The Generalized Tate Construction

by

Olga Stroilova

Bachelor of Science, Massachusetts Institute of Technology (2006)

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Author . . . .
Department of Mathematics
August 10, 2012

Certified by . . . .
Haynes R. Miller
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Paul Seidel, Gigliola Staffilani, Peter Shor
Department co-Chairs, Graduate Committee

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Abstract

The purpose of this work is give some field notes on exploring the idea that a generalized Tate construction t_k reduces chromatic level in stable homotopy theory.

The first parts introduce the construction and discuss chromatic reduction. The next section makes a computation and gives the duals of $L(n) = L(n)_1$. The last part looks ahead, mentioning how this computation could be extended to finding the duals of Steinberg summands in corresponding Thom spectra of negative representations, $L(n)_{-q}$, and presents an equivariant loopspace machine. Finally, observations made are pulled together and brought back to compute the base case of the generalized Tate construction, evaluated on a sphere.

Results parallel work of A. Cathcart, B. Guillou and P. May, and N. Stapleton, among others.

Thesis Supervisor: Haynes R. Miller
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction

Let the k-th generalized Tate construction, t_k, on an arbitrary spectrum X denote the direct limit of function spectra from Steinberg summands in Thom spectra of classifying spaces of a fixed cyclic group of rank k at the prime $p > 0$, twisted by (an increasingly negative) number $-q$ of copies of the regular real representation, into the spectrum X.

That is, in the notation of Chapter 2, define $t_k X$ to be

Definition 1.0.1.

$$t_k(X) = \operatorname{dirlim}_q F(\epsilon_k(B(Z/p)^k)^{-q\bar{\rho}}, X).$$

The existence of such a construction, and most of the properties presented below, have been known for some time. I was introduced to t_k in this form by Mark Behrens.

The origins of this construction come from a generalized Tate cohomology known as Mahowald's inverse limit, or the t construction of Ando-Morava-Sadofsky. The property of the t construction of interest to us is that it is transchromatic in nature: the result of [AMS] is that a completion of $tE(n)$ splits as a wedge of $E(n-1)$'s. Furthermore, the homotopy type of $tE(n)$ has been determined explicitly.

We show that the generalized Tate construction behaves similarly, at least after suitable completion.

Theorem 1.0.2. *[Algebraic version] The element v_{n-k} becomes invertible in $\pi_* t_k E_n$, after completion at I_{n-k}.*

This statement is perhaps more illuminating when viewed from a toplogical perspective, in the language of the stable homotopy category.

[Topological version – generalizing AMS] The construction t_k sends v_n-periodic complex oriented complete spectra to v_{n-k}-periodic spectra.

Another version of this statement, taken from the perspective of algebraic geometry, appears in the work of Nathaniel Stapleton and is given in the language of p-divisible groups:

[Algebro-geometric version – Stapleton, generalizing HKR] The étale part in the connected-étale decomposition for the p-divisible group associated to E_n, base changed to $L_{n-k}E_n$, becomes constant upon suitable further base change to a ring $C_{n,k}$. This ring has to do with a generalization of $\pi_* t_k E_n$ to cover all p-power torsion. This is a generalization of [HKR00] to intermediate heights.

The most obvious thing lacking from Theorem 1.0.2 is either a nicer topological statement, or a more complete algebraic statement.

For example, what is the actual homotopy type of $t_k E_n$? This is known for $k = 1$, and is more difficult for $k > 1$.

We now give an outline of how a proof of the main Theorem may be obtained, and how some generalizing directions of inquiry may be approached.

The process of exploration will be framed by a series of questions, some of which needed to be understood for the Theorem, some of which are answered for their own sakes, and some of which remain unresolved and tie into areas of current interest in stable homotopy theory.

First off, to get more context for the situation at hand, we look for insight into the generalized Tate construction itself and the spectra that appear in its definition:

0. What are the Steinberg summands $\epsilon_k(B(Z/p)^k)^{-q\bar{\rho}}$?

It turns out that these spectra are reoccurring building blocks in stable homotopy theory. They are the spectra $L(k,-q)$ at the prime p. Further notation: $L(k,1)$ is known as just $L(k)$. Alternative and popular notation is to denote the rank of the group by n and the suspension twist by k, and write $L(n)_k = L(n,k)$ and $L(n) = L(n)_1$. This subscript notation is the one that we adopt here.

The properties of these building blocks are of self-contained interest.

1. What is the dual of the most basic building block $D(L(n)) = D(L(n)_1) = F(L(n)_1, S)$?

These can be computed as an application of the Segal machine, the splitting formula, and the Segal conjecture:

Theorem 1.0.3. *[Cathcart], Ph. D. Thesis, Cambridge*

$$D(L(n)) = L(n) \vee L(n-1).$$

A natural generalization gives the following question, which, amazingly enough, is not yet answered in full generality.

2. What are the duals $D(L(n,k))$?

It turns out this may be easier to compute for negative twists k. Using an extended equivariant Segal machine, and hoping that basepoint issues are not an issue, extend the computation methods of the previous theorem to obtain

Claim 1.0.4.

$$D(L(n)_{-k}) = L(n)_k \vee L(n-1)_k, \qquad for - k \leq 0.$$

Since the splitting formula holds only for suspension spectra of spaces with a group action, new methods may be required to try to get an expression for the duals of the $L(n)_k$ with $k > 1$. Such methods may include either studying defining cofiber sequences, as in some work of Takayasu and, separately, Behrens, or using more advanced equivariant methods or Segal machines. This question seems equivalent to determining an expression for the fixed points of a negative equivariant sphere.

The form such an expression might take would be a homotopy colimit.

3. What is the base case of the generalized Tate construction, $t_k S$?

Even though we have an hypothesized answer to perhaps only the easier part of the first question, this is enough to get a hypothesized answer to this second question.

Claim 1.0.5. *Let $k > 1$. Then*

$$t_k S = \varinjlim_q DL(k, -q) = pt.$$

Note that $t_1 S = t_0 S = S$.

Now this gives us some momentum to try to approach the original question. This consists of several steps and tangents, which approach power operations, and results of Behrens-Rezk.

4. What is $F(L(k)_q, E_n)$?

This is an attempt to compute $E_n^* L(k)_q$.

Apparently the dimensions of these cohomology groups were already known, and a generating formula can be determined from [Rezk], [Behrens-Rezk].

Further, it seems that explicit generators may be inductively determined, using methods of spectral sequence "embroidery." Some work has been done in this direction, but has not been properly written up.

Such results might be relevant for several directions.

5. What is the homotopy type of $t_k E_n$?

No practical work has been done on this beyond $k = 1$.

6. What is the tie between the generalized Tate construction/generalized characters, rings of power operations, and the Bousfield-Kuhn functor?

Charles Rezk computes the value of the Bousfield-Kuhn functor on the total power operation. Can anything be said about what happens if one goes through height reduction?

7. Do these computations shed light on beacon conjectures? What is t_k of the $K(n)$-local sphere? What is t_k of the E_n-local sphere?

The conjecture in question is due to Hopkins-Mahowald, and is written up as conjecture (2) in section I of [Hovey-Sadofsky].

Mark Behrens has also suggested that $t_k S_{K(n)}$ might be approached through an application of Rezk's congruence criterion (an extended Wilkersons criterion) to compute $F(L(k)_q, E_n)^{hG_n}$, the homotopy fixed points of the extended Morava stabilizer group on the function spectrum above.

The E_n-local version of the question is both less computation and more mysterious.

These questions are some first steps in attempting to understand the $K(n)$-local and E_n-local categories.

Chapter 2

Background for the generalized Tate construction

The key concept that is used in both $n = 1$ case of the Tate construction, and in all generalizations, is the construction of Thom complexes – twistings of a classifying space of a group by a representation.

The Thom construction can be thought of as a a nontrivial, or "twisted," suspension in spaces by the action of a finite group. Being able to nontrivially suspend at all in the category of spaces is of interest. The key point in this work is that in the stable setting we are allowed to nontrivially desuspend. We will recall the definition of virtual Thom spectra, and see that these are suspensions by sums of possibly negative representations.

In the following work we try to reduce chromatic height nonequivariantly by going to an equivariant setting, performing some work with finite groups, and coming back to the nonequivariant, classical context. It might seem strange that some information is won from this turn; however, the existence of constructions like virtual Thom spectra suggests that equivariant stable homotopy theory is not disjoint from stable homotopy theory, and could shed light on underlying structure.

To put it another way, as was suggested by a recent lecture of Stefan Schwede, going to an equivariant setting allows the action of a finite group to unravel, stretch, and realize its full potential, before coming back to reality.

2.1 Projective spaces. Mahowald's inverse limit. $P^{\infty}_{-\infty}$.

The technical details of the basic Thom construction, including properties, stabilization, and duality are presented in M. F. Atiyah's "Thom Complexes" ([Atiyah]). This work also introduces stunted projective spaces, which are the pieces that appear in the $n = 1$ case.

Further properties of the base case construction are presented both in [Lin], from a hands-on, algebro-topological perspective, and in [AMS], from a homotopy-theoretic perspective.

[Lin] introduces $P_k = P^{\infty}_{k}$, $P^{k}_{-\infty}$, and some defining cofiber sequences for $p = 2$. The odd primary case, constructed from stunted $B\Sigma_p$'s, is discussed in in [Sadofsky]. The P_k at some fixed prime p are the components of the t-construction, or "Mahowald's inverse limit", which is defined as

$$tX := \mathrm{invlim}(P_{-k} \wedge X).$$

[AMS] goes on to identify the homotopy type of $tE(n)$, after a completion.

The first remark to be made here is that the base case of the t-construction is, by Lin's theorem for $p = 2$ and Gunawardena's theorem for $p > 2$, up to suspension and p-completion, just a sphere:

$$tS = \mathrm{invlim}\, P_{-k} = \mathrm{invlim}\ \mathrm{dirlim}\, P^{s}_{-k} = P^{\infty}_{-\infty} \approx (S^{-1})^{\hat{}}_{p}.$$

The second remark is that there is a big difference between

$$P^{\infty}_{-\infty} \wedge X$$

and

$$tX = \mathrm{invlim}(P_{-k} \wedge X).$$

The first construction does not lower chromatic height. As [AMS] shows, the second does. So inverse limits and smashing really do not commute in this case.

Now, making the observation that stunted projected spaces are, up to a suspension, self-dual, one could alternatively define another functor, t_1, based on a direct limit construction, as follows.

$$t_1 X = \operatorname{dirlim} F(P_{-q}, X).$$

This is the base case $k = 1$ of the generalized t_k construction presented above.

The advantage of this direct limit is that it has good algebraic properties. In particular, it attains, in the end, for $k = n$, the direct limit for the p-torsion part of the ring that appears in [HKR].

It will turn out that the algebraic construction t_k for $k \geq 1$ has the desired chromatic properties. This computation is discussed in the next section.

Before that though, we give some discussion of the relationship between the two constructions.

To begin with, there is the question of whether t_1 and t are actually, up to suspension, the same.

The original construction of Mahowald's inverse limit can be written out as

$$tX = \operatorname{invlim}(P_{-k} \wedge X) = \operatorname{invlim}_k((\operatorname{dirlim}_s P^s_{-k}) \wedge X) = \operatorname{invlim}_k \operatorname{dirlim}_s(P^s_{-k} \wedge X).$$

Now, at least at $p = 2$, stunted projective spaces are, up to a suspension, self-dual: $DP^s_{-k} = \Sigma P^{k-1}_{-s-1}$. So this last expression can be rewritten as

$$\begin{aligned} tX &= \operatorname{invlim}_k \operatorname{dirlim}_s(\Sigma^{-1} DP^{k-1}_{-s-1} \wedge X) \\ &= \operatorname{invlim}_k \operatorname{dirlim}_s F(S, \Sigma^{-1} DP^{k-1}_{-s-1} \wedge X) \\ &= \operatorname{invlim}_k \operatorname{dirlim}_s F(\Sigma P^{k-1}_{-s-1}, X) \\ &= \operatorname{invlim}_r \operatorname{dirlim}_q \Sigma^{-1} F(P^r_{-q}, X), \end{aligned}$$

while the base case of the construction being generalized is

$$
\begin{aligned}
t_1 X &= \operatorname{dirlim}_q F(\operatorname{dirlim}_r P^r_{-q}, X) \\
&= \operatorname{dirlim}_q \operatorname{invlim}_r F(P^r_{-q}, X).
\end{aligned}
$$

This reduces the comparison question to a question of commutativity of limits and colimits.

Mark Behrens pointed out to me that Stephen Mitchell discusses some cases in which limits and colimits commute, and explained how the criteria of (3.4)-(3.5) in [Mitchell] might apply here.

Explore applying the criteria for the very special case $X = H\mathbb{F}_p$. Then the homotopy groups of $X_{r,q} = F(P^r_{-q}, H\mathbb{F}_p)$ compute mod-p cohomology of stunted projective spaces – something we know both at $p = 2$ and at odd primes.

In particular,

(i) the $X_{r,q}$ are "uniformly bounded above:" For a fixed r, the homotopy groups $\pi_s X_{r,q} = H\mathbb{F}_p^{-s}(P^r_{-q})$ vanish above $s > q$ ($-s < -q$), and this bound is independent of r.

(ii) in fact, for a fixed s, and all $r > s$, we have that $\pi_s X_{r,q} = \pi_s X_{r,q+1}$ ($H\mathbb{F}_p^{-s}(P^r_{-s-1}) = H\mathbb{F}_p^{-s}(P^r_{-s-2}) = H\mathbb{F}_p^{-s}(P^r_{-s-3}) = \dots$) for all $q > s$, and this bound is independent of r, so the $X_{r,q}$ are "uniformly stable."

So it seems that, at least for $X = H\mathbb{F}_p$, the two conditions introduced in (3.4) and expanded on in the following paragraphs are satisfied, so corollary (3.5) applies, and the limits commute here. However, in this case, such machinery is not needed to make the statement, as both sides can be computed directly.

Chapter 3

The generalized Tate construction reduces chromatic level

In this part we examine a higher Tate spectrum as defined above, but without taking into account the action of the Steinberg idempotent.

That is, as before, let $(B(\mathbb{Z}/p\mathbb{Z})^k)^{s\rho_0}$ denote the Thom construction in spectra on s copies of the reduced regular representation ρ_0, where s is allowed to be negative. These Thom spectra form an inverse system on s. Taking function spectra into an arbitrary spectrum X, obtain a direct system. For this section, define $t_k X$ to be the colimit of the resulting system of function spectra:

$$t_k X := \varinjlim_{s} F((B(\mathbb{Z}/p\mathbb{Z})^k)^{-s\rho_0}, X).$$

We are examining this less refined construction here to simplify the algebra needed for the statement below, and also to underline that the observed phenomenon of chromatic reduction is not governed by the presence of the Steinberg idempotent. It might be useful to consider the more refined definition of t_k if, for example, one were to go further and try to compute the actual homotopy type of $t_k X$.

