/T \end{array}$$

where we are using the symbol μ to also denote the group multiplication on K since it shall be clear from the context which map we are referring to. This diagram along with theorems 2.6.8 and 2.6.9 says

$$\mu^* \pi^*(\delta_R^w) = \pi'^* \mu^*(\delta_R^w) = \sum \pi'^*(\delta_R^u) \smile \pi'^*(\sigma_R^v) = \sum \pi_1^* \pi^* \delta_R^u \smile \pi_2^* \pi^* \delta_R^v$$

where the sum is being taken over all distinct pairs u, v with the property $uv = w$ and $l(u) + l(v) = l(w)$. This is exactly the statement of theorem 2.6.2.

Remark 2.6.10. Notice that nowhere in the proof of theorem 2.6.2 have we used the fact that R is a field. Indeed, we notice that $\pi^* H(K/T; R)$ is a Hopf-algebra for any coefficient ring R. Now define $\mathbb{P}$ to be the ideal of $H^*(K/T; \mathbb{Z})$ generated by elements of homogeneous degree 2. It is easy to see that

$$\pi^* H(K/T; \mathbb{Z}) = H^*(K/T; \mathbb{Z})/\mathbb{P}$$

and that

$$\pi^* H(K/T; \mathbb{F}) = \mathbb{F} \otimes_{\mathbb{Z}} \pi^* H(K/T; \mathbb{Z}).$$

Hence the Hopf-algebra $\pi^*H(K/T;\mathbb{Z})$ is an integral lift of the Hopf-algebras $\pi^*H(K/T;\mathbb{F})$.

7. $H^*(K/T;\mathbb{F}_p)$ as a module over the Steenrod algebra

In this section we show that $H^*(K/T;\mathbb{F}_p)$ is locally finite as a module over the mod p Steenrod algebra, $\mathcal{A}_p$.

Recall that the image, $R_{\mathbb{F}_p}$, of the characteristic homomorphism is a finite-dimensional $\mathbb{F}_p$-vectorspace. By equation 2.3.7, $R_{\mathbb{F}_p}$ is closed under the action of the operators $A_i^{\mathbb{F}_p}$. It is also clear that $R_{\mathbb{F}_p}$ is an $\mathcal{A}_p$-submodule of $H^*(K/T;\mathbb{F}_p)$. Note that $R_{\mathbb{F}_p}$ contains all the elements of degree 2.

Let $2m$ be the highest degree among the homogeneous elements of $R_{\mathbb{F}_p}$. For any homogeneous subset $X \subseteq H^*(K/T;\mathbb{F}_p)$, define $d(X)$ to be the biggest degree among the homogeneous elements in X. Hence $d(X) \leq \infty$. Given a homogeneous element $z \in H^*(K/T;\mathbb{F}_p)$, let $M(z)$ be the smallest $R_{\mathbb{F}_p}$-submodule of $H^*(K/T;\mathbb{F}_p)$ containing z such that $M(z)$ is also a $\mathcal{A}_p$-submodule of $H^*(K/T;\mathbb{F}_p)$. We define $d(z)$ to be $d(M(z))$. We have

THEOREM 2.7.1. *If z is a homogeneous element of degree $2k$ for $k > 0$, then*

$$d(z) \leq 2k(m+1) - 2.$$

PROOF. We proceed by induction on the degree of z. Since $R_{\mathbb{F}_p}$ contains all degree 2 elements, the theorem is true for $k = 1$. Let z be a homogeneous element of degree $2n+2$. Consider an element $x \in M(z)$. Using the Cartan formula we can write x as a finite sum

$$x = \sum_I r_I \mathcal{P}^I(z) \tag{2.7.2}$$

where r_I are elements of $R_{\mathbb{F}_p}$, I ranges over finite sequences of positive integers $i_1, \ldots, i_k$ and $\mathcal{P}^I$ denotes the composite $\mathcal{P}^{i_1} \ldots \mathcal{P}^{i_k}$. Now consider some element of the form $\mathcal{P}^I(z)$. By repeated application of the formula given in theorem 2.4.5, we observe that

$$A_i^{\mathbb{F}_p}\mathcal{P}^I(z) \in M(A_i^{\mathbb{F}_p} z) \quad \forall i.$$

Hence by the inductive hypothesis any element of the form $\mathcal{P}^I(z)$ can have degree at most $2n(m+1) - 2 + 2 = 2n(m+1)$. Using equation 2.7.2 we observe that

$$d(z) = d(M(z)) \leq 2m + 2n(m+1) = 2(n+1)(m+1) - 2$$

and so we are done by induction. □

As an immediate corollary we have

COROLLARY 2.7.3. *$H^*(K/T;\mathbb{F}_p)$ and $H^*(K;\mathbb{F}_p)$ are locally finite as a modules over $\mathcal{A}_p$.*

CHAPTER 3

Cohomology of rank 2 Kač-Moody groups

1. Introduction

Let $G(a,b)$ be the Kač-Moody group corresponding to the generalized Cartan matrix of the form

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$

Let $K(a,b)$ be the unitary form of $G(a,b)$. If $ab < 4$, then $K(a,b)$ is a simply-connected compact Lie group of rank 2 with the corresponding Cartan matrix. If $ab \geq 4$, then these groups are infinite dimensional and the purpose of this chapter is to compute their cohomology. We shall use the program described in chapter 2 to carry out this computation. The main results of this chapter are as follows:

Define a sequence of integers c_i and d_i by the recurrence

$$c_0 = d_0 = 0, \quad c_1 = d_1 = 1, \quad c_{j+1} = ad_j - c_{j-1}, \quad d_{j+1} = bc_j - d_{j-1}.$$

Let $g_i = (c_i, d_i)$ be the g.c.d. of the pair c_i, d_i. Then the additive structure of $H^*(K(a,b);\mathbb{Z})$ is given by

$$H^{2n}(K(a,b);\mathbb{Z}) = H^{2n+3}(K(a,b);\mathbb{Z}) = \mathbb{Z}/g_n\mathbb{Z}.$$

Now let p be an odd prime and let k be the smallest positive integer such that p divides g_k. Then there is an isomorphism of graded Hopf-algebras

$$H_*(K(a,b);\mathbb{F}_p) = \mathbb{F}_p[x_{2k}] \otimes E(x_3, x_{2k-1}).$$

Moreover if n is the exponent of p in g_k, then there is a Bockstein homomorphism of height n connecting the class x_{2k} with the class x_{2k-1}.

Let $BK(a,b)$ denote the classifying space of $K(a,b)$. Then there is an isomorphism of graded rings

$$H^*(BK(a,b);\mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2k}] \otimes E(x_{2k+1})$$

and a Bockstein homomorphism of height n connects the class x_{2k} with the class x_{2k+1}. The integer k can be made more explicit as follows.

