Koszul duality and the bar spectral sequence

by

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Abstract

The bar spectral sequence for algebras over a spectral operad relates Koszul duality phenomena in several contexts. In this thesis, we apply this classical tool to the Koszul dual pair given by the (non-unital) $\mathbb{E}_\infty$-operad and the spectral Lie operad over $\mathbb{F}_p$. The bar spectral sequence for $\mathbb{E}_\infty$-algebras yields the structure of operations on mod p Topological André-Quillen cohomology and the homotopy groups of spectral partition Lie algebras, building on the work of Brantner-Mathew. In the colimit, the unary operations are Koszul dual to the Dyer-Lashof algebra. On the other hand, the bar construction against certain spectral Lie algebras models labeled configuration spaces by a theorem of Knudsen. The associated bar spectral sequence yields new results on their mod p homology at low weights, as well as interesting patterns of universal differentials. We also record an attempt with Andrew Senger on detecting these differentials via deformation of the bar comonad.

Thesis Supervisor: Haynes R. Miller
Title: Emeritus Professor of Mathematics

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Chapter 1

Introduction

The study of Koszul duality in classical algebra dates back to Moore [Moo70] and Quillen [Quui68], who examined adjunctions between the category of dg-Lie algebras and the category of cocommutative coaugmented dg-coalgebras over $\mathbb{Q}$ via a certain bar construction. Quillen further showed that this adjunction restricts to an equivalence of homotopy categories between the full subcategories of connected dg-Lie algebras and 1-connected cocommutative coaugmented dg-coalgebras over $\mathbb{Q}$. Later on, Ginzburg-Kapranov [GK94] and Getzler-Jones [GJ94] observed that this adjunction of categories of algebras reflects the *Koszul duality* of the quadratic operads Comm and Lie in the category of chain complexes over $\mathbb{Q}$ via the bar construction. On the other hand, Priddy [Pri70] developed the notion of Koszul duality for augmented quadratic algebras over a field k. For A a Koszul algebra, its Koszul dual $H^*(A) = \mathrm{Ext}^*_A(k,k)$ has a presentation with generators and relations the linear dual of those of A.

The phenomenon of Koszul duality of operads in higher algebra was first studied by Ching [Chi05] and later vastly generalized by Lurie in [Lur17]. For any non-unital augmented operad $\mathcal{O}$ in a stable presentable symmetric monoidal ∞-category $\mathscr{C}$ with geometric realizations, its *Koszul dual* is the operad given by the Spanier Whitehead dual $\mathbb{D}B(\mathcal{O})$ of the operadic bar construction $B(\mathcal{O}) := |\mathrm{Bar}(1,\mathcal{O},1)|$.

It is then natural to ask if the Koszul duality of operads gives rise to a Koszul duality of algebras in higher algebra, i.e., if the bar construction yields an adjunction at the level of algebras over operads. In [FG12], Francis and Gaistgory showed that the bar construction

$|\mathrm{Bar}_\bullet(\mathrm{id},\mathcal{O},-)|$ on $\mathcal{O}$-algebras refines to a bar-cobar adjunction

$$\mathrm{Alg}_{\mathcal{O}}(\mathscr{C}) \underset{\mathrm{coBar}_{B(\mathcal{O})}}{\overset{\mathrm{Bar}_{\mathcal{O}}}{\rightleftarrows}} \mathrm{coAlg}^{\mathrm{d.p.}}_{B(\mathcal{O})}(\mathscr{C})\ .$$

Here the superscript d.p. stands for divided power coalgebras, indicating that the $B(\mathcal{O})$-coalgebras C in the image of the bar construction are equipped with structure maps that factor through the norm maps

$$(B(\mathcal{O})(n)\otimes C^{\otimes n})_{h\Sigma_n} \to (B(\mathcal{O}(n))\otimes C^{\otimes n})^{h\Sigma_n}.$$

In the case where $\mathcal{O}$ is the non-unital $\mathbb{E}_\infty$-operad, Francis-Gaitsfory [FG12], Ching-Harper [CH19] and Brantner-Mathew [BM19] examined on which subcatogries do the bar-cobar adjunction restricts to an equivalence in several contexts. Their results are subsumed by the upcoming work of Heuts [Heu], who showed that if $\mathcal{O}(1)\simeq 1$, the monoidal unit of $\mathscr{C}$, the bar-cobar adjunction restricts to an equivalence of ∞-categories on the subcategory of complete $\mathcal{O}$-algebras and cocomplete $B(\mathcal{O})$-coalgebras with divided power structure. Furthermore, these subcategories are the optimal for a general operad $\mathcal{O}$. Here completeness is with respect to the n-truncated operads $\tau_{\leq n}(\mathcal{O})$, and cocompleteness is defined dually.

The next natural question to one could ask is whether the Koszul duality of algebras over operads is reflected in some form of duality between natural operations on the homology groups of algebras over operads. In this thesis, we use the bar spectral sequence to investigate this question when the operad is Koszul dual pair given by the non-unital $\mathbb{E}_\infty$-operad and the spectral Lie operad over $H\mathbb{F}_p$. For $\mathcal{O}$ a non-unital augmented operad in the ∞-category of $H\mathbb{F}_p$-module spectra and an $\mathcal{O}$-algebra A, the bar spectral sequence

$$E^2_{s,t} = \pi_s\pi_t\mathrm{Bar}_\bullet\big(\mathrm{id},\mathcal{O},A\big) = \pi_{s,t}\mathrm{Bar}_\bullet\big(\mathrm{id},\hat{O},\pi_*(A)\big) \Rightarrow \pi_{s+t}|\mathrm{Bar}_\bullet(\mathrm{id},\mathcal{O},A)|$$

is obtained by skeletal filtration of the geometric realization of the bar construction. Here $\hat{O}$ is the monad on the 1-category of $\mathbb{F}_p$-modules that parametrizes natural operations on the homotopy groups of $\mathcal{O}$-algebras.

1.1 The bar spectral sequence for non-unital $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebras

Partition Lie algebras are the key objects in the emerging field of formal moduli problems in characteristic p. Work of Brantner and Mathew [BM19] showed that there is an equivalence of ∞-categories between spectral formal moduli problems over $\mathbb{F}_p$ and spectral partition Lie algebras, generalizing the characteristic 0 phenomenon studied by Drinfeld [Dri], Pridham [Pri10], Lurie [Lur11], and many others. A restricted version of this equivalence establishes spectral partition Lie algebras over $\mathbb{F}_p$ as divided power algebras Koszul dual to non-unital $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebras, implementing the Koszul duality between the non-unital $\mathbb{E}_\infty$-operad and the spectral Lie operad.

Since spectral partition Lie algebras are algebras over a certain monad $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$, the homotopy groups of free spectral partition Lie algebras $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^{i_1}H\mathbb{F}_p\oplus\cdots\oplus\Sigma^{i_k}H\mathbb{F}_p)$ parametrize all natural k-ary operations on the homotopy groups of spectral partition Lie algebras as $(i_1,\dots,i_k)$ varies. In [BM19], Brantner and Mathew obtained bases for homotopy groups of free spectral partition Lie algebras. Nonetheless, their method did not provide explicit descriptions of the nature of the operations, nor were the relations among the operations clarified. On the other hand, spectral partition Lie algebras are closely related to topological André-Quillen objects introduced by Kriz [Kri93] and Basterra [Bas99].

Definition 1.1.1. For any object R in the category of $\mathbb{E}_\infty$-$\mathbb{S}$-algebras with a map to $H\mathbb{F}_p$, the *topological André-Quillen object* of R is given by

$$\mathrm{TAQ}(R,\mathbb{S};H\mathbb{F}_p)\simeq|\mathrm{Bar}_\bullet(H\mathbb{F}_p\otimes(-),\mathbb{E}_\infty,R)|.$$

The nth $\mathbb{F}_p$-linear TAQ cohomology of an $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebra R is given by

$$\mathrm{TAQ}^n(R,H\mathbb{F}_p;H\mathbb{F}_p)=[\Sigma^{-n}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{E}_\infty\otimes H\mathbb{F}_p,R)|,H\mathbb{F}_p]_{\mathrm{Mod}_{H\mathbb{F}_p}}.$$

The $\mathbb{F}_p$-linear TAQ cohomology $\mathrm{TAQ}^*(R,H\mathbb{F}_p;H\mathbb{F}_p)$ of $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebras R has representing objects trivial square-zero extensions, and the reduced $\mathbb{F}_p$-linear TAQ cohomology groups of trivial algebras $H\mathbb{F}_p\oplus\Sigma^{i_1}H\mathbb{F}_p\oplus\cdots\oplus\Sigma^{i_k}H\mathbb{F}_p$ parametrize all natural k-ary operations. By [BM19], there is an isomorphism

$$\begin{aligned}
&\pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^{i_1}H\mathbb{F}_p\oplus\cdots\oplus\Sigma^{i_k}H\mathbb{F}_p))\oplus\mathbb{F}_p\\
&\cong \mathrm{TAQ}^{-*}(H\mathbb{F}_p\oplus\Sigma^{-i_1}H\mathbb{F}_p\oplus\cdots\oplus\Sigma^{-i_k}H\mathbb{F}_p, H\mathbb{F}_p; H\mathbb{F}_p).
\end{aligned}$$

Hence natural operations on the homotopy groups of spectral partition Lie algebras agree with cohomology operations on the (reduced) $\mathbb{F}_p$-linear TAQ cohomology of $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebras. In unpublished work, Kriz computed the $\mathbb{F}_2$-linear TAQ cohomology on a generator in non-negative degree [Kri93]. Around the same time, Basterra and Mandell announced a computation of unary operations and their relations as the Koszul dual to Dyer-Lashof operations on $\mathbb{F}_p$-linear TAQ cohomology of connective objects for $p > 2$ and observed a shifted restricted Lie algebra structure, but a proof never appeared.

In Chapter 3, we use the dual of the bar spectral sequence (2.2) with $\mathcal{O} = \mathbb{E}_\infty^{\mathrm{nu}} \otimes H\mathbb{F}_p$ the nonunital $\mathbb{E}_\infty$-operad in $\mathrm{Mod}_{H\mathbb{F}_p}$ to identify the structure of the homotopy groups of spectral partition Lie algebras and $\mathbb{F}_p$-linear TAQ cohomology, noting that the spectral sequence collapses on the E^2-page when A is a trivial algebra. The unary operations are parametrized by a *power ring* $\mathcal{P}$ (Definition 3.5.4), which is a collection of unstable Ext groups over the Dyer-Lashof algebra, with composition product given by a sheared Yoneda product. The verification of the law of composition makes use of a general result of Brantner [Bra17] that demonstrates the compatibility of the algebraic Koszul duality on the E^2-page of the (dual) bar spectral sequence with the monadic Koszul duality that the E^∞-page assembles to when there are no higher differentials in the spectral sequence. As the degree of a homotopy class gets arbitrarily large, the colimit of the algebra of additive unary operations on that class is the Koszul dual algebra of the Dyer-Lashof algebra. Then we construct a shifted Lie bracket on the homotopy groups of spectral partition Lie algebras, and used a homotopy fixed points spectral sequence to detect a restriction map on the shifted Lie bracket.

Theorem 1.1.2. *(Theorem 3.5.5 and 3.6.6)*

1. *The homotopy groups of a spectral partition Lie algebra over $H\mathbb{F}_2$, or the reduced TAQ cohomology of an $\mathbb{E}_\infty$-$H\mathbb{F}_2$-algebra, form a module over the power ring $\mathcal{P}$ of additive unary operations.*

2. *The weight 2 additive operations are given by the collection* $R^i \in \mathcal{P}_j^{j-i}[1]$ *for all* $i, j \in \mathbb{Z}$ *satisfying* $i > -j+1$*, subject to the Adem relations*

$$R^a R^b = \sum_{a+b-c \geq 2c,\, c > -j+1} \binom{b-c-1}{a-2c} R^{a+b-c} R^c$$

in $\mathcal{P}_j^{j-a-b}[2]$ *for all* $a, b \in \mathbb{Z}$ *satisfying* $b - j < a < 2b$ *and* $b > -j+1$.

3. *There is a nonadditive unary operation* $R^{-|x|+1}(x)$ *for any homotopy class* x *that serves as the restriction* $x^{[2]}$ *on* x*. The restriction on a sum of classes* x *and* y *in different degrees is given by*

$$(x+y)^{[2]} = R^{-|x|+1}(x) + R^{-|y|+1}(y) + [x, y].$$

The bracket is compatible with the unary operations in the sense that $[y, \alpha(x)] = 0$ *for any homotopy class* x, y *and unary operation* α *of weight greater than* 1 *that is not an iteration of the restriction.*

4. *The operations* R^i *and the shifted restricted Lie bracket generate all natural operations under the above relations. A basis for unary operations on a degree* j *class is given by the collection of all monomials* $R^{i_1} R^{i_2} \cdots R^{i_l}$ *such that* $i_l > -j$ *and* $i_m \geq 2i_{m+1}$ *for* $1 \leq m < l$.

Theorem 1.1.3. *(Theorem 3.5.6 and 3.6.6)*

1. *The homotopy groups of a spectral partition Lie algebra over* $H\mathbb{F}_p$*, or the reduced TAQ cohomology of any* $\mathbb{E}_\infty$*-*$H\mathbb{F}_p$*-algebra, form a module over the power ring* $\mathcal{P}$.

2. *The weight* p *unary operations are given by the collection* $\beta^\varepsilon R^i \in \mathcal{P}_j^{j-2(p-1)i-\varepsilon}[1]$ *for* $\varepsilon = 0, 1$ *and any* $2i > -j$*, subject to the Adem relations*

$$\beta R^a \beta R^b = \sum_{a+b-c > pc, 2c > -j} (-1)^{a-c+1} \binom{(p-1)(b-c)-1}{a-pc-1} \beta R^{a+b-c} \beta R^c$$

in $\mathcal{P}_j^{j-2(p-1)(a+b)-2}[2]$ *for all* $a, b \in \mathbb{Z}$ *satisfying* $a \leq pb$*,* $2b > -j$*,* $2a > 2(p-1)b - j$

,

$$R^a \beta R^b = \sum_{a+b-c \geq pc, 2c > -j} (-1)^{a-c} \binom{(p-1)(b-c)}{a-pc} \beta P^{a+b-c} R^c$$
$$- \sum_{a+b-c > pc, 2c > -j} (-1)^{a-c} \binom{(p-1)(b-c)-1}{a-pc-1} R^{a+b-c} \beta R^c$$

in $\mathcal{P}_j^{j-2(p-1)(a+b)-1}[2]$ *for all* $a, b \in \mathbb{Z}$ *satisfying* $a \leq pb$, $2b > -j$, $2a > 2(p-1)b + 1 - j$,

$$\beta^\varepsilon R^a R^b = \sum_{a+b-c \geq pc, 2c > -j} (-1)^{a-c} \binom{(p-1)(b-c)-1}{a-pc} \beta^\varepsilon R^{a+b-c} R^c$$

in $\mathcal{P}_j^{j-2(p-1)(a+b)-\varepsilon}[2]$ *for all* $a, b \in \mathbb{Z}$ *satisfying* $a < pb$, $2b > -j$, $2a > 2(p-1)b - j$, *and* $\varepsilon \in \{0, 1\}$.

3. *For all odd* j *and* x *a homotopy class in degree* j*, the restriction* $x^{[p]}$ *is the bottom operation* $R^{(-j+1)/2}(x)$ *up to a unit* λ_j*, i.e.,* $[y, \lambda_j R^{(-j+1)/2}(x)] = [[\cdots[[y,x],x]\cdots],x]$ *for any class* y*, where bracketing with* x *is iterated* p *times on the right hand side. The restriction map on a sum of classes* x *and* y *in odd degrees* $j \neq k$ *is given by*

$$(x+y)^{[p]} = \lambda_j R^{(-j+1)/2}(x) + \lambda_k R^{(-k+1)/2}(y) + \sum_{i=1}^{p-1} \frac{s_i}{i}(x,y),$$

where s_i *is the coefficient of* t^{i-1} *in the formal expression* $\mathrm{ad}(tx+y)^{p-1}(x)$*. Furthermore,* $[y, \alpha(x)] = 0$ *for any homotopy class* x, y *and* α *a unary operation of weight greater than* 1*, unless* x *is in odd degree and* α *an iteration of the restriction.*

4. *The operations* $\beta^\varepsilon R^i$ *and the shifted restricted Lie bracket generate all natural operations under the above relations. A basis for unary operations on a degree* j *class with* j *odd is given by all monomials* $\beta^{\varepsilon_1} R^{i_1} \beta^{\varepsilon_2} R^{i_2} \cdots \beta^{\varepsilon_l} R^{i_l}$ *such that* $2i_l > -j$ *and* $i_m \geq p i_{m+1} + \varepsilon_{m+1}$ *for* $1 \leq m < l$*. If* j *is even, a basis is given by* $\beta^{\varepsilon_1} R^{i_1} \beta^{\varepsilon_2} R^{i_2} \cdots \beta^{\varepsilon_l} R^{i_l} B^\varepsilon$ *such that* $2i_l > -(1+\varepsilon)j - \varepsilon$ *and* $i_m \geq p i_{m+1} + \varepsilon_{m+1}$ *for* $1 \leq m < l$.

As an immediate application, we obtain a computation of natural operations and relations on the mod p TAQ cohomology $\mathrm{TAQ}^*(R,\mathbb{S};H\mathbb{F}_p)$ of $\mathbb{E}_\infty$-$\mathbb{S}$-algebras R, which is based on conversations with Tyler Lawson.

Since the functor $\mathrm{TAQ}^i(-,\mathbb{S};H\mathbb{F}_p)$ has representing object the trivial square-zero extension $\mathbb{S}\oplus\Sigma^i H\mathbb{F}_p$ for all i, operations and relations are again parametrized by the mod p TAQ cohomology on the trivial square-zero extensions $\mathbb{S}\oplus\Sigma^{i_1}H\mathbb{F}_p\oplus\cdots\oplus\Sigma^{i_k}H\mathbb{F}_p$. Using the base change formula

$$\mathrm{TAQ}(-,\mathbb{S};H\mathbb{F}_p)\underset{\mathbb{S}}{\otimes}H\mathbb{F}_p\simeq\mathrm{TAQ}(-\underset{\mathbb{S}}{\otimes}H\mathbb{F}_p,H\mathbb{F}_p;H\mathbb{F}_p),$$

we deduce immediately from Theorem 1.1.2 and 1.1.3 the structure of natural operations on the mod p TAQ cohomology $\mathbb{E}_\infty$-$\mathbb{S}$-algebras.