The goal of these sections is to start with a complex oriented v_n-periodic spectrum E, for some fixed height $n > 0$, and for an appropriate definition of v_n-periodicity, and show that the construction t_k "reduces chromatic level" - that is, that the resulting spectrum $t_k E$ is v_{n-k}-periodic. We make such a statement for Morava E-theory, and say that it might hold more generally for a complex orientable appropriately complete E_n, depending on some questions of algebra and distinctness of roots.

Proposition 3.0.1. *Fix a prime $p > 0$. Let $n \geq k \geq 1$, and suppose that E_n is Morava E-theory of height n. Then*

$$t_k E_n := \varinjlim_{s} F((B(\mathbb{Z}/p\mathbb{Z})^k)^{-s\rho_0}, E_n)$$

is v_{n-k}-periodic.

The exploration of whether the desired generalization of this statement holds has not been finished. It would have been nice to say that this result holds for any v_n-periodic complex orientable p-local spectrum, complete in the sense of [HKR]. Some steps taken in this direction are recorded at the end of the chapter, together with some outline of the missing parts.

The way the proof of Proposition 3.0.1 is structured is as follows. We will show that for a given prime $p > 0$ and complex orientable complete E_n of height n (so, for which v_n is a unit in homotopy), under some additional algebraic conditions, which have been shown to hold for Morava E_n, v_{n-k} becomes invertible in the homotopy groups of the construction $t_k E_n$ after completing at the invariant ideal $I_{n-k} = \langle p, v_1, \ldots, v_{n-k-1} \rangle$. Algebraic completion on homotopy groups is realized in homotopy theory by localizing at a finite p-local spectrum of type n. So the finite spectrum definition of v_n-periodicity that appears in [GS] can then be used to translate our algebraic computations into the topological statement above.

3.1 Notation and Intermediate Computations

3.1.1 Definition of v_n-periodicity

We recall the finite-spectrum definition of periodicity, as it appears in [GS].

For sufficiently large multi-indices $I = (i_0, \ldots, i_{n-1})$, the periodicity theorem promises the existence of finite complexes of type n with BP homology $BP_*/(p^{i_0}, \dots, v_{n-1}^{i_{n-1}})$. Such complexes are called generalized Moore spectra, and are denoted $M_I = M(p^{i_0}, \dots, v_{n-1}^{i_{n-1}})$.

Definition 3.1.1. *Let E be a complex-oriented spectrum.*

E is called v_n-periodic if v_n is a unit on $E \wedge M(p^{i_0}, \dots, v_{n-1}^{i_{n-1}})$.

This definition is independent of I and chosen M_I.

Note that, depending on whether we are okay with viewing the trivial map on the point as a unit, this definition could allow a contractible spectrum to be viewed as v_n-periodic trivially.

3.1.2 Homotopy of the Tate construction

We recall classical statements of complex-oriented cohomology of finite groups, and find an expression for the homotopy of $t_k E$, under nice assumptions on E.

Proposition 3.1.2. *When E is complex-orientable, p-local, complete, as in [HKR], and of positive height $n > 0$ (so that $[p](x)$ is not a zero divisor),*

$$\pi_*(t_k E) = {e_k}^{-1} E^*[[x_1, \dots, x_k]]/\langle [p](x_1), \dots [p](x_k) \rangle,$$

where e_k is the Euler class of the reduced regular representation ρ_0 of the group $(\mathbb{Z}/p\mathbb{Z})^k$.

Inverting e_k is equivalent to inverting the multiplicatively closed set generated by the set

$$\{[a_1](x_1) +_\bullet \cdots +_\bullet [a_k](x_k) \mid (a_1, \dots, a_k) \in \{(\mathbb{Z}/p\mathbb{Z})^k \setminus 0\}\}.$$

Here $+_\bullet$ denotes the sum with respect to the formal group law on $E^*[[x]]$ induced by a choice of complex orientation x,

Proof. Taking homotopy groups commutes with direct limits, so

$$\pi_*(t_k E) = \pi_*(\varinjlim F((B(\mathbb{Z}/p\mathbb{Z})^k)^{-s\rho_0}, E)) = \varinjlim(\pi_* F((B(\mathbb{Z}/p\mathbb{Z})^k)^{-s\rho_0}, E))$$

$$= \varinjlim E^{-*}(B(\mathbb{Z}/p\mathbb{Z})^k)^{-s\rho_0}) = \varinjlim E^{-*-sr_0}(B(\mathbb{Z}/p\mathbb{Z})^k)$$

The last equality follows from the generalized Thom isomorphism, with $r_0 = \dim(\rho_0)$.

We know both the rings and maps in this direct system when E is nice enough. Using a Gysin sequence argument, we can set up a long exact sequence for the E-cohomology of the classifying space of a finite cyclic group for any complex-oriented cohomology theory E. If $[p](x)$ is not a zero divisor in $E^*[[x]]$, this gives us that

$$E^*(B\mathbb{Z}/p\mathbb{Z}) = E^*[[x]]/([p](x)),$$

and, furthermore, if E has a Kunneth formula,

$$E^*(B(\mathbb{Z}/p\mathbb{Z})^k) = E^*[[x_1, \ldots, x_k]]/\langle [p](x_1), \ldots, [p](x_k)\rangle,$$

[HKR] shows that both of these assumptions hold when E^* is a local ring, complete in the topology of its maximal ideal $\mathfrak{m}$, with residue characteristic $p > 0$, and the mod $\mathfrak{m}$ reduction of the formal group law associated with E has height $n > 0$, and so – under our assumptions on E.

The maps in this directed system $\{E^{*-sr_0}(B(\mathbb{Z}/p\mathbb{Z})^k))\}$ are induced by the maps in the generalized Thom isomorphism, and are multiplication by (a multiple of) e_k, the Euler class of ρ_0. Taking the direct limit amounts to inverting e_k.

The reduced regular representation breaks up into line bundles indexed by nonzero elements of $(\mathbb{Z}/p\mathbb{Z})^k$. For complex oriented E,

$$e_k = \prod_{(a_1,\ldots,a_k)\in\{(\mathbb{Z}/p\mathbb{Z})^k \setminus 0\}} ([a_1](x_1) +_{\bullet} \cdots +_{\bullet} [a_k](x_k)),$$

in $E^*(B(\mathbb{Z}/p\mathbb{Z})^k)$, where, as mentioned before, $+_{\bullet}$ denotes the sum with respect to the formal group law induced by a choice of coordinate (complex orientation), giving the last claim in the statement of the proposition.

□

3.2 Proof of Proposition 3.0.1

We are working at a prime $p > 0$ and a height $n > 0$.

Start with a complex-orientable cohomology theory E_n of height n and choose a BP orientation. This BP orientation on E_n gives a p-typical formal group law with p-series

$$[p](x) = v_0 x +_\bullet v_1 x^{p^1} +_\bullet \cdots +_\bullet v_n x^{p^n} +_\bullet h.o.t.,$$

where, by assumption, (the image of) v_n is a unit in $E_n = E_n^* = E_n^*(pt)$. Assume further that E_n is a local ring of residue characteristic p, complete in the topology of its maximal ideal.

Under these assumptions that E_n is complete and of height n, we can apply the Weierstrass Preparation Theorem to the p-series of E_n and obtain a factorization

$$[p](x) = g(x)u(x)$$

for some unique monic polynomial $g(x)$ of degree p^n and some unit power series $u(x)$.

As usual, let I_{n-k} denote the ideal $\langle p = v_0, v_1, \ldots, v_{n-k-1}\rangle$ in E_n, and let π_k denote the projection map $E_n \to E_n/I_{n-k}$. Introduce a new power series

$$\alpha_k(x) := (\pi_k^*[p])(x) = v_{n-k} x^{p^{n-k}} +_\bullet \cdots +_\bullet v_n x^{p^n} +_\bullet h.o.t.$$

This is actually a power series in $x^{p^{n-k}}$, so there is some power series $\beta_k(y) \in E_n/I_{n-k}$ such that

$$\alpha_k(x) = \beta_k(x^{p^{n-k}})$$

The WPT applies to both $\alpha_k(x)$ and $\beta_k(y)$ over E_n/I_{n-k}, so there is a polynomial $h(x)$ of degree $p^n/p^{n-k} = p^k$, such that, by uniqueness of the WP decomposition,

$$\pi_k^*(g(x)) = h(x^{p^{n-k}}).$$

Now continue base changing. The picture in mind is

$$\begin{array}{ccccc}
\boxed{E_n} & \hookrightarrow & E_n[[x_0,\ldots,x_{k-1}]]/[p](x_i) =: \boxed{E_{n,k}} & \hookrightarrow & E_n[[x_0,\ldots,x_{n-1}]]/[p](x_i) =: \boxed{E_{n,n}} \\
 & & \downarrow & & \downarrow \\
 & & E_{n,k}/K_k =: \boxed{{}_kE_n} & & E_{n,n}/K_n = \boxed{{}_nE_n}
\end{array}$$

where K_j is the kernel of inverting the top Euler class e_j in the cohomology of $B(\mathbb{Z}/p\mathbb{Z})^j$

$$K_j := \mathrm{Ker}\{E_n^*(B(\mathbb{Z}/p\mathbb{Z})^j) \to \varinjlim_s E_n^*((B(\mathbb{Z}/p\mathbb{Z})^j)^{s\rho_0})\}$$

Also, the kernel of the composition $E_{n,k} \to {}_nE_n$ contains K_k. So the compositions $E_{n,k} \to {}_nE_n$ and $E_n \to {}_nE_n$ factor through the surjection $E_{n,k} \twoheadrightarrow {}_kE_n$. Furthermore, we know $E_n \hookrightarrow {}_nE_n$ is an inclusion from [HKR]: the target ring is just the Drinfel'd ring of level-$(\mathbb{Z}/p)^n$-structures for E_n, or, in the notation of [HKR], it is $D_1(E^*)$, the image of $E^*(B\Lambda_1)$ in $L_1(E^*)$. There $L_1(E^*) = p^{-1}D_1(E^*)$ was shown to be finite and faithfully flat over $p^{-1}(E^*)$. This means that each intermediate map

$$E_n \hookrightarrow {}_kE_n$$

is also an inclusion, and so is its reduction modulo I_{n-k}.

So far we have $\pi_k^*(g(x)) = h(x^{p^{n-k}})$ over E_n/I_{n-k}, and $E_n/I_{n-k} \hookrightarrow {}_kE_n/I_{n-k}$ is injective, so the same relation holds over ${}_kE_n/I_{n-k}$. But over ${}_kE_n$, and, therefore, over ${}_kE_n/I_{n-k}$, we have additional information. Over ${}_kE_n$, the original Weierstrass polynomial $g(x)$ has (at least) p^k roots, which we have adjoined by taking the kernel of inverting e_k. The ones we know are indexed by $(\mathbb{Z}/pZ)^k$:

$$S_k = \left\{\alpha_{(a_0,\ldots,a_{k-1})}\right\} := \left\{[a_0](x_0) +_\bullet \cdots +_\bullet [a_{k-1}](x_{k-1}), \quad \forall (a_0,\ldots,a_{k-1}) \in (\mathbb{Z}/pZ)^k\right\},$$

which, except for the one which is identically 0, are all non-zero-divisors in ${}_kE_n$, by construction.

We need to be able to factor over the target ring, so we would like some form of the following claim to hold.

Claim 3.2.1. *The elements of the set $S_k \backslash 0$ are not zero modulo I_{n-k} and remain non-zero-divisors over ${}_kE_n/I_{n-k}$. We also need them to be distinct and remain distinct everywhere.*

This Claim can be shown to hold for Morava E-theory. The exploration of whether it holds more generally is not finished, and some discussion of the general case is given below. For Morava E-theory, this Claim should be a corollary of the argument in Proposition 7.10 in [Strickland], where Strickland states that, in his notation, the scheme "Level(A, $\mathbf{G}$)" for a universal deformation formal group $\mathbf{G}$ of height n and abelian group A of rank $r = k$ is a smooth scheme of dimension n, so therefore the rings of interest are an integral domain and factoring given enough roots is not a problem.

In general, one would need to make either some similarly strong geometric statement, or some more delicate algebraic argument. Work in this direction has not been finished. An outline is given below, as a place holder.

So, the following is a survey of some steps that have been taken in exploring Claim 3.2.1 in further generality. We want to say that the first part is obvious because the roots are not in K_k or I_{n-k}, by looking at linear terms.

We would also like to say that a non-zero-divisor in ${}_kE_n$ that is not zero in ${}_kE_n/I_{n-k}$ remains a non-zero-divisor in ${}_kE_n/I_{n-k}$.

We do know that I_{n-k} is a prime ideal of E_n, and $E_n \to {}_kE_n$ is well behaved. What kind of (submodule/ideal) I_{n-k} is in ${}_kE_n$?

It would be nice to say that ${}_kE_n$ is a ring, an extension of the ring E_n, and that, further, this extension is an integral extension.

Then, hypothetically, by the going-up theorem, this ring extension would satisfy the lying over property. Namely, there is some prime ideal in ${}_kE_n$ lying over E_n. Can it then be deduced that this ideal is somehow the ideal we are modding out by when we reduce modulo I_{n-k}? (For example, is the ideal generated by I_{n-k} in ${}_kE_n$ then necessarily prime? Or is only some larger ideal?)

These p^k roots are distinct in ${}_kE_n$, and, because, up to a unit, the difference of two roots is another root, which is nonzero by the above claim after reduction modulo I_{n-k}, they remain distinct over ${}_kE_n/I_{n-k}$. So the set $S_k\backslash 0$ contains p^k-1 distinct nonzerodivisors in ${}_kE_n/I_{n-k}$.

But we actually need something more. We would like to make a statement about the set where each element has been raised to a certain power of p:

Claim 3.2.2. *The operation of raising to the power p^{n-k} preserves the properties of being distinct non-zero-divisors here.*

(Remark: this holds for the case we are arguing to have shown, as the p-power map is monic on an integral domain.)

More generally, if some element b is a non-zero-divisor, then so is b^m, for any power m. Now if $\alpha_{(a_0,\ldots,a_{k-1})} \neq \alpha_{(b_0,\ldots,b_{k-1})}$, then, in this characteristic p ring (the prime $p \in I_{n-k}$, unless $k=n$, but then the question is not relevant),

$$(\alpha_{(a_0,\ldots,a_{k-1})})^{p^{n-k}} - (\alpha_{(b_0,\ldots,b_{k-1})})^{p^{n-k}} = (\alpha_{(a_0,\ldots,a_{k-1})} - \alpha_{(b_0,\ldots,b_{k-1})})^{p^{n-k}} \sim (\alpha_{(a_0-b_0,\ldots,a_{k-1}-b_{k-1})})^{p^{n-k}},$$

i.e. that, up to a unit, the difference between two elements of this new set is still an element of the set, jusat as before, and so, nonzero.

So distinctness would again be implied by reduction modulo I_{n-k} not killing any elements of the set.