If $ab = 4 \bmod p$, then $k = p$; if p divides a or b but not both, then $k = 2p$; in all other cases k is the multiplicative order of the element r, where r is the root of the polynomial

$$x^2 - (ab-2)x + 1$$

defined over the field of p^2 elements. Note that the above polynomial always splits over the field of p^2 elements. If $\mathbb{F}_p$ is the splitting field of this polynomial, then it is easy to see that k will be a divisor of $p-1$, otherwise k will be a divisor of $p+1$. Moreover, for a fixed prime p, any divisor of $p+1$ or $p-1$ occurs as a k for a suitable pair of integers a and b.

From now on we fix the integers a, b with $ab \geq 4$ and denote the unitary form $K(a,b)$ by K.

To prove the results described above, we shall need the multiplicative structure constants for the cohomology of the flag variety K/T in terms of the Schubert basis δ^w. To compute these constants, we will use the formulas derived in [**14**]. The reader should bear in mind that the Schubert basis, δ^w, used in [**14**] is what we call $\delta^{(w^{-1})}$ and so the formulas in [**14**] have to be suitably altered before being used.

2. The computation of $H^*(K/T;\mathbb{Z})$

Let $A = (a_{ij})$ be a generalized, Cartan matrix. The corresponding Weyl group W acts on the Cartan subalgebra via

$$w_j(h_i) = h_i - a_{ij}h_j.$$

If we consider Cartan matrices given by

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}, \qquad ab \geq 4$$

then the Weyl group generators w_1 and w_2 have the following matrix representations on the Cartan subalgebra with basis h_1 and h_2:

$$w_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}.$$

The representation of W on the root-space spanned by the simple roots α_1 and α_2 is generated by the matrices

$$w_1 = \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 1 & 0 \\ b & -1 \end{pmatrix}.$$

The Weyl group is isomorphic to the infinite Dihedral group and consequently every element of W can be uniquely written as $(w_1w_2)^r w_1^s$ where r is any integer and s is defined modulo 2. It is an easy induction argument to show that the representation of W on the root space is given by the matrices

$$(3.2.1) \quad (w_1w_2)^n = \begin{pmatrix} c_{2n+1} & -c_{2n} \\ d_{2n} & -d_{2n-1} \end{pmatrix}, \qquad (w_1w_n)^n w_1 = \begin{pmatrix} -c_{2n+1} & c_{2n+2} \\ -d_{2n} & d_{2n+1} \end{pmatrix}$$

where the integers c_i and d_i are defined as follows:

$$c_0 = d_0 = 0, \quad c_1 = d_1 = 1, \quad c_{j+1} = ad_j - c_{j-1}, \quad d_{j+1} = bc_j - d_{j-1}.$$

The reader should bear in mind that c_i and d_i are defined for *all* integers i by the recurrence. It is easy to check that $c_{-i} = -c_i$ and $d_{-i} = -d_i$.

Similarly, the representation of W on the Cartan subalgebra with basis h_1 and h_2 is given by the matrices

$$(3.2.2) \quad (w_1w_2)^n = \begin{pmatrix} d_{2n+1} & -d_{2n} \\ c_{2n} & -c_{2n-1} \end{pmatrix}, \qquad (w_1w_n)^n w_1 = \begin{pmatrix} -d_{2n+1} & d_{2n+2} \\ -c_{2n} & c_{2n+1} \end{pmatrix}.$$

Now notice that the Weyl group has exactly two elements of length n for $n > 0$. Let δ_n denote the element in the Schubert basis for $H^*(K/T;\mathbb{Z})$ corresponding to the unique element $w \in W$ with the property $l(w) = n$ and $l(ww_1) < l(w)$. Similarly let σ_n denote the other element of degree $2n$ in the Schubert basis corresponding to

$w \in W$ with the property $l(w) = n$ and $l(ww_2) < l(w)$. Define $\sigma_0 = \delta_0 = 1$. We can use the formulas given by equation 3.2.1 along with lemma (5.9) in [**14**] to compute the action of the Weyl group on the Schubert basis:

$$w_1\delta_n = -\delta_n + (d_{n+1} - d_{n-1})\sigma_n, \qquad w_1\sigma_n = \sigma_n \tag{3.2.3}$$

$$w_2\sigma_n = -\sigma_n + (c_{n+1} - c_{n-1})\delta_n, \qquad w_2\delta_n = \delta_n.$$

These formulas can now be used in conjunction with proposition (4.30) in [**14**] to compute the following identities:

$$\delta \smile \delta_n = d_{n+1}\delta_{n+1}, \quad \delta \smile \sigma_n = \delta_{n+1} + d_n\sigma_{n+1} \tag{3.2.4}$$

$$\sigma \smile \sigma_n = c_{n+1}\sigma_{n+1}, \quad \sigma \smile \delta_n = \sigma_{n+1} + c_n\delta_{n+1}$$

where we have denoted the elements δ_1 and σ_1 by δ and σ respectively. The computation of these identities from the formulas in [**14**] is easy but tedious and has been left to the enthusiastic reader.

The study of the multiplicative structure constants is interesting in its own right. Define the *generalized binomial coefficients*[1] $C(n,m)$ and $D(n,m)$ by

$$C(n,m) = \frac{c_{m+n}c_{m+n-1}\dots c_1}{c_nc_{n-1}\dots c_1c_mc_{m-1}\dots c_1}, \qquad D(n,m) = \frac{d_{m+n}d_{m+n-1}\dots d_1}{d_nd_{n-1}\dots d_1d_md_{m-1}\dots d_1}.$$

Notice that when $a = b = 2$, these generalized binomial coefficients reduce to the usual binomial coefficients. It is a priori non-obvious that these generalized binomial coefficients are integers. Using 3.2.4 we can write

$$\delta_n \smile \delta_m = D(n,m)\delta_{m+n}, \quad \sigma_n \smile \sigma_m = C(n,m)\sigma_{m+n}, \tag{3.2.5}$$

from which it becomes clear that $C(n,m)$ and $D(n,m)$ are indeed integers!

Now since $H^*(K/T;\mathbb{Z})$ has no torsion, the formulas 3.2.4 uniquely determine the ring structure in cohomology. Notice that the subalgebra generated by the elements δ_n or by the elements σ_n is reminiscent of a divided polynomial algebra and hence we shall call it a *generalized divided polynomial algebra*. If we work with rational coefficients, a simple verification of Poincaré series yields

$$H^*(K/T;\mathbb{Q}) = \mathbb{Q}[\delta,\sigma]/I \tag{3.2.6}$$

where I is the ideal generated by the quadratic relation

$$a\delta^2 + b\sigma^2 - ab\delta\sigma.$$

3. The Hopf algebra $\pi^*H(K/T;\mathbb{F}_p)$

By remark 2.6.10 from chapter 2 we know that $\pi^*H(K/T;\mathbb{Z})$ is a Hopf algebra, where π is the projection map

$$\pi : K \longrightarrow K/T.$$

This Hopf algebra is an integral lift of Hopf algebras $\pi^*H(K/T;\mathbb{F})$ in the sense that

$$\pi^*H(K/T;\mathbb{F}) = \mathbb{F} \otimes_{\mathbb{Z}} \pi^*H(K/T;\mathbb{Z})$$

where $\mathbb{F}$ is any field.