Theorem 1.1.4. *(Corollary 3.7.1, Proposition 3.7.2) For any tuple $(i_1,\dots i_k)$ of integers, the k-ary cohomology operations*

$$\prod_{i=1}^{k}\mathrm{TAQ}^{i_l}(-,\mathbb{S};H\mathbb{F}_p)\to\mathrm{TAQ}^m(-,\mathbb{S};H\mathbb{F}_p).$$

are parametrized by the homological degree $-m$ part of $\mathrm{Free}^{\mathrm{sLie}^\rho_{\mathcal{P}}}(\Sigma^{-i_1}\mathcal{A}\oplus\cdots\oplus\Sigma^{-i_k}\mathcal{A})$, where $\mathcal{A}$ is the Steenrod algebra graded homologically. All operations vanish on the unit except for scalar multiplication. The Steenrod operations commute with the bracket via the Cartan formula and the $\mathbb{F}_p$-linear TAQ cohomology operations via the Nishida relations on cohomology of the second extended power:

1. *For $p=2$ we have*

$$Sq^a[x,y]=\sum_i[Sq^i(x),Sq^{a-i}(y)],$$

$$Sq^aR^{-|x|+1}(x)=\sum\binom{|x|-c}{a-2c}R^{a+|x|+1-c}Sq^c(x)+\sum_{l<k,l+k=a}[Sq^l(x),Sq^k(x)],$$

$$Sq^aR^b(x)=\sum\binom{b-1-c}{a-2c}R^{a+b-c}Sq^c(x),\ \ b>-|x|+1.$$

2. *For $p > 2$ we have*

$$P^a[x,y] = \sum_i [P^i(x), P^{a-i}(y)], \;\; \beta P^a[x,y] = \sum_i ([\beta P^i(x), P^{a-i}(y)] + [P^i(x), \beta P^{a-i}(y)]).$$

For any class x and all $2j > -|x|+1$, the Nishida relations are

$$P^n \beta R^j = (-1)^{n-i} \sum_i \binom{(j-i)(p-1)}{n-pi} \beta R^{n+j-i} P^i + (-1)^{n-i} \sum_i \binom{(j-i)(p-1)-1}{n-pi-1} R^{n+j-i} \beta P^i,$$

$$P^n R^j = (-1)^{n-i} \sum_i \binom{(j-i)(p-1)-1}{n-pi} R^{n+j-i} P^i,$$

as well as

$$\begin{aligned} P^n R^j(x) = &(-1)^{n-i} \sum_i \binom{(j-i)(p-1)-1}{n-pi} R^{n+j-i} P^i(x) \\ &+ \frac{1}{\lambda_{|x|}} \sum_{I, \sigma \in \Sigma_p, \sigma(1)=1} [[\cdots[[P^{i_{\sigma(1)}}(x), P^{i_{\sigma(2)}}(x)], P^{i_{\sigma(3)}}(x)] \cdots], P^{i_{\sigma(p)}}(x)] \end{aligned}$$

when the degree of x is odd and $2j = -|x|+1$, where the bracket term sums over all nondecreasing sequences $I = (0 \le i_1 \le i_2 \le \ldots \le i_p)$ with $i_1 + i_2 + \cdots + i_p = n$, and $\lambda_{|x|}$ is a fixed unit given in Theorem 1.1.3.(3).

1.2 The bar spectral sequence for spectral Lie algebras

Spectral Lie algebras are algebras over the spectral Lie operad $s\mathscr{L}$, generalizing the notion of Lie algebras over a field k to the $(\infty-)$category of spectra. The homology operad $\{H_*(\partial_n(\mathrm{Id}); k)\}_n$ of the spectral Lie operad recovers the ordinary Lie operad over k up to a shift [GK94][Fre00][Chi05].

In Chapter 4, we study the bar spectral sequence for spectral Lie algebras in $\mathrm{Mod}_{H\mathbb{F}_p}$. To compute the E^2-page of the bar spectral sequence

$$E^2_{s,t} = \pi_s \pi_t \mathrm{Bar}_\bullet\big(\mathrm{id}, s\mathscr{L}, A \otimes H\mathbb{F}_p\big) \Rightarrow \pi_{s+t} |\mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, A) \otimes H\mathbb{F}_p| \tag{1.1}$$

for A a spectral Lie algebra, it is necessary to understand the structure of the mod p homology of spectral Lie algebras. In [Beh12], Behrens constructed Dyer-Lashof-type unary operations $\bar{Q}^j$ on the mod 2 homology of spectral Lie algebras and determined the relations among these operations. Building on the work of Behrens, Antolín-Camarena [AC20] showed that the structure of the mod 2 homology of spectral Lie algebras is parametrized by a monad $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$. An algebra over $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$ is an unstable module over the algebra $\bar{\mathcal{R}}$ of Behrens' operations, along with a shifted Lie algebra structure such that brackets of operations always vanish and the self-bracket on an element x is identified with the bottom nonvanishing operation $\bar{Q}_0 := \bar{Q}^{|x|}$ on x. Following the approach of Behrens and Antolín-Camarena, Kjaer [Kja18] constructed Dyer-Lashof-type unary operations $\overline{\beta^{\varepsilon}Q^j}$ on the mod p homology of spectral Lie algebras for $p > 2$ and proved that brackets of operations always vanish. Recently, Konovalov [Kon23] completed the study of the structure of these operations by computing the relations among the unary relations. Hence the E^2-page of the bar spectral sequence is given by the following algebraic object:

Definition 1.2.1. The *Quillen homology* of a $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra $\mathfrak{g}$ is the total left derived functor

$$\mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{*,*}(\mathfrak{g}) := H_{*,*}\mathbb{L}Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\mathfrak{g}) \simeq \pi_{*,*}\mathrm{Bar}_\bullet\left(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g}\right).$$

The main challenge in computing the Quillen homology of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras when $p = 2$ arises from the identification of the self-bracket with the bottom operation $\bar{Q}_0$, which precludes a factorization of the free $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra functor as a composition of the free $\mathrm{Lie}^s_{\mathbb{F}_2}$-algebra functor followed by the free $\bar{\mathcal{R}}$-algebra functor. Furthermore, since the category of $\mathrm{Lie}^s_{\mathbb{F}_2}$-algebras is nonabelian, we cannot resort to the usual Grothendieck spectral sequence and the generalized Grothendieck spectral sequence becomes unwieldy very fast.

To get around these obstacles, we construct a May spectral sequence with respect to a length filtration on $\bar{\mathcal{R}}$-module. The E^1-page is bounded above by the Quillen homology of a variant of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras whose the unary and binary operations are disentangled, thus admitting a factorization as the homotopy group of the total complex of a double complex. The homotopy groups of these total complexes can be computed with the machinery of Koszul duality for additive Koszul algebras [Pri70] and Lie algebras [BHK19][CE48][May66A][Pri70], as well as explicit understanding of the Bousfield-

Cartan-Dwyer operations

$$\gamma_i : \pi_{h+r,t}(\Lambda^h(V_\bullet)) \to \pi_{2h+1+r+i,2t-1}(\Lambda^{2h+1}(V_\bullet)), 1 \leq i \leq r$$

on the homotopy group of the free simplicial shifted graded exterior algebra $\Lambda^\bullet(V_\bullet)$ on a simplicial $\mathbb{F}_2$-module $V_\bullet$ [Bou68][Dwy80a]. This allows us to obtain general upper bounds for the Quillen homology of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras and precise formulae in low weights.

Furthermore, we are able to provide a full computation of the Quillen homology of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras in universal cases. Denote by $\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}$ the free allowable $\bar{\mathcal{R}}$-module functor. The category $\mathrm{Mod}_{\bar{\mathcal{R}}}$ is stable under the desuspension functor Σ^{-1} of $\mathbb{F}_2$-modules. Then for $1 \leq n \leq \infty$, the $\bar{\mathcal{R}}$-module $\Sigma^{-n}\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2)$ is an $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra whose Lie^s-structure is trivial. Note that when $n = \infty$, this is the trivial $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra

$$\mathrm{colim}_{i\to\infty}\Sigma^{-n}\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2) \cong \Sigma^k\mathbb{F}_2.$$

Theorem 1.2.2. *[Zha21] (Theorem 4.2.27) The Quillen homology*

$$\mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{*,*}(\Omega^n\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2)) \cong \pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \Omega^n\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2))$$

of the $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$*-algebra* $\Omega^n\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2), 1 \leq n \leq \infty$ *is isomorphic as a bigraded vector space to the shifted graded exterior algebra on generators* $\gamma_I\bar{Q}_J(x_k)$ *satisfying the following conditions:*

1. $I = (i_1, \dots, i_m)$ *satisfies* $i_l \geq 2i_{l+1}$ *for* $l < m$, $i_m \geq 2$, *and* $i_1 - i_2 - \cdots - i_m \leq r$;

2. $J = (j_1, \dots, j_r)$ *satisfies* $0 \leq j_l \leq j_{l+1} + 1$ *for* $l < r$, $0 \leq j_r < n$, *and if* $j_1 = 0$ *then either* $r = 1$ *or* $i_m = 2$.

Note in particular that in natural operations on a class of degree k in the Quillen homology of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras are given by the Quillen homology of the trivial $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras $\Sigma^k\mathbb{F}_2$, and the above theorem gives us a dimension count.

1.2.1 Application to labeled configuration spaces

One application of the Quillen homology of $\mathrm{Lie}^s_{\mathcal{R}}$-algebras is the computation of the mod p homology of labeled configuration spectrum

$$B_k(M,X) := \Sigma^\infty_+ \mathrm{Conf}_k(M) \underset{h\Sigma_k}{\otimes} X^{\otimes k}$$

of k points in a parallelizable manifold M with labels in a spectrum X. The study of labeled configuration spaces dates back to as early as Segal [Seg73] and McDuff [McD75] as generalizations of the unordered configuration space $\Sigma^\infty_+ B_k(M) = B_k(M;\mathbb{S}^0)$ of k points in M. The rational homology groups of labeled configuration spaces are well understood in cases of interests via classical methods, see for instance [BC88][BCT89][Tot96]. Nonetheless, the mod p homology groups have remained mostly intractable. Classically, the only known cases are the following:

(1). $M = \mathbb{R}^\infty$ with arbitrary labeling spectra by May [May72] and McClure [BMMS88, IX], and $M = \mathbb{R}^n$ by F. Cohen [CLM76, III]. Then $\bigoplus_{k\geq 0} B_k(M;X)$ is the free $\mathbb{E}_n$-algebra on X. Its mod p homology is captured by Dyer-Lashof operations and Browder brackets as a functor of $H_*(X;\mathbb{F}_p)$.

(2). Arbitrary manifold M with labeling spectrum $X = \Sigma^\infty S^r$, where either $p = 2$ or $p > 2$ and $n + r$ is odd [BCT89][ML88][BCM93]. In these cases, there is a homology decomposition

$$H_*(\bigoplus_{k\geq 0} B_k(M;\mathbb{S}^r)) \cong \bigotimes_i H_*(\Omega^i S^{n+r})^{\otimes \dim H_i(M)}. \tag{1.2}$$

In particular, the homology depends only on the $\mathbb{F}_p$-module $H_*(M;\mathbb{F}_p)$.

The most recent developments in the computation of the homology of labeled configuration spaces originate from a result of Knudsen [Knu18]. Using factorization homology, he established an equivalence of spectra

$$\bigoplus_{k\geq 1} B_k(M;X) \simeq |\ \mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, \mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)^{M^+})\ |. \tag{1.3}$$

Here M is a parallelizable n-manifold, $s\mathscr{L}$ is the monad associated to the free spectral Lie algebra functor $\mathrm{Free}^{s\mathscr{L}}$, and $(-)^{M^+}$ the cotensor with the one-point compactification of M in the ∞-category of spectral Lie algebras. Knudsen's result opens up a path for extracting information about the homology of labeled configuration spaces. In [Knu17], Knudsen provided a general formula for the Betti numbers of unordered configuration spaces by observing that the bar spectral sequence for the bar construction (1.3) with rational coefficients, which we abbreviate as Knudsen's spectral sequence, collapses at the E^2-page. Building on Knudsen's work, Drummond-Cole and Knudsen [DCK17] computed the Betti numbers of unordered configuration spaces of surfaces. In [BHK19], Brantner, Hahn, and Knudsen studied Knudsen's spectral sequence with coefficients in Morava E-theory at an odd prime. They computed the weight p part of the labeled configuration spaces in $\mathbb{R}^n$ and punctured genus g surfaces $\Sigma_{g,1}$ for $g \geq 1$ with coefficient in a sphere.

In the second half of Chapter 4, we adapt their approach to study the mod p homology of $B_k(M,X)$ for M a parallelizable n-manifold and X any spectrum by examining the mod p Knudsen's spectral sequence, i.e., the bar spectral sequence (1.1) with coefficients in $\mathbb{F}_p$ applied to the bar construction (1.3).

When $p=2$, our general understanding of the E^2-page, i.e., the Quillen homology of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras, allows us to obtain an upper bound for $H_*(B_k(M,X);\mathbb{F}_2)$ in Theorem 4.4.5 for arbitrary parallelizable manifold M and spectrum X. In the universal case $M=\mathbb{R}^\infty$ and $X=\mathbb{S}^r$, the bar spectral sequence has E^2-page given by Theorem 4.2.27. Comparing with the computation of the homology of free $\mathbb{E}_\infty$-algebras [Ade52, DL62, May70, BMMS88], we see that there are infinitely many higher differentials and conjecture the following universal pattern, which can be verified in low weight by sparsity arguments:

Conjecture 1.2.3 (Conjecture 4.3.5)**.** *Each page of the spectral sequence*

$$E^2_{s,t} = \mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{s,t}(\Sigma^k\mathbb{F}_2) \Rightarrow \pi_{s+t}\mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, \Sigma^k\mathbb{F}_2)$$

is an exterior algebra. The higher differentials act on the exterior generators of the E^2-page as follows, see Figure 4-1:

1. *For an exterior generator* $\alpha = \bar{Q}_{j_1} \cdots \bar{Q}_{j_m}(x_k)$ *on the* E^2*-page, we have*

$$d^{r+1}\gamma_{r+1}(\alpha) = \bar{Q}_r(\alpha)$$

for $r < m$ *and* $r \leq j_1 + 1$.

2. *For an exterior generator* $\beta = \gamma_{n+1}\bar{Q}_{j_1} \cdots \bar{Q}_{j_m}(x_k)$ *on the* E^2*-page, we have*

   (a) $d^{n+1}(\beta) = \bar{Q}_n\bar{Q}_{j_1} \cdots \bar{Q}_{j_m}(x_k)$,

   (b) $d^{n+1}\gamma_{m+n+1}(\beta) = d^{n+1}(\beta) \otimes \beta$,

   (c) $\gamma_{l-2}d^{n+1}(\beta) = d^{2n+1}\gamma_{n+l+1}(\beta)$ *for* $n+1 < l < m$.

These generate all higher differentials under further applications of the γ_i *operations in accordance with (2).(b) and (2).(c), as well as the exterior product.*

For an arbitrary M, sparsity arguments show that the weight k part of Knudsen's spectral sequence with $\mathbb{F}_2$ coefficients always collapses on the E^2-page for small k. In particular, we observe that the $\mathbb{F}_2$-module $H_*(B_k(M;X))$ depends on and only on the cohomology *ring* $H^*(M^+;\mathbb{F}_2)$ when $k = 2,3$ and $H_*(X;\mathbb{F}_2)$ has at least two generators. This is in contrast to the case when $X = \mathbb{S}^r$, in that the equivalence (1.2) depends only on the $\mathbb{F}_2$-module $H^*(M;\mathbb{F}_2)$ [BCT89].

When $p > 2$, the weight $k \leq p$ part of the E^2-page of Knudsen's spectral sequence with $\mathbb{F}_p$ coefficients can be described in terms of $\mathrm{Lie}^s_{\mathbb{F}_p}$-algebra homology. In particular, the spectral sequence collapses when $k = 2$ or $k = 3$ and $p \geq 5$ (Corollary 4.5.9). As a corollary, we deduce the following:

Corollary 1.2.4. *(Remark 4.5.10) When* $X = \mathbb{S}^r$ *and* $k = 2,3$*, the* $\mathbb{F}_p$*-module* $H_*(B_k(M;\mathbb{S}^r);\mathbb{F}_p)$ *depends on and only on the cohomology ring* $H^*(M^+;\mathbb{F}_p)$ *when* $r+l$ *is even.*

This is in contrast to the case when $r+l$ is odd in the equivalence (1.2) [BCT89].

1.3 Work in progress: bar spectral sequence via deformation of comonads

In the universal case $M = \mathbb{R}^\infty$ and $X = \mathbb{S}^k$, Knudsen's spectral sequence

$$E^2_{s,t} = \pi_{s,t}\mathrm{Bar}_\bullet\big(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \Sigma^k\mathbb{F}_2\big) \cong \mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{s,t}(\Sigma^k\mathbb{F}_2) \Rightarrow \pi_{s+t}|\mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, \Sigma^k H\mathbb{F}_2)| \tag{1.4}$$

has E^2-page given by Theorem 4.2.27. Comparing with the homology of free $\mathbb{E}_\infty$-algebras [May66A, BMMS88], we see that there are infinitely many higher differentials and observe the following pattern:

Conjecture 1.3.1. *[Zha21] Each page of the spectral sequence (1.4) is an exterior algebra. The higher differentials act on the exterior generators of the E^2-page as follows:*

1. *$d^{r+1}\gamma_{r+1}(\alpha) = \bar{Q}_r(\alpha)$ for an exterior generator $\alpha = \bar{Q}_{j_1}\cdots\bar{Q}_{j_m}(x_k)$ on the E^2-page with $r < m$ and $r \leq j_1 + 1$.*
2. *For an exterior generator $\beta = \gamma_{n+1}\bar{Q}_{j_1}\cdots\bar{Q}_{j_m}(x_k)$ on the E^2-page, we have $d^{n+1}\gamma_{m+n+1}(\beta) = d^{n+1}(\beta)\otimes\beta$ and $\gamma_l d^{n+1}(\beta) = d^{2n+1}\gamma_{n+l-1}(\beta)$ for $n+2 < l \leq m$.*

These generate all higher differentials under further applications of γ_i and the exterior product.