Thus the set $\{(\alpha_{(a_0,\ldots,a_{k-1})})^{p^{n-k}}\}$ give p^k distinct roots of $h(y)$ over ${}_kE_n/I_{n-k}$.

Now this polynomial $h(y)$ is itself of degree p^k, so we would like to say that $h(y)$ factors completely over ${}_kE_n/I_{n-k}$.

In order to be able to make such a statement, we need some additional information about ${}_kE_n/I_{n-k}$.

Here we need ${}_kE_n/I_{n-k}$ to be an integral domain, or, at least to be able to factor $h(y)$ over it, knowing some of its roots. We do get that E_n, ${}_kE_n$, and ${}_kE_n/I_{n-k}$ are integral domains for the universal Morava E_n : level-A structures on the universal deformation formal group of height n, for an arbitrary abelian group A of rank $r \leq n$, is represented by a smooth scheme of dimension n. See [Strickland], section 7, and, in particular, proposition 7.10 for a discussion.

If E_n is not universal Morava E-theory, we would still like to factor $h(y)$ over ${}_kE_n/I_{n-k}$, but without using that anything is a domain.

Here is what we know about these rings:

1. ${}_kE_n$ is the universal ring of level-k structures on E_n.

2. We have an injection from $E_n \hookrightarrow {}_kE_n$. So, in particular, modding out by K_k did not kill everything. (The colimit of the direct system for inverting the Euler class e_k, $0 < k \leq n$ is not zero.)

3. At ${}_kE_n$, and in ${}_kE_n/I_{n-k}$, as mentioned above, we have adjoined some roots of $g(x)$. Therefore, over ${}_kE_n/I_{n-k}$, we have some roots of $h(y)$. The roots of $h(y)$ that we have are the projections of the p^k distinct elements $\{(\alpha_{(a_0,\ldots,a_{k-1})})^{p^{n-k}}\}$ in ${}_kE_n$ – namely, their p^k distinct projections in ${}_kE_n/I_{n-k}$. Moreover, and most importantly, we know that each of these elements is a non-zero-divisor in ${}_kE_n$ and, by the above claim, also a non-zero-divisor in ${}_kE_n/I_{n-k}$ (This key point heavily relies on the above two claims and being able to raise to a power.)

Now, we would like to combine all this information to that $h(y)$, a degree p^k polynomial, factors as the product of corresponding linear terms.

We use the additional piece of information that their pairwise differences are also nonzerodivisors:

Up to a unit, $x - y \sim x -_{\bullet} y$, the difference with respect to the formal group law. Therefore, up to a unit, as mentioned before,

$$(\alpha_{(a_0,\ldots,a_{k-1})})^{p^{n-k}} - (\alpha_{(b_0,\ldots,b_{k-1})})^{p^{n-k}} = (\alpha_{(a_0,\ldots,a_{k-1})} - \alpha_{(b_0,\ldots,b_{k-1})})^{p^{n-k}} \sim (\alpha_{(a_0-b_0,\ldots,a_{k-1}-b_{k-1})})^{p^{n-k}},$$

which is another nonzero root, so also a non-zero-divisor.

For this insight, I would like to thank Nathaniel Stapleton, who was going off of a paper of Charles Rezk.

Now say a_1 and a_2 are distinct roots of some polynomial $f_m(x) \in R[x]$ of degree m, and that their difference $a_1 - a_2$ is a nonzerodivisor. Then, since a_1 is a zero of

$f_m(x)$, by a long division algorithm, we can factor $f_m(x)$ as $f_m(x) = (x - a_1)f_{m-1}(x)$ for some polynomial $f_{m-1}(x)$ of degree $m-1$. But now we can say more: a_2 is a zero of this $f_{m-1}(x)$ since $0 = f_m(a_2) = (a_2 - a_1)f_{m-1}(a_2)$ and we are assuming $a_2 - a_1$ is not a zerodivisor.

In our case, we have a degree p^k polynomial $h(y)$ and p^k distinct roots, whose mutual differences are all non-zero-divisors. So, proceed inductively to factor $h(y)$ as desired.

This means, that, setting $y = x^{p^{n-k}}$, we have that, up to a unit power series,

$$\prod_{(a_0,\ldots,a_{k-1})\in(\mathbb{Z}/pZ)^k} (x^{p^{n-k}} - (\alpha_{(a_0,\ldots,a_{k-1})})^{p^{n-k}}) = h(x^{p^{n-k}}) \equiv v_{n-k}x^{p^k} +_{\bullet} \cdots +_{\bullet} v_n x^{p^n}$$

By degree argument, v_{n-k} must be a product of powers of the nonzero roots of $h(y)$ over ${}_kE_n/I_{n-k}$. This means that, over ${}_kE_n$,

$$v_{n-k} \equiv \prod_{(a_0,\ldots,a_{k-1})\in(\mathbb{Z}/pZ)^k \ 0} \alpha_{(a_0,\ldots,a_{k-1})}^{p^{n-k}}, \qquad \text{modulo } \mathrm{I_{n-k}}.$$

Now, to the situation at hand. We wanted to show that v_{n-k} becomes invertible in appropriately completed homotopy groups of the k-th higher Tate construction applied to E_n. The key point is that in $\pi_*(t_kE_n) = (e_k)^{-1}({}_kE_n)$, we are inverting the Euler class e_k of the reduced regular representation, which splits as a sum of line bundles, which means that all the nonzero roots $\alpha_{(a_0,\ldots,a_{k-1})}$, which are exactly the nonzero Chern classes of line bundles, are inverted and become units. Thus, over $\pi_*(t_kE_n)$, v_{n-k} is a product of units, modulo I_{n-k}. This gives us the claim that v_{n-k} becomes a unit in $\pi_*(L_{F(n-k)}(t_kE_n)) = (\pi_*(t_kE_n))^{\wedge}_{I_{n-k}}$. Here $F(n-k)$ denotes the Bousfield class of finite p-local spectra of type $n-k$.

To connect to our chosen definition of periodicity, let $I = \langle p^{i_0}, \ldots, v_{n-k-1}^{i_{n-k-1}} \rangle$ be large enough multiindex for which there exists a generalized Moore spectrum M_I of type $n-k$. The homotopy of $\pi_*(t_kE_n \wedge M_I)$ is $(\pi_*(t_kE_n))/I$. The above shows that v_{n-k} is a unit on $(\pi_*(t_kE_n))/I_{n-k}$. Therefore, it is a unit on $(\pi_*(t_kE_n))/I$, and so t_kE_n is v_{n-k}-periodic.

Remark. Here we use that v_{n-k} is not a zero-divisor modulo I_{n-k}. This seems to be an additional assumption. But it might follow from the equality, since it is a product of non-zero-divisors.

The case that v_{n-k} becomes zero after applying the Tate construction might be a special case that fits into the definition trivially, if we then allowed $t_k E_n$ to be trivially v_{n-k}-periodic. This might happen, for example, for the Morava K-theories.

However, it is then of value to demonstrate that the construction gives a nontrivial answer for some of the basic examples. For this it would be interesting to study the homotopy type of t_k applied to Morava E_n or I_n-complete Johnson-Wilson $E(n)$.

Chapter 4

The dual of $L(n)$

The generalized Tate construction takes a colimit of function spectra from spectra which arise in many other ways in stable homotopy theory. These spectra $L(n)_q$, and, in particular, $L(n)_1 = L(n)$ are building-block spectra in stable homotopy theory.

The spectra $L(n)$ arose classically as desuspended successive quotients in the symmetric powers filtration of the sphere spectrum at the prime p. They also comprise the minimal projective resolution of $H\mathbb{Z}_{(p)}$ that appears in the Whitehead conjecture ([Kuhn]). As discussed above, $L(n)$ and its generalizations constructed from Thom spectra of regular representations are linked to machinery governing chromatic phenomena. The properties of these spectra are of interest both for computational purposes and for guiding intuition. A basic question that arose from trying to understand more about the generalized Tate construction is to determine their functional dual, at least up to p-completion.

Let ϵ_n denote the Steinberg idempotent at the prime p, and let $M(n) = \epsilon_n\, B(\mathbb{Z}/p)^n_+$ denote the Steinberg factor in the classifying space of the n-torus over $\mathbb{F}_p$. By a theorem of [Mitchell-Priddy], $L(n)$ also appears as a summand in this Steinberg factor - in fact, $M(n)$ splits as the wedge $M(n) = L(n) \vee L(n-1)$. We use this as a toehold, work in p-complete spectra, and study the action of the Steinberg idempotent ϵ_n on the decomposition of $F(B(\mathbb{Z}/p)^n_+, S)$ given by the Segal conjecture. We recall some details of Segal's infinite loopspace machine to determine the action of automorphisms on this decomposition of the functional dual. From here, we get to show $F(M(n), S) =$

$M(n) \vee M(n-1)$ by an exercise in linear algebra, and then use stable cancellation to deduce the main result:

Theorem 4.0.3. *Let $n > 0$. After p-completion, there is an equivalence*

$$L(n) \vee L(n-1) \cong F(L(n), S).$$

This identifies the functional dual of $L(n)$ in the category of p-complete spectra.

For $n = 0$, $L(0)$ is just the (p-complete) sphere spectrum S, which is self-dual.

This question was suggested by Haynes Miller and discussed with H. Miller and Mark Behrens, arriving at the conclusions presented below. I have since learned that this result had also been previously discovered by Alan George Cathcart, as part of his doctoral thesis under the guidance of J. Frank Adams. So we might not have been there first, but perhaps the material would not mind another exposition.

4.1 Bases and permutations. Setting up the computation.

Throughout the paper, G will denote a finite group.

Fix a prime p. We will be most interested in the case when $G = V$, an elementary abelian p-group. Remark that if V is of rank $n > 0$, a choice of ordered basis identifies V with the vector space $(\mathbb{Z}/p)^n$. We will refer to this vector space $(\mathbb{Z}/p)^n$, together with an explicit choice of ordered basis, $\mathcal{B} = \{e_1, \ldots, e_n\}$, by subscript notation V_n. Also, subgroups of an arbitrary finite group G will be denoted by letters like H, while subgroups of an elementary abelian p-group V (effectively subspaces of a vector space) will be marked by letters like U.

The left action of $\mathrm{Aut}(G)$ on G induces a left action of the automorphism group of G on the classifying space BG. This action lifts to a left action on the suspension spectrum. Spectra form an additive category, and BG is p-complete, so we obtain an action of the integral group ring and the p-adic group ring of the automorphism group on BG. We extend this to an action on BG_+ by a trivial action on the basepoint.

In the case we are especially interested in - namely when $G = V_n = (\mathbb{Z}/p)^n$, the fixed choice of ordered basis allows us to identify the automorphism group as $\mathrm{Aut}(G) = \mathrm{Aut}(V_n) = GL_n(\mathbb{F}_p)$, and obtain an action of $\mathbb{Z}_p[GL_n(\mathbb{F}_p)]$ on BV_n and BV_{n+}.

The action of $\mathrm{Aut}(G)$ is usually studied as a right action in mod-p cohomology. Idempotents $\epsilon \in (\mathbb{Z}/p)[\mathrm{Aut}(G)]$ split off summands in cohomology, lift to the p-adic group ring, and provide wedge summands, $\epsilon\, BG_+$, by a telescope construction.

In particular, both $L(n)$ and $L(n-1)$ live in BV_{n+}. Together they make up the piece corresponding to the Steinberg idempotent $\epsilon_n \in (\mathbb{Z}/p)[GL_n(\mathbb{F}_p)]$, which is usually denoted by $M(n)$ ([Mitchell-Priddy], [Nishida78]):

$$M(n) = \epsilon_n\, BV_{n+} = L(n) \vee L(n-1).$$

The left action of the automorphism group on BG_+ induces a right action of $\mathrm{Aut}(G)$ on the functional dual $DBG_+ = F(BG_+, S)$. We study this action in general, but find that it has a nicer form in the case $G = V_n = (\mathbb{Z}/p)^n$. We use this to compute the dual of $M(n) = \epsilon_n\, BV_{n+}$ and deduce the dual of $L(n)$ as a corollary. The computation is made by studying the decomposition of $F(BV_+, S)$ as a wedge of classifying spaces of quotients of V given by the Segal conjecture and the splitting formula ([AGM], [Carlsson], [May]). Namely we recall that the splitting formula came from taking the "spectrification" of an appropriate symmetric monoidal category (also known as an infinite loop space machine, Segal gamma spaces, or equivariant Barratt-Priddy-Quillen), and use functoriality to determine how automorphisms of V and their formal sums act on the Segal decomposition.

4.1.1 The Segal conjecture. The induced action of $\mathrm{Aut}(G)$ on $F(BG_+, S)$.

The Segal conjecture gives that for finite groups G, this functional dual $DBG_+ = F(BG_+, S)$ can be presented as the completion of a wedge of classifying spaces

$$F(BG_+, S) = (F(EG_+, S))^G = ((S_G)\hat{}_I)^G = (\bigvee_{[H<G]} BW_G H_+)\hat{}_I,$$

where the middle two pieces are taking fixed points of G-equivariant spectra, the completion is at the augmentation ideal I of the representation ring of G, the last equality is given by the splitting theorem for G-fixed point spectra, and the wedge is taken over conjugacy classes of subgroups $H < G$.

Some reference for what completion of a spectrum at an ideal is can be found in ([GS]) and in the more recent work of [Ragnarsson]. For us, it will suffice to know that when G is a p-group, completing at the augmentation ideal here turns out to be just p-completion away from the basepoint term. Now, we are working in the p-complete setting. Also, when $G = V$, an elementary abelian p-group, each conjugacy class of subgroups contains only one element, and Weyl groups of subgroups $U \subset V$ are just quotients V/U. So in this case, the Segal decomposition of the functional dual simplifies to

$$F(BV_+, S) = \bigvee_{U \subseteq V} B(V/U)_+,$$

where the wedge is taken over all subspaces U of V.

This decomposition is of interest to us because we wish to study the action of $\mathrm{Aut}(G)$ on DBG_+. The classifying space construction gives an left action of $\gamma \in \mathrm{Aut}(G)$ on BG_+ by classifying maps

$$B\gamma : BG_+ \longrightarrow BG_+$$

and an induced right action on the dual $DBG_+ = F(BG_+, S)$, which can be thought of as pullback or precomposition:

$$\gamma^* : F(BG_+, S) \leftarrow F(BG_+, S).$$

The left action of $\gamma \in \mathrm{Aut}(G)$ on BG_+ is well understood in cohomology. However, to say something concrete about the induced right action on the dual, we will need to do some work. We will try to give an explicit description of the action of γ^* in terms of the Segal decomposition, as a large "matrix" of stable maps between the classifying spaces of quotients by subgroups that appear in the right hand side above.

This might seem daunting as the decompositions involved have bases of dimension given by the number of conjugacy classes of G, or distinct subspaces of V. Fortunately, it will turn out that the effect of just one automorphism of G is not hard to describe by remembering where the splitting formula came from.