[1] This terminology is due to H. Miller

Now let $\mathbb{P} \subset H^*(K/T;\mathbb{Z})$ be the ideal generated by elements of homogeneous degree 2. Again by 2.6.10 we see that

$$\pi^* H(K/T;\mathbb{Z}) = H^*(K/T;\mathbb{Z})/\mathbb{P}. \tag{3.3.1}$$

Henceforth we shall use $A_{\mathbb{Z}}$ and $A_{\mathbb{F}}$ to denote the Hopf algebras $\pi^* H(K/T;\mathbb{Z})$ and $\pi^* H(K/T;\mathbb{F})$ respectively.

Let $g_n = (c_n, d_n)$ be the g.c.d. of the pair c_n, d_n. Using equations 3.3.1 and 3.2.4 we see that, in degree $2n$, the evenly-graded algebra $A_{\mathbb{Z}}$ is isomorphic to a cyclic group of order g_n.

$$\pi^* H^{2n}(K/T;\mathbb{Z}) = \mathbb{Z}/g_n\mathbb{Z}. \tag{3.3.2}$$

Both δ_n and σ_n serve as generators of this cyclic group. The co-algebra structure on $A_{\mathbb{Z}}$ was explicitly given in 2.6.1. Using our terminology, this co-algebra structure is given by the following equations:

$$\Delta(\delta_n) = \sum_0^n \delta'_{n-i} \otimes \delta_i \tag{3.3.3}$$

where $\delta'_{n-i} = \delta_{n-i}$ if i is even and $\delta'_{n-i} = \sigma_{n-i}$ if i is odd. Similarly,

$$\Delta(\sigma_n) = \sum_0^n \sigma'_{n-i} \otimes \sigma_i \tag{3.3.4}$$

where $\sigma'_{n-i} = \sigma_{n-i}$ if i is even and $\sigma'_{n-i} = \delta_{n-i}$ if i is odd.

Now fix an odd prime p. Let $\mathbb{F}_p$ denote the field of p elements. Recall that

$$\pi^* H(K/T;\mathbb{F}_p) = A_{\mathbb{F}_p} = \mathbb{F}_p \otimes_{\mathbb{Z}} A_{\mathbb{Z}}$$

and so to understand $A_{\mathbb{F}_p}$ we need to know when the prime p divides an integer of the form g_n.

Let k be the smallest positive integer such that p divides g_k. It will be shown in the Appendix that k always exists and that p divides g_j if and only if k divides j. Hence $A_{\mathbb{F}_p}$ is isomorphic to a cyclic group of order p in dimensions which are multiples of $2k$ and is trivial in all other dimensions. If n is a multiple of k then the elements δ_n and σ_n both serve as generators of $A_{\mathbb{F}_p}$ in dimension $2n$. Now using equations 3.3.3 and 3.3.4 it is easy to see that the dual Hopf-algebra $A^*_{\mathbb{F}_p}$ is isomorphic to a polynomial algebra on a class in dimension $2k$

$$A^*_{\mathbb{F}_p} = \mathbb{F}_p[x_{2k}].$$

It follows that we have an isomorphism of Hopf-algebras

$$\pi^* H(K/T;\mathbb{F}_p) = A_{\mathbb{F}_p} = \Gamma[x_{2k}]. \tag{3.3.5}$$

where Γ stands for a divided polynomial algebra.

4. The additive structure of $H^*(K;\mathbb{Z})$

In this section we wish to establish the additive structure of $H^*(K;\mathbb{Z})$. We shall use this in the next section to understand the Hopf-algebra $H^*(K;\mathbb{F}_p)$.

Consider the Serre spectral sequence for the fibration

$$T \longrightarrow K \xrightarrow{\pi} K/T$$

with the E_2 term given by $H^*(K/T;\mathbb{Z})\otimes H^*(T;\mathbb{Z})$ and converging to $H^*(K;\mathbb{Z})$. Note that

$$E_3^{*,0} = E_\infty^{*,0} = H^*(K/T;\mathbb{Z})/\mathbb{P} = \pi^* H(K/T;\mathbb{Z}) \tag{3.4.1}$$

where $\mathbb{P}$ was defined as the ideal generated by $H^2(K/T;\mathbb{Z})$. Since T has rank 2, this spectral sequence does not support any nontrivial differentials of length bigger than two. It follows that $E_3 = E_\infty$. We now proceed to compute the E_3 term of this spectral sequence.

The E_2 term of this spectral sequence is the cochain complex

$$H^*(K/T;\mathbb{Z})\otimes H^*(T;\mathbb{Z});\quad d_2(x\otimes\alpha) = x\smile\delta,\quad d_2(x\otimes\beta) = x\smile\sigma$$

where $\alpha,\beta\in H^1(T;\mathbb{Z})$ are generators and d_2 is a derivation. Using formulas 3.2.4, it is easy to see that the kernel of the differential

$$d_2 : H^{2n}(K/T;\mathbb{Z})\otimes H^1(T;\mathbb{Z}) \longrightarrow H^{2n+2}(K/T;\mathbb{Z})$$

is the subgroup of all elements of the form

$$(R\delta_n + P\sigma_n)\otimes\alpha + (S\delta_n + Q\sigma_n)\otimes\beta \tag{3.4.2}$$

where P, Q, R and S are integers related via

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} -d_{n+1} & -c_n \\ \frac{b}{a}c_n & d_{n-1} \end{pmatrix}\begin{pmatrix} R \\ S \end{pmatrix}. \tag{3.4.3}$$

Similarly one can show that the map

$$d_2 : H^{2n-2}(K/T;\mathbb{Z})\otimes H^2(T;\mathbb{Z}) \longrightarrow H^{2n}(K/T;\mathbb{Z})\otimes H^1(T;\mathbb{Z})$$

is injective and its image is the set of all elements given by 3.4.2 satisfying 3.4.3 with the additional requirement that c_{n-1} divides R.