While the pattern of universal differentials is similar to classical ones studied by Dwyer [Dwy80b], the operations $\bar{Q}_j$ on coalgebras over the comonad $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, -)$ increase filtration and hence cannot be constructed using classical methods. In joint work in progress with Andrew Senger, we use a suitable deformation of the comonad associated to the bar construction $|\mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, -)|$ on spectral Lie algebras in $\mathrm{Mod}_{H\mathbb{F}_2}$ to propagate weight two differentials to higher weights.

More generally, let k be a field and $\mathcal{O}$ a spectral operad. The comonad $|\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{O}, -)|$ on Mod_{Hk} arises from the adjunction $\mathrm{cot} \dashv \mathrm{sqz} : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_{Hk}) \to \mathrm{Mod}_{Hk}$, and admits a lift to a comonad $\mathrm{cot}^* \circ \mathrm{cot}_*$ on the ∞-category $\mathcal{P}_\Sigma(\mathrm{Mod}^{\mathrm{ff}}_{Hk})$ of product-preserving presheaves over the ∞-category of finite-free Hk-modules. We note that there is an equivalence Φ :

$\mathcal{P}_\Sigma(\mathrm{Mod}^{\mathrm{ff}}_{Hk}) \xrightarrow{\simeq} (\mathrm{Mod}^{\mathrm{Fil}}_{Hk})_{\geq 0}$, where the target is the ∞-category of Postnikov-connective filtered Hk-modules. An object in $(\mathrm{Mod}^{\mathrm{Fil}}_{Hk})_{\geq 0}$ is a diagram $C_\bullet = \cdots \to C_1 \to C_0 \to C_{-1} \to \cdots$ in Mod_{Hk} such that C_n is n-connective for all n.

There is a realization functor $\mathrm{Re} : \mathrm{Mod}^{\mathrm{Fil}}_{Hk} \to \mathrm{Mod}_{Hk}$ sending $C_\bullet$ to $\operatorname{colim}_n C_{-n}$ and an associated graded functor Gr sending $C_\bullet$ to $\{C_n/C_{n+1}\}_n \in \mathrm{Fun}(\mathbb{Z}^{op}, \mathrm{Mod}_{Hk})$. The two are related by a spectral sequence

$$E^2_{p,q} = \pi_{p+q}\mathrm{Gr}(C_\bullet)_p \Rightarrow \pi_{p+q}(\mathrm{Re}(C_\bullet)),$$

which recovers the spectral sequence associated to a filtered object [Lur17]. This allows us to identify bar spectral sequences as coalgebras over the deformation $\cot^* \circ \cot_*$ of the comonad $|\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{O}, -)|$:

Theorem 1.3.2 (Senger-Zhang)**.** *For A an $\mathcal{O}$-algebra in the category of Hk-modules, the bar spectral sequence*

$$E^2_{s,t} = \pi_s\pi_t(\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{O}, A) \Rightarrow \pi_{s+t}|\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{O}, A)|$$

is naturally isomorphic to the spectral sequence for $\pi_*\mathrm{Re}(\Phi(\cot^* \circ \nu(A))) \in (\mathrm{Mod}^{\mathrm{Fil}}_k)_{\geq 0}$. If $A = \mathrm{sqz}(X)$ with X a finite Hk-module, then the bar spectral sequence is isomorphic to the spectral sequence for $\pi_*\mathrm{Re}(\Phi(\cot^* \circ \cot_*(\nu(X)))$.

Here $\nu : \mathrm{Mod}^{(\mathrm{ff})}_{Hk} \to \mathcal{P}_\Sigma(\mathrm{Mod}^{\mathrm{ff}}_{Hk})$ is the (restricted) Yoneda embedding. Furthermore, the weight decomposition $\cot \circ \mathrm{sqz}(X) = \bigoplus_{i \geq 1} \mathrm{Bar}(\mathcal{O})(i) \otimes_{h\Sigma_i} X^{\otimes n}$ lifts to a weight decomposition of the deformed comonad $\cot^* \circ \cot_*(-) \simeq \bigoplus_{i \geq 1} \mathbb{D}_i(-)$. In the case $k = \mathbb{F}_2$ and $\mathcal{O} = s\mathscr{L}$, the degeneration of the weight two part of the bar spectral sequence associated with the trivial algebra $\Sigma^k H\mathbb{F}_2$ allows us to compute $\mathbb{D}_2(X)$ for any $X \in \mathcal{P}_\Sigma(\mathrm{Mod}^{\mathrm{ff}}_{H\mathbb{F}_2})$. This allows us to detect all differentials in weight two of the bar spectral sequence in the universal case. Our hope is to use the structure map of the comonad $\cot^* \circ \cot_*(-)$ and the computation of the E^2-page (Theorem 4.2.27) to inductively propagate and pull back universal differentials along the weight. This will be explored in future endeavors.

1.3.1 Conventions

We assume that every object is graded and weighted whenever it makes sense. For instance, $\mathrm{Mod}_{\mathbb{F}_p}$ stands for the ordinary category of weighted graded $\mathbb{F}_p$-modules. A weighted graded $\mathbb{F}_2$-module $M_\bullet$ is an $\mathbb{N}$-indexed collection of $\mathbb{Z}$-graded $\mathbb{F}_p$-modules $\{M(w)_\bullet\}_{w\in\mathbb{N}}$. The weight grading of an element $x \in M(w)_n$ is w, and the internal grading is $|x| = n$. Morphisms are weight preserving morphisms of graded $\mathbb{F}_p$-modules. The Day convolution $\otimes$ makes $\mathrm{Mod}_{\mathbb{F}_p}$ a symmetric monoidal category. The Koszul sign rule $x \otimes y = (-1)^{|x||y|} y \otimes x$ for the symmetric monoidal product $\otimes$ depends only on the internal grading and not the weight grading.

Similarly, a shifted Lie algebra L over $\mathbb{F}_p$ is a weighted graded $\mathbb{F}_p$-module equipped with a shifted Lie bracket $[-,-] : L_m \otimes L_n \to L_{m+n-1}$ that adds weights, as well as satisfying graded commutativity $[x,y] = (-1)^{|x||y|}[y,x]$ and the graded Jacobi identity

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0.$$

When $p = 3$ we further require that $[[x,x],x] = 0$ for all $x \in L$. Denote by $\mathrm{Lie}^s_{\mathbb{F}_p}$ the category of shifted weighted graded Lie algebras over $\mathbb{F}_p$, as well as the monad associated to the free $\mathrm{Lie}^s_{\mathbb{F}_p}$-algebra functor. When $p = 2$, we use the abbreviation $\mathrm{Lie}^s = \mathrm{Lie}^s_{\mathbb{F}_2}$. We further consider the category $\mathrm{Lie}^{s,\mathrm{ti}}$ of totally-isotropic Lie^s-algebras, i.e., Lie^s-algebras that have vanishing self-brackets. We use the notation $\langle -,- \rangle$ exclusively for $\mathrm{Lie}^{s,\mathrm{ti}}$ brackets.

We mean by shifted graded exterior algebra over $\mathbb{F}_p$ a graded $\mathbb{F}_p$-module $M_\bullet$ together with a graded commutative product $M_m \wedge M_n \to M_{m+n-1}$ such that $x \wedge x = 0$ for all $x \in M_\bullet$. We will often omit the adjectives shifted graded for the exterior algebra.

We use $\pi_n(-)$ to denote the following functors: the functor taking the nth homotopy group of a spectrum, an $H\mathbb{F}_p$-module spectrum, or a simplicial $\mathbb{F}_p$-module, as well as the functor taking the nth homology group of a chain complex over $\mathbb{F}_p$.

We use $\pi_{*,*}(-)$ to denote the functor taking the bigraded homotopy group of a (weighted graded) bisimplicial $\mathbb{F}_p$-module, which is equivalent to taking the homology of the total complex of the associated double complex via the generalized Eilenberg-Zilber theorem. The bidegree (s,t) is given by the pair (simplicial degree, internal degree).

Chapter 2

Preliminaries

2.1 The spectral Lie operad

The Koszul dual pair we are interested in involves the non-unital $\mathbb{E}_\infty$-operad and the *spectral Lie operad*, which we recall in this section.

Ching [Chi05] and Salvatore showed that the Goowillie derivatives $\partial_n(\mathrm{Id})$ of the identity functor $\mathrm{Id} : \mathrm{Top}_* \to \mathrm{Top}_*$ form an operad $s\mathscr{L} := \{\partial_n(\mathrm{Id})\}_n$ in Spectra. The nth-derivative $\partial_n(\mathrm{Id})$ admits an explicit description due to Arone and Mahowald [AM99], following the work of Johnson [Joh95]. Let $\mathcal{P}_n$ be the poset of partitions of the set $\underline{n} = \{1, 2, \ldots, n\}$ ordered by refinements, equipped with a Σ_n-action induced from that on $\underline{n}$. Denote by $\hat{0}$ the discrete partition and $\hat{1}$ the partition $\{\underline{n}\}$. Set $\Pi_n = \mathcal{P}_n - \{\hat{0}, \hat{1}\}$. Regarding a poset $\mathcal{P}$ as a category, we obtain via the nerve construction a simplicial set $N_\bullet(\mathcal{P})$. The *partition complex* $\Sigma|\Pi_n|^\diamond$, the reduced-unreduced suspension of the realization $|\Pi_n|$, is modeled by the simplicial set

$$N_\bullet(\mathcal{P}_n)/(N_\bullet(\mathcal{P}_n - \hat{0}) \cup N_\bullet(\mathcal{P}_n - \hat{1}))$$

for $n \geq 2$ and the simplicial 0-circle S^0 for $n = 1$. Then there is an equivalence

$$\partial_n(\mathrm{Id}) \simeq \mathbb{D}(\Sigma|\Pi_n|^\diamond) \simeq \mathbb{D}\mathrm{Bar}(1, \mathbb{E}_\infty^{\mathrm{nu}}, 1)(n)$$

of spectra with Σ_n-action, where $\mathbb{D}$ denotes the Spanier-Whitehead dual of a spectrum. This identifies $s\mathscr{L}$ as the Koszul dual to the nonunital commutative operad $\mathbb{E}_\infty^{\mathrm{nu}}$, i.e.,

$$s\mathscr{L} \simeq \mathbb{D}\mathrm{Bar}(1, \mathbb{E}_\infty^{\mathrm{nu}}, 1).$$

For a description of the operadic bar construction and a proof of the compatibility of the operadic structure on both sides, see [Chi05] for a topological model using trees and [Bra17, Appendix D] for an ∞-categorical construction along with a comparison with the topological model.

2.2 Theory of operations

Given an operad, or more generally a monad in $\mathrm{Mod}_{H\mathbb{F}_p}$, one can ask for *natural operations* on the homotopy groups of algebras over the monad. The following is adapted from Lawson's excellent survey [Law20, section 1.4] on the theory of operations for algebras over operads.

Given a monad $\mathbf{T}$ on $\mathrm{Mod}_{H\mathbb{F}_p}$, we define an *operation* on $\mathbf{T}$-algebras to be a natural transformation $\pi_m(-) \to \pi_n(-)$ of functors $h\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{H\mathbb{F}_p}) \to \mathrm{Sets}$ for some m, n. Here $h\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{H\mathbb{F}_p})$ is the homotopy category of $\mathbf{T}$-algebras over $\mathrm{Mod}_{H\mathbb{F}_p}$. Let $\mathrm{Op}(m;n)$ be the set of operations for fixed m, n. It follows from the universal property of free algebras that for any $\mathbf{T}$-algebra A,

$$\pi_m(A) \cong \mathrm{Map}_{\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{H\mathbb{F}_p})}(\mathrm{Free}^{\mathbf{T}}(\Sigma^m H\mathbb{F}_p), A).$$

Hence $\mathrm{Free}^{\mathbf{T}}(\Sigma^m H\mathbb{F}_p)$ is the representing object for the functor $\pi_m(-)$ on $h\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{H\mathbb{F}_p})$.

By the Yoneda Lemma, the set of operations $\mathrm{Op}(m;n)$, or equivalently natural transformations $\pi_m(-) \to \pi_n(-)$ in $h\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{H\mathbb{F}_p})$, is isomorphic to $\pi_n(\mathrm{Free}^{\mathbf{T}}(\Sigma^m H\mathbb{F}_p))$. Explicitly, given an operation $\alpha \in \pi_n(\mathrm{Free}^{\mathbf{T}}(\Sigma^m H\mathbb{F}_p))$ and a class $x \in \pi_m(A)$ with A a $\mathbf{T}$-algebra, we obtain a class $\alpha(x)$ in $\pi_n(A)$ via the pullback

$$\pi_m(A) \cong \mathrm{Map}_{\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{H\mathbb{F}_p})}(\mathrm{Free}^{\mathbf{T}}(\Sigma^m H\mathbb{F}_p), A) \xrightarrow{\alpha^*} \mathrm{Map}_{\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{H\mathbb{F}_p})}(\mathrm{Free}^{\mathbf{T}}(\Sigma^n H\mathbb{F}_p), A) \cong \pi_n(A).$$

Therefore, to understand the unary operations on $\mathbf{T}$-algebras and their relations, we need to first compute $\pi_*(\mathrm{Free}^{\mathbf{T}}(\Sigma^m H\mathbb{F}_p))$ as an algebra for all m. Then we need to understand the composition product on unary operations

$$\pi_n(\mathrm{Free}^{\mathbf{T}}(\Sigma^m H\mathbb{F}_p)) \times \pi_m(\mathrm{Free}^{\mathbf{T}}(\Sigma^l H\mathbb{F}_p)) \to \pi_n(\mathrm{Free}^{\mathbf{T}}(\Sigma^l H\mathbb{F}_p))$$

for all l,m,n, which corresponds to composing two natural transformations $\pi_l(-) \to \pi_m(-)$ and $\pi_m(-) \to \pi_n(-)$ of functors on $h\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{H\mathbb{F}_p})$. In general, natural k-ary operations $\prod_{l=1}^k \pi_{i_l}(-) \to \pi_n(-)$ are parametrized by the homotopy groups

$$\pi_n(\mathrm{Free}^{\mathbf{T}}(\Sigma^{i_1} H\mathbb{F}_p \oplus \cdots \oplus \Sigma^{i_k} H\mathbb{F}_p))$$

for all k-tuples $(i_1, \ldots, i_k)$.

2.3 The bar spectral sequence

To investigate how Koszul duality of algebras manifest itself at the level of operations, we make use of the *bar spectral sequence*. Given an operad $\mathcal{O}$ in $\mathrm{Mod}_{H\mathbb{F}_p}$ and an algebra A over $\mathcal{O}$, there is a spectral sequence

$$E^2_{s,t} = \pi_s \pi_t \mathrm{Bar}_\bullet\big(\mathrm{id}, \mathcal{O}, A) \Rightarrow \pi_{s+t}|\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{O}, A)| \tag{2.1}$$

obtained by skeletal filtration of the geometric realization in the bar construction. Note that $\mathcal{O}$ has an analytic approximation in the sense that there is a monad $\hat{O}$ on $\mathrm{Mod}_{\mathbb{F}_p}$ such that $\pi_*(\mathrm{Free}^{\mathcal{O}}(X)) = \hat{O}(\pi_*(X))$ for any $X \in \mathrm{Mod}_{H\mathbb{F}_p}$. (cf. [AC20, Proposition 2.1]) Hence we can rewrite the E^2-page as the bigraded homotopy group

$$E^2_{s,t} = \pi_{s,t} \mathrm{Bar}_\bullet\big(\mathrm{id}, \hat{O}, \pi_*(A)) \Rightarrow \pi_{s+t}|\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{O}, A)| \tag{2.2}$$

Note that the E^2-page is also the total left derived functor that takes the indecomposables of the $\hat{O}$-algebra structure, as we recall below.

2.3.1 The derived indecomposable functor

We briefly record without proof the homotopy theory of monads on the category of weighted graded $\mathbb{F}_p$-modules and especially the two-sided bar construction for simplicial objects, following closely Sections 3.1, 4.2 and 4.3 in [BHK19]. For the general theory, see for instance Sections 3.1 and 3.2 of [JN14].

Let $\mathbf{T}$ be an augmented monad on the category $\mathrm{Mod}_{\mathbb{F}_p}$ of weighted graded $\mathbb{F}_p$-modules. Denote by $\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{\mathbb{F}_p})$ the category of $\mathbf{T}$-algebras. The forgetful functor $U : \mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{\mathbb{F}_p}) \to \mathrm{Mod}_{\mathbb{F}_p}$ admits a left adjoint, the free functor $\mathrm{Free}^{\mathbf{T}} : \mathrm{Mod}_{\mathbb{F}_p} \to \mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{\mathbb{F}_p})$.

Denote by $s\mathrm{Mod}_{\mathbb{F}_p}$ the category of simplicial weighted graded $\mathbb{F}_p$-modules. Levelwise application of the adjunction $\mathrm{Free}^{\mathbf{T}} \dashv U$ gives rise to an adjunction between the corresponding categories of simplicial objects

$$\mathrm{Free}^{\mathbf{T}} \dashv U : \mathrm{Alg}_{\mathbf{T}}(s\mathrm{Mod}_{\mathbb{F}_p}) \to s\mathrm{Mod}_{\mathbb{F}_p},$$

as well as a monad $\mathbf{T}$ on $s\mathrm{Mod}_{\mathbb{F}_p}$. We equip $s\mathrm{Mod}_{\mathbb{F}_p}$ with the standard cofibrantly generated model structure. Suppose that the path objects of $s\mathrm{Mod}_{\mathbb{F}_p}$ lifts to $s\mathrm{Alg}_{\mathbf{T}}$, the category of simplicial $\mathbf{T}$-algebras. Then this adjunction induces a right transferred model structure on the category of simplicial $\mathbf{T}$-algebras, with weak equivalences and fibrations defined on the underlying simplicial weighted graded $\mathbb{F}_p$-modules by [JN14, Theorem 3.2, Remark 3.3]. In particular, this is true for all the monads that we will encounter in this thesis.

Denote by $T^{\mathbf{T}} : \mathrm{Mod}_{\mathbb{F}_p} = \mathrm{Alg}_{\mathrm{Id}}(\mathrm{Mod}_{\mathbb{F}_p}) \to \mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{\mathbb{F}_p})$ the inclusion of trivial $\mathbf{T}$-algebras, which is induced by the augmentation. It has a left adjoint $Q^{\mathbf{T}} : \mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{\mathbb{F}_p}) \to \mathrm{Mod}_{\mathbb{F}_p}$, the indecomposable functor with respect to the $\mathbf{T}$-algebra structure. Applying this adjunction levelwise to the corresponding categories of simplicial objects, we obtain a Quillen adjunction

$$Q^{\mathbf{T}} \dashv T^{\mathbf{T}} : s\mathrm{Alg}_{\mathbf{T}} \to s\mathrm{Mod}_{\mathbb{F}_p}.$$

The total left derived functor $\mathbb{L}Q^{\mathbf{T}}$ of $Q^{\mathbf{T}}$ can be computed by the following standard recipe.