So in the next section we recall Segal's equivariant loopspace machine and show that each γ^* is given by a block-diagonal matrix with "iso-permutation" blocks.

4.1.2 Categories and spectrification. A description of γ^*.

We set up a way to determine γ^* by recalling how the decomposition in the Segal conjecture arises from an infinite loop space machine.

Let $\mathcal{G}$ be the category of finite G-sets and G-isomorphisms. This is a symmetric monoidal category under disjoint union. By Segal's theory of infinite loop spaces ([Segal74]), the algebraic K-theory, or "spectrification," $Sp(\mathcal{G})$, of this category gives a generalized cohomology theory, and so corresponds to some spectrum. Furthermore, the equivariant analogue of the Barratt-Priddy-Quillen theorem for finite sets identifies $Sp(\mathcal{G})$ as the G-fixed-point spectrum of the equivariant sphere, $(S_G)^G$.

The details of this construction are "essentially due to Segal" ([Segal70]), and are written out by Nishida in the appendix to [Nishida78]. The result is that the zero space of the infinite loop space given by $Sp(\mathcal{G})$ is a product of infinite loop spaces of classifying spaces of the automorphism groups of irreducible objects in the category. From finite group theory, the irreducible objects in $\mathcal{G}$ are transitive G-sets.

These are G-isomorphic to cosets G/H_+ with automorphism groups $\mathrm{Aut}_G(G/H_+) = N_G H/H_+ = W_G H_+$, where $N_G H$ denotes the normalizer of H in G and $W_G H$ is the corresponding Weyl group. So the zero space is

$$Sp(\mathcal{G})_0 = Q(S_G)^G = \prod_{[G/H]} QB\,\mathrm{Aut}_G(G/H_+) = \prod_{[H<G]} QBW_G H_+.$$

Hopefully, it was okay to extend the action of $\mathrm{Aut}\,G$ from BG to BG_+ by a trivial action on the basepoint...

Taking suspension spectra then gives the familiar decomposition in Segal's conjecture (actually this is a special case of the splitting formula for fixed points of an equivariant suspension spectrum - see, e.g., [May], Chapter XIX):

$$Sp(\mathcal{G}) = (S_G)^G = \bigvee_{[H<G]} BW_G H_+.$$

(Here, and in other places, the same notation is used for the classifying space of a finite group and the suspension spectrum of this space - the classifying spectrum of the group.)

As stated in the previous section, the outcome of the conjecture is that the map from this wedge of suspension spectra to the functional dual of BG_+ is completion at the augmentation ideal. The important thing for us is that the map can be made from the geometric realization of the category $\mathcal{G}$ and factors through group completion.[1] This will allow us to get a handle on the action of $\mathrm{Aut}(G)$ on DBG_+ by studying the action on the underlying category $\mathcal{G}$.

An element $\gamma \in \mathrm{Aut}(G)$ gives an endofunctor of $\mathcal{G}$ by pulling back the G-action on a G-set X by precomposition and leaving isomorphisms fixed on the set level. The action on objects can be described explicitly on the irreducible objects - cosets G/H, which are determined by subgroups $H < G$. An automorphism γ sends a coset G/H

[1] Remark that the original Segal conjecture was made as a statement on π_0. The spectrum version in full generalization makes a statement identifying the functional dual spectrum $F(BG_{+,}BK_+)$ with the completion of $Sp(\mathcal{A}_{G,H})$ for a Burnside category of (G,K)-sets. This statement is known to people in the area and written out in a sequence of papers by May. Details were also exposed as necessary in a recent paper of [Ragnarsson].

to $\gamma^*(G/H)$. This is a G-set which has the same underlying set as G/H, but with the action of G precomposed by γ : elements $g \in G$ act by left multiplication by $\gamma(g)$. Furthermore, this is an irreducible object in the category: it can be checked that this G-set $\gamma^*(G/H)$ is $G-$isomorphic to the coset $G/\gamma^{-1}(H)$ under the isomorphism induced from γ^{-1}. That is,

Lemma 4.1.1. *Take any element $\gamma \in \mathrm{Aut}(G)$. The action of γ on the irreducible objects in $\mathcal{G}$, the category of G-sets and G-isomorphisms, can be described as follows.*

Let $H < G$. Then

$$\gamma : [G/H] \mapsto [\gamma^*(G/H)],$$

Furthermore, if we denote by $\underline{\gamma}_H^{-1}$ the isomorphism induced by the following diagram,

$$\begin{array}{ccc} G & \xrightarrow{\gamma^{-1}} & G \\ \downarrow & & \downarrow \\ G/H & \xrightarrow{\underline{\gamma}_H^{-1}} & G/\gamma^{-1}(H), \end{array}$$

we can identify $\gamma^*(G/H)$ in $\mathcal{G}$ with the the coset $G/\gamma^{-1}(H)$:

$$\gamma^*(G/H) \overset{\underline{\gamma}_H^{-1}}{\underset{\sim}{\longrightarrow}} G/\gamma^{-1}(H).$$

That is, $\underline{\gamma}_H^{-1}$ gives a G-isomorphism from the G-set $\gamma^*(G/H)$ to the coset $G/\gamma^{-1}(H)$.

This means that an automorphism of G permutes the basis of the Segal decomposition of DBG_+, sending the conjugacy class of a subgroup $[H < G]$ to the conjugacy class of its preimage under γ, $[\gamma^{-1}(H) < G]$.

This result could have been also computed indirectly by using the double coset formula for the action of $\mathrm{Aut}(G) \subset [BG_+, BG_+]$ on $\pi_0 F(BG_+, S) = [BG_+,S]$. But we want to know more than just the action on π_0 - we would also like to know the induced morphism of spectra on each summand in the decomposition, and for this the double coset formula by itself is not enough. From the perspective of examining the action on the category $\mathcal{G}$, we also want to determine what $\mathrm{Aut}(G)$ does to the automorphism groups of irreducible objects. So we push the discussion above a little further.

An automorphism of G fixes G-isomorphisms as maps of sets, so each γ identifies $\mathrm{Aut}(G/H)$ with $\mathrm{Aut}(\gamma^*(G/H))$. But we want to know the map on automorphism groups for our chosen representatives - left G cosets. So we have to determine the induced map

$$\gamma^* : \mathrm{Aut}(G/H) \longrightarrow \mathrm{Aut}(G/\gamma^{-1}(H)).$$

First recall the standard identification of $\mathrm{Aut}(G/H)$ with $W_G(H) = N_G(H)/H$: for any $\phi \in \mathrm{Aut}(G/H)$, $\phi([1])$ determines $\phi([g]) = g\phi([1])$, so ϕ can be thought of as right multiplication by some element of G that commutes with H, i.e., lies in the normalizer of H in G. This gives a map $N_G(H) \twoheadrightarrow \mathrm{Aut}(G/H)$ that sends $a \in N_G(H)$ to ϕ_a with $\phi_a[1] = [a]$, and the kernel of this map is H itself.

So take any $\phi \in \mathrm{Aut}(G/H)$. There is an $a \in N_G(H)$ such that $\phi = \phi_a$ as above. We draw a large diagram, where the top square is the action of $\gamma \in \mathrm{Aut}(G)$ on objects and morphisms in $\mathcal{G}$, and bottom square is a commutative diagram, induced by the isomorphism $\underline{\gamma}_H^{-1}$:

$$\begin{tikzcd}
G/H \arrow[r, "\phi_a"] \arrow[d, mapsto] & G/H \arrow[d, mapsto] \\
\gamma^*(G/H) \arrow[r, "\gamma_{\mathcal{G}}^*(\phi_a)=\phi_a"] \arrow[d, "\underline{\gamma}_H^{-1}"] \arrow[dr, phantom, "\curvearrowright"] & \gamma^*(G/H) \arrow[d, "\underline{\gamma}_H^{-1}"] \\
G/\gamma^{-1}(H) \arrow[r, "\gamma^*(\phi_a)"] & G/\gamma^{-1}(H)
\end{tikzcd},$$

and deduce $\gamma^*(\phi) = \gamma^*(\phi_a)$ from the commutative square on the bottom. Composing across and down gives that $[1] \in \gamma^*(G/H)$ gets sent to $[\gamma^{-1}(a)] \in G/\gamma^{-1}(H)$, so

$\gamma^*(\phi(a))$ must come from right multiplication by $\gamma^{-1}(a)$. That is,

$$\gamma^*(\phi_a) = \phi_{\gamma^{-1}(a)}.$$

So the map induced by γ from $\mathrm{Aut}(G/H) = N_G H/H$ to $\mathrm{Aut}(G/\gamma^{-1}(H)) = N_G\gamma^{-1}(H)/\gamma^{-1}(H)$ is given by $\underline{\gamma}_H^{-1}$, restricted to $N_G(H)$.

In particular, for $G = V$, the basis for the decomposition is indexed by subspaces $U \subset V$, and γ^* permutes this basis by sending

$$\gamma^* : \; U' \;\mapsto\; \gamma^{-1}(U') = U,$$

i.e., mapping the coset V/U' to $V/\gamma^{-1}(U')$. The action on the classifying spectra of the corresponding automorphism groups is the classifying map of

$$(\gamma^*)_{U'} : W_V U' = V/U' \longrightarrow W_V U = V/U = V/\gamma^{-1}(U').$$

Since in this special case the Weyl group as a set coincides with the coset itself, this is just the map $\underline{\gamma}_{U'}^{-1}$ from the above lemma. It is obtained by taking γ^{-1} as a self-map of the vector space V and projecting it modulo U' on its domain (so modulo U on the range). We will need to work with such maps a lot, and it is most straightforward to think of them as maps of vector spaces, so we choose to adopt the slightly more suggestive and explicit notation for $(\gamma^*)_{U'} = \underline{\gamma}_{U'}^{-1}$ of $\mathrm{proj}_{U'}(\gamma^{-1})$:

$$(\gamma^*)_{U'} = \mathrm{proj}_{U'}(\gamma^{-1}) : \; V/U' \longrightarrow V/U.$$

Note that U' and U have to be of the same dimension since γ is an automorphism of vector spaces, so $\mathrm{proj}_{U'}(\gamma^{-1})$ is an isomorphism between vector spaces of dimension $\dim(V) - \dim(U)$.

We have thus determined that γ^* acts on the wedge decomposition of $F(BV_+, S)$ by isomorphisms between classifying spectra of groups of the same rank. To set up for the computations that follow, we will present such information by using indexed matrices (matrices between unordered bases).

Work in the case $V = V_n$, let $c(n, k, p)$ denote the number of k-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_p$, and let

$$C = \sum_{0 \le k \le n} c(n, k, p).$$

Then

$$F(BV_{n+}, S) = \bigvee_{0 \le k \le n} \ \bigvee_{c(n,k,p)} B(\mathbb{Z}/p)_+^{n-k},$$

and we will say how automorphisms of G act on this wedge decomposition by saying what the action is on each term - by giving a large $C \times C$ (indexed) matrix M.

What we have determined above is that the matrix for γ^*, M_γ, is block-diagonal, with blocks of size $c(n, k, p)$, varying over $0 \le k \le n$. Each such block in the matrix is given by specifying a permutation of the k-dimensional subspaces, $U'_k \mapsto \gamma^{-1}(U'_k) =: U_k$, and replacing identities in the resulting permutation matrix on the basis indexed by $\{U_k\}$ with isomorphisms $B\gamma^*_{(U_k, U'_k)} : B(V_n/U'_k)_+ \to B(V_n/U_k)_+$, which we determined to come from projections $\gamma^*_{(U_k,U'_k)} = \mathrm{proj}_{U'_k}(\gamma^{-1}) \in GL_{n-k}(\mathbb{F}_p)$. That is, γ^* is a block-diagonal matrix with "iso-permutation" blocks. It might need to be remarked that M would really be a true matrix only after choosing an ordering for the basis of subspaces; however, we will always refer to these matrices in indexing notation, so what we do here suffices for our needs.

The indexing notation chosen presents the right action of γ^* on $F(BV_{n+}, S)$ as a left action by the transpose (we are intending M_γ to act on the left).

4.2 The image of the Steinberg idempotent.

4.2.1 Passing to the group algebra. The Steinberg idempotent.

We seek to determine the splitting of DBG_+ in the category of p-complete spectra by idempotents arising in $\mathbb{Z}/p[\mathrm{Aut}(G)]$. As described, for example, in the Preliminaries to [Mitchell-Priddy84], such splittings are really constructed by taking the telescope of lifts of the idempotent to the p-adic group ring. Since we check the image of an idempotent in mod-p cohomology, and we are working in the p-complete setting, it is sufficient to work with mod-p coefficients.

We work in the special case of an elementary abelian p-group of rank n, $G = V_n = (\mathbb{Z}/p)^n$, with a fixed choice of ordered basis, $\mathcal{B} = \{e_1, \ldots, e_n\}$. Let $B \subset GL_n(\mathbb{F}_p) = \mathrm{Aut}(V_n)$ be the corresponding Borel subgroup of upper-triangular matrices, and Σ - the corresponding subgroup of permutation matrices. Up to a unit, the Steinberg idempotent $\epsilon_n \in \mathbb{Z}/p[GL_n(\mathbb{F}_p)]$ is defined to be

$$\epsilon_n = \sum_{b \in B} b \sum_{\sigma \in \Sigma} \mathrm{sign}(\sigma)\sigma,$$

i.e, a signed sum of products $b\sigma$ of an upper triangular matrix and a permutation.

The notation set up in the previous section can be extended to the action of the group ring $\mathbb{Z}/p[GL_n(\mathbb{F}_p)]$: if morphisms $\{\gamma_\alpha\}$ act on the Segal decomposition by matrices $\{M_{\gamma_\alpha}\}$, then linear combinations $\sum c_\alpha \gamma_\alpha$ act by the formal sum $\sum c_\alpha M_{\gamma_i}$, or, equivalently, as the matrix $M_{\sum c_\alpha \gamma_\alpha}$, with entries in the group ring.

In particular, each $b\sigma$ gives an automorphism of V_n, so acts on $F(BV_{n+}, S)$ as a $C \times C$ block-diagonal matrix as described above, with each entry giving an explicit action in mod p cohomology after a choice of basis for the quotient subspace source and target, and we compute the action of ϵ_n by summing the block-diagonal matrices corresponding to each $b\sigma$.

Denote

$$B_k := \bigvee_{c(n,k,p)} B(\mathbb{Z}/p)_+^{n-k}$$

so that

$$F(BV_{n+}, S) = \bigvee_{0 \le k \le n} B_k.$$

Let $M_{\gamma,k}$ denote the matrix representation of γ acting on B_k. In this notation,

$$M_\gamma = \begin{bmatrix} M_{\gamma,0} & 0 & & \dots & 0 \\ 0 & M_{\gamma,1} & 0 & \dots & 0 \\ & & & & \\ & & \dots & & \\ & & & & \\ 0 & \dots & 0 & M_{\gamma,n-1} & 0 \\ 0 & \dots & & 0 & M_{\gamma,n} \end{bmatrix},$$

where each $M_{\gamma,k} : B_k \to B_k$ is a $c(n,k,p) \times c(n,k,p)$ matrix. Note that $c(n,n,p) = c(n,0,p) = 1$, and $M_{\gamma,0}$ and $M_{\gamma,n}$ are just automorphisms of BV_{n+} and $B0_+ = S^0$, respectively.