Now consider the group

$$G = \frac{(\frac{a}{bc_n}\mathbb{Z})\cap\mathbb{Z}}{(\frac{a}{bc_n}\mathbb{Z})\cap c_{n-1}\mathbb{Z}}.$$

Corollary 3.7.4 in the Appendix shows that G is a cyclic group of order g_{n-1}. Using this fact and the description of the E_2 term given above it is easy to see that

$$E_3^{2n,1} = \mathbb{Z}/g_{n-1}\mathbb{Z}. \tag{3.4.4}$$

Putting equations 3.3.2, 3.4.1 and 3.4.4 together we obtain

$$H^{2n}(K;\mathbb{Z}) = H^{2n+3}(K;\mathbb{Z}) = \mathbb{Z}/g_n\mathbb{Z}. \tag{3.4.5}$$

5. The Hopf algebras $H_*(K;\mathbb{F}_p)$ and $H^*(K;\mathbb{F}_p)$

¿From chapter 2 we know that there is an extension of Hopf algebras

$$1 \longrightarrow A_{\mathbb{F}_p} \longrightarrow H^*(K;\mathbb{F}_p) \longrightarrow E(x_1,x_2) \longrightarrow 1 \tag{3.5.1}$$

where the Hopf algebra $E(x_1,x_2)$ is an exterior algebra on primitive generators x_1 and x_2 of odd homogeneous degree. To find the degrees of these generators it is sufficient to know the Poincaré series of the graded algebra $H^*(K;\mathbb{F}_p)$.

Recall that there exists a positive integer k with the property that p divides g_j if and only if k divides j. Now using 3.4.5, we observe that the Poincaré series of

$H^*(K;\mathbb{F}_p)$ is given by

$$P(H(K;\mathbb{F}_p),t) = \frac{(1+t^3)(1+t^{2k-1})}{1-t^{2k}}. \tag{3.5.2}$$

¿From 3.3.5, the Poincaré series for $A_{\mathbb{F}_p}$ is

$$P(A_{\mathbb{F}_p},t) = \frac{1}{1-t^{2k}}. \tag{3.5.3}$$

Putting 3.5.1, 3.5.2 and 3.5.3 together, we notice that the degrees of the elements x_1 and x_2 are 3 and $2k-1$ respectively. It is easy to see that the extension 3.5.1 must be trivial and hence we obtain an isomorphism of Hopf algebras

$$H^*(K;\mathbb{F}_p) = \Gamma[x_{2k}] \otimes E(x_3, x_{2k-1}) \tag{3.5.4}$$

$$H_*(K;\mathbb{F}_p) = \mathbb{F}_p[x_{2k}] \otimes E(x_3, x_{2k-1}).$$

If n is the exponent of the prime p in g_k, then 3.4.5 tells us that a Bockstein homomorphism of height n connects the classes x_{2k} and x_{2k-1}.

6. Cohomology of the classifying spaces

Let BG denote the classifying space of a topological group G and let $\mathbb{F}$ be any field. Then one has a multiplicative spectral sequence converging to $H^*(BG;\mathbb{F})$ with the E_2 term given by

$$E_2^{p,q} = Ext^{p,q}_{H_*(G;\mathbb{F})}(\mathbb{F};\mathbb{F}).$$

For the case in hand let G be the unitary form K and let $\mathbb{F}$ be the field of p elements. The E_2 term for the corresponding spectral sequence is

$$E_2 = \mathbb{F}_p[x_4, x_{2k}] \otimes E(x_{2k+1})$$

where the elements x_4, x_{2k} and x_{2k+1} have bi-degrees $(1,3)$,$(1,2k-1)$ and $(1,2k)$ respectively. The elements x_4 and x_{2k} are permanent cycles for dimensional reasons. To see that x_{2k+1} is a permanent cycle, observe using 3.4.5 that K is a rational 3-sphere. Thus BK is a rational BS^3 and so it follows that the element represented by x_{2k} must be the source of a higher Bockstein homomorphism with x_{2k+1} representing its target. It is clear from what we have so far that this higher Bockstein homomorphism has height n where n is the exponent of the prime p in g_k.

So we finally obtain an isomorphism of graded algebras

$$H^*(BK;\mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2k}] \otimes E(x_{2k+1}), \qquad \beta^{(n)}x_{2k} = x_{2k+1}. \tag{3.6.1}$$

Remark 3.6.2. So far we have only concentrated on odd primes. The results for $p = 2$ are similar and are as follows:

The homology algebra of the groups $K = K(a,b)$ can be written as

$$H_*(K;\mathbb{F}_2) = \mathbb{F}_2[x_{2k}] \otimes E(x_3, x_{2k-1})$$

with a Bockstein homomorphism of height n connecting the class x_{2k} with the class x_{2k-1}. The number k is defined as the first positive integer for which 2 divides g_k with n being the exponent of 2 in g_k. The classes x_3 and x_{2k-1} are primitive; the class x_{2k} is primitive in all cases except the case $k = 3, n = 1$. The values that k is allowed to take are $2, 3$ and 4. For a fixed value of k, one can vary the pair of integers a, b so as to give n the value of any positive integer. Furthermore, the cohomology of BK is given by

$$H^*(BK;\mathbb{F}_2) = \mathbb{F}_2[x_4, x_{2k}] \otimes E(x_{2k+1}), \qquad \beta^{(n)} x_{2k} = x_{2k+1}.$$

7. Appendix

We now derive some properties of the integers c_i, d_i defined by the recurrence

$$(3.7.1) \qquad c_0 = d_0 = 0, \quad c_1 = d_1 = 1, \quad c_{j+1} = ad_j - c_{j-1}, \quad d_{j+1} = bc_j - d_{j-1}.$$

We begin with a simple observation

Claim 3.7.2. *For all non-negative integers n we have the equalities*

$$ad_{2n} = bc_{2n} \in ab\mathbb{Z},$$

$$c_{2n+1} = d_{2n+1}.$$

Proof. We shall prove both statements simultaneously by induction on n. The case $n = 1$ is clear. Assume that the claim is true for some integer n. From 3.7.1 we have the following equations:

$$bc_{2n+2} = abd_{2n+1} - bc_{2n},$$

$$ad_{2n+2} = abc_{2n+1} - ad_{2n}.$$

By assumption we know that $c_{2n+1} = d_{2n+1}$ and $bc_{2n} = ad_{2n} \in ab\mathbb{Z}$. Hence we verify from the above equations that $ad_{2n+2} = bc_{2n+2} \in ab\mathbb{Z}$. Next consider the equations

$$c_{2n+3} = ad_{2n+2} - c_{2n+1},$$

$$d_{2n+3} = bc_{2n+2} - d_{2n+1}.$$

We have verified that $ad_{2n+2} = bc_{2n+2}$ and by assumption we know $c_{2n+1} = d_{2n+1}$. Thus we obtain $c_{2n+3} = d_{2n+3}$. The proof is complete by induction. □

Now define g_n to be the integer (c_n, d_n) where (x, y) denotes the greatest common divisor of the pair x, y. We have

Claim 3.7.3.

$$g_n = \frac{c_n(a, bc_{n+1})}{a}.$$

Proof. Using recurrence 3.7.1 one can verify that $(a, bc_{n+1}) = a$ if n is odd and $(a, bc_{n+1}) = (a, b)$ if n is even. The result now follows from claim 3.7.2. □

Corollary 3.7.4. *Let G be the group*

$$G = \frac{(\frac{a}{bc_n}\mathbb{Z}) \cap \mathbb{Z}}{(\frac{a}{bc_n}\mathbb{Z}) \cap c_{n-1}\mathbb{Z}}.$$

Then G is a cyclic group of order g_{n-1}.