Construction 2.3.1. Given a right module $R : \mathrm{Mod}_{\mathbb{F}_p} \to \mathscr{D}$ over $\mathbf{T}$, and a simplicial object A in $\mathrm{Alg}_{\mathbf{T}}(\mathrm{Mod}_{\mathbb{F}_p})$, one can apply the two-sided bar construction $\mathrm{Bar}_\bullet(R, \mathbf{T}, -)$ levelwise to A. The diagonal of the resulting bisimplicial complex is a simplicial object in $\mathscr{D}$, denoted by $\mathrm{Bar}_\bullet(R,\mathbf{T},A)$.

In particular, if we regard a $\mathbf{T}$-algebra A as the constant simplicial object on $U(A)$ equipped with a simplicial $\mathbf{T}$-algebra structure, denoted also as A by abuse of notation, then $\mathrm{Bar}_\bullet(R,\mathbf{T},A)$ agrees with the usual two-sided bar construction.

Since the free resolution $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{T}},\mathbf{T},A)$ is a cofibrant replacement of A in the category of simplicial $\mathbf{T}$-algebras, the left derived functor of a functor F can be computed by applying F levelwise to a cofibrant replacement, so

$$\mathbb{L}Q^{\mathbf{T}}(A) \simeq Q^{\mathbf{T}}\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{T}},\mathbf{T},A) = \mathrm{Bar}_\bullet(\mathrm{id},\mathbf{T},A).$$

Now suppose that we have a composite monad $\mathbf{L}\circ\mathbf{R}$ with distributive law the natural transformation $\mathbf{L}\circ\mathbf{R}\Rightarrow\mathbf{R}\circ\mathbf{L}$ in the sense of Beck [Bec69, Section 1]. Suppose in addition that $\mathbf{L},\mathbf{R}$ and $\mathbf{L}\circ\mathbf{R}$ are all compatibly augmented and each admit a cofibrant replacement given by the free resolution. Let $\mathrm{Alg}_{\mathbf{L}},\mathrm{Alg}_{\mathbf{R}},\mathrm{Alg}_{\mathbf{L}\circ\mathbf{R}}$ be the respective categories of algebras. Then an $\mathbf{L}\circ\mathbf{R}$-algebra A is an $\mathbf{R}$-algebra via the forgetful map $U_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}:\mathrm{Alg}_{\mathbf{L}\circ\mathbf{R}}\to\mathrm{Alg}_{\mathbf{R}}$ induced by the augmentation of $\mathbf{L}$, and an $\mathbf{L}$-algebra via the augmentation of $\mathbf{R}$. Furthermore, we have adjunctions

$$\mathrm{Mod}_{\mathbb{F}_p} \underset{T^{\mathbf{R}}}{\overset{Q^{\mathbf{R}}}{\rightleftarrows}} \mathrm{Alg}_{\mathbf{R}} \underset{T_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}}{\overset{Q_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}}{\rightleftarrows}} \mathrm{Alg}_{\mathbf{L}\circ\mathbf{R}},$$

Construction 2.3.2. For A an algebra over $\mathbf{L}\circ\mathbf{R}$, the free resolution $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{R}},\mathbf{R},A)$ has the structure of a simplicial $\mathbf{L}\circ\mathbf{R}$-algebra given as follows. Levelwise, the $\mathbf{L}\circ\mathbf{R}$-algebra structure map is given by

$$\mathbf{L}\circ\mathbf{R}\circ\mathbf{R}^{\circ n}(A)\to\mathbf{R}\circ\mathbf{L}\circ\mathbf{R}^{\circ(n-1)}(A)\to\cdots\to\mathbf{R}^{\circ n}\circ\mathbf{L}\circ(\mathbf{R})(A)\to\mathbf{R}^{\circ n}(A),$$

where the rightmost arrow is the $\mathbf{L}\circ\mathbf{R}$-algebra structure map on A and the other arrows are induced from the distributive law $\mathbf{L}\circ\mathbf{R}\Rightarrow\mathbf{R}\circ\mathbf{L}$. The face and degeneracy maps are structure maps of the monad $\mathbf{R}$ and hence compatible with the levelwise $\mathbf{L}\circ\mathbf{R}$-algebra structure maps by naturality of the distributive law.

Levelwise application of $Q_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}$ to $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{R}},\mathbf{R},A)$ yields a simplicial $\mathbf{L}$-algebra structure on the bar construction $\mathrm{Bar}_\bullet(\mathrm{id},\mathbf{R},A) = Q_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{R}},\mathbf{R},A)$.

We record the following result about the total left derived functor of the indecomposable functor of a composite monad, which generalizes [BHK19, Proposition 4.19].

Lemma 2.3.3. *Let A be an $\mathbf{L}\circ\mathbf{R}$-algebra. The homotopy group of* $\mathrm{Bar}_\bullet(\mathrm{id},\mathbf{L}\circ\mathbf{R},A)$ *is computed by the homotopy group of the bisimplicial object* $\mathrm{Bar}_\bullet(\mathrm{id},\mathbf{L},\mathrm{Bar}_\bullet(\mathrm{id},\mathbf{R},A))$.

Recall that the homotopy group of a bisimplicial $\mathbb{F}_p$-module can be computed via the Eilenberg-Zilber theorem, i.e. by first taking associated chain complexes in both directions and then forming the total complex of the double complex. See for instance [GJ09, Chapter 4].

Proof. The augmentation $\mathbf{L}\circ\mathbf{R}\to\mathbf{R}$ induces a map of simplicial $\mathbf{L}\circ\mathbf{R}$-algebras

$$\Psi : \mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{L}\circ\mathbf{R}},\mathbf{L}\circ\mathbf{R},A)\to\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{R}},\mathbf{R},A),$$

where the simplicial $\mathbf{L}\circ\mathbf{R}$-algebra structure are the target is given by Construction 2.3.2. This is an equivalence since both are free resolutions of A of L as an $\mathbf{L}\circ\mathbf{R}$-algebra and an $\mathbf{R}$-algebra respectively, and weak equivalences in $s\mathrm{Alg}_{\mathbf{L}\circ\mathbf{R}}$ are detected by the underlying simplicial $\mathbb{F}_p$-modules. We want to show that $Q_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}$ preserves this weak equivalence. Since $U^{\mathbf{L}}$ preserves weak equivalences, it suffices to show that $U^{\mathbf{L}}\circ Q_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}\circ\Psi$ is a weak equivalence.

Note that there is an isomorphism

$$Q^{\mathbf{R}}\circ U_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}\cong U^{\mathbf{L}}\circ Q_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}.$$

Hence $U^{\mathbf{L}}\circ Q_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}\circ\Psi$ is the map

$$Q^{\mathbf{R}}\circ U_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{L}\circ\mathbf{R}},\mathbf{L}\circ\mathbf{R},A)\to Q^{\mathbf{R}}\circ U_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{R}},\mathbf{R},A) = Q^{\mathbf{R}}\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{R}},\mathbf{R},A).$$

Since both $U_{\mathbf{R}}^{\mathbf{L}\circ\mathbf{R}}\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{L}\circ\mathbf{R}},\mathbf{L}\circ\mathbf{R},A)$ and $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathbf{R}},\mathbf{R},A)$ are free resolutions of A in $s\mathrm{Alg}_{\mathbf{R}}$ and $Q^{\mathbf{R}}$ is a left Quillen functor, this is indeed a weak equivalence. □

Chapter 3

Spectral partition Lie algebras and TAQ cohomology

3.1 Spectral partition Lie algebras

Motivated by the theory of classical operadic Koszul duality [GK94], the natural next step is to formulate a Koszul duality theorem between suitable categories of algebras over the Koszul dual pair $\mathbb{E}_\infty^{\mathrm{nu}}$ and $\partial_*(\mathrm{Id})$. Partial progress was achieved by Ching and Harper in [CH19], following a general conjecture by Francis and Gaitsgory [FG12]. Recent work of Brantner and Mathew [BM19] on spectral partition Lie algebras completely resolved the question over $H\mathbb{F}_p$, and we will give a very brief summary of their results.

Let $\mathrm{Mod}^{\mathrm{ft}}_{H\mathbb{F}_p} \subset \mathrm{Mod}_{H\mathbb{F}_p}$ be the subcategory spanned by $H\mathbb{F}_p$-modules of *finite type*, i.e. $H\mathbb{F}_p$-modules with degree-wise finite homotopy groups. Denote by $\mathrm{Mod}^{\mathrm{ft}}_{H\mathbb{F}_p,\leq 0} \subset \mathrm{Mod}^{\mathrm{ft}}_{H\mathbb{F}_p}$ the subcategory spanned by coconnective objects. Let $\mathbb{P}$ be the nonunital commutative operad in $\mathrm{Mod}_{H\mathbb{F}_p}$. There is an adjunction

$$\mathrm{Alg}_{\mathbb{P}}(\mathrm{Mod}_{H\mathbb{F}_p}) \underset{\mathrm{sqz}}{\overset{\mathrm{cot}}{\rightleftarrows}} \mathrm{Mod}_{H\mathbb{F}_p}\ ,$$

where the functor sqz sends an object M to the $\mathbb{P}$-algebra M as a trivial square-zero extension. The restriction of this adjunction to the subcategory $\mathrm{Mod}^{\mathrm{ft}}_{H\mathbb{F}_p,\leq 0}$ defines a sifted-colimit-preserving monad $(M \mapsto \mathrm{cot}(\mathrm{sqz}(M)^\vee)^\vee)$ on $\mathrm{Mod}^{\mathrm{ft}}_{H\mathbb{F}_p,\leq 0}$.

Definition 3.1.1. [BM19, Definition 5.32] The *spectral partition Lie monad* $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$ is the unique sifted-colimit-preserving monad

$$\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty} : \mathrm{Mod}_{H\mathbb{F}_p} \to \mathrm{Mod}_{H\mathbb{F}_p}$$

extending the monad $(M \mapsto \mathrm{cot}(\mathrm{sqz}(M)^\vee)^\vee)$ on $\mathrm{Mod}^{\mathrm{ft}}_{H\mathbb{F}_p,\leq 0}$,

Algebras over the monad $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$ are called *spectral partition Lie algebras*. The free spectral partition Lie algebras on bounded above objects admit an explicit description.

Proposition 3.1.2. *[BM19, Proposition 5.35] For V a bounded above $H\mathbb{F}_p$-module,*

$$\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(V) \simeq \mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},V^\vee)| \simeq \bigoplus_{n\geq 1}\left((\partial_n(\mathrm{id})\otimes H\mathbb{F}_p)\otimes(V)^{\otimes n}\right)^{h\Sigma_n}.$$

The above formula makes it clear that spectral partition Lie algebras are *not* algebras over the spectral Lie operad, as the structural map of an algebra L over the spectral Lie operad in $\mathrm{Mod}_{H\mathbb{F}_p}$ is given by

$$\mathrm{Free}^{\partial_*(\mathrm{Id})\otimes H\mathbb{F}_p}(L) \simeq \bigoplus_{n\geq 1}\left((\partial_n(\mathrm{id})\otimes H\mathbb{F}_p)\otimes(L)^{\otimes n}\right)_{h\Sigma_n} \to L.$$

Heuristically, spectral partition Lie algebras are the dual of divided power coalgebras over the cooperad $\mathrm{Bar}(1,\mathbb{P},1)$, and hence candidates for the Koszul dual of $\mathbb{E}^{\mathrm{nu}}_\infty$-$H\mathbb{F}_p$-algebras. To formulate the precise Koszul duality statement, we need to introduce one more technical condition.

Definition 3.1.3. An $\mathbb{E}_\infty$-H$\mathbb{F}_p$-algebra A is *complete local Noetherian* if

(1). $\pi_0(A)$ is a complete local Noetherian ring;

(2). A is connective and $\pi_n(A)$ is a finitely-generated module over $\pi_0(A)$ for all $n \geq 0$.

Now we can state a restricted version of of the main results by Brantner and Mathew.

Theorem 3.1.4. *[BM19, Theorem 1.19] There is an equivalence of ∞-categories between complete local Noetherian $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebras and the ∞-category of coconnective spectral partition Lie algebras of finite type.*

3.1.1 Relation to TAQ cohomology

Spectral partition Lie algebras are closely related to the $\mathbb{F}_p$-linear TAQ spectrum. Inspired by the unpublished work of Kriz [Kri94], Basterra constructed the *topological André-Quillen homology object* $\mathrm{TAQ}(R,A;B)$ for a fixed map of $\mathbb{E}_\infty$-algebras $A \to B$ and any object R in the category of $\mathbb{E}_\infty$-algebras between A and B [Bas99]. For any object R in the category of $\mathbb{E}_\infty$-$\mathbb{S}$-algebras with a map to $H\mathbb{F}_p$, we obtain the TAQ spectrum

$$\mathrm{TAQ}(R,\mathbb{S};H\mathbb{F}_p) \simeq |\mathrm{Bar}_\bullet(H\mathbb{F}_p \otimes (-), \mathbb{E}_\infty, R)|,$$

cf. [Bas99, section 5] and [Law20, Proposition 1.8.9]. There is a base change formula to $\mathbb{F}_p$-linear TAQ spectrum

$$\mathrm{TAQ}(R,\mathbb{S};H\mathbb{F}_p) \underset{\mathbb{S}}{\otimes} H\mathbb{F}_p \simeq \mathrm{TAQ}(R \underset{\mathbb{S}}{\otimes} H\mathbb{F}_p, H\mathbb{F}_p; H\mathbb{F}_p)$$

for R any $\mathbb{E}_\infty$-$\mathbb{S}$-algebra. The nth $\mathbb{F}_p$-*linear TAQ cohomology* is defined to be

$$\mathrm{TAQ}^n(R,H\mathbb{F}_p;H\mathbb{F}_p) = [\Sigma^{-n}\mathrm{TAQ}(R,H\mathbb{F}_p;H\mathbb{F}_p), H\mathbb{F}_p]_{\mathrm{Mod}_{H\mathbb{F}_p}}$$

for R any $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebra.

In this paper we work with the nonunital $\mathbb{F}_p$-linear version

$$\overline{\mathrm{TAQ}}(A) \simeq |\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{P}, A)|,$$

where $\mathbb{P}$ is the nonunital $\mathbb{E}_\infty$-operad in $\mathrm{Mod}_{H\mathbb{F}_p}$ and A a $\mathbb{P}$-algebra. We call this the *reduced mod p TAQ spectrum* of A, since

$$\overline{\mathrm{TAQ}}(A) \oplus H\mathbb{F}_p \simeq \mathrm{TAQ}(H\mathbb{F}_p \oplus A, H\mathbb{F}_p; H\mathbb{F}_p).$$

Thus the reduced mod p TAQ cohomology group $\overline{\mathrm{TAQ}}^n(A) := [\Sigma^{-n}\overline{\mathrm{TAQ}}(A), H\mathbb{F}_p]_{\mathrm{Mod}_{H\mathbb{F}_p}}$ differ from the $\mathbb{F}_p$-linear TAQ cohomology group $\mathrm{TAQ}^n(A \oplus H\mathbb{F}_p, H\mathbb{F}_p; H\mathbb{F}_p)$ only when $n = 0$ by a copy of $\mathbb{F}_p$. By Proposition 3.1.2, when A is a bounded above $H\mathbb{F}_p$-module of finite type considered as a trivial $\mathbb{P}$-algebra, there is an equivalence

$$\overline{\mathrm{TAQ}}^n(A^\vee) \cong \pi_{-n}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},A^\vee)|) \cong \pi_{-n}(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A)).$$

Going forward we will often omit $\mathbb{F}_p$-linear and mod p when there is no ambiguity regarding which version of TAQ cohomology is concerned.

3.2 Operations on spectral partition Lie algebras and TAQ cohomology

Now we specialize the general theory of operations in 2.2 to the monad $\mathbf{T} = \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$. The decomposition of the free algebra over $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$ into homogeneous pieces in Proposition 3.1.2 allows us to impose a weight grading on the operations on the homotopy groups of $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$-algebras in the usual sense.

On the other hand, the mod p TAQ cohomology functor $\mathrm{TAQ}(-,H\mathbb{F}_p;H\mathbb{F}_p)$ on $\mathrm{Mod}_{H\mathbb{F}_p}$ has as representing objects the trivial square-zero extensions [Law20, section 1.8]. Therefore, for any m and tuple $(i_1,\dots,i_k)$, the group of cohomology operations

$$\prod_{i=1}^{k} \mathrm{TAQ}^{i_l}(-,H\mathbb{F}_p;H\mathbb{F}_p) \to \mathrm{TAQ}^m(-,H\mathbb{F}_p;H\mathbb{F}_p)$$

is the given by $\mathrm{TAQ}^m(H\mathbb{F}_p \oplus \Sigma^{i_1}H\mathbb{F}_p \oplus \cdots \Sigma^{i_k}H\mathbb{F}_p, H\mathbb{F}_p; H\mathbb{F}_p)$.

Note that all operations vanish on the unit except for scalar multiplication. Since

$$\overline{\mathrm{TAQ}}^n(A^\vee) \cong \pi_{-n}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},A^\vee)|) \cong \pi_{-n}(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A))$$

for all n when A is a bounded above $H\mathbb{F}_p$-module of finite type considered as a trivial $\mathbb{P}$-algebra, natural operations and their relations on the reduced mod p TAQ cohomology, or equivalently, the mod p TAQ cohomology away from the unit, agree with those on the homotopy group of spectral partition Lie algebras up to a change of grading conventions.

Brantner and Mathew obtained bases of the homotopy groups of free spectral partition Lie algebras on $\Sigma^j H\mathbb{F}_p$ via an isotropy spectral sequence as in [ADL13, Example 1.3]. Then they propagated the result to any direct sum of shifts of $H\mathbb{F}_p$ using a Hilton-Milnor-type decomposition of the partition complex and an EHP sequence developed in [AB21].

Definition 3.2.1. We say a word w in letters $\{x_1,\ldots,x_k\}$ is a *Lyndon word* if it is smaller than any of its cyclic rotations in the lexicographic order with $x_1 < \cdots < x_k$. Write $B(n_1,\ldots,n_k)$ for the set of Lyndon words in which the letter x_i appears precisely n_i times.