We will make the computation one block at a time, and compute $M_{\epsilon_n,k} : B_k \to B_k$. So far we have seen that, up to a unit,

$$M_{\epsilon_n,k} = \sum_{b \in B, \sigma \in \Sigma} \operatorname{sign}(\sigma) M_{b\sigma,k},$$

where, for a given automorphism γ, the nonzero entries of $M_{\gamma,k}$ are

$$(M_{\gamma,k})_{\gamma^{-1}(U'_k),U'_k} = \operatorname{proj}_{U'_k}(\gamma^{-1}).$$

(As before, $\{U_k\}$ denotes the indexing set of B_k by the k-dimensional subspaces of V_n, and M is acting on the left).

This entry can be rewritten as the inverse of a projection. Denote $U_k = \gamma^{-1}(U'_k)$. Then $U'_k = \gamma(U_k)$ and

$$(M_{\gamma,k})_{\gamma^{-1}(U'_k),U'_k} = (M_{\gamma,k})_{U_k,\gamma(U_k)} = \operatorname{proj}_{U'_k}(\gamma^{-1}) = (\operatorname{proj}_{U_k} \gamma)^{-1}.$$

4.2.2 Computations. Three cases.

Using the notation set up in the previous section, we compute the images of $M_{\epsilon_n,k} : B_k \to B_k$, and combine these results to get the image of $M_{\epsilon n} : DBV_{n+} \longrightarrow DBV_{n+}$, obtaining $DM(n)$. We will make the computation separately for three cases, based on the dimension k of the indexing subspaces: $k = 0$, $k = 1$, and $1 < k \leq n$.

4.2.2.1 The action of ϵ_n on the highest dimension. $M_{\epsilon_n,0}$.

For $k = 0$, we are at the Weyl group of highest dimension, and have just one factor that is acted on. Here we are modding out by the zero subspace, and the projection is trivial:

$$M_{b\sigma,0} = \mathrm{proj}_0(b\sigma)^{-1} = \sigma^{-1}b^{-1}.$$

So,

$$\begin{aligned}
M_{\epsilon_n,0} &= \sum_{b\in B,\sigma\in\Sigma} \mathrm{sign}(\sigma)M_{b\sigma} \\
&= \sum_{b\in B,\sigma\in\Sigma} \mathrm{sign}(\sigma)\sigma^{-1}b^{-1} \\
&= \sum_{b\in B,\sigma\in\Sigma} \mathrm{sign}(\sigma^{-1})\sigma^{-1}b^{-1} \\
&= \sum_{b'\in B,\sigma'\in\Sigma} \mathrm{sign}(\sigma')\sigma'b'.
\end{aligned}$$

Here we reindexed the inverse of each element in the group. We are still summing over all elements of both groups.

Now, this latter sum also gives a Steinberg idempotent, which is sometimes denoted by ϵ'_n (see, for example, Section 1 in [Nishida86] for this particular case or the more general discussion in Section 2 of [Mitchell-Priddy]), which breaks off an equivalent summand stably, so the image of $M_{\epsilon_n,0}$ acting on $B_0 = BV_{n+}$ is $\epsilon'_n BV_{n+} = \epsilon'_n BV_{n+}$.

4.2.2.2 Next-highest dimension. Computing $M_{\epsilon_n,1}$.

At $k = 1$ we are working with dimension-one subspaces, so with classifying spectra of Weyl groups of dimension $n-1$. This case is the most important part of the computation. The Steinberg idempotent and, less visibly for us right now, chromatic homotopy phenomena seems to be most delicate at one level below the height. It is as though the heights are not just stratified out, but are locked in, one on the next lowest in a jigsaw way.

In this case we will have to compute each matrix entry in the indexed matrix $M_{\epsilon_n,1}$ of morphisms of spectra explicitly.

To illustrate the technique, compute the entry of $M_{\epsilon_n,1}$ indexed by (U_1, U_1') for $U_1 = U_1' = \langle e_n \rangle$, the span of the basis element in highest filtration.

For $b \in B = B_n$ and $\sigma \in \Sigma = \Sigma_n$,

$$(b\sigma)(e_k) = b(\sigma(e_k)) = b(\sigma(k)) = b_{\sigma(k)},$$

where $b_{\sigma(k)}$ is the $\sigma(k)$-th column of b. Since b is upper triangular, this is of the form $a_1e_1 + \cdots + a_{\sigma(k)}e_{\sigma(k)}$, where $a_{\sigma(k)}$ is a unit (nonzero), and the other a_i are zero for $i > \sigma(k)$ and arbitrary for $i < \sigma(k)$. So if the span of $(b\sigma)(e_n)$ is $\langle e_n \rangle$, have $\sigma(n) = n$ and $b_n = u_n e_n$, for a unit $u_n \in F_p^\times$. Thus the admissible pairs (b, σ) are of the form

$$b\sigma = \begin{bmatrix} * & & \dots & * & \\ & * & & * & 0 \\ & & & & \\ & 0 & & * & \\ & & & & u_n \end{bmatrix} \begin{bmatrix} & & & 0 \\ & & \sigma' & \vdots \\ & & & \\ 0 & \dots & & 1 \end{bmatrix} = \begin{bmatrix} b' & 0 \\ 0 & u_n \end{bmatrix} \begin{bmatrix} \sigma' & 0 \\ 0 & 1 \end{bmatrix},$$

where $b' \in B_{n-1}$ and $\sigma' \in \Sigma_{n-1}$ in $GL_{n-1}(\mathbb{F}_p)$. The matrix entry for this automorphism is

$$(\mathrm{proj}_{\langle e_n \rangle}(b\sigma))^{-1} = (b'\sigma')^{-1} = (\sigma')^{-1}(b')^{-1},$$

which will be taken with coefficient $\mathrm{sign}(\sigma^{-1}) = \mathrm{sign}(\sigma) = \mathrm{sign}(\sigma')$ in the total sum.

All pairs in $B_{n-1} \times \Sigma_{n-1}$ can be obtained in this way, and each b' has $|F_p^\times| = p-1$ preimages, so each distinct pair (b', σ') appears with multiplicity $p-1$. Thus

$$\begin{aligned}(M_{\epsilon_n,1})_{\langle e_n\rangle,\langle e_n\rangle} &= \sum_{\text{admissible } (b,\sigma)} \operatorname{sign}(\sigma) \operatorname{proj}_{\langle e_n\rangle}(b\sigma)^{-1} \\ &= \sum_{b' \in B_{n-1},\, \sigma' \in \Sigma_{n-1}} (p-1) \operatorname{sign}(\sigma')(\sigma')^{-1}(b')^{-1} = (p-1)\, \epsilon'_{n-1}\end{aligned}$$

This also gives an answer for all one-dimensional target subspaces that live in highest filtration, that is, for all target subspaces of the form $\langle w\rangle$ for $w = a_1e_1 + \cdots + e_n$, and source subspaces spanned by one of the basis elements, e_i, $1 \leq i \leq n$. As above, admissible pairs (b, σ) are those that satisfy $\sigma(i) = n$ and $b_n = u_n w$ for some unit u_n. Since the bottom row and last column in b and bottom row and one column in σ are forced up to a unit, as before, the counting argument is also the same, except now $\operatorname{sign}(\sigma) = (-1)^{i+n} \operatorname{sign}(\sigma')$ and

$$(M_{\epsilon_n,1})_{\langle e_i\rangle,\langle \sum_{l<n} a_l e_l + e_n\rangle} = (p-1)(-1)^{i+n}\, \epsilon'_{n-1} \underset{p}{\equiv} (-1)^{i+n+1}\, \epsilon'_{n-1}\,.$$

Now for target spanned by $w = \sum_{l<k} a_l e_l + e_j$, when $j < n$, the situation is different. As before, when the source is the span of a basis vector e_i, admissible pairs (b, σ) are those with $\sigma(i) = j$ and $b = u_j w$ for a unit u_j. Now there are p^{n-j} times as many admissible b since $n - j$ is the difference in the number of entries above the diagonal (pivot) in the nth and jth columns. Also, we can compute the projection of b mod $\langle e_j\rangle$ by column-reducing by the jth column across the jth row, and then crossing both out. This shows that there are $n - j$ degrees of freedom, exhibited by the entries on the jth row to the right of the pivot. That is, the projection $\bar{b}\bar{\sigma}$ of any admissible pair (b, σ) now has $(p-1)p^{n-j}$ preimages (b', σ) with fixed σ, so the coefficient of each morphism $(\bar{b}\bar{\sigma})^{-1}$ in the total sum has a factor of p^{n-j}, and

$$(M_{\epsilon_n,1})_{\langle e_i\rangle,\langle \sum_{l<j} a_l e_l + e_j\rangle} \underset{p}{\equiv} 0, \quad \text{for } j < n.$$

In general, the source U and target U' of dimension $k = 1$ are the spans of arbitrary nonzero vectors w and w', respectively. Write $w = \sum_{i_l} a_{i_l} e_{i_l}$, with nonzero coefficients a_{i_l} for ordered indices $i_1 < \cdots < i_t$, and $w' = \sum_{i<k} a'_i e_i + e_k,\ k \leq n$.

We want to compute the (U, U') entry of $M_{\epsilon_n,1}$. Do this in cases, based on the value of the pivot position l of the target U', i.e., on which filtration U' is in.

If U' is completely contained in lower filtration ($l < n$), we will get overrepresentation of projections, as before. That is, say (b, σ) is an admissible pair ($b\sigma(U) = U'$). Then $\sigma(\{i_j\}) \subset \{1, \ldots, l\}$, and $\sigma(i_s) = l$ for some $1 \leq s \leq t$. Examine the fiber above $\mathrm{proj}(b\sigma) = \bar{b}\bar{\sigma}$ for this fixed σ. These are admissible pairs (b', σ), such that $\mathrm{proj}_U(b'\sigma) = \mathrm{proj}_U(b\sigma) = \bar{b}\bar{\sigma}$. That is, want b' for which $b'\sigma$ takes U to U' and $\bar{b'} = \bar{b}$. Then the form of b' is forced by the fixed choice of σ as follows. By column,

$$b'_i = \begin{cases} b_i & 1 \leq i < l \\ \\ a_{i_s}^{-1}(-\sum_{j \neq s} a_{i_j} b'_{\sigma(i_j)} + uw'),\ u \in \mathbb{F}_p^\times\,, & i = l \\ \\ b_i + c_i w',\ c_i \in \mathbb{F}_p\,, & l < i \leq n. \end{cases}$$

So there are $(p-1)(p^{n-l})$ preimages (b', σ) above $\bar{b}\bar{\sigma}$ for a fixed σ, and all the corresponding morphisms $b'\sigma$ originally appear in ϵ_n with the same coefficient $\mathrm{sign}(\sigma)$, so $(\bar{b}\bar{\sigma})^{-1}$ appears in the projection sum with coefficient divisible by p for $l < n$.

Finally, if the target U' lives in highest filtration of the flag, we do not have this trick of overcounting by p in each fiber, and we must compute the projections explicitly. To make this easier, we compare summed admissible morphisms $\{b\sigma\}$ at position (U, U') by making a local choice of basis for $V = V_n$, denoted ${}_U\mathcal{B}$, presenting all the morphisms as matrices from this new basis to the standard globally chosen $\mathcal{B}$ and computing their projections as maps from a local basis, conveniently chosen to make taking the quotient of the morphism $b\sigma$ modulo the source U easier. That is, before taking inverses, express $\mathrm{proj}_U(b\sigma)$ for each admissible (b, σ) as a map from the locally chosen basis ${}_U\mathcal{B}$ with a basis vector omitted at index m corresponding to the filtration index m of U (the pivot position of a spanning vector of U) to the globally chosen basis $\mathcal{B}$ with the omitted basis vector at the index of the filtration of U' (which in this case will always amount to omitting e_n, since U' is now assumed to live in highest filtration $l = n$). Note that the locally chosen basis ${}_U\mathcal{B}$ and the subsequent projected local basis ${}_U\mathcal{B}_{\hat{m}}$ is fixed not only per matrix entry, but also across rows. The globally chosen projected basis $\mathcal{B}_{\hat{n}}$ is fixed down columns. Also note that the morphisms that will actually appear in the final sum giving the group algebra element matrix entry $(M_{\epsilon_n,1})_{U,U'}$ are inverses of these computed projections. So at this position (U, U'), the entry will be expressed as a sum of isomorphisms given as $n-1 \times n-1$ matrices from $\mathcal{B}_{\bar{n}}$ to ${}_{\mathcal{U}}\mathcal{B}_{\bar{m}}$, where m is the filtration of U and U' lives in highest filtration $l = n$.

Make the local choice of basis on the domain U by replacing the mth basis vector in $\mathcal{B}$ with a spanning vector for U. Explicitly, we have denoted $U = \langle w \rangle$ for $w = \sum_{j=1}^{t} a_{i_j} e_{i_j}$, written as an ordered linear combination of t basis vectors e_{i_j} with $1 \leq i_1 < \cdots < i_t = m$, nonzero coefficients $\{a_{i_j}\}$ and pivot position $i_t = m$. Set ${}_U\mathcal{B}$ to be $\{'e_1, \ldots, 'e_n\}$ for

$$'e_i = \begin{cases} e_i, & i \neq i_t \\ w & i = i_t \end{cases}$$

Let ${}_UA$ be the upper-triangular change of basis matrix from ${}_U\mathcal{B}$ to $\mathcal{B}$:

$${}_UA = \begin{bmatrix} I_{i_t-1} & | & 0 \\ 0 & w & 0 \\ 0 & 0 & I_{n-i_t} \end{bmatrix}.$$

For each admissible pair (b, σ), compute $\mathrm{proj}_U(b\sigma)$ as a map from ${}_U\mathcal{B}_{\hat{i_t}}$ to $\mathcal{B}_{\hat{n}}$. That is, compute $\mathrm{proj}_{\langle 'e_{i_t} \rangle}(b\sigma {}_UA)$. Project the second morphism, b, by crossing out its nth row and column. σ has pivot in the nth row at some position $i_s = \sigma^{-1}(n)$. If we were working in the original basis on the domain, this would be inconvenient, as it would make projecting and comparing projections of different admissible σ difficult. But precomposing with ${}_WA$ allows us to choose a convenient intermediate basis for determining the projection of σ based on the pivot position in the nth row, i_s. So project σ by crossing out nth row and pivot i_s column, - effectively, compute this projection as a map from $\mathcal{B}_{\hat{i_s}}$ to $\mathcal{B}_{\hat{n}}$. Finally, project ${}_UA$ as a map from ${}_U\mathcal{B}_{\hat{i_t}}$ to $\mathcal{B}_{\hat{i_s}}$. Thus ${}_UA$ will take on all the work of the projection and hide the nonuniform choice of intermediate basis.