PROOF. It is a matter of simple arithmetic to verify that

$$\frac{(\frac{a}{bc_n}\mathbb{Z}) \cap \mathbb{Z}}{(\frac{a}{bc_n}\mathbb{Z}) \cap c_{n-1}\mathbb{Z}} = \frac{(a, bc_n c_{n-1})\mathbb{Z}}{c_{n-1}(a, bc_n)\mathbb{Z}}.$$

Using claim 3.7.2 one sees that $(a, bc_n c_{n-1}) = a$ and then the result follows from claim 3.7.3. □

We now turn our attention to the question of when an odd prime p divides the integer g_n. To answer this question we have to introduce the generating function

$$F(x) = \sum_{i=0}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} x^i.$$

This generating function is to be thought of as a formal power series with coefficients in the 2-dimensional vectorspace over the field, $\mathbb{F}_{p^2}$, of p^2 elements. Thus our question inquires when the coefficient of x^n is zero.

Using equations 3.7.1 we obtain a functional equation

$$\left(x^2 - \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} x + \mathrm{I}\right) F(x) = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can rewrite this equation as

$$(x - M)\,(x - M^{-1})\,F(x) = x \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.7.5}$$

where M is the matrix given by

$$M = 1/2 \begin{pmatrix} \tau & a \\ b & \tau \end{pmatrix}, \qquad \tau = \sqrt{ab-4} \in \mathbb{F}_{p^2}.$$

Notice that $M = M^{-1}$ if and only if $ab = 4 \bmod p$. Our main theorem in this section is

THEOREM 3.7.6. *Let p be an odd prime. Let k be the smallest positive integer with the property that p divides g_k. Then p divides g_n if and only if k divides n. Moreover the integer k can be made explicit as follows:*

If $ab = 4 \bmod p$, then $k = p$; if p divides a or b but not both, then $k = 2p$; in all other cases k is the multiplicative order of the element r, where r is the root of the polynomial

$$x^2 - (ab-2)x + 1$$

defined over the field of p^2 elements.

PROOF. First consider the case $ab = 4 \bmod p$. In this case $M = M^{-1}$ and so using 3.7.5 we can write

$$F(x) = \sum_{i=0}^{\infty} i x^i M^{i+1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since M is an invertible matrix, the coefficient of x^n is zero if and only if n is a multiple of p. So we are done for this case.

Now for all other cases $M - M^{-1}$ is an invertible matrix and using 3.7.5 the generating function is given by

$$F(x) = \sum_{i=0}^{\infty} x^i (\frac{M^i - M^{-i}}{M - M^{-1}}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We are interested in finding those n for which

$$(\frac{M^n - M^{-n}}{M - M^{-1}}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $M - M^{-1}$ is an invertible matrix, we are interested in finding n for which

$$M^{2n} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.7.7}$$

Consider the case when p divides a or b but not both. Assume without loss of generality that p divides b. Then M is the matrix

$$M = \begin{pmatrix} \tau & a/2 \\ 0 & \tau \end{pmatrix}, \quad \tau = \sqrt{-1} \in \mathbb{F}_{p^2}. \tag{3.7.8}$$

Using 3.7.8 we notice that

$$M^{2n} = \begin{pmatrix} (-1)^n & (-1)^{n+1} na\tau \\ 0 & (-1)^n \end{pmatrix}, \quad \tau = \sqrt{-1} \in \mathbb{F}_{p^2}.$$

It follows that 3.7.7 holds if and only if n is a multiple of $2p$.

For all other cases not considered above, it is easy to verify that the matrix M is diagonalizable over $\mathbb{F}_{p^2}$. Now since M^2 has determinant equal to one, the only way 3.7.7 can hold is if M^{2n} is the identity matrix. Which is equivalent to saying that the eigenvalues of M^2 have multiplicative order dividing n. These eigenvalues are exactly the roots of the characteristic polynomial

$$x^2 - (ab - 2)x + 1$$

defined over the field $\mathbb{F}_{p^2}$. This completes our proof. □

Remark 3.7.9. For a fixed odd prime p, let k be one the integers: p, $2p$, a divisor of $p \pm 1$. Then it follows easily from theorem 3.7.6 that we can find positive integers a, b which satisfy

$$H_*(K; \mathbb{F}_p) = \mathbb{F}_p[x_{2k}] \otimes E(x_3, x_{2k-1})$$

where $K = K(a, b)$.

The next result identifies the image of W under mod p reduction.

Theorem 3.7.10. *Let W be considered as a subgroup of $Gl_2(\mathbb{Z})$ via its action on the integral lattice spanned by h_1 and h_2. Let W_p be the image of W under the mod p reduction map*

$$\psi : Gl_2(\mathbb{Z}) \longrightarrow Gl_2(\mathbb{F}_p).$$

Then W_p is a dihedral group of order $2k$ generated by the elements $\psi(w_1)$ and $\psi(w_2)$.

Proof. It is clear that the elements $\psi(w_1)$ and $\psi(w_2)$ have order 2 and generate the group $W_p = \psi(W)$. Thus W_p must be a finite dihedral group of order $2n$ where

n is the order of the element $\psi(w_1 w_2)$. It remains to show that $n = k$. The element $w_1 w_2$ acts on the integral lattice spanned by h_1 and h_2 via the representation

$$w_1 w_2 = \begin{pmatrix} ab-1 & -b \\ a & -1 \end{pmatrix}.$$

Hence we are interested in finding the order of this matrix considered as an element of $Gl_2(\mathbb{F}_p)$. As in theorem 3.7.6, we proceed by cases. If $ab = 4 \bmod p$, then it is easy to verify using 3.7.1 that mod p we have the equalities

$$c_{2m-1} = d_{2m-1} = 2m-1, \qquad c_{2m} = am, \qquad d_{2m} = bm.$$

Using the representation for $(w_1 w_2)^n$ given by 3.2.2, it follows that the order of the element $w_1 w_2$ is $n = p$. The next case we consider is the case when p divides a or b but not both. Assume without loss of generality that p divides b. It is trivial to verify that the order of the resulting matrix is $n = 2p$.