Note that the collection of all Lyndon words in letters $x_1,\ldots,x_k$ produces a basis for the free *totally-isotropic* Lie algebra over $\mathbb{F}_p$ on k generators, where totally-isotropic means there are no non-vanishing self-brackets.

Theorem 3.2.2. *[BM19, Theorem 1.20] The $\mathbb{F}_p$-vector space $\pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^{l_1}\mathbb{F}_p \oplus \cdots \oplus \Sigma^{l_k}\mathbb{F}_p))$ has a basis indexed by sequences $(i_1,\ldots,i_k,e,w)$. Here $w \in B(n_1,\ldots,n_m)$ is a Lyndon word. We have $e \in \{0,\iota\}$, where $\iota = 1$ if p is odd and $\deg(w) := \sum_i (l_i-1)n_i + 1$ is even. Otherwise, $\iota = 0$. The integers $i_1,\ldots,i_k$ satisfy:*

(1). Each i_j is congruent to 0 or 1 modulo $2(p-1)$.

(2). For all $1 \le j < k$, we have $i_j < p i_{j+1}$.

(3). We have $i_k \le (p-1)(1+e)\deg(w) - \iota$. The homological degree of $(i_1,\ldots,i_k,e,w)$ is $((1+e)\deg(w)-e) + i_1 + \ldots + i_k - k$.

Nonetheless, their method does not yield explicit descriptions of the nature of the operations, nor are the composition products or the relations among the operations clarified. Here, we resolve the problem using the dual of the bar spectral sequence (2.2) to compute the relations among the unary operations.

3.3 The dual bar spectral sequence for $\mathbb{E}_\infty^{\mathrm{nu}}$-algebras

This section serves as a preliminary examination of the dual of the bar spectral sequence (2.2) for a $\mathbb{P}$-algebra A, where $\mathbb{P}$ is the $\mathbb{E}_\infty^{\mathrm{nu}}$-operad in $\mathrm{Mod}_{H\mathbb{F}_p}$. The dual bar spectral sequence is given by

$$E^2_{s,t} = \pi_s(\pi_t(\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},A)^\vee)) \Rightarrow \pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},A)|).$$

When $A = \Sigma^{-j}H\mathbb{F}_p$ is a trivial $\mathbb{P}$-algebra, the E^∞-page records unary operations on a degree j class in the homotopy group of any spectral partition Lie algebra and those on a degree $-j$ cohomology class in the reduced mod p TAQ cohomology. We begin by reviewing the analytic approximation of $\mathbb{P}$, which parametrizes operations on the homotopy groups of $\mathbb{P}$-algebras.

3.3.1 The Dyer-Lashof algebra

Dyer-Lashof operations are natural unary operations on the mod p homology of infinite loop spaces and $\mathbb{E}_\infty$-algebras in Spectra. These operations and their relations were computed by Araki-Kudo [KA56], Dyer-Lashof [DL62], Cohen-Lada-May [CLM76], and Bruner-May-McCLure-Steinberger [BMMS88]. Denote by $\mathcal{R}$ the non-unital mod p Dyer-Lashof algebra.

Proposition 3.3.1. *[Ade52, I.1], [BMMS88, III.1] At $p = 2$, the Dyer-Lashof algebra $\bar{\mathcal{R}}$ is generated by operations Q^i in degree i and weight 2 subject to the Adem relations*

$$Q^rQ^s = \sum_{r+s-i\leq 2i} \binom{i-s-1}{2i-r} Q^{r+s-i}Q^i$$

for $r > 2s$.

For p an odd prime, the mod p Dyer-Lashof algebra is generated by operations $\beta^\varepsilon Q^i$ in degree $2(p-1)i-\varepsilon$ and weight p for $\varepsilon \in \{0,1\}$ and all i, subject to the Adem relations

$$Q^rQ^s = \sum_{r+s-i\leq pi} (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} Q^{r+s-i}Q^i,$$

$$\beta Q^rQ^s = \sum_{r+s-i\leq pi} (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} \beta Q^{r+s-i}Q^i$$

for $r > ps$ and

$$Q^r\beta Q^s = \sum_{r+s-i<pi} (-1)^{r+i}\binom{(p-1)(i-s)}{pi-r}\beta Q^{r+s-i}Q^i - \sum_{r+s-i<pi} (-1)^{r+i}\binom{(p-1)(i-s)-1}{pi-r-1}Q^{r+s-i}\beta Q^i,$$

$$\beta Q^r\beta Q^s = -\sum_{r+s-i<pi} (-1)^{r+i}\binom{(p-1)(i-s)-1}{pi-r-1}\beta Q^{r+s-i}\beta Q^i$$

for $r \geq ps$.

We say that a modules M over the Dyer-Lashof algebra $\bar{\mathcal{R}}$ is *unstable* (or *allowable*) if the following conditions holds:

1. When $p = 2$, for any nonempty sequence of operation $Q^I = Q^{i_1}\cdots Q^{i_k}$ and $x \in M$ of degree j, if $i_l - i_{l+1} - \ldots - i_k < j$ for some $1 \leq l \leq k$ then $Q^I(x) = 0$.

2. When $p > 2$, for any $x \in M$ of degree j and any nonempty sequence of operation $\alpha = \beta^{\varepsilon_1}Q^{i_1}\cdots\beta^{\varepsilon_k}Q^{i_k}$, if $2i_m - \varepsilon_m < j + 2(p-1)i_{m+1} + \ldots 2(p-1)i_k - \varepsilon_1 - \cdots \varepsilon_k$ for some $1 \leq m \leq k$ then $\alpha(x) = 0$.

Denote by $\mathrm{Mod}_{\mathcal{R}}$ the category of unstable $\mathcal{R}$-modules.

Definition 3.3.2. A $\mathrm{Poly}_{\mathcal{R}}$-*algebra M* is a (graded weighted) polynomial algebra over $\mathbb{F}_p$ with an unstable $\bar{\mathcal{R}}$-module structure that is compatible with the commutative product $\otimes$ in the sense that

(1) The Cartan formula is satisfied: $Q^i(x\otimes y) = \sum_j Q^j(x)\otimes Q^{i-j}(y)$;

(2) We further require that $Q^{|x|/2}(x) = x^{\otimes p}$ for all even degree $x \in M$ when $p > 2$, and $Q^{|x|}(x) = x^{\otimes 2}$ when $p = 2$.

Let $\mathrm{Poly}_{\mathcal{R}}$ be the category of $\mathrm{Poly}_{\mathcal{R}}$-algebras. A classical result by May and McClure tells us that this is the target category for the mod p homology of non-unital $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebras.

Theorem 3.3.3. *[May72, BMMS88] For any $\mathbb{P}$-algebra X, there is an isomorphism*

$$\pi_*(\mathrm{Free}^{\mathbb{P}}(X)) \cong \mathrm{Free}^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathbb{F}_p}}(\pi_*(X))$$

of free $\mathrm{Poly}_{\mathcal{R}}$*-algebras.*

The free $\mathrm{Poly}_{\mathcal{R}}$-algebra on an $\mathbb{F}_p$-module M can be obtained by taking the polynomial algebra on $\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Mod}_{\mathcal{R}}}(M)$, identifying $Q^{|x|/2}(x) = x^{\otimes p}$ for all even $x \in M$ when $p > 2$ and $Q^{|x|}(x) = x^{\otimes 2}$ when $p = 2$, and finally imposing the Cartan formula. Denote again by $\mathrm{Poly}_{\mathcal{R}}$ the monad coming from the free-forgetful adjunction

$$\mathrm{Mod}_{\mathbb{F}_p} \underset{U_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}}}{\overset{\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}}}{\rightleftarrows}} \mathrm{Poly}_{\mathcal{R}} .$$

Hence the bar spectral sequence for a $\mathbb{P}$-algebra A with $M = \pi_*(A)$ can be rewritten as

$$\widetilde{E}^2_{s,t} = \pi_s(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathcal{R}}, M))_t \Rightarrow \pi_{s+t}(|\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{P}, A)|) = \overline{\mathrm{TAQ}}_{s+t}(A).$$

Similarly, the dual bar spectral sequence takes the form

$$E^2_{s,t} = \pi_s(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathcal{R}}, M)^\vee)_t \Rightarrow \pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{P}, A)|).$$

Since we will only be concerned with objects of finite type over $H\mathbb{F}_p$, we can switch freely between the two version by taking linear dual when computing the second page. It is easier to work with the bar construction, so we will focus on the bar spectral sequence.

Remark 3.3.4. Note that the E^2-page of the bar spectral sequence

$$\widetilde{E}^2_{s,t} = \pi_s(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathcal{R}}, M)_t = \pi_s(\mathbb{L}Q_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}}(M))_t$$

is the *André-Quillen homology* of M with respect to the monad $\mathrm{Poly}_{\mathcal{R}}$.

3.3.2 A smaller complex for the E^1-page

Our plan is to find a suitable factorization of the indecomposable functor $Q_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}}$ to separate the unary and binary structures of the monad $\mathrm{Poly}_{\mathcal{R}}$. This will allow us to replace the bar construction computing its total left derived functor by a smaller double complex that is amenable to Koszul-duality-type computations. The subtlety lies in the bottom non-vanishing Dyer-Lashof operations, which appear in both the unary and binary structures of $\mathrm{Poly}_{\mathcal{R}}$. We disentangle the unary and binary structure by defining a structure that heuristically discard the bottom non-vanishing Dyer-Lashof operations on any element in a module over the Dyer-Lashof algebra when $p=2$ and on any even class when $p>2$.

Let $\mathrm{Mod}_{\mathcal{R}'}$ be category of modules over the Dyer-Lashof algebra $\bar{\mathcal{R}}$ with unstability conditions $Q^i(x)=0$ for $i\leq |x|$ when $p=2$ and $2i\leq |x|$ when $p>2$. Then the indecomposables functor $Q^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathbb{F}_p}}$ factors as $Q^{\mathrm{Mod}_{\mathcal{R}'}}_{\mathrm{Mod}_{\mathbb{F}_p}}\circ Q^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathcal{R}'}}$ sitting in the composite adjunction with the inclusion functor:

$$\mathrm{Mod}_{\mathbb{F}_p} \underset{T^{\mathrm{Mod}_{\mathcal{R}'}}_{\mathrm{Mod}_{\mathbb{F}_p}}}{\overset{Q^{\mathrm{Mod}_{\mathcal{R}'}}_{\mathrm{Mod}_{\mathbb{F}_p}}}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{R}'} \underset{T^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathcal{R}'}}}{\overset{Q^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathcal{R}'}}}{\rightleftarrows}} \mathrm{Poly}_{\mathcal{R}}.$$

In particular, there is an isomorphism

$$U^{\mathrm{Mod}_{\mathcal{R}'}}_{\mathrm{Mod}_{\mathbb{F}_p}}\circ Q^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathcal{R}'}}\cong Q^{\mathrm{Poly}_{\mathbb{F}_p}}_{\mathrm{Mod}_{\mathbb{F}_p}}\circ U^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Poly}_{\mathbb{F}_p}}.$$

Denote by $\mathrm{Poly}_{\mathbb{F}_p}$ the monad corresponding to the free graded polynomial algebra functor on $\mathrm{Mod}_{\mathbb{F}_p}$. We want to use the factorization $Q^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathbb{F}_p}}\cong Q^{\mathrm{Mod}_{\mathcal{R}'}}_{\mathrm{Mod}_{\mathbb{F}_p}}\circ Q^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathcal{R}'}}$ to obtain a double complex that is more computable than the bar complex $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Poly}_{\mathcal{R}},M)$.

Lemma 3.3.5. *There is a weak equivalence of simplicial $\bar{\mathcal{R}}'$-modules*

$$\mathbb{L}Q^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathcal{R}'}}(M)\simeq \mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Poly}_{\mathbb{F}_p},M)$$

for any $\mathbb{F}_p$-module M considered as a trivial $\mathrm{Poly}_{\mathcal{R}}$*-algebra.*

Proof. There is a map of augmented monads $\mathrm{Poly}_{\mathcal{R}}\to\mathrm{Poly}_{\mathbb{F}_p}\to\mathrm{id}$, the first of which kills all Dyer-Lashof operations that are not the bottom operations $Q^{|x|}(x)=x^{\otimes 2}$ when $p=2$ or the bottom operations $Q^{/2}(x)=x^{\otimes p}$ on even classes when $p>2$. When M is an $\mathbb{F}_p$-module considered as a trivial $\mathrm{Poly}_{\mathcal{R}}$-algebra, the map $\mathrm{Poly}_{\mathcal{R}}(M)\to\mathrm{Poly}_{\mathbb{F}_p}(M)\to M$ is a map of $\mathrm{Poly}_{\mathcal{R}}$-algebras if we regard $\mathrm{Poly}_{\mathbb{F}_p}(M)$ as a $\mathrm{Poly}_{\mathcal{R}}$-algebra where all Dyer-Lashof operations vanish except for the bottom operations when $p=2$ and the bottom operations on even classes when $p>2$. Therefore we obtain a map of free bar resolutions

$$\Psi : \mathrm{Bar}_\bullet(\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}}, \mathrm{Poly}_{\mathcal{R}}, M) \to \mathrm{Bar}_\bullet(\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathbb{F}_p}}, \mathrm{Poly}_{\mathbb{F}_p}, M),$$

which is a weak equivalence of simplicial $\mathrm{Poly}_{\mathcal{R}}$-algebras.

Next we want to show that applying $Q_{\mathrm{Mod}_{\mathcal{R}'}}^{\mathrm{Poly}_{\mathcal{R}}}$ preserves this weak equivalence, i.e.

$$\mathbb{L}Q_{\mathrm{Mod}_{\mathcal{R}'}}^{\mathrm{Poly}_{\mathcal{R}}}(M) \simeq Q_{\mathrm{Mod}_{\mathcal{R}'}}^{\mathrm{Poly}_{\mathcal{R}}} \mathrm{Bar}_\bullet(\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}}, \mathrm{Poly}_{\mathcal{R}}, M) \to Q_{\mathrm{Mod}_{\mathcal{R}'}}^{\mathrm{Poly}_{\mathcal{R}}} \mathrm{Bar}_\bullet(\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathbb{F}_p}}, \mathrm{Poly}_{\mathbb{F}_p}, M)$$

is a weak equivalence of simplicial $\bar{\mathcal{R}}'$-modules. This is equivalent to showing that the underlying map of simplicial $\mathbb{F}_p$-modules $U_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Mod}_{\mathcal{R}'}}(\Psi)$ is a weak equivalence.

Using the isomorphism

$$U_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Mod}_{\mathcal{R}'}} \circ Q_{\mathrm{Mod}_{\mathcal{R}'}}^{\mathrm{Poly}_{\mathcal{R}}} \cong Q_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathbb{F}_p}} \circ U_{\mathrm{Poly}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}},$$

we can rewrite $U_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Mod}_{\mathcal{R}'}}(\Psi)$ as

$$\begin{aligned}
&U_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Mod}_{\mathcal{R}'}} \circ Q_{\mathrm{Mod}_{\mathcal{R}'}}^{\mathrm{Poly}_{\mathcal{R}}} \mathrm{Bar}_\bullet(\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Mod}_{\mathcal{R}'}}, \mathrm{Poly}_{\mathcal{R}}, M) \simeq Q_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathbb{F}_p}} \circ U_{\mathrm{Poly}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}} \mathrm{Bar}_\bullet(\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}}, \mathrm{Poly}_{\mathcal{R}}, M) \\
\to &U_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Mod}_{\mathcal{R}'}} \circ Q_{\mathrm{Mod}_{\mathcal{R}'}}^{\mathrm{Poly}_{\mathcal{R}}} \mathrm{Bar}_\bullet(\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathbb{F}_p}}, \mathrm{Poly}_{\mathbb{F}_p}, M) \\
\simeq &Q_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathbb{F}_p}} \circ U_{\mathrm{Poly}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}} \mathrm{Bar}_\bullet(\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathbb{F}_p}}, \mathrm{Poly}_{\mathbb{F}_p}, M) \simeq \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathbb{F}_p}, M).
\end{aligned}$$

This is indeed a weak equivalence since we are applying $Q_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathbb{F}_p}}$ to a free simplicial $\mathrm{Poly}_{\mathbb{F}_p}$-algebra on both sides. Therefore we obtain a weak equivalence of simplicial $\bar{\mathcal{R}}'$-modules

$$\mathbb{L}Q_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Poly}_{\mathcal{R}}}(M)) \cong \mathrm{Bar}_\bullet(\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_p}}^{\mathrm{Mod}_{\mathcal{R}'}}, \mathrm{Poly}_{\mathcal{R}}, M) \simeq \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathbb{F}_p}, M)$$

as desired. □

Let $\mathcal{A}_{\mathcal{R}'}$ be the additive monad associated to the free $\bar{\mathcal{R}}'$-module functor. Therefore the André-Quillen homology of an algebra M over the monad $\mathrm{Poly}_{\mathcal{R}}$, i.e. the E^2-page of the bar spectral sequence

$$\begin{aligned}
\widetilde{E}^2_{*,*} \cong \pi_*(\mathbb{L}Q^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathbb{F}_p}}(M)) &\cong \pi_*\big(Q^{\mathrm{Mod}_{\mathcal{R}'}}_{\mathrm{Mod}_{\mathbb{F}_p}} \circ Q^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathcal{R}'}} \mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Poly}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathbb{F}_p}}, \mathrm{Poly}_{\mathcal{R}}, M)\big) \\
&\cong \pi_*\big(Q^{\mathrm{Mod}_{\mathcal{R}'}}_{\mathrm{Mod}_{\mathbb{F}_p}} \mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Mod}_{\mathcal{R}'}}_{\mathrm{Mod}_{\mathbb{F}_p}}, \mathrm{Poly}_{\mathcal{R}}, M)\big) \\
&\cong \pi_*\big(\mathbb{L}Q^{\mathrm{Mod}_{\mathcal{R}'}}_{\mathrm{Mod}_{\mathbb{F}_p}}(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathbb{F}_p}, M))\big),
\end{aligned}$$

can be computed as the homotopy group of the double complex $\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\mathcal{R}'}, \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathbb{F}_p}, M))$.