Then, over all admissible pairs for this matrix entry, we have t different projection representations of ${}_UA$ that represent the t different basis vectors that admissible $b\sigma$ can carry to highest filtration. These are t different matrices that all represent the same morphism $\mathrm{proj}_{\langle' e_{i_t}\rangle}\, {}_WA$, but are presented by choosing different bases $\mathcal{B}_{\hat{i_s}}$, $1 \leq s \leq t$, on the target quotient space. Denote these matrix representations ${}_UA_{i_1}, \ldots, {}_UA_{i_t}$, and compute them explicitly. Projection mod U is obtained by adding the relation $0 = w = \sum_{j=1}^{t} a_{i_j} e_{i_j}$. Choosing to present the morphism as a matrix to the basis $\mathcal{B}_{\hat{i_s}}$ means crossing out the i_sth row and i_tth column, and, if $i_s < i_t$, using this relation to replace e_{i_s}.

$$e_{i_s} = -a_{i_s}^{-1} \sum_{j \neq s} a_{i_j} e_{i_j} = \sum_{j \neq s} (-a_{i_s}^{-1} a_{i_j}) e_{i_j}$$

So ${}_UA_{i_t} = I_{n-1}$, the $n-1$ identity matrix, and, if $t > 1$, each of the other ${}_UA_{i_s}$ for $1 \leq s < t$ is an $n-1$ identity matrix with the i_sth column replaced by $-a_{i_s}^{-1} w_{\hat{i_s}}$. It is not necessarily upper triangular, but it is block diagonal, in two blocks of size $i_t - 1$ and $n - i_t$.

$${}_UA_{i_s} = \left[\begin{array}{c|cc} I_{i_s-1} & & \\ & & \\ & -a_{i_s}^{-1} w_{\hat{i_s}} \quad I_{i_t-i_s-1} & \\ & 0 & I_{n-i_t} \end{array}\right], \quad \text{defined for each } i_s \in \{i_1, \ldots, i_{t-1}\},\ t > 1.$$

Then the projection of admissible $b\sigma$ can be represented explicitly as a matrix (acting on the left) from ${}_U\mathcal{B}_{\hat{i_t}}$ to $\mathcal{B}_{\hat{n}}$ by choosing the appropriate ${}_U A_{i_j}$:

$$\operatorname{proj}_U(b\sigma) = \bar{b}\bar{\sigma}_U A_{\sigma^{-1}(n)} = \left[\begin{array}{cc|c} & \bar{b} & \\ & & \\ - & - & (n,n) \end{array}\right] \left[\begin{array}{cc|cc} \bar{\sigma} & & & \bar{\sigma} \\ & & & \\ - & & (n,\sigma^{-1}(n)) & - \end{array}\right] \left[\begin{array}{c} \\ {}_U A_{\sigma^{-1}(n)} \\ \\ \end{array}\right],$$

$$\text{where } \bar{b} \in B_{n-1},\ \bar{\sigma} \in \Sigma_{n-1},\ \operatorname{sign} \bar{\sigma} = (-1)^{n+\sigma^{-1}(n)} \operatorname{sign}(\sigma).$$

Again, for a fixed σ, have $p-1$ preimages above each $\bar{b}\bar{\sigma}_U A_{\sigma^{-1}n}$. So, in this case (when U' in highest filtration n), obtain

$$\begin{aligned}
(M_{\epsilon_n,1})_{U,U'} &= \sum_{\text{admissible } (b,\sigma)\in B_n\times\Sigma_n} \operatorname{sign}(\sigma)(\operatorname{proj}_U(b\sigma))^{-1} \\
&= \sum_{\text{adm. } (b,\sigma)} \operatorname{sign}(\sigma)(p-1)(\bar{b}\bar{\sigma}_U A_{\sigma^{-1}(n)})^{-1} \\
&= \sum_{\text{adm. } (b,\sigma)} \operatorname{sign}(\sigma)(p-1)({}_U A_{\sigma^{-1}(n)})^{-1}(\bar{\sigma})^{-1}(\bar{b})^{-1} \\
&= \sum_{\text{nonzero j in U}} \left(({}_U A_j)^{-1} \sum_{\text{adm. } (b,\sigma), \sigma^{-1}(n)=j} \operatorname{sign}(\sigma)(p-1)(\bar{\sigma})^{-1}(\bar{b})^{-1} \right) \\
&= \sum_{\text{nonzero j in U}} \left(({}_U A_j)^{-1} \sum_{\text{adm. } (b,\sigma), \sigma^{-1}(n)=j} \operatorname{sign}(\bar{\sigma})(-1)^{n+j}(p-1)(\bar{\sigma})^{-1}(\bar{b})^{-1} \right) \\
&= \sum_{\text{nonzero j in U}} \left(({}_U A_j)^{-1} \sum_{\sigma'\in\Sigma_{n-1},\, b'\in B_{n-1}} \operatorname{sign}(\sigma')(-1)^{n+j}(p-1)\sigma' b' \right) \\
&= \sum_{\text{nonzero j in U}} \left((-1)^{n+j}(p-1)({}_U A_j)^{-1} \sum_{\sigma'\in\Sigma_{n-1},\, b'\in B_{n-1}} \operatorname{sign}(\sigma')\sigma' b' \right) \\
&= \sum_{\text{nonzero j in U}} \big((-1)^{n+j}(p-1)({}_U A_j)^{-1}\, \epsilon'_{n-1} \big) \\
&= \left(\sum_{\text{nonzero j in U}} (-1)^{n+j}(p-1)({}_U A_j)^{-1} \right) \epsilon'_{n-1}\,.
\end{aligned}$$

Combining all the cases above for one-dimensional source $U_1 = U$ and target

$U_1' = U'$, what this computes is an expression for each entry of $M_{\epsilon_n,1}$, which is either 0 mod p if U_1' is in lower filtration, or, if U_1' is in highest nth filtration and U_1 is in some mth filtration, is a sum of transformations which, when written as matrices from $\mathcal{B}_{\hat{n}}$ to ${}_{U_1}\mathcal{B}_{\hat{m}}$, is a left multiple of ϵ'_{n-1} by an element of the group ring that depends on the row index, U_1. As mentioned before, this presentation is in a local basis on the target, ${}_{U_1}\mathcal{B}_{\hat{m}}$, which also depends on the row.

$$(M_{\epsilon_n,1})_{U_1,U_1'} = \begin{cases} 0 \bmod p, & \text{pivot}(U_1') < n, \\ \\ \left(\sum_{\text{nonzero j in } U_1} (-1)^{n+j}(p-1)({}_{U_1}A_j)^{-1}\right)\epsilon'_{n-1}, & \text{pivot}(U_1') = n \end{cases}$$

So, if we can ignore that matrix entries are written in different, row-dependent, bases, this says that, at least mod p, we can factor out the ϵ'_{n-1} from all entries. The nonzero columns in the remaining matrix are repeating copies of the vector $(\sum_{\text{nonzero j in } U_1} (-1)^{n+j}(p-1)({}_{U_1}A_j)^{-1})$, which has unit entries in all rows indexed by subspaces spanned by a single basis vector. If we can do this, obtain that, mod p, $M_{\epsilon_n,1}$ is a rank-1 matrix, scaled by ϵ'_{n-1} on the right.

Actually, we could probably go further and compute $(M_{\epsilon_n,1})_{U_1,U_1'}$ explicitly (not just mod p) even when U_1' is in some lower filtration $l < n$. I think you would get an expression like the one for highest filtration, except with the factor $(p-1)$ in the multiple of ϵ'_{n-1} replaced by $(p-1)p^{n-l}$.

4.2.2.3 The remaining dimensions. $M_{b\sigma,k}$, $1 < k \leq n$.

Fix k, $1 < k \leq n$. Let two k-dimensional subspaces U_k and U_k' of V_n be given.

Let (b, σ) be an admissible pair for these subspaces. That is, $(b\sigma)(U_k) = U_k'$.

Put spanning sets of vectors for V_k and V_k' in row-echelon form to determine their ordered pivot position sets $I = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$, and $I' = \{i_1', \ldots, i_k'\}$, $i_1' < \cdots < i_k'$, respectively. Suppose such an ordered spanning set of vectors for the target U_k' is some $\{w_1', \ldots, w_k'\}$.

Compute the projection of $b\sigma$ as a matrix from $\mathcal{B}_{\hat{I}}$ to $\mathcal{B}_{\hat{I}'}$.

$$\operatorname{proj}_{U_k}(b\sigma) = \operatorname{proj}_{\sigma(U_k)}(b)\operatorname{proj}_{U_k}(\sigma) = \bar{b}\operatorname{proj}_{U_k}(\sigma),$$

where the intermediate basis is always chosen to be $\mathcal{B}_{\hat{I}'}$ for convenience of projection of the upper triangular b. So $\operatorname{proj}_{U_k}(\sigma)$ is presented as a matrix from $\mathcal{B}_{\hat{I}}$ to $\mathcal{B}_{\hat{I}'}$, and $\bar{b}$ is a matrix from $\mathcal{B}_{\hat{I}'}$ to itself.

Since b is upper triangular, $\sigma(U_k)$ and $U'_k = b\sigma(U_k)$ have the same ordered pivot position set I', and each target w'_j is a linear combination of the first i'_j columns of b. So $\bar{b}$ can be obtained from b by iteratively for $j = 1, \ldots, k$ crossing out the i'_jth column, column reducing by w'_j to reflect the added relation $w'_j = 0$, and crossing out the by-now-zero i'_jth row.

This gives a way to determine all admissible (b', σ) that have the same projection $\bar{b}\operatorname{proj}_{U_k}(\sigma)$. Iteratively, we must have

$$\begin{aligned} b'\sigma((b\sigma)^{-1}(w'_1)) &= u'_1 w'_1, && u'_1 \in \mathbb{F}_p^{\times}, \\ b'\sigma((b\sigma)^{-1}(w'_2)) &= u'_2 w'_2 + a'_{21} w'_1, && u'_2 \in \mathbb{F}_p^{\times},\ a'_{21} \in \mathbb{F}_p, \\ &\cdots \end{aligned}$$

In particular, since $k > 1$, $b'_{i'_2}$ is determined by a choice of arbitrary unit multiple of w'_2, an arbitrary multiple of w'_1, and a forced combination of previously chosen columns of b'. So the number of such preimages for a fixed σ is divisible by p, in fact, by $(p-1)^2 p$. (Probably we can go further to determine it explicitly, to come out to be something like $|B_k| = (p-1)^k p^{(k^2-k)/2}$.) This means that, for the case $1 < k \leq n$, $(M_{\epsilon_n,k})_{U_k,U'_k} \equiv 0 \bmod p$ for all U_k, U'_k.

4.2.3 Stable cancellation. The dual of $L(n)$.

Combining the cases above, we have shown that

$$F(M(n), S) = F(\epsilon_n\, BV_+, S) = F(BV_+, S)\,\epsilon_n = M(n) \vee M(n-1)$$

in the category of p-complete spectra.

Our goal is to obtain a result for the dual of $L(n)$.

A unique factorization theorem holds for spectra in the p-complete setting (see, for example, Chapter 10 of [Margolis]). This allows us to use "stable cancellation" and induction on n to show the main result.

We have two known base cases $DL(0)$ and $DL(1)$:

$$F(L(0), S) = F(S, S) = S = L(0)$$

and

$$F(L(1), S) = L(1) \vee L(0)$$

(since

$$\begin{aligned} F(L(1) \vee L(0), S) &= F(M(1), S) = F(\epsilon_1 \, BZ/p_+, S) = F(B\Sigma_p \vee S, S) \\ &= B\Sigma_{p+} \vee S = M(1) \vee M(0) = L(1) \vee L(0) \vee L(0) \end{aligned}$$

).

We then deduce the main result by stable cancellation on the inductive formula

$$\begin{aligned} F(L(n), S) \vee F(L(n-1), S) &= F(L(n) \vee L(n-1), S) \\ &= F(M(n), S) \\ &= M(n) \vee M(n-1) \\ &= L(n) \vee L(n-1) \vee L(n-1) \vee L(n-2). \end{aligned}$$

Chapter 5

The dual of $L(n)_{-k}$

The methods used to compute $DL(n)$ can be pushed further to get at the duals of Thom spectra of negative multiples of the reduced regular representation and deduce expressions for the $DL(n)_{-k}$. The formulas for these duals can then be used to evaluate the generalized Tate construction at the sphere spectrum.

The key insight of how to pass through the equivariant setting to make this further step was suggested from the outside. Also, it appears that the mildly ad-hoc categorification used later to get a hold of the splitting formula decomposition, which is a generalization of the category in the equivariant Barratt-Priddy-Quillen theorem, may be the G-fixed points of some equivariant loop space machines of Shimakawa or Guillou-May, modulo having to figure out a basepoint issue.

5.1 Some equivariant methods and the Segal conjecture.

5.1.1 Virtual Thom spectra and their duals. DBG^{-V}.

Let V be a finite-dimensional vector space and $\rho : G \to \mathrm{Aut}(V)$ a representation of our finite group G. Let BG^V denote the Thom space of the bundle induced by this representation. Let BG^V also denote the suspension spectrum of this space - i.e., the Thom spectrum corresponding to this representation, and write BG^{-V} for the Thom spectrum of the negative (virtual) bundle.

Then the equivariant Segal conjecture states that the arrow below is a completion.

$$\begin{aligned} DBG^{-V} &:= F(BG^{-V}, S) \\ &= F(EG_+ \wedge_G S^{-V}, S) \\ &= F_G(EG_+ \wedge S^{-V}, S)^G \leftarrow F_G(S^{-V}, S)^G \\ &= F_G(S, S^V)^G \\ &= (S^V)^G. \end{aligned}$$

Here the spectra in the middle are G-fixed points of function spectra in the equivariant category. I am using the fact that the equivariant spectrum S^{-V} is the equivariant dual of the equivariant suspension spectrum S^V. A reference for equivariant methods, the Segal conjecture, and the splitting formula below is provided in [May].

Such a duality statement holds for any virtual representation bundle. However, if we start with the Thom spectrum of a negative representation, we get to the G-fixed points of a positive representation sphere, i.e., something to which we can apply the splitting formula for suspension spectra:

$$(S^V)^G = \bigvee_{[H<G]} EW_GH_+ \wedge_{W_GH} (S^V)^H.$$

So, the dual of the Thom spectrum of a negative orthogonal representation is, up to a certain completion,

$$DBG^{-V} = \bigvee_{[H<G]} BW_GH^{V^H}.$$

5.1.2 The reduced regular representation of an abelian p-group. $D(B(\mathbb{Z}/p)^n)^{-k\bar{\rho}}$.

Turn to the case of interest. Let $G = V_n = (\mathbb{Z}/p)^n$ be the elementary abelian p-group of rank n, and let $\rho = \rho_n = \rho_{V_n}$ be its regular real representation.