For the remaining cases, the matrix given above is diagonalizable and its eigenvalues are inverses of each other. The order of this matrix is exactly the multiplicative order of the eigenvalues considered as elements of $\mathbb{F}_{p^2}$. These eigenvalues are roots of the characteristic polynomial

$$x^2 - (ab-2)x + 1.$$

A quick comparison of the values for n we have just computed with the integers k in theorem 3.7.6 completes the proof. □

CHAPTER 4

Homotopy decompositions of Classifying Spaces

1. Introduction

In this chapter we will show that for a Kač-Moody group G which is not of finite type, the canonical map from the homotopy-colimit of the classifying spaces of its proper standard parabolics to BG is an equivalence. We will show a similar result for the Weyl group W. Let K denote the unitary form of G and let $N(T) \subset K$ be the normalizer of the maximal torus, then using the above results we will be able to show that the obvious map induced by the inclusion of a subgroup

$$BN(T) \longrightarrow BK$$

is an equivalence at all primes not appearing in the torsion of the Weyl group. The key ingredient in this chapter is a result of S. Mitchell [**15**] on the contractibility of certain Tits buildings.

2. Topological Tits buildings

We shall begin this chapter by introducing some useful notation. Let G be a Kač-Moody group of rank n, with unitary form K. Let B be the positive Borel subgroup and let $N \subset G$ denote the normalizer of the (complex) maximal torus in B. Let $N(T) = N \cap K$ be the normalizer of the maximal torus in K. Let W denote the Weyl group and let $S = \{\bar{r}_1, \ldots, \bar{r}_n\}$ be the set of elements of G that generate the group $\bar{W}$ described in section 5 of chapter 1. For a subset $I \subset S$, let $\bar{W}_I$ denote the subgroup generated by I and let W_I denote the subgroup of the Weyl group generated by the elements $\{r_i : \bar{r}_i \in I\}$. The parabolic subgroup $P_I \subset G$ is defined to be the subgroup generated by B and I. We can similarly define the group $N_I = P_I \cap K$ to be the group generated by T and I.

The data (G, B, N, S) forms a topological Tits system in the sense of [**15**] (§2). Let $\mathcal{C}$ be the category defined by the poset of proper subsets of S (including the empty set). We have a functor from $\mathcal{C}$ to topological spaces given by $I \mapsto G/P_I$. Similarly we have a functor from $\mathcal{C}$ to topological spaces given by $I \mapsto W/W_I$. It was shown in [**15**] that the homotopy colimit of these functors is contractible whenever the Weyl group W is infinite

$$\text{hocolim}_{\mathcal{C}}\, G/P_I \xrightarrow{\sim} *, \qquad \text{hocolim}_{\mathcal{C}}\, W/W_I \xrightarrow{\sim} *. \tag{4.2.1}$$

NOTE 4.2.2. Strictly speaking, the result as stated in [**15**] requires an assumption on the topological Tits system which our system may not satisfy. The only reason for this assumption is to ensure that the parabolic subgroups are closed in G. This condition is known to be true for Kač-Moody groups.

Theorem 4.2.3. *If the Weyl group W of G is infinite, then the following canonical maps are homotopy equivalences:*

(i) $\mathrm{hocolim}_{\mathcal{C}}\, BP_I \xrightarrow{\sim} BG\,,$ (ii) $\mathrm{hocolim}_{\mathcal{C}}\, BN_I \xrightarrow{\sim} BN(T)\,,$ (iii) $\mathrm{hocolim}_{\mathcal{C}}\, BW_I \xrightarrow{\sim} BW.$

Proof. We shall only prove the first of these equivalences, the rest follow using a similar argument. Consider the Borel construction

$$EG \times_G (\mathrm{hocolim}_{\mathcal{C}}\, G/P_I) = \mathrm{hocolim}_{\mathcal{C}}\, EG \times_G (G/P_I) = \mathrm{hocolim}_{\mathcal{C}}\, BP_I.$$

But from 4.2.1 it follows that we have a homotopy equivalence

$$EG \times_G (\mathrm{hocolim}_{\mathcal{C}}\, G/P_I) \xrightarrow{\sim} EG \times_G (*) = BG.$$

The above equations give us a homotopy equivalence. We leave it to the reader to verify that the equivalence just obtained is induced by the canonical map. □

Let $H_I \subset P_I$ be the Levi subgroup defined as $H_I = P_I \cap K$. The Levi subgroups satisfy

$$K/H_I = G/P_I$$

and hence we get a version of theorem 4.2.3 (i) on the level of unitary forms:

Theorem 4.2.4. *If the Weyl group W of K is infinite, then the following canonical map is a homotopy equivalence:*

$$\mathrm{hocolim}_{\mathcal{C}}\, BH_I \xrightarrow{\sim} BK.$$

Now the Levi subgroups H_I can be written as

$$H_I = K_I \rtimes T_I \tag{4.2.5}$$

where K_I is a unitary form of a Kač-Moody group with Weyl group W_I. T_I denotes the subtorus of T with codimension equal to the cardinality of I generated by the elements $\exp h_j$ for every $\bar{r}_j$ that lies in the complement of I in S. The group N_I is a subgroup of H_I and the decomposition 4.2.5 induces a similar decomposition

$$N_I = N(I) \rtimes T_I \tag{4.2.6}$$

where $N(I)$ is the normalizer of the maximal torus in K_I.

Theorem 4.2.7. *The obvious map induced by inclusion of a subgroup*

$$BN(T) \longrightarrow BK$$

is an equivalence at all primes not appearing in the torsion of W.

Proof. This result is a well known result for the class of compact connected Lie groups. Hence we can assume that the Weyl group is infinite in our case. We proceed by induction on the rank n of K. The result clearly holds if the rank is 1 since the only Kač-Moody group of rank 1 is $SL_2(\mathbb{C})$. Since K_I is the unitary form of a Kač-Moody group and the cardinality of I is strictly less than n, the induction assumption implies that the map induced by the inclusion of subgroups

$$BN(I) \longrightarrow BK_I$$

is an equivalence at all primes not appearing in the torsion of $W_I \subset W$. Using the decompositons given in equations 4.2.5 and 4.2.6 it follows that the map induced by the inclusion of subgroups

$$BN_I \longrightarrow BH_I$$

is an equivalence at all primes not appearing in the torsion of W. The result now follows from naturality using theorems 4.2.3 (ii) and 4.2.4. $\square$

A similar induction argument using theorem 4.2.3 (iii) gives a proof of the following theorem:

THEOREM 4.2.8. *The space BW is acyclic at all primes not appearing in the torsion of W.*

Let $T_{(2)}$ be the subgroup of T consisting of the square-roots of unity. It is easy to see using 1.5.1 that one has a short exact sequence

$$1 \longrightarrow T_{(2)} \longrightarrow N \longrightarrow T \rtimes W \longrightarrow 1.$$

Since $T_{(2)}$ is a 2-group and W is generated by reflections, it follows that

COROLLARY 4.2.9. *For a Kač-Moody group with Weyl group W and unitary form K, there is an equivalence*

$$B(T \rtimes W) \cong BK$$

at all primes not appearing in the torsion of W.