3.4 Computing the dual bar spectral sequence

In this section, we compute the E^2-page of the (dual) bar spectral sequences in the universal case, i.e., when $A = \Sigma^{i_1} H\mathbb{F}_p \oplus \cdots \oplus \Sigma^{i_l} H\mathbb{F}_p$. Then a comparison with Theorem 3.2.2 allows us to deduce the degeneration of the spectral sequence in these cases. There is a non trivial distinction between the cases $p = 2$ and $p > 2$ regarding the restriction on shifted Lie algebra structures and the notations are rather different, so we record them separately.

3.4.1 The E^2-page at $p = 2$

We will utilize Priddy's machinery on algebraic Koszul duality in [Pri70, Theorem 2.5]. Since the Dyer-Lashof algebra $\bar{\mathcal{R}}$ is a Koszul algebra, the Ext group

$$\mathrm{Ext}^{*,*}_{\mathcal{R}}(\mathbb{F}_2, \mathbb{F}_2) = \pi_*((\mathrm{Bar}_\bullet(\mathbb{F}_2, \mathcal{R}, \mathbb{F}_2))^\vee)$$

is the Koszul dual algebra of $\bar{\mathcal{R}}$. The Koszul generators are given by the collection

$$(Q^i)^* := [(Q^i)^\vee]1 \in \mathrm{Ext}^{-1,*}_{\mathcal{R}}(\mathbb{F}_2, \mathbb{F}_2)$$

with homological bidegree $(-1, -i)$ and weight 2, with composition given by juxtaposition, which corresponds to the Yoneda product on Ext groups, cf. [Pri70, p.42] and [McC01, Theorem 9.8]. The quadratic relations among the generators are the Koszul dual of the Adem relations, i.e.

$$(Q^a)^*(Q^b)^* = \sum_{a+b-c>2c} \binom{b-c-1}{a-2c-1} (Q^{a+b-c})^*(Q^c)^* \tag{3.1}$$

for $a \leq 2b$.

We are interested in the *unstable* Ext group

$$\mathrm{UnExt}^{*,*}_{\mathcal{R}'}(\mathbb{F}_2, \Sigma^j \mathbb{F}_2) = \pi_*(\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\mathcal{R}'}, \Sigma^{-j}\mathbb{F}_2)^\vee),$$

which is a variant of the Ext group

$$\mathrm{Ext}^{*,*}_{\mathcal{R}}(\mathbb{F}_2,\mathbb{F}_2) = \pi_*((\mathrm{Bar}_\bullet(\mathbb{F}_2,\mathcal{R},\mathbb{F}_2))^\vee)$$

obtained by regardding $\Sigma^{-j}\mathbb{F}_2$ as an unstable trivial module over $\bar{\mathcal{R}}$ and imposing the unstablitly conditions $[Q^j|\alpha] = 0$ for $j \le |\alpha|$ in the bar complex, cf. [BC70, §3]. To incorporate the Koszul dual of the unstability conditions as well as the simplicial grading, we introduce the following ringoid.

Definition 3.4.1. Let $\mathcal{F}$ be the ringoid with objects the $\mathbb{Z}_{\le 0}\times\mathbb{Z}$ and morphisms freely generated over $\mathbb{F}_2$ under juxtaposition by the following elements: for any $s \le 0$ and all i, j satisfying $i > -j$, there is an element $(Q^i)^* \in \mathcal{F}\big((s,j),(s-1,j-i)\big)$ of weight 2. Let $(\mathcal{R}')^!$ be the quotient of $\mathcal{F}$ by the ideal generated by the relations

$$(Q^a)^*(Q^b)^* = \sum_{a+b-c>2c,c>-j}\binom{b-c-1}{a-2c-1}(Q^{a+b-c})^*(Q^c)^* \tag{3.2}$$

for all a, b satisfying $a \le 2b$, $b > -j$ and $a > b-j$ in $\mathcal{F}\big((s,j),(s-2,j-a-b)\big)$.

The first grading corresponds to the homological degree in Ext, or equivalently the filtration degree in the dual bar spectral sequence. The second grading is the topological degree.

Remark 3.4.2. There is an evident isomorphism $(\mathcal{R}')^!((s,i),(s',j)) \cong (\mathcal{R}')^!((s-r,i),(s'-r,j))$ for any i,j,s,s' and r such that $s-r<0$. For any $t>0$, there is an injection

$$\mathrm{susp}^{\mathrm{t}} : (\mathcal{R}')^!((s,i),(s',j)) \hookrightarrow (\mathcal{R}')^!((s,i+t),(s',j+t)),$$

since more operations are defined on classes with higher homological degree.

Remark 3.4.3. Note the relations are always well defined on both sides: if $b \ge -j+1$, then $a \ge b-j+1 \ge -2j+2$, so $\lfloor\frac{a+b}{3}\rfloor \ge -j+1$ and the right hand side is never empty.

The unstable Ext group $\mathrm{UnExt}^{*,*}_{\mathcal{R}'}(\mathbb{F}_2,\Sigma^j\mathbb{F}_2)$ is thus the underlying bigraded $\mathbb{F}_2$-module of the free $(\mathcal{R}')^!$-module $(\mathcal{R}')^!((0,j),-)$. We grade the Ext groups homologicially.

On the other hand, the Tor group $\mathrm{Tor}^*_{\mathcal{R}'}(\mathbb{F}_2,\mathbb{F}_2)$ is a coalgebra $(\bar{\mathcal{R}})^\vee$ generated by classes in

$$\mathrm{Tor}^{1,*}_{\mathcal{R}'}(\mathbb{F}_2,\mathbb{F}_2) = \mathbb{F}_2\{[Q^i]1, i\in\mathbb{N}\}.$$

The unstable Tor group

$$\mathrm{UnTor}^{*,*}_{\mathcal{R}'}(\mathbb{F}_2,\Sigma^j\mathbb{F}_2) = \pi_*(\mathrm{Bar}_\bullet(\mathrm{id},\mathcal{A}_{\mathcal{R}'},\Sigma^j\mathbb{F}_2))$$

is thus a coalgebra generated under juxtaposition by elements in

$$\mathrm{UnTor}^{1,*}_{\mathcal{R}'}(\mathbb{F}_2,\Sigma^j\mathbb{F}_2) = \{[Q^i]1, i > j\},$$

which is a comodule over the co-ringoid $(\mathcal{R}')^\vee$.

Back to computing the E^2-page. Denote by $co\mathrm{Free}^{(\mathcal{R}')^\vee}$ the functor that takes the cofree $(\mathcal{R}')^\vee$-comodule. Similarly, $\mathrm{Free}^{(\mathcal{R}')^!}$ is the functor that takes the free $(\mathcal{R}')^!$-module with a simplicial grading, i.e. the class $(Q^{i_1})^*(Q^{i_2})^*\cdots(Q^{i_k})^*(x)\in\mathrm{Free}^{(\mathcal{R}')^!}(M)$ has simplicial degree $-k$ for $x\in M$.

Since $\mathcal{A}_{\mathcal{R}'}$ is an additive monad, there is no nontrivial simplicial operations on the total left derived functor. Furthermore, the homological (vertical) and simplicial (horizontal) differentials do not mix, i.e. the targets of the vertical differential never involve elements from the inner bar complex and vice versa. Therefore we can deduce the following:

Lemma 3.4.4. *Suppose that $V_\bullet$ is a trivial simplicial $\mathcal{R}'$-module. Then*

$$\pi_*(\mathrm{Bar}_\bullet(\mathrm{id},\mathcal{A}_{\mathcal{R}'},V_\bullet)) = co\mathrm{Free}^{(\mathcal{R}')^\vee}(\pi_*(V_\bullet)).$$

In our case, we are interested in the trivial simplicial $\mathcal{R}'$-module $V_\bullet = \mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Poly}_{\mathbb{F}_2},M)$ where M is a direct sum of shifts of $\mathbb{F}_2$ as trivial $\mathrm{Poly}_{\mathbb{F}_2}$-modules.

Definition 3.4.5. (cf. [Jac41], [Fre00] for the unshifted version.) A shifted *restricted* Lie algebra over $\mathbb{F}_2$, denoted as a $\mathrm{sLie}^\rho_{\mathbb{F}_2}$-algebra, is a graded $\mathbb{F}_2$-module $L = L_\bullet$ with a shifted Lie bracket $L_m\otimes L_n\to L_{m+n-1}$ and a restriction map $x\mapsto x^{[2]}$ with $x^{[2]}\in L_{2|x|-1}$, satisfying the following identities:

1. $\mathrm{ad}(x^{[2]}) = \mathrm{ad}(x)^2$ for all $x \in L$;

2. For all $x, y \in L$, $(x+y)^{[p]} = x^{[p]} + y^{[p]} + [x,y]$.

Here $\mathrm{ad}(x)$ stands for the adjoint representation, i.e., the self-map $y \mapsto [y,x]$ on L.

Let $\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_2}}$ be the associated free functor. Given an $\mathbb{F}_p$-module M with basis $\{x_1, \dots, x_k\}$, a basis for $\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_2}}(M)$ is given by

$$\{u, u^{[2]}, (u^{[2]})^{[2]}, \dots\},$$

where u ranges over Lyndon words in letters $x_1, \dots, x_k$. (See, for instance, [BKS05, section 2].)

Proposition 3.4.6. *The dual bar spectral sequence converging to $\pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_2,\mathbb{E}_\infty}(A^\vee)) \cong \overline{\mathrm{TAQ}}^*(A)$ has E^2-page $E^2 \cong \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_2}}(M^\vee)$ for A an $H\mathbb{F}_2$-module considered as a trivial $\mathbb{P}$-algebra and $M = \pi_*(A)$.*

Proof. Note that the monad $\mathrm{Poly}_{\mathbb{F}_2}$ is the monad associated with the free functor of the commutative operad Comm in $\mathrm{Mod}_{\mathbb{F}_2}$. Denote by sLie the shifted Lie operad in $\mathrm{Mod}_{\mathbb{F}_2}$, so $\mathrm{sLie}(n)$ has internal degree $1-n$. Recall from [Fre04, Section 5.2.3] that the shifted Harrison complex of an algebra M over an operad Comm in $\mathrm{Mod}_{\mathbb{F}_2}$ is defined as follows. Let

$$K(\mathrm{Comm})_n := \ker(\mathrm{Bar}_n(I,\mathrm{Comm},I)_{(n)} \to \mathrm{Bar}_{n-1}(I,\mathrm{Comm},I)_{(n)}),$$

where $\mathrm{Bar}_r(I,\mathrm{Comm},I)_{(s)}$ is the weight s part of the simplicial degree r piece of the bar construction $\mathrm{Bar}_\bullet(I,\mathrm{Comm},I)$ in the category of symmetric sequences with I the unit. The linear dual $K(\mathrm{Comm})^\vee_\bullet$ forms an operad isomorphic to the shifted Lie operad $\{\mathrm{sLie}(n)\}$ in graded $\mathbb{F}_2$-modules, i.e., equipped with the sign representation (cf. [Fre04, Fact 6.2]). The Harrison complex of M is given by $\bigoplus_n (K(\mathrm{Comm})_n \otimes M^{\otimes n})_{\Sigma_n} \simeq \bigoplus_n (\mathrm{sLie}(n)^\vee \otimes M^{\otimes n})_{\Sigma_n}$. When M is an $\mathbb{F}_2$-module considered as a trivial algebra over Comm, the inclusion of subcomplex induces a comparison morphism

$$\bigoplus_n (\mathrm{sLie}(n)^\vee \otimes M^{\otimes n})_{\Sigma_n} \to \mathrm{Bar}_\bullet(I,\mathrm{Comm},I) \circ M \simeq \mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Comm},M)$$

of the shifted Harrison complex and the bar construction of M (cf. [Fre04, 5.2.3, 6.6]). This is an isomorphism on homotopy: any cycle on the right hand side is a sum of all way to put $n-1$ nested parenthesis on a fixed sequence $x_1, \dots x_n$ of n classes in M, such that each nesting represents taking the polynomial product with one more class on a different simplicial level and no nesting is trivial, and this cycle has preimage the bracket $[\cdots[[x_1,x_2],x_3]\dots],x_n]$. Taking linear dual, self-brackets become restrictions by [Fre00, Theorem 0.1], so we have

$$\pi_*(\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Comm},M)) = \pi_*(\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Poly}_{\mathbb{F}_2},M)) \cong co\mathrm{Free}^{\mathrm{co-sLie}_{\mathbb{F}_2}}(M),$$

$$\pi_*(\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Comm},M)^\vee) = \bigoplus_n (\mathrm{sLie}(n)\otimes(M^\vee)^{\otimes n})^{\Sigma_n} \cong \mathrm{Free}^{\mathrm{sLie}^\rho_{\mathbb{F}_2}}(M^\vee)$$

for any trivial algebra M over Comm. On the other hand, a shifted Lie coalgebra over $\mathbb{F}_2$ has a shifted Lie cobracket $L_{m+n+1} \to L_m \otimes L_n$ satisfying the co-Jacobi identity. The shift reflects the simplicial degree in the (co)bar resolution. Similarly a co-$\mathrm{sLie}_{\mathbb{F}_2}$-algebra stands for shifted coLie-algebra over $\mathbb{F}_2$. □

3.4.2 The E^∞-page for $p=2$

In the case where $A = \Sigma^j H\mathbb{F}_p$, the E^∞-page records all unary operations on a degree $-j$ class in the homotopy groups of spectral partition Lie algebras. Since

$$\pi_*(\mathbb{D}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Poly}_{\mathbb{F}_2},\pi_*(\Sigma^j H\mathbb{F}_p))) = \mathrm{Free}^{\mathrm{sLie}^\rho_{\mathbb{F}_2}}(\Sigma^{-j}\mathbb{F}_2)$$

has exactly one class $x^{[2]^s} \in \pi_s$ of weight 2^s for all $s \geq 0$, the dual bar spectral sequence simplifies to

$$E^2_{s,t} = \mathrm{Free}^{(\mathcal{R}')^!}(\mathbb{F}_2\{x^{[2]^s}, s\geq 0\}) \Rightarrow \pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},\Sigma^j H\mathbb{F}_p)|)$$

Note that the E^2-page is concentrated in weight 2^k for $k\in\mathbb{N}$, and the weight 2^k part is concentrated on a single line $s=-k$. Hence the spectral sequences collapses on the second page and there are no extension problems. Therefore we have found all the unary operations on a degree j class for any j.

In particular, we see that the restriction $x^{[2]}$ is represented by the cycle $(Q^{|x|})^*|x$ on the E^1-page $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathcal{R}}, \Sigma^j\mathbb{F}_2)^\vee$. On the other hand, the relations in (3.1) never involve the bottom operation: on a class x of degree j, if $c = -j$ then the coefficient $\binom{b+j-1}{a+2j-1}$ of $(Q^{a+b-c})^*(Q^c)^*$ is nonzero only if $a+2j-1 \leq b+j-1$, which is impossible since $a > b-j$.

Hence the ringoid $(\mathcal{R}')^!$ is the ringoid of *additive* operations. Now we define a new ringoid that takes into account the restriction being an unary operation.

Definition 3.4.7. Let $\mathcal{F}$ be the ringoid with objects $\mathbb{Z}_{\leq 0} \times \mathbb{Z}$ whose morphisms are freely generated over $\mathbb{F}_2$ under juxtaposition by the following elements: for all i, j satisfying $i \geq -j$ and $s \leq 0$, there is an element $(Q^i)^* \in \mathcal{F}\big((s,j),(s-1,j-i)\big)$ of weight 2.

Let $\bar{\mathcal{R}}^!$ be the quotient of $\mathcal{F}$ by the ideal generated by the relations $(Q^{a-j})^*(Q^a)^* = 0$ for all j,s and $(Q^a)^* \in \mathcal{F}\big((s,j),(s-1,j-a)\big)$ with $a > -j$, and the Adem relations

$$(Q^a)^*(Q^b)^* = \sum_{a+b-c>2c, c>-j} \binom{b-c-1}{a-2c-1}(Q^{a+b-c})^*(Q^c)^* \tag{3.3}$$

for all a,b satisfying $a \leq 2b$, $b > -j$ and $a > b-j$ in $\mathcal{F}\big((s,j),(s-2,j-a-b)\big)$.

The E^2-page of the dual bar spectral sequence on one generator has the following structure.

Definition 3.4.8. An $\mathrm{sLie}^{\rho}_{\bar{\mathcal{R}}^!}$-algebra is an $\mathbb{F}_2$-module M with an $\bar{\mathcal{R}}^!$-module structure and a shifted Lie bracket

$$[\ ,\ ] : M_{s,t} \otimes M_{s',t'} \to M_{s+s'-1,t+t'}$$

with restriction $(-)^{[2]}$ satisfying the following conditions:

(1) The bottom operation $(Q^{-|x|})^*(x) = x^{[2]}$ is the restriction for any x;

(2) $[x, \alpha(y)] = 0$ for any x, y and non-empty sequence α of $(Q^i)^*$'s unless α is an iteration of the restriction map.

Let $\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\bar{\mathcal{R}}^!}}$ denote the functor that first takes the free $\mathrm{sLie}^{\rho}_{\mathbb{F}_2}$-algebra on a bigraded $\mathbb{F}_2$-module M, then takes the free $\bar{\mathcal{R}}^!$-module on the underlying graded $\mathbb{F}_2$-module of

$\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_2}}(M)$, and finally identifies the restriction with the bottom operation $x^{[2]} = (Q^{-|x|})^*(x)$ for all $x \in \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_2}}(M)$. Therefore we can deduce that:

Proposition 3.4.9. *For $A = \Sigma^j H\mathbb{F}_2$, the bar spectral sequence has E^∞-page*

$$\widetilde{E}^{\infty}_{*,*} \cong \widetilde{E}^{2}_{*,*} \cong co\mathrm{Free}^{(\mathcal{R}')^!} co\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_2}}(\Sigma^j \mathbb{F}_2).$$

The dual bar spectral sequence for A has E^∞-page

$$E^{\infty}_{*,*} \cong E^{2}_{*,*} \cong \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\tilde{\mathcal{R}}^!}}(\Sigma^{-j}\mathbb{F}_2).$$

A basis is given by the monomials $(Q^{i_1})^*(Q^{i_2})^* \cdots (Q^{i_s})^*(x)$, where $i_s \geq |x| = -j$ and $i_l > 2i_{l+1}$ for all $1 \leq l < s$.

Now we can compute the E^∞-page of the (dual) bar spectral sequences in the universal case, and deduce the set of k-ary natural operations of all k. A priori, knowing the composition product and relations among operations on the André-Quillen cohomology $\mathrm{AQ}^*_{\mathrm{Poly}_{\mathcal{R}}}(-)$ does not imply knowledge of the relations on the homotopy groups of spectral partition Lie algebra and mod 2 TAQ cohomology. The composition product on the later differs from that on the former, as we will see in Theorem 3.5.5.