The regular representation of a group is a permutation representation of the group on a vector space with basis elements indexed by elements of the group. The action is by left-multiplication. The action of the group fixes the formal sum of all the group elements, splitting off a one-dimensional trivial representation. The remaining representation, $\rho - 1$, is called $\bar{\rho}$, the reduced regular real representation of $(\mathbb{Z}/p)^n$. It has dimension $p^n - 1$.

In this section we will work out more explicitly the dual of the Thom spectrum of $\bar{\rho}$. We will have to compute fixed-points of this representation. Over the complex numbers we could leverage complete splitting of representations, but over the reals we will instead go ahead and make use of the fact that this is a permutation representation of an abelian group.

As in the computation of $DL(n)$, the completion appearing in the Segal conjecture for $V_n = (\mathbb{Z}/p)^n$ is p-completion away from the basepoint, subgroups are subspaces of some dimension $0 \leq m \leq n$, and conjugacy classes contain one element each. For any subgroup $H = U_m \cong (\mathbb{Z}/p)^m < V_n$, we can compute the U_m-fixed points of the regular representation. Pick a basis of V_n so that U_m occupies the first m positions. Then basis elements of the regular representation of V_n can be written as $x_{I_n}^{a_{I_n}}$, where $I_n = \{1, ..., n\}$, $a_{I_n} \in (\mathbb{Z}/p)^n$, and the first $\{x_i, 1 \leq i \leq m\}$ give a basis for U_m. Then a basis for the fixed points ρ^{U_m} is given by $(\sum_{a_{I_m} \in (\mathbb{Z}/p)^m} x_{I_m}^{a_{I_m}}) x_{J_{n-m}}^{b_{J_{n-m}}}$, where $J_{n-m} = \{m+1, \ldots, n\}$ and $b_{J_{n-m}}$varies over $(\mathbb{Z}/p)^{n-m}$. This basis is of dimension p^{n-m} and gives the regular representation of V_n/U_m, the Weyl group of this subgroup. That is,

$$(\rho_{V_n})^{U_m} = \rho_{V_n/U_m} = \rho_{W_{V_n}U_m} \cong \rho_{n-m}.$$

The trivial representation in the regular representation of V_n is fixed by all of V_n, and so lies inside the fixed points under any subgroup. So the fixed points of the reduced regular representation is also a reduced regular representation of the Weyl group:

$$(\bar{\rho}_n)^{U_m} = (\bar{\rho}_{V_n})^{U_m} = (\rho_{V_n} - 1)^{U_m} = (\rho_{V_n})^{U_m} - 1 = \bar{\rho}_{V_n/U_m} = \bar{\rho}_{W_{V_n}U_m} \cong \bar{\rho}_{n-m}.$$

So up to p-completion,

$$
\begin{aligned}
D(B(\mathbb{Z}/p)^n)^{-k\bar{\rho}_n} &= D(BV_n)^{-k\bar{\rho}_n} \\
&= \bigvee_{0\leq m\leq n} \bigvee_{U_m<(\mathbb{Z}/p)^n} EW_{V_n}U_{k+} \wedge_{W_{V_n}U_k} \left(S^{k\bar{\rho}_{V_n}}\right)^{U_m} \\
&= \bigvee_{0\leq m\leq n} \bigvee_{U_m<(\mathbb{Z}/p)^n} (BW_{V_n}U_m)^{k\bar{\rho}_{W_{V_n}U_m}} \\
&= \bigvee_{0\leq m\leq n} \bigvee_{c(n,m,p)} (BV_{n-k})^{q\bar{\rho}_{n-m}}.
\end{aligned}
$$

5.2 Group actions and categorification

It turns out that Thom spectra of copies of the (reduced) regular representation of V_n carry an action of $\mathrm{Aut}(V_n) = GL_n(\mathbb{F}_p)$ and of $\mathbb{Z}_p[GL_n(\mathbb{F}_p)]$, which contains a lift of the Steinberg idempotent ϵ_n. By definition, $L(n)_{-k} = \epsilon_n \, BV_n{}^{-k\bar{\rho}}$.

We will discuss this action and compute the duals

$$D(L(n)_{-k}) = D(\epsilon_n \, BV_n^{-k\bar{\rho}}) = D(BV_n^{-k\bar{\rho}}) \, \epsilon_n$$

by determining the image of the Steinberg idempotent in the decomposition of the dual given by the splitting formula.

In order to make this computation, we first try to understand how $GL_n(\mathbb{F}_p)$ acts on this decomposition. Then we draw a parallel to the action of $GL_n(\mathbb{F}_p)$ on DBV_n, and use the linear algebra computations of the action of ϵ_n on mod-p cohomology from that story to deduce the answer here.

5.2.1 The action of $\mathrm{Aut}(G)$ on the Borel construction.

Let X be a G-space. Although $\mathrm{Aut}(G)$ does not act on X, any automorphism $\gamma \in \mathrm{Aut}(G)$ can be used to pull back the G-action on X and give a new G-space, γ^*X. Similarly, even though $\mathrm{Aut}(G)$ lifts by the classifying construction to provide automorphisms of BG, one does not in general get an action of $\mathrm{Aut}(G)$ on the Borel construction $EG \times_G X$. However, in the very special case that X can be identified equivariantly and functorially with its pullbacks γ^*X, an action of $\mathrm{Aut}(G)$ on the Borel construction $EG \times_G X$ can be defined.

Perhaps this is best thought about in a two-category setting.

The classifying space of a finite group is the geometric realization of the category with one object and a G-worth of morphisms:

$$BG = | *^{\curvearrowright G} |.$$

This is the geometric realization of $[G.pt]$, the translation category of G acting on a point.

In general, the translation category $[G.X]$ of the action of G on X is defined to be the category with objects the points of X and morphisms labeled by elements of the group $g \in G$ and connecting points in X with their image under g:

$$x \xrightarrow{g} gx, \quad x \in X,\ g \in G.$$

(Thus the translation category spreads out a topological space over its points, and then underlines its G-structure by connecting up each G-orbit.)

The Borel construction can be obtained as the geometric realization of this category,

$$EG \times_G X = |[G.X]|.$$

This categorification of the Borel construction allows us to imagine forcing a G-action on the Borel construction on the level of categories.

Fix any $\gamma \in \mathrm{Aut}(G)$. By fixing objects and transforming morphisms, we get an induced functor $F_\gamma : [G.\gamma^*X] \mapsto [G.X]$, given by the following commutative diagram:

$$\begin{array}{ccc} x \in \gamma^*X & \mapsto & x \in X \\ g\downarrow & & \downarrow \gamma(g) \\ \gamma(g)x \in \gamma^*X & \mapsto & \gamma(g)x \in X. \end{array}$$

Equivalently, since γ is invertible, this gives a functor $[G.X] \to [G.(\gamma^{-1})^*X]$. Now, if there existed an equivariant identification $\alpha : (\gamma^{-1})^*X \cong X$, this construction could be extended to give an endofunctor of the translation category:

$$\begin{array}{cccc} F_{\gamma,\alpha} : & [G.X] & \mapsto & [G.X], \\ & & & \\ & x \in X & \mapsto & \alpha(x) \in X \\ & g\downarrow & & \downarrow \gamma(g) \\ & gx & \mapsto & \alpha(gx) = \alpha(\gamma^{-1}(\gamma(g))\, x) = \gamma(g)\alpha(x), \end{array}$$

which would realize to an endomorphism of $EG \times_G X$ as a space and as a spectrum, and, also give an endomorphism of the spectrum $\Sigma^\infty(EG \times_G X)_+ = \Sigma^\infty EG_+ \wedge_G X_+$. If X were also the total space of a representation of G, this would give an action of $\mathrm{Aut}(G)$ on the Thom space (or spectrum) of the representation, and could be extended to virtual spectra of positive and negative copies of the representation.

This approach gives the action of $GL_nF_p = \mathrm{Aut}(V_n)$ on $BV_n^{\bar{\rho}_n}$ that is referred to when people say that $L(n)_k$ can be identified with the Steinberg summand $e_nBV_n^{k\bar{\rho}_n}$ in a Thom spectrum. The details of this special case are written out in the next section.

5.2.2 The action of $\mathrm{Aut}(V_n)$ on $BV_n^{\rho_n}$.

Return to the case of interest, $G = V_n$ and $X = [\rho_n]$ (the total space of the regular real representation - a real vector space). For any $\gamma \in GL_n\mathbb{F}_p = \mathrm{Aut}(V_n)$, $(\gamma^{-1})^*([\rho_n])$ can be identified with $[\rho_n]$ by the isomorphism α_γ of vector spaces induced from acting by γ on the basis of the regular representation labeled by elements of V_n:

$$\alpha_\gamma : (\gamma^{-1})^*([\rho_n]) \xrightarrow{\simeq} [\rho_n].$$

This identification is equivariant with respect to the two actions of V_n, and, supposedly, is appropriately functorial in $\mathrm{Aut}(V_n)$. (I did not check this last point, and it may not be strictly necessary for the following; it would be needed to make a clean statement that we are actually constructing a two-morphism - a functor and a natural transformation).

Following the construction in the previous section, we get an endofunctor of the translation category of V_n acting on $[\rho_n]$, that we will label $F(\gamma, \alpha_\gamma)$:

$$F(\gamma, \alpha_\gamma) : [V_n.[\rho_n]] \longrightarrow [V_n.[\rho_n]]$$

This functor realizes to an endomorphism of the Borel construction and suspends to a morphism of the Thom spectrum $BV_n^{\rho_n} = \Sigma^\infty(EV_{n+} \wedge_{V_n} S^{\rho_n}) = \Sigma^\infty(EV_n \times_{V_n} [\rho_n])_+$. We are identifying S^{ρ_n} with $[\rho_n]_+$.

Since γ is unital, α_γ restricts to an identification for the reduced regular representation, $\bar{\rho}_n$, giving an endomorphism of $BV_n^{\bar{\rho}_n}$ and $BV_n^{k\bar{\rho}_n}$ in the same way, for which we will use the same notation.

This action of $GL_n\mathbb{F}_p$ extends to an action of the group algebra $\mathbb{Z}_p[GL_n\mathbb{F}_p]$ by formal sums on the Thom spectrum, and, in particular, gives the summand $\epsilon_n\, BV_n^{k\bar{\rho}_n} = L(n)_k$ corresponding to a (lift of the) Steinberg idempotent $\epsilon_n \in \mathbb{Z}/p[GL_n\mathbb{F}_p]$.

We have been so explicit in the exposition of this construction because the goal is to identify restrictions of the action of $\mathrm{Aut}(V_n)$ on components of the duals of Thom spectra of negative representations. This is approached in the next section.

5.2.3 A categorification for the splitting formula. The action of $\mathrm{Aut}(V_n)$ on $DBV_n^{-k\rho_n}$.

We know from the previous section that, up to p-completion, $DBV_n^{-k\rho_n}$ breaks up as a large wedge of Thom spectra of k (positive) copies of the reduced regular representations of Weyl groups of subgroups of V_n. To determine how $\mathrm{Aut}(V_n)$ acts on this decomposition, we need a handle for where it may have come from.

We get such a handle, perhaps in a somewhat ad-hoc fashion, by finding a believable categorification for the fixed-point spectrum $\left(S^{k\rho_n}\right)^{V_n}$ that would produce the splitting formula. The inspiration for this category comes from mixing the category of finite G-sets and G-isomorphisms that lies behind the equivariant Barratt-Priddy-Quillen theorem for identifying the G-fixed points of the equivariant sphere $(S^0)^G$ with Segal's classical loop space machine, which goes through producing a (Γ-) category whose geometric realization spectrifies (group completes on components) to the suspension spectrum $\sum^\infty X_+$ for an arbitrary space X.

Proceed in general. Fix a finite group G. Let $\mathcal{IG}_f$ denote the category mentioned above, of finite G-sets and G-isomorphisms. This is the maximal groupoid in the category of finite G-sets and all G-morphisms, which, in turn, is the "G-fixed-point category" of the G-category of finite G-sets and all morphisms, which carries a G-action on the hom-sets by conjugation.

Let Y be any G-space. Define a contravariant functor

$$P_Y : \mathcal{IG}_f^{op} \to Top$$

whose action on objects is to send a finite G-set S to all the G-maps from S to Y,

$$S \mapsto \mathrm{map}_G(S, Y),$$

and whose action on morphisms is mapping to the precomposition (pullback).

Note that the image of a transitive G-set of type $H < G$ can be identified with points in the fixed-point set of Y under H:

$$P_Y(G/H) = \mathrm{map}_G(G/H, Y) \cong Y^H.$$

Proposition 5.2.1. *The geometric realization of the translation category of P_Y spectrifies to the G-fixed points of the equivariant suspension spectrum of Y_+.*

That is,

$$Sp(|[P_Y.\mathcal{IG}_f]|) = \bigvee_{[H<G]} \Sigma^\infty EW_GH_+ \wedge_{W_GH} \Sigma^\infty (Y^H)_+ = (\Sigma^\infty Y_+)^G,$$

where the last equality is given by the splitting formula.

The reasoning here is along the following lines. The geometric realization of the category $[P_Y.\mathcal{IG}_f]$ breaks up into a product of components, along the types of G-sets. Let $\{[H<G]\}$ index the conjugacy classes of subgroups of G. Then the category $\mathcal{IG}_f$ breaks up into the product of full subcategories $\mathcal{IG}_{f,H}$ of copies of transitive G-sets of the same type H, for each conjugacy class, and

$$\begin{aligned}
|[P_Y.\mathcal{IG}_f]| &= |\prod_{\{[H<G]\}} [P_Y.\mathcal{IG}_{f,H}]| \\
&= \prod_{\{[H<G]\}} \coprod_{n\geq 0} |[P_Y.[\Sigma_n \wr W_GH.[nG/H]]| \\
&= \prod_{\{[H<G]\}} \coprod_{n\geq 0} |[\Sigma_n \wr W_GH.(P_Y(G/H))^n]| \\
&= \prod_{\{[H<G]\}} \coprod_{n\geq 0} |[\Sigma_n \wr W_GH.\left(Y^H\right)^n]| \\
&= \prod_{\{[H<G]\}} \coprod_{n\geq 0} E\Sigma_n \times_{\Sigma_n} \left(EW_GH \times_{W_GH} Y^H\right)^n.
\end{aligned}$$

From this last identification, by Segal's loop space machine [Segal74], the spectrification of each of the pieces is

$$Sp(|[P_Y.\mathcal{IG}_{f,H_i}]|) = \Sigma^\infty(EW_GH_i \times_{W_GH_i} Y^H)_+ = \Sigma^\infty EW_GH_{i+} \wedge_{W_GH_i} \Sigma^\infty {Y^H}_+.$$

Passing to the entire product gives the identification in the proposition since ${Y^H}_+ = (Y_+)^H$. The splitting formula appears, for example, in [May].