REMARK 4.2.10. Let W be the Weyl group of a Kač-Moody group of rank n. Since $W \subset GL_n(\mathbb{Z})$, it follows that any prime that appears in the torsion of W is at most $n + 1$.

CHAPTER 5

An interesting fibration

1. Introduction

In this chapter we use certain rank 2 Kač-Moody groups to construct an interesting fibration. Recall the cohomology of the classifying space of the unitary form of a rank 2 Kač-Moody group which is not of finite type:

$$H^*(BK, \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2k}] \otimes E(x_{2k+1}), \qquad \beta^{(n)} x_{2k} = x_{2k+1}$$

where the numbers k and n were described in chapter 3. If we assume that $(2k, p) = 1$, then there is a p-compact group P with the following cohomology:

$$H^*(BP, \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2k}].$$

After p-completing all the spaces involved, we will construct a map

$$B\lambda : BP \longrightarrow \hat{BK}$$

which is surjective in mod p cohomology. Let $\widehat{K/P}$ be the fiber of this map. One can also construct a map from $\hat{BS^3}$ into BP which is surjective in mod p cohomology and we can similarly define spaces $\widehat{P/S^3}$ and $\widehat{K/S^3}$. It is easy to see that there is an equivalence $\widehat{P/S^3} \cong \hat{S}^{2k-1}$. These spaces fit into a fibration sequence

$$\hat{S}^{2k-1} \longrightarrow \widehat{K/S^3} \longrightarrow \widehat{K/P}. \tag{5.1.1}$$

We will give evidence to suggest that the space $\widehat{K/P}$ is equivalent to $\Omega \hat{S^{2k+1}}$. If our conjecture is indeed true, then the fibration 5.1.1 resembles an interesting fibration constructed by D. Anick [**1**] of the form

$$\hat{S}^{2k-1} \longrightarrow T_\infty(p^n) \longrightarrow \Omega \hat{S^{2k+1}}. \tag{5.1.2}$$

2. A p-compact subgroup

Let K be the compact form of a rank 2 Kač-Moody with an infinite Weyl group. Since this Weyl group is isomorphic to an infinite Dihedral group, 4.2.7 says that at all odd primes there is an equivalence

$$BN(T) \xrightarrow{\sim} BK$$

where $N(T)$ stands for the normalizer of the maximal torus in K. Now fix an odd prime p with the property that $(p, 2k) = 1$. Since $W \subset GL_2(\mathbb{Z})$, one can define W_p to be the image of W under the mod p reduction map

$$\psi : GL_2(\mathbb{Z}) \longrightarrow GL_2(\mathbb{F}_p).$$

Recall from 3.7.10 that the group W_p is isomorphic to D_{2k}. Let $H \subset N(T)$ be the subgroup of $N(T)$ defined by the short exact sequence

$$1 \longrightarrow H \longrightarrow N(T) \xrightarrow{\alpha} W_p \longrightarrow 1 \tag{5.2.1}$$

where the map α is defined as the composite

$$\alpha : N(T) \longrightarrow W \longrightarrow W_p.$$

It is not hard to see that H is isomorphic to a semidirect product

$$H \cong T \rtimes \mathbb{Z}$$

where $\mathbb{Z} \subset W$ is the kernel of the map onto W_p with the action on T induced by the action of W. Notice that in the extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow W \longrightarrow W_p \longrightarrow 1$$

the action of W_p on $\mathbb{Z}$ corresponds to the sign representation of W_p. The inclusion $\mathbb{Z} \subset GL_2(\mathbb{Z})$ extends to a unique continuous inclusion $\mathbb{Z}_p \subset GL_2(\mathbb{Z}_p) = \mathrm{Aut}(\hat{T})$ where $\hat{X}$ denotes the p-completion of a space X. From this remark it is not hard to see that

$$\hat{BH} = B(\hat{T} \rtimes \mathbb{Z}_p). \tag{5.2.2}$$

Thinking of BH as the space $EN(T)/H$, it is clear that W_p acts on BH, inducing and action of W_p on $\hat{BH}$. Let $[Y, Y]$ denote the monoid of homotopy classes of unpointed self-maps of Y, and let $[Y, Y]_*$ denote the monoid of homotopy classes of pointed self-maps of Y. From the above remarks, we have a map of monoids

$$\gamma : W_p \longrightarrow [\hat{BH}, \hat{BH}]. \tag{5.2.3}$$

Claim 5.2.4. *There exists a map of monoids*

$$\bar{\gamma} : W_p \longrightarrow [\hat{BH}, \hat{BH}]_*$$

such that γ factors as

$$\gamma : W_p \xrightarrow{\bar{\gamma}} [\hat{BH}, \hat{BH}]_* \longrightarrow [\hat{BH}, \hat{BH}]$$

where the latter map is the canonical map that disregards the fact that elements in $[\hat{BH}, \hat{BH}]_*$ preserve basepoints.

Proof. Consider the space given by the Borel construction $X = EW_p \times_{W_p} \hat{BH}$. X fits into a fibration sequence

$$\hat{BH} \longrightarrow X \longrightarrow BW_p \tag{5.2.5}$$

giving an extension

$$1 \longrightarrow \mathbb{Z}_p \longrightarrow \pi_1(X) \longrightarrow W_p \longrightarrow 1.$$

Since $(p, 2k) = 1$, this extension splits and we get an inclusion $W_p \subset \pi_1(X)$. Now for any fibration

$$F \longrightarrow E \longrightarrow B$$

it is well known that there is a pointed homotopy action of $\pi_1(E)$ on F, i.e. a map of monoids

$$\pi_1(E) \longrightarrow [F, F]_*.$$

Using this fact for the fibration 5.2.5 one constructs $\bar{\gamma}$ as the following composite:

$$W_p \subset \pi_1(X) \longrightarrow [\hat{BH}, \hat{BH}]_*.$$

Once we have $\bar{\gamma}$, to prove the rest of the theorem is straightforward and is left to the reader. □

Notice that $\pi_1(\hat{BH}) = \mathbb{Z}_p$ and $\pi_2(\hat{BH}) = \mathbb{Z}_p^2$. Claim 5.2.4 gives us a map of monoids on the level of homotopy groups $W_p \longrightarrow \mathrm{End}(\pi_2\hat{BH})$ where the monoid structure on $\mathrm{End}(\pi_2\hat{BH})$ is composition. Hence we get an honest action of W_p on the space $\hat{BT} = K(\pi_2(\hat{BH}), 2)$. Now let

$$\iota : \hat{BT} \longrightarrow \hat{BH}$$

be induced by the inclusion $T \hookrightarrow H$. We have

Claim 5.2.6. *For every group element $g \in W_p$, the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \hat{BT} & \xrightarrow{\iota} & \hat{BH} \\ \downarrow g & & \downarrow g \\ \hat{BT} & \xrightarrow{\iota} & \hat{BH}. \end{array}$$