Proposition 3.4.10. *Let $A = \Sigma^{j_1} H\mathbb{F}_2 \oplus \Sigma^{j_2} H\mathbb{F}_2 \oplus \cdots \oplus \Sigma^{j_k} H\mathbb{F}_2$ be a trivial $\mathbb{P}$-algebra. Then the dual bar spectral sequence for $\pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_2,\mathbb{E}_\infty}(A^\vee)) \cong \overline{\mathrm{TAQ}}^*(A)$ has E^∞-page*

$$E^{\infty}_{*,*} \cong E^{2}_{*,*} \cong \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\tilde{\mathcal{R}}^!}}(\Sigma^{-j_1}\mathbb{F}_2 \oplus \cdots \oplus \Sigma^{-j_k}\mathbb{F}_2).$$

Proof. The dual bar spectral sequence simplifies to

$$E^{2}_{*,*} = \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\tilde{\mathcal{R}}^!}}(\Sigma^{-j_1}\mathbb{F}_2 \oplus \cdots \oplus \Sigma^{-j_k}\mathbb{F}_2) \Rightarrow \pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{P}, A)|).$$

A priori we can't deduce that the spectral sequence collapses using a sparsity argument when $k > 1$, since the E^2-page is concentrated on multiple lines at p^m when $m > 1$. However, the E^∞-page of the dual bar spectral sequence is the homotopy group of

$$\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A^\vee) \simeq \mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},A)|,$$

the free spectral partition Lie algebra on $A^\vee$. Comparing the basis of the E^2-page with Theorem 3.2.2, which is a basis of $\mathrm{Lie}^{\pi}_{\mathbb{F}_2,\mathbb{E}_\infty}(A^\vee)$, we deduce that $E^2 \cong E^\infty$, so the spectral sequence collapses on the second page and there are no extension problems. $\square$

3.4.3 The dual bar spectral sequence for odd primes

In the section, we apply the same analysis to the odd primary case.

Again utilizing Priddy's machinary on algebraic Koszul duality in [Pri70, Theorem 2.5], we deduce that $\mathrm{Ext}_{\mathcal{R}}(\mathbb{F}_p,\mathbb{F}_p) = \pi_*((\mathrm{Bar}_\bullet(\mathbb{F}_p,\mathcal{R}',\mathbb{F}_p))^\vee)$ is isomorphic as an algebra to the Koszul dual $(\mathcal{R}')^!$ of $\mathcal{R}'$. The Koszul generators are given by the collection

$$(\beta^\varepsilon Q^i)^* := [(\beta^\varepsilon Q^i)^\vee]1 \in \mathrm{Ext}^1_{\mathcal{R}'}(\mathbb{F}_p,\mathbb{F}_p), \varepsilon \in \{0,1\}, i \in \mathbb{N}$$

where $(\beta^\varepsilon Q^i)^*$ has homological bidegree $(-1,-2(p-1)i+\varepsilon)$ and weight p. Composition is given by juxtaposition corresponding to the Yoneda product. The quadratic relations are the Koszul dual of the Adem relations (Proposition **??**), i.e.

$$(Q^a)^*(Q^b)^* + \sum_{a+b-c>pc} (-1)^{a-c}\binom{(p-1)(b-c)-1}{a-pc-1}(Q^{a+b-c})^*(Q^c)^* = 0$$

for $a \le pb$,

$$\begin{aligned}(\beta Q^a)^*(Q^b)^* - \sum_{a+b-c\ge pc} (-1)^{a-c}\binom{(p-1)(b-c)}{a-pc}(Q^{a+b-c})^*(\beta Q^c)^* \\ + \sum_{a+b-c>pc} (-1)^{a-c}\binom{(p-1)(b-c)-1}{a-pc-1}(\beta Q^{a+b-c})^*(Q^c)^* = 0\end{aligned}$$

for $a \le pb$,

$$(\beta^\varepsilon Q^a)^*(\beta Q^b)^* - \sum_{a+b-c\ge pc} (-1)^{a-c}\binom{(p-1)(b-c)-1}{a-pc}(\beta^\varepsilon Q^{a+b-c})^*(\beta Q^c)^* = 0$$

for $\varepsilon \in \{0,1\}$ and $a < pb$, cf. [KL83].

Analogous to the case $p = 2$, we dualize the unstability condition and record the simplicial grading using a ringoid.

Definition 3.4.11. Let $\mathcal{F}$ be the ringoid with objects $\mathbb{Z}_{\leq 0} \times \mathbb{Z}$ and morphisms freely generated over $\mathbb{F}_p$ under juxtaposition by the following elements: for $2i > -j$ and any $s \leq 0$ there are elements $(Q^i)^* \in \mathcal{F}\big((s,j),(s-1,j-2(p-1)i)\big)$ and $(\beta Q^i)^* \in \mathcal{F}\big((s,j),(s-1,j-2(p-1)i+1)\big)$. We suppress the first grading for ease of notation when there is no ambiguity.

Let $(\mathcal{R}')^!$ be the quotient of $\mathcal{F}$ by the ideal generated by the following quadratic relations for all $s \leq 0$:

$$(Q^a)^*(Q^b)^* = -\sum_{a+b-c>pc,\ 2c>-k} (-1)^{a-c}\binom{(p-1)(b-c)-1}{a-pc-1}(Q^{a+b-c})^*(Q^c)^*,$$

in $\mathcal{F}\big(j, j-2(p-1)a-2(p-1)b\big)$ for all $a,b \in \mathbb{Z}$ and satisfying $a \leq pb$, $2b > -j$, $2a > 2(p-1)b - j$,

$$\begin{aligned}(\beta Q^a)^*(Q^b)^* = &\sum_{a+b-c\geq pc, 2c>-k} (-1)^{a-c}\binom{(p-1)(b-c)}{a-pc}(Q^{a+b-c})^*(\beta Q^c)^* \\ &- \sum_{a+b-c>pc, 2c>-k} (-1)^{a-c}\binom{(p-1)(b-c)-1}{a-pc-1}(\beta Q^{a+b-c})^*(Q^c)^* = 0\end{aligned}$$

in $\mathcal{F}\big(j, j-2(p-1)a-2(p-1)b+1\big)$ for all $a,b \in \mathbb{Z}$ satisfying $a \leq pb$, $2b > -j$, $2a > 2(p-1)b - j$, and

$$(\beta^\varepsilon Q^a)^*(\beta Q^b)^* = \sum_{a+b-c\geq pc, 2c>-k} (-1)^{a-c}\binom{(p-1)(b-c)-1}{a-pc}(\beta^\varepsilon Q^{a+b-c})^*(\beta Q^c)^*$$

in $\mathcal{F}\big(j, j-2(p-1)a-2(p-1)b+\varepsilon+1\big)$ for $\varepsilon \in \{0,1\}$ and $a,b \in \mathbb{Z}$ satisfying $a < pb$, $2b > -j$, $2a > 2(p-1)b - j$.

A basis for $(\mathcal{R}')^!\big((s,j),(s-k,-)\big)$ is given by sequences $(\beta^{\varepsilon_1}Q^{i_1})^*(\beta^{\varepsilon_2}Q^{i_2})^*\cdots(\beta^{\varepsilon_k}Q^{i_k})^*$ where $2i_k > -j$ and $i_l > pi_{l+1} - \varepsilon$ for $1 \leq l < k$.

Remark 3.4.12. Similar to the case $p=2$, there is an isomorphism

$$(\mathcal{R}')^!((s,i),(s',j)) \cong (\mathcal{R}')^!((s-r,i),(s'-r,j))$$

for any i,j,s,s' and r such that $s-r<0$. For any $t>0$, there is an injection

$$\mathrm{susp}^t : (\mathcal{R}')^!((s,i),(s',j)) \hookrightarrow (\mathcal{R}')^!((s,i+t),(s',j+t)),$$

since more operations are defined on classes with higher homological degree.

For x in degree $2d+1$, the relations never involve the bottom operations on x, i.e. the terms $(\beta^\varepsilon Q^{a+b+d})^*(\beta Q^{-d})^*$: the coefficient is nonzero only if $(p-1)(b+d)-1 \geq a+pd$, i.e., $(p-1)b-d-1 \geq a$. But $2a \geq 2(p-1)b-2d$ by assumption.

Hence the unstable Ext group

$$\mathrm{UnExt}^{*,*}_{\mathcal{R}}(\mathbb{F}_p,\Sigma^{-j}\mathbb{F}_p) = \pi_*(\mathrm{Bar}_\bullet(\mathrm{id},\mathcal{A}_{\mathcal{R}'},\Sigma^{-j}\mathbb{F}_p)^\vee)$$

is the free $(\mathcal{R}')^!$-module $(\mathcal{R}')^!((0,j),-)$. Whereas the Tor group $\mathrm{Tor}^{*,*}_{\mathcal{R}'}(\mathbb{F}_p,\mathbb{F}_p) \cong (\mathcal{R}^!)^\vee$ is a coalgebra generated by classes in

$$\mathrm{Tor}^{1,*}_{\mathcal{R}'}(\mathbb{F}_p,\mathbb{F}_p) = \mathbb{F}_p\{[\beta^\varepsilon Q^i]1, i\in\mathbb{N}, \varepsilon=0,1\}.$$

The unstable Tor groups are cofree comodules over the co-ringoid $((\mathcal{R}')^!)^\vee$.

The functor $\mathrm{Free}^{(\mathcal{R}')^!}$ takes the free $(\mathcal{R}')^!$-module with a dual simplicial grading that counts the number of generators, i.e., the element $(\beta^{\varepsilon_1}Q^{i_1})^*(\beta^{\varepsilon_2}Q^{i_2})^*\cdots(\beta^{\varepsilon_k}Q^{i_k})^*(x) \in \mathrm{Free}^{\mathcal{R}^!}(M)$ with $x\in M$ has simplicial degree $-k$. Denote by $co\mathrm{Free}^{(\mathcal{R}')^\vee}$ the functor that takes the cofree $(\mathcal{R}')^\vee$-comodule. Hence $\pi_*(\mathrm{Bar}_\bullet(\mathrm{id},\mathcal{A}_{\mathcal{R}'},\Sigma^{-j}\mathbb{F}_p)^\vee)$ is the free $\bar{\mathcal{R}}^!$-module on a single generator x in degree j, and $\pi_*(\mathrm{Bar}_\bullet(\mathrm{id},\mathcal{A}_{\mathcal{R}'},\Sigma^{j}\mathbb{F}_p))$ is the free $(\mathcal{R}')^\vee$-comodule on x.

Since $\mathcal{A}_{\mathcal{R}'}$ is an additive monad, there is no nontrivial simplicial operations on the total left derived functor. Furthermore the vertical and horizontal differentials act strictly within their complexes, i.e., the targets of the vertical differential never involve elements from the inner bar complex and vice versa. Therefore we can deduce the following:

Lemma 3.4.13. *Suppose that $V_\bullet$ is a trivial simplicial $\mathcal{R}'$-module. Then*

$$\pi_*(\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\mathcal{R}'}, V_\bullet)) = co\mathrm{Free}^{((\mathcal{R}')^!)^\vee}(\pi_*(V_\bullet)).$$

Now we run the spectral sequence for a trivial $\mathbb{P}$-algebra A. First we look at the dual bar spectral sequences for $A = \Sigma^j H\mathbb{F}_p$, which parametrizes unary operations on a degree $-j$ class the homotopy groups of spectral partition Lie algebra.

Proposition 3.4.14. *Let $A = \Sigma^j H\mathbb{F}_p$. If j is odd, then $\pi_*(\mathrm{Lie}^\pi_{\mathbb{F}_p,\mathbb{E}_\infty}(A^\vee)) \cong \mathrm{Free}^{(\mathcal{R}')^!}(\Sigma^{-j}\mathbb{F}_p)$. If j is even, then $\pi_*(\mathrm{Lie}^\pi_{\mathbb{F}_p,\mathbb{E}_\infty}(A^\vee)) \cong \mathrm{Free}^{(\mathcal{R}')^!}(\Sigma^{-j}\mathbb{F}_p \oplus \Sigma^{-2j-1}\mathbb{F}_p)$.*

Proof. If $A = \Sigma^j H\mathbb{F}_p$ with j odd, then $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathbb{F}_p}, \Sigma^j\mathbb{F}_p)$ is the constant simplicial object on $\Sigma^j\mathbb{F}_p$ due to graded commutativity. The dual bar spectral sequence simplifies to

$$E^2_{*,*} = \mathrm{Free}^{(\mathcal{R}')^!}(\Sigma^{-j}\mathbb{F}_p) \Rightarrow \pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{P}, A)|) \cong \pi_{s+t}(\mathrm{Lie}^\pi_{\mathbb{F}_p,\mathbb{E}_\infty}(A^\vee)).$$

Note that the E^2-page is concentrated in weight p^k for $k \in \mathbb{N}$, and the weight p^k part is concentrated on a single line $s = -k$. Hence the spectral sequence collapses on the second page and there are no extension problems.

If $A = \Sigma^j H\mathbb{F}_p$ with j even, then

$$\pi_*(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Poly}_{\mathbb{F}_p}, \Sigma^j\mathbb{F}_p)^\vee) \cong \mathbb{F}_p\{x, [x,x]\}$$

with a weight 1 class $x \in \pi_0$ and a weight 2 class $[x,x] \in \pi_{-1}$. On the E^2-page of the dual bar spectral sequence

$$E^2_{*,*} = \mathrm{Free}^{(\mathcal{R}')^!}(\mathbb{F}_p\{x, [x,x]\}) \Rightarrow \pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{P}, A)|)$$

is concentrated in weights p^k and $2p^k$, and at each k concentrated on the line $s = -k$ at weight p^k and the line $s = -k-1$ at weight $2p^k$. Again there are no further differentials or extension problems. $\square$

Remark 3.4.15. (1) As in the case $p = 2$, knowing the relations among unary operations on the André-Quillen cohomology $\mathrm{AQ}^*_{\mathrm{Poly}_{\mathcal{R}}}(-)$ does not immediately yield knowledge of the relations on the homotopy groups of spectral partition Lie algebra and mod p TAQ cohomology. The composition product on the later differs from that on the former, as we will see in Theorem 3.5.6.

(2) While we expect to see a shifted restricted Lie algebra, the restriction on an odd class is not detected on the E^2-page for filtration reasons, as we shall see in Lemma 3.6.5 in the next section.

In general, we have the odd primary counterpart of Proposition 3.4.10.

Proposition 3.4.16. *Suppose that $A = \Sigma^{j_1} H\mathbb{F}_p \oplus \Sigma^{j_2} H\mathbb{F}_p \oplus \cdots \oplus \Sigma^{j_l} H\mathbb{F}_p$ is a trivial $\mathbb{P}$-algebra. Then the E^∞-page of the dual bar spectral sequence for $\pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A)) \cong \overline{\mathrm{TAQ}}^{-*}(A)$ has E^∞-page*

$$E^\infty_{*,*} \cong E^2_{*,*} \cong \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}^{\mathbb{F}_p}}(\Sigma^{j_1}\mathbb{F}_p \oplus \cdots \oplus \Sigma^{j_l}\mathbb{F}_p).$$

Proof. The comparison morphism $\bigoplus_n (\mathrm{sLie}(n)^\vee \otimes M)^{\otimes n})_{\Sigma_n} \to \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Comm}, M)$ of the shifted Harrison complex and the bar construction (cf. [Fre04, 5.2.3, 6.2]) is surjective on homotopy when M is an $\mathbb{F}_p$-module considered as a trivial algebra: any cycle on the left-hand side is an alternating sum of all way to put $n-1$ nested parenthesis on a fixed sequence $x_1, \ldots x_n$ of n classes in M, such that each nesting represents taking the polynomial product with one more class on a different simplicial level and no nesting is trivial, and this cycle has preimage the bracket $[\cdots[[x_1,x_2],x_3]\ldots],x_n]$. The graded commutativity and the vanishing of self-brackets on odd degree classes correspond to the graded commutativity of the polynomial product. Since the restriction on odd classes are zero by Proposition 3.4.14, taking linear dual yields

$$\pi_*(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Comm}, M)^\vee) \cong \mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(M^\vee)$$

for any trivial algebra M over Comm and the dual bar spectral sequence simplifies to

$$E^2_{*,*} = \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^{j_1}\mathbb{F}_p \oplus \cdots \oplus \Sigma^{j_l}\mathbb{F}_p) \Rightarrow \pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},A^\vee)|).$$

As in the case $p = 2$, a priori we can't deduce that the spectral sequence collapses using a sparsity argument when $k > 1$, since the E^2-page is concentrated on multiple lines at p^m when $m > 1$. Nonetheless, comparing the basis of the E^2-page with Theorem 3.2.2, we deduce that $E^2 \cong E^\infty$, so the spectral sequence collapses on the second pages. □

Now we proceed to construct all natural operations on the homotopy groups of spectral partition Lie algebras and mod p TAQ (co)homology. It follows from a general result of Brantner ([Bra17, Theorem 3.5.1 and 4.3.2]) that composition product of additive operations on the homotopy groups of spectral partition Lie algebras agrees, up to a shearing, with the Yoneda product on the E^2-page of the dual bar spectral sequence. This allows us to deduce all relations among the unary operations in Theorem 3.5.5 and 3.5.6. Then we construct a shifted Lie algebra structure and prove the existence of a restriction map in Lemma 3.6.5 when $p > 2$. Finally we deduce all relations among unary operations and the bracket in Theorem 3.6.6.

3.5 The structure of unary operations

In the dual bar spectral sequence

$$E^2_{s,t} \cong \mathrm{Free}^{\mathrm{sLie}^\rho_{\bar{\mathcal{R}}^!}}(\Sigma^j\mathbb{F}_2)_{s,t} \Rightarrow \pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},\Sigma^{-j}H\mathbb{F}_2)|) \simeq \pi_{s+t}(\mathrm{Lie}^\pi_{\mathbb{F}_2,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_2)),\ \ p=2$$

$$E^2_{s,t} \cong \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j\mathbb{F}_p)_{s,t} \Rightarrow \pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},\Sigma^{-j}H\mathbb{F}_p)|) \simeq \pi_{s+t}(\mathrm{Lie}^\pi_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p)),\ \ p>2,$$

the E^2-page is generated by a single class $x \in E_2^{0,j}$ under unary operations

$$(Q^i)^* : E^2_{s,t} \to E^2_{s-1,t-i},\ \ i > -t$$

for $p=2$, and

$$(\beta^{\varepsilon}Q^i)^*: E^2_{s,t} \to E^2_{s-1,t-2(p-1)i+\varepsilon}, \ \ 2i > -t$$

where $\varepsilon \in \{0,1\}$, as well as a shifted self-bracket if j is even for $p>2$. The additive unary operations, excluding the self-bracket, and their relations are encoded by the ringoid $(\mathcal{R}')^!$ in Definition 3.4.1 and 3.4.11.