Note that when Y is a point, $[P_{pt}.\mathcal{IG}_f] = (\mathcal{IG}_f^{op})^{op} = \mathcal{IG}_f$, and the statement reduces to the equivariant Barratt-Priddy-Quillen theorem ([Nishida78]). When G is the trivial group, this is just the first level of the Γ-category behind the classical statement in [Segal74].

Return to the case of interest $G = V_n, Y = S^{\rho_n}$ (or, later, $Y = S^{k\bar{\rho}_n}$). Take any $\gamma \in \mathrm{Aut}(V_n) = GL_n(\mathbb{F}_p)$.

Actually, directly, this would give the action of $\mathrm{Aut}(V_n)$ on $(Y_+)^G$, whereas we would like to know the action on $(Y)^G$. We need to try to figure out how to work around the basepoint issue, and there are some attempts made at this below.

The action of γ on the category $[P_Y.\mathcal{IG}_f]$ is induced by the action on $\mathcal{IG}_f$ described in 3.2 and the equivariant identification $\alpha_{\gamma^{-1}} : \gamma^*Y \cong Y$ described before for this Y. On irreducible objects, this gives the composite

$$\begin{aligned}
&[[V_n/U_m],\ x \in \mathrm{map}_{\mathrm{V_n}}(V_n/U_m, Y) \cong Y^{U_m}] \\
&\qquad \mapsto [[\gamma^*(V_n/U_m) \cong V_n/\gamma^{-1}(U_m)],\ x \in \mathrm{map}_{V_n}(\gamma^*(V_n/U_m), \gamma * Y) \cong (\gamma^*Y)^{\gamma^{-1}(U_m)}] \\
&\qquad \mapsto [[V_n/\gamma^{-1}(U_k)],\ \alpha_{\gamma^{-1}}(x) \in \mathrm{map}_{V_n}(V_n/\gamma^{-1}(U_k) \cong (Y)^{\gamma^{-1}(U_m)}].
\end{aligned}$$

On morphisms, the action is also induced from the action on $\mathcal{IG}_f$ that sends

$$W_{V_n}U_k \cong V_n/U_k \overset{\mathrm{proj}_{U_k}\gamma^{-1}}{\mapsto} W_{V_n}\gamma^{-1}(U_k) \cong V_n/\gamma^{-1}(U_k)$$

So, on each first component of the geometric realization, this map is

$$F(\mathrm{proj}_{U_k}\gamma^{-1}, \alpha_{\gamma^{-1}}) :\ [W_{V_n}U_m, Y^{U_m}] \ \to\ [W_{V_n}\gamma^{-1}(U_m), Y^{\gamma^{-1}(U_m)}]$$

Now, $Y = \rho_n$ (or $\bar{\rho}_n$), and we have computed that $\rho_n^{U_m} = \rho_{W_{V_n}U_m} \cong \rho_{n-m}$ (respectively, $\bar{\rho}_n^{U_m} = \bar{\rho}_{W_{V_n}U_m} \cong \bar{\rho}_{n-m}$), so this geometrically realizes to the morphism on Borel constructions given by $F(\mathrm{proj}_{U_k}\gamma^{-1}, \alpha_{\gamma^{-1}})$. Thus, the action on this entire component of the product spectrifies to the morphism

$$BW_{V_n}U_m^{\rho_{W_{V_n}U_m}} \;\to\; BW_{V_n}\gamma^{-1}(U_m)^{\rho_{W_{V_n}\gamma^{-1}(U_m)}}$$

given by $F(\mathrm{proj}_{U_k}\gamma^{-1}, \alpha_\gamma^{-1})$, and somehow magically we hope that this is exactly the morphism induced on these summands by "$\mathrm{proj}_{U_k}\gamma^{-1}$" on these Thom spectra.

This would allow us to say that the subsequent computation of the action of ϵ_n proceeds as in the case of the trivial representation.

Okay, here is the proposed identification.

We have a map on subcategories

$$[V_n/U_m \,.\, ([\rho_{V_n}])^{U_m}] \;\longrightarrow\; [V_n/\gamma^{-1}(U_m) \,.\, ([\rho_{V_n}])^{\gamma^{-1}(U_m)}]$$

given by the tuple $[\mathrm{proj}_{U_m}\gamma^{-1}\,,\ \alpha_{\gamma^{-1}}|_{([\rho_{V_n}])^{U_m}}]$.

After changing bases, each Weyl group V_n/U_m can be identified with V_{n-m}, and each fixed point set acted on is identified with the regular representation $\rho_{V_{n-m}} = \rho_{n-m}$ of the acting group. That is, in the spectrification, both the source and the target subcategory category will contribute to a copy of isomorphic Thom spectra. We want to determine the morphism in spectra induced from this tuple. In particular, we want to know if this functor of categories spectrifies to an identified map of Thom complexes, coming from an automorphism of V_{n-m}.

That is, $\mathrm{proj}_{U_m}\gamma^{-1} \in \mathrm{Iso}[V_n/U_m, V_n/\gamma^{-1}(U_m)]$ is identified with some $\varphi \in \mathrm{Aut}[V_{n-m}]$. The question being asked is whether the following diagram "commutes."

$$
\begin{array}{ccc}
[V_n/U_m \,.\, ([\rho_{V_n}])^{U_m}] & \xrightarrow{[\mathrm{proj}_{U_m}\gamma^{-1}\,,\ \alpha_{\gamma^{-1}}|_{([\rho_{V_n}])^{U_m}}]} & [V_n/\gamma^{-1}(U_m) \,.\, ([\rho_{V_n}])^{\gamma^{-1}(U_m)}] \\
i_1 \simeq & \circlearrowright ? & \simeq i_2 \\
[V_{n-m} \,.\, [\rho_{n-m}]] & \xrightarrow[F(\varphi,\alpha_\varphi)]{} & [V_{n-m} \,.\, [\rho_{n-m}]]
\end{array}
$$

Here i_1 and i_2 are the two identifications of representations induced from the identifications of groups, and we need to check whether for any $x \in ([\rho_{V_n}])^{U_m}$, $\alpha_\varphi(i_1(x))$ matches $i_2(\alpha_{\gamma^{-1}}(x))$.

Any $x \in ([\rho_{V_n}])^{U_m}$ is a formal sum $x = \sum_{a_v \in \mathbb{R}, v \in V_n} a_v v$, such that $ux = x$ for all $u \in U_m$. That is, $a_v = a_{uv}$, for all $u \in U_m$, and there is a well-defined identification of x under i_1 as $i_1(x) = \sum_{[v] \in V_n/U_m} a_v[v]$. Also,

$$
\begin{aligned}
\alpha_\varphi(i_1(x)) &= \alpha_\varphi\Big(\sum_{[v]\in V_n/U_m} a_v[v]\Big) \\
&= \sum_{[v]\in V_n/U_m} a_v \varphi([v]) = \sum_{[v]\in V_n/U_m} a_v \,\mathrm{proj}_{U_m}\,\gamma^{-1}([v]) \\
&= \sum_{[v]\in V_n/U_m} a_v[\gamma^{-1}(v)],
\end{aligned}
$$

while

$$
\begin{aligned}
i_2(\alpha_{\gamma^{-1}}(x)) = i_2\Big(\alpha_{\gamma^{-1}}\Big(\sum_{v\in V_n} a_v v\Big)\Big) = i_2\Big(\sum_{v\in V_n} a_v \gamma^{-1}(v)\Big) \\
= \sum_{[\gamma^{-1}(v)]\in V_n/\gamma^{-1}(U_m)} a_v[\gamma^{-1}(v)],
\end{aligned}
$$

where the last equality is well-defined because $a_v = a_{uv}$ for $u \in U_m$ implies the coefficient of $\gamma^{-1}(v)$ is equal to the coefficient of $u'\gamma^{-1}(v) = \gamma^{-1}(uv)$ for all $u' = \gamma^{-1}(u) \in \gamma^{-1}(U_m)$.

These two expressions are the same, as ω identifies their indexing set.

Thus we seem to be okay writing that the right action of $\gamma \in \mathrm{Aut}(V_n)$ on $DBV_n^{-\rho_n}$ restricts to a left action on components sending

$$B(V_n/U_m)^{\rho_{n-m}} \longrightarrow B(V_n/\gamma^{-1}(U_m))^{\rho_{n-m}}$$

by the morphism of spectra $\varphi \in \mathrm{Aut}(V_{n-m})$ corresponding to $\mathrm{proj}_{U_m}\gamma^{-1}$ after identifying both sides with $BV_{n-m}^{\rho_{n-m}}$.

The same reasoning gives the action of $\gamma \in \mathrm{Aut}(V_n)$ on $DBV_n^{-k\rho_n}$ as sending

$$B(V_n/U_m)^{k\rho_{n-m}} \longrightarrow B(V_n/\gamma^{-1}(U_m))^{k\rho_{n-m}}$$

by the induced morphism corresponding to $\mathrm{proj}_{U_m}\gamma^{-1}$ after identifying both sides with $BV_{n-m}^{k\rho_{n-m}}$.

This gives that the matrix of the action of $\gamma \in \mathrm{Aut}(V_n)$ on the decomposition of $D(BV_{n+}^{-k\rho_n})$ indexed by $U_m < V_n$ is the same as that computed for DBV_n in 3.2.

We would like to be able to deduce that the action on the decomposition of $DBV_n^{-k\rho_n}$ is therefore also the same.

Recap: For our given Y and G, we have determined that there is an action of $\mathrm{Aut}(G)$ on $(\sum^\infty Y_+)^G$ and, as we knew from before, there is also an action of $\mathrm{Aut}(G)$ on $(S^0)^G$. It happens that we also know that there is an action of $\mathrm{Aut}(G)$ on $(\sum^\infty Y)^G$, because this happens to be, up to completion, the dual of a spectrum with an action of $\mathrm{Aut}(G)$, and so should have some abstract induced action. Our goal was to determine this action explicitly. So far, we were only able to determine it explicitly for $(\sum^\infty Y_+)^G$.

Now, using the splitting formula, we can get the following identification of (ordinary) spectra:

$$
\begin{aligned}
(\Sigma^\infty(Y_+))^G &= \bigvee_{[H<G]} \Sigma^\infty(EWH_+ \wedge_{WH} (Y_+)^H) \\
&= \bigvee_{[H<G]} \Sigma^\infty(EWH_+ \wedge_{WH} Y_+^H) \\
&= \bigvee_{[H<G]} \Sigma^\infty(EWH_+ \wedge_{WH} Y_+^H) \\
&= \bigvee_{[H<G]} EWH_+ \wedge_{WH} \Sigma^\infty(Y_+^H) \\
&= \bigvee_{[H<G]} EWH_+ \wedge_{WH} \Sigma^\infty(Y_+^H) \\
&= \bigvee_{[H<G]} EWH_+ \wedge_{WH} (\Sigma^\infty Y^H \vee S^0) \\
&= (\bigvee_{[H<G]} EWH_+ \wedge_{WH} \Sigma^\infty Y^H) \vee (\bigvee_{[H<G]} EWH_+ \wedge_{WH} S^0) \\
&= (\Sigma^\infty Y)^G \vee (S^0)^G
\end{aligned}
$$

If we could say that this decomposition was equivariant with respect to the actions of $\operatorname{Aut} G$ mentioned, we could read off the desired action of $\operatorname{Aut}(G)$ on $(\Sigma^\infty Y)^G$ from the known action on the categorifications of the other two components.

What this seems to be saying is that the categorification machinery has a left-over basepoint trail, and we have to work around it. Namely, we have two tangible categories, which spectrify to a spectrum and its summand, and we hope to be able to subtract off the basepoint trail of the machinery, in an appropriately equivariant fashion, by subtracting off the category for a point:

$$
\begin{array}{ccccc}
|[P_Y.\mathcal{IG}_f]| & & & & |[P_{pt}.\mathcal{IG}_f]| \\
\downarrow & & & & \downarrow \\
(\Sigma^\infty(Y_+))^G & = & (\Sigma^\infty Y)^G \vee & & (S^0)^G
\end{array}
$$

This is clumsy and tenuous. It would be better to have a category for $(\Sigma^\infty Y)^G$ directly. What does Thomason do?

Note also, that in our case talking about $Y_+ = S^\rho_+$ is slightly strange. This space has two points fixed by the action of G: the point at infinity, and the disjoint base point.

5.2.4 The image of the Steinberg idempotent. $D(L(n)_{-k})$.

The result of the previous section means that the matrix of the action of $\gamma \in \mathrm{Aut}(V_n)$ on the decomposition of $DBV_n^{-k\rho_n}$ indexed by $U_m < V_n$ is the same as the matrix M_γ of the action of γ on DBV_n computed in 3.2. Put another way, the action on different Thom spectra have the same action on morphisms in the underlying translation category , and differ only by what they do to objects; also they act on objects predictably and compatibly. So we can hope that the work done in 3.2 to compute the image of a sum of matrices on the decomposition transfers directly to this more general case.

The work of 3.2 computed the image of the matrix of a formal sum of such automorphism corresponding to the Steinberg idempotent ϵ_n, by working only with that matrix M_{ϵ_n} and making computations mod p. The result there was that $D(BV_n)\,\epsilon_n \simeq \epsilon_n(B(\mathbb{Z}/p)^n) \vee \epsilon_{n-1}(B(\mathbb{Z}/p)^{n-1})$.

By direct translation, we get that up to p-completion,

$$D(BV_n^{-k\bar{\rho}})\,\epsilon_n \simeq \epsilon_n(B(\mathbb{Z}/p)^n)^{k\bar{\rho}_n} \vee \epsilon_{n-1}(B(\mathbb{Z}/p)^{n-1})^{k\bar{\rho}_{n-1}}.$$

That is, the duals of the $L(n)_{-k}$'s are identified as

$$D(L(n)_{-k}) = D(\epsilon_n\, BV_n^{-k\bar{\rho}}) \simeq L(n)_k \vee L(n-1)_k,$$

in the p-complete setting.

5.3 Application to the generalized Tate construction. $t_n S^0$.

We are interested in saying something about the generalized Tate construction discussed in 2.1. The n-th Tate construction on a spectrum E is defined as the colimit of Steinberg idempotents of function spectra from the Steinberg pieces in Thom spectra of classifying spaces of the elementary abelian p-group of rank n, twisted by increasingly negative copies of the reduced regular real representation:

$$t_n(E) := \lim_{\vec{k}} F(L(n)_{-k}, E)$$

Use the results of the previous sections to get homotopy types for the base case: t_n evaluated at the sphere spectrum.

$$\begin{aligned} t_n(S^0) &:= \lim_{\vec{k}} F(L(n)_{-k}, S^0) \\ &= \lim_{\vec{k}} DL(n)_{-k}, \\ &= \lim_{\vec{k}} L(n)_{k,} \vee L(n-1)_k \\ &= \begin{cases} S^0, & n = 1, \\ pt, & n > 1. \end{cases} \end{aligned}$$

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