Proof. For a given element $g \in W_p$ consider the following commutative diagram where the rows are fibrations:

$$\begin{array}{ccccc} \hat{BT} & \xrightarrow{\iota} & \hat{BH} & \longrightarrow & B\mathbb{Z}_p \\ \vdots f & & \downarrow g & & \downarrow (-1)^g \\ \hat{BT} & \xrightarrow{\iota} & \hat{BH} & \longrightarrow & B\mathbb{Z}_p. \end{array}$$

The commutativity of the right hand square implies the existence of the dotted map f making the left hand square commute. To see that f and g are homotopic maps on $\hat{BT}$ we need only check their effect in π_2. Notice that the map ι is an isomorphism on π_2. On identifying $\pi_2(\hat{BT})$ with $\pi_2(\hat{BH})$ and recalling the action of W_p on $\hat{BT}$, it follows immediately that both f and g are homotopic self-maps of $\hat{BT}$. □

Notice that the map

$$\hat{BH} \longrightarrow B\hat{N}(T)$$

is W_p equivariant with respect to the trivial W_p action on $B\hat{N}(T)$. On composing with the map ι and using claim 5.2.6 we have

Theorem 5.2.7. *The map induced via the inclusion*

$$j : \hat{BT} \longrightarrow B\hat{N}(T)$$

is W_p homotopy equivariant with respect to the trivial W_p action on $B\hat{N}(T)$.

Since $(p, 2k) = 1$ one can use [**21**] to assume that the map j is honestly W_p equivariant. The map j extends to the Borel construction on $\hat{BT}$

$$B\lambda : EW_p \times_{W_p} \hat{BT} \longrightarrow B\hat{N}(T).$$

The p-completion of $EW_p \times_{W_p} \hat{BT}$ is the space BP for a p-compact group P, and its cohomology is given by

$$H^*(BP; \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2k}].$$

The map thus constructed

$$B\lambda : BP \longrightarrow B\hat{N(T)} \tag{5.2.8}$$

is seen to be surjective in mod p cohomology. Define $\hat{K/P}$ to be the fiber of this map. It is not hard to construct a map $\hat{BS^3} \longrightarrow BP$ which is surjective in mod p cohomology. The fiber of the latter map is seen to be $\hat{S}^{2k-1}$ by a simple cohomology calculation. Let $\hat{K/S^3}$ denote the fiber of the composite map $\hat{BS^3} \longrightarrow B\hat{N(T)}$. It is an easy exercise to show that we have a fibration

$$\hat{S}^{2k-1} \longrightarrow \hat{K/S^3} \longrightarrow \hat{K/P}. \tag{5.2.9}$$

THEOREM 5.2.10. *The p-adic cohomology of $\hat{K/P}$ is isomorphic to a divided polynomial algebra on a class in dimension $2k$*

$$H^*(\hat{K/P}; \mathbb{Z}_p) = \Gamma[x_{2k}].$$

PROOF. Recall the integers g_m defined as the g.c.d. of the pair (c_m, d_m) described in equation 3.7.1. Let k, p and n be as discussed in the beginning of this section. For any integer m define $\eta(m)$ to be the exponent of p in g_{mk}. By equation 3.4.5 we have

$$H^{2mk}(K; \mathbb{Z}_p) = H^{2mk+3}(K; \mathbb{Z}_p) = \mathbb{Z}/p^{\eta(m)}\mathbb{Z}. \tag{5.2.11}$$

Now by Browder's results on torsion in H-spaces (cf. [**6**] pg. 103), one can show that

$$\eta(m) = n + \nu(m) \tag{5.2.12}$$

where $\nu(m)$ is the exponent of p in m. Now the map $\hat{S}^3 \longrightarrow \hat{K}$ is surjective in p-adic cohomology and thus from the Serre spectral sequence for the fibration

$$\hat{S}^3 \longrightarrow \hat{K} \longrightarrow \hat{K/S^3}$$

it follows easily that

$$H^{2mk}(\hat{K/S^3}; \mathbb{Z}_p) = \mathbb{Z}/p^{\eta(m)}\mathbb{Z}. \tag{5.2.13}$$

Now using the Eilenberg-Moore spectral sequence for the fibration

$$\hat{K/P} \longrightarrow BP \longrightarrow B\hat{N(T)}$$

it follows that $H^*(\hat{K/P}; \mathbb{F}_p)$ is a vector space of dimension 1 in degrees which are a multiple of $2k$. Hence $H^*(\hat{K/P}; \mathbb{Z}_p)$ is a free $\mathbb{Z}_p$ module of rank 1 in degrees which are a multiple of $2k$. At this point the Serre spectral sequence for the fibration

$$\hat{S}^{2k-1} \longrightarrow \hat{K/S^3} \longrightarrow \hat{K/P}$$

along with 5.2.12 and 5.2.13 forces the p-adic cohomology of $\hat{K/P}$ to have the structure of a divided polynomial algebra on a class in dimension $2k$ completing the proof. □

THEOREM 5.2.14. *The mod p Steenrod algebra acts trivially on $H^*(\hat{K/P}; \mathbb{F}_p)$.*

Proof. The map $\hat{K} \longrightarrow \hat{K/P}$ is injective in mod p cohomology and by equation 3.5.4 it is easy to see that the mod p Steenrod algebra acts trivially on the sub Hopf-algebra of $H^*(K; \mathbb{F}_p)$ given by the image of $H^*(\hat{K/P}; \mathbb{F}_p)$. □

Theorems 5.2.10 and 5.2.14 give us evidence to conjecture

Conjecture 5.2.15. There is a homotopy equivalence

$$\Omega \hat{S^{2k+1}} \cong \hat{K/P}$$

and the corresponding fibration

$$\hat{S}^{2k-1} \longrightarrow \hat{K/S^3} \longrightarrow \Omega \hat{S^{2k+1}}$$

is equivalent to the Anick fibration [**1**] of the form

$$\hat{S}^{2k-1} \longrightarrow T_\infty(p^n) \longrightarrow \Omega \hat{S^{2k+1}}.$$

Note that K contains two standard Levi subgroups each containing a canonical copy of S^3. So one can construct a homogeneous space K/S^3 which has the homotopy type of a CW-complex of finite type. It is not hard to see that, on p-completion, this homogeneous space K/S^3 is equivalent to the fiber of the map $B\hat{S^3} \longrightarrow B\hat{N}(T)$ (which we have been calling $\hat{K/S^3}$). Hence if 5.2.15 is answered in the affirmative, then we obtain a "nice" integral lift of the spaces $T_\infty(p^n)$ constructed by D. Anick.

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