The E^∞-page is the homotopy groups of the free spectral partition Lie algebra on $\Sigma^j H\mathbb{F}_p$. Hence it parametrizes unary operations on a degree j homotopy class of a spectral partition Lie algebra and a degree $-j$ cohomology class in mod p TAQ cohomology. Now we give a concrete description of unary operations on the the homotopy groups of any spectral partition Lie algebra A.

Construction 3.5.1. Suppose that $\xi : \Sigma^j H\mathbb{F}_p \to A$ represents a homotopy class $x \in \pi_j(A)$.

(1) Suppose that $p=2$. For any sequence

$$\alpha = (Q^{i_1})^*(Q^{i_2})^* \cdots (Q^{i_k})^* \in \bar{\mathcal{R}}^!((0,j),(-k,j-m))$$

with $i_1+\cdots+i_k = m$, there is a unique class $R^{(i_1+1,i_2+1,\dots,i_k+1)}(x) \in \pi_{j-m-k}(A)$ given by

$$\begin{aligned}\Sigma^{j-m-k}\mathbb{F}_2 \xrightarrow{\alpha} \mathrm{Free}^{\mathcal{R}^!}(\Sigma^j\mathbb{F}_2) \hookrightarrow \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\bar{\mathcal{R}}^!}}(\Sigma^j\mathbb{F}_2)\\ \cong \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_2,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_2)) \xrightarrow{\xi_*} \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_2,\mathbb{E}_\infty}(A)) \to \pi_*(A).\end{aligned}$$

(2) Suppose that $p>2$. For any sequence

$$\alpha = (\beta^{\varepsilon_1}Q^{i_1})^*(\beta^{\varepsilon_2}Q^{i_2})^* \cdots (\beta^{\varepsilon_k}Q^{i_k})^* \in (\mathcal{R}')^!((0,j),(-k,j-m))$$

with $m = 2(p-1)i_1+\cdots+2(p-1)i_k-\varepsilon_1-\cdots-\varepsilon_k$, there is a unique class $R^{(i_1,\dots,i_k,1-\varepsilon_1,\dots,1-\varepsilon_k)}(x) \in \pi_{j-m-k}(A)$ given by

$$\begin{aligned}\Sigma^{j-m-k}\mathbb{F}_p \xrightarrow{\alpha} \mathrm{Free}^{(\mathcal{R}')^!}(\Sigma^j\mathbb{F}_p) \hookrightarrow \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j\mathbb{F}_p)\\ \cong \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p)) \xrightarrow{\xi_*} \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A)) \to \pi_*(A).\end{aligned}$$

If j is even, then there is a unique class $B(x) \in \pi_{2j-1}(A)$ given by

$$\Sigma^{2j-1} \xrightarrow{[x,x]} \mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j \mathbb{F}_p) \hookrightarrow \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j \mathbb{F}_p) \cong \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p)) \to \pi_*(A)$$

Translating to cohomological grading, for any $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebra A, there are unary operations

$$R^{(i_1,i_2,\dots,i_k)} : \overline{\mathrm{TAQ}}^j(A) \to \overline{\mathrm{TAQ}}^{j+i_1+\cdots+i_k}(A), \;\; i > j$$

for $p = 2$ and

$$R^{(i_1,\dots,i_k,\varepsilon_1,\dots,\varepsilon_k)} : \overline{\mathrm{TAQ}}^j(A) \to \overline{\mathrm{TAQ}}^{j+2(p-1)(i_1+\cdots+i_k)+\varepsilon_1+\cdots+\varepsilon_k}(A), \;\; 2i > j$$

where $\varepsilon \in \{0,1\}$ as well as a self-bracket

$$B : \overline{\mathrm{TAQ}}^j(A) \to \overline{\mathrm{TAQ}}^{2j+1}(A)$$

when j is even for $p > 2$.

Remark 3.5.2. These operations are stable in the sense that any cohomology operation $\alpha : \overline{\mathrm{TAQ}}^m(A) \to \overline{\mathrm{TAQ}}^{m+|\alpha|}(A)$ agrees with $\alpha : \overline{\mathrm{TAQ}}^{m-1}(A) \to \overline{\mathrm{TAQ}}^{m-1+|\alpha|}(A)$ under cohomological desuspension, or equivalently $\alpha : \pi_m(L) \to \pi_{m-|\alpha|}(L)$ agrees with $\alpha : \pi_{m+1}(L) \to \pi_{m+1-|\alpha|}(L)$ for any spectral partition Lie algebra L. This is straightforward to check on the E^2-page of the dual bar spectral sequence in the universal cases using the fact that Dyer-Lashof operations satisfy this notion of stability.

A convenient way to encode the structure of additive operations is via a power ring, as was introduced in [Bra17] to encode additive unary operations on the Lubin-Tate theory of spectral Lie algebras. Note that our convention differs in that we switch to a logarithmic grading convention for the weight grading.

Definition 3.5.3. [Bra17, Definition 3.17][BHK19, Definition 4.5] A *power ring* is a collection $P = \{P^k_j(w)\}_{(j,k,w)\in\mathbb{Z}^2\times\mathbb{Z}_{\geq 0})}$ of abelian groups with elements $\iota_i \in P^i_i[0]$ for all i, equipped with associative and unital composition maps $P^i_j[v] \times P^j_k[w] \to P^i_k[v+w]$.

A *module* over the power ring P is a (weighted) $\mathbb{F}_p$-module $M = \bigoplus_{(j,w)\in\mathbb{Z}\times\mathbb{Z}_{\geq 0}} M_j(w)$ with structure maps $P^i_j[v] \otimes M_j(w) \to M_i(p^v w)$ that are compatible with the composition maps in P.

Definition 3.5.4. The collection $\{\mathcal{P}^j_k[w] := (\mathcal{R}')^!((0,k),(-w,j+w)), w > 0\}$, along with $\mathcal{P}^i_i[0] := \mathbb{F}_p\{\iota_i\}$ for all i and $\mathcal{P}^j_i[0] = \emptyset$ for $i \neq j$, defines a power ring $\mathcal{P}$, with composition product given by the sheared Yoneda product

$$\begin{tikzcd}
\mathcal{P}^i_j[v] \times \mathcal{P}^j_k[w] \arrow[r, "\cong"] \arrow[ddd, dashed] & (\mathcal{R}')^!((0,j),(-v,i+v)) \times (\mathcal{R}')^!((0,k),(-w,j+w)) \arrow[d, hook, "\mathrm{susp}^w \times \mathrm{id}"] \\
& (\mathcal{R}')^!((0,j+w),(-v,i+v+w)) \times (\mathcal{R}')^!((0,k),(-w,j+w)) \arrow[d, "\cong"] \\
& (\mathcal{R}')^!((-w,j+w),(-v-w,i+v+w)) \times (\mathcal{R}')^!((0,k),(-w,j+w)) \arrow[d, "\text{juxtaposition}"] \\
\mathcal{P}^i_k[v+w] \arrow[r, "\cong"] & (\mathcal{R}')^!((0,k),(-v-w,i+v+w)),
\end{tikzcd}$$

for $v,w > 0$, as well as isomorphisms $\mathcal{P}^j_i[w] \times \mathcal{P}^i_i[0] \xrightarrow{\cong} \mathcal{P}^j_i[w]$ and $\mathcal{P}^j_j[0] \times \mathcal{P}^j_i[w] \xrightarrow{\cong} \mathcal{P}^j_i[w]$ exhibiting ι_i as a two-sided unit.

The first map is an injection on the left factor because operations are stable under suspension and here $w \geq 0$, cf. Remark 3.4.2 and 3.4.12. The last map is the composition in the ringoid $(\mathcal{R}')^!$, i.e., juxtaposition corresponding to the Yoneda product on Ext groups.

Explicitly, when $p = 2$, the $\mathbb{F}_2$-module $\mathcal{P}^k_j[w]$ consists of operations $R^{i_1,\dots,i_w}$ such that $j - i_1 - \dots i_w = k$ and $i_l - 1 > i_{l+1} + \dots + i_w - j - (w-l)$ for all $1 \leq l \leq w$, subject to the relations in $(\mathcal{R}')^!((0,j),(-w,k+w))$. The composition product sends $R^{(i_1,\dots,i_v)} \circ R^{(j_1,\dots,j_w)}$ to the juxtaposition $R^{(i_1,\dots,i_v,j_1,\dots,j_w)}$. The weight 2 additive operations are given by the collection $R^i \in \mathcal{P}^{j-i}_i[1]$ for any $i > -j+1$.

Theorem 3.5.5. *The homotopy groups of a spectral partition Lie algebra over $H\mathbb{F}_2$, or the reduced TAQ cohomology of an $\mathbb{E}_\infty$-$H\mathbb{F}_2$-algebra form a left module over the power ring $\mathcal{P}$ of additive unary operations. The relations among the weight* 2 *additive operations are* *given by the Adem relations*

$$R^a R^b = \sum_{a+b-c\geq 2c,\; c>-j+1} \binom{b-c-1}{a-2c} R^{a+b-c} R^c$$

in $\mathcal{P}_j^{j-a-b}[2]$ *for all* $a,b \in \mathbb{Z}$ *satisfying* $b-j < a < 2b$ *and* $b > -j+1$.

A basis for unary operations on a degree j *class is given by all monomials* $R^{i_1}R^{i_2}\cdots R^{i_l}$ *such that* $i_l > -j$ *and* $i_m \geq 2i_{m+1}$ *for* $1 \leq m < l$.

When $p > 2$, the weight p unary operations are given by the collection of elements $\beta^\varepsilon R^i := R^{(i,\varepsilon)} \in \mathcal{P}_j^{j-2(p-1)i-\varepsilon}$ for $\varepsilon = 0,1$ and any $2i > -j$.

Theorem 3.5.6. *The homotopy groups of a spectral partition Lie algebra* A *over* $H\mathbb{F}_p$*, or the reduced TAQ cohomology of any* $\mathbb{E}_\infty$*-*$H\mathbb{F}_p$*-algebra, form a module over the power ring* $\mathcal{P}$ *of unary operations. The relations among the weight* p *operations are given by the Adem relations*

$$\beta R^a \beta R^b = \sum_{a+b-c>pc, 2c>-j} (-1)^{a-c+1} \binom{(p-1)(b-c)-1}{a-pc-1} \beta R^{a+b-c} \beta R^c$$

in $\mathcal{F}_p(j, j-2(p-1)a-2(p-1)b-2)$ *for all* $a,b \in \mathbb{Z}$ *satisfying* $a \leq pb$, $2b > -j$, *and* $2a > 2(p-1)b-j$,

$$\begin{aligned} R^a \beta R^b = &\sum_{a+b-c\geq pc, 2c>-j} (-1)^{a-c} \binom{(p-1)(b-c)}{a-pc} \beta P^{a+b-c} R^c \\ &- \sum_{a+b-c>pc, 2c>-j} (-1)^{a-c} \binom{(p-1)(b-c)-1}{a-pc-1} R^{a+b-c} \beta R^c \end{aligned}$$

in $\mathcal{F}_p(j, j-2(p-1)a-2(p-1)b-1)$ *for all* $a,b \in \mathbb{Z}$ *satisfying* $a \leq pb$, $2b > -j$, *and* $2a > 2(p-1)b+1-j$,

$$\beta^\varepsilon R^a R^b = \sum_{a+b-c\geq pc, 2c>-j} (-1)^{a-c} \binom{(p-1)(b-c)-1}{a-pc} \beta^\varepsilon R^{a+b-c} R^c$$

in $\mathcal{P}_j^{j-2(p-1)a-2(p-1)b-\varepsilon}[2]$ *for all* $a,b \in \mathbb{Z}$ *satisfying* $a < pb$, $2b > -j$, $2a > 2(p-1)b-j$, *and* $\varepsilon \in \{0,1\}$.

A basis for unary operations on a degree j class with j odd is given by the collection of all monomials $\beta^{\varepsilon_1}R^{i_1}\beta^{\varepsilon_2}R^{i_2}\cdots\beta^{\varepsilon_l}R^{i_l}$ such that $2i_l > -j$ and $i_m \geq pi_{m+1}+\varepsilon_{m+1}$ for $1 \leq m < l$. If j is even, a basis is given by the collection $\beta^{\varepsilon_1}R^{i_1}\beta^{\varepsilon_2}R^{i_2}\cdots\beta^{\varepsilon_l}R^{i_l}B^{\varepsilon}$ such that $2i_l > -(1+\varepsilon)j-\varepsilon$ and $i_m \geq pi_{m+1}+\varepsilon_{m+1}$ for $1 \leq m < l$.

Proof of Theorem 5.5 and 5.6. By construction, the set $\mathcal{P}^i_j[w]$ is isomorphic to the image of

$$(\mathcal{R}')^!((0,j),(-w,w+i)) \subset E^2_{*,*}$$

in $\pi_i(\mathcal{L}(\Sigma^j H\mathbb{F}_p))$ via the collapse of the dual bar spectral sequence

$$E^2_{*,*} \cong \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\bar{\mathcal{R}}^!}}(\Sigma^j\mathbb{F}_p) \Rightarrow \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_2,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p))$$

and for $p > 2$

$$E^2_{*,*} \cong \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j\mathbb{F}_p) \Rightarrow \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_2,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p)).$$

We need to verify that compositions of unary operations on the homotopy groups of spectral partition Lie algebras is reflected by the composition product of the power ring $\mathcal{P}$.

For ease of notations, we will use $\mathcal{L}$ to denote the monad $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$ throughout this proof. The unary operations on the homotopy groups of algebras over $\mathcal{L}$, other than the self-brackets on even classes when $p > 2$, are concentrated in weights p^n for $n \geq 1$ by Proposition 3.4.9 and 3.4.14. When A is bounded above, they live in the homotopy groups of the summands

$$\mathcal{L}[n](A) := (\partial_{p^n}(\mathrm{Id}) \otimes H\mathbb{F}_p) \overset{h\Sigma_{p^n}}{\otimes} (A)^{\otimes p^n} \xhookrightarrow{\iota_{p^n}} \mathcal{L}(A)$$

by Proposition 3.1.2. The composition $\beta \circ \alpha$ of two unary operations

$$\alpha \in \mathcal{P}^j_k[w] \subseteq \pi_j(\mathcal{L}[w](\Sigma^k H\mathbb{F}_p)),\;\; \beta \in \mathcal{P}^i_j[v] \subseteq \pi_i(\mathcal{L}[v](\Sigma^j H\mathbb{F}_p)),$$

considered as maps $\alpha : \Sigma^j H\mathbb{F}_p \to \mathcal{L}[w](\Sigma^k H\mathbb{F}_p)$ and $\beta : \Sigma^i H\mathbb{F}_p \to \mathcal{L}[v](\Sigma^j H\mathbb{F}_p)$, is given by

$$\Sigma^i H\mathbb{F}_p \xrightarrow{\beta} \mathcal{L}[v](\Sigma^j H\mathbb{F}_p) \xrightarrow{\mathcal{L}[v](\alpha)} \mathcal{L}[v] \circ \mathcal{L}[w](\Sigma^k H\mathbb{F}_p) \to \mathcal{L}[v+w](\Sigma^k H\mathbb{F}_p). \tag{3.4}$$

The last map is induced by the weight p^{v+w} summand

$$(\partial_{p^v}(\mathrm{Id}) \otimes H\mathbb{F}_p) \overset{h\Sigma_{p^v}}{\otimes} \left((\partial_{p^w}(\mathrm{Id}) \otimes H\mathbb{F}_p) \overset{h\Sigma_{p^w}}{\otimes} A^{\otimes p^w})\right)^{\otimes p^v} \to (\partial_{p^{(v+w)}}(\mathrm{Id}) \otimes H\mathbb{F}_p) \overset{h\Sigma_{p^{v+w}}}{\otimes} A^{\otimes p^{v+w}}$$

of the structure map of the monad $\mathcal{L} \circ \mathcal{L} \to \mathcal{L}$ on any bounded above object A.

Let $a \in \bar{\mathcal{R}}^!((0,k),(-w,j+w)), b \in \bar{\mathcal{R}}^!((0,j),(-v,i+v))$ be the unique preimages under the isomorphisms in Proposition 3.4.9 and 3.4.14 of α and β on the E^2-pages of the dual bar spectral sequences converging respectively to $\pi_*(\mathcal{L}(\Sigma^k H\mathbb{F}_p))$ and $\pi_*(\mathcal{L}(\Sigma^j H\mathbb{F}_p))$.

Since $\pi_*(\mathcal{L}(\Sigma^k H\mathbb{F}_p))$ is bounded above and of finite type, we can run the dual bar spectral sequence for the $H\mathbb{F}_p$-module $A = \mathcal{L}(\Sigma^k H\mathbb{F}_p)$ converging to $\pi_*(\mathcal{L} \circ \mathcal{L}(\Sigma^k H\mathbb{F}_p))$. The spectral sequence collapses on the E^2-page

$$E^2 \cong \mathrm{Free}^{\mathrm{sLie}^\rho_{\bar{\mathcal{R}}^!}}\left(\pi_*(\mathcal{L}(\Sigma^k \mathbb{F}_p))\right) \cong \mathrm{Free}^{\mathrm{sLie}^\rho_{\bar{\mathcal{R}}^!}} \circ \mathrm{Free}^{\mathrm{sLie}^\rho_{\bar{\mathcal{R}}^!}}(\Sigma^k \mathbb{F}_p),\ p=2,$$

$$E^2 \cong \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}\left(\pi_*(\mathcal{L}(\Sigma^k \mathbb{F}_p))\right) \cong \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}} \circ \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^k \mathbb{F}_p),\ p>2$$

by comparing with Theorem 3.2.2 in the limiting case.

The map $\mathcal{L} \circ \mathcal{L}(\Sigma^k H\mathbb{F}_p) \to \mathcal{L}(\Sigma^k H\mathbb{F}_p)$ coming from the monad composition induces a map of the E^2-pages of the dual bar spectral sequences

$$\mathrm{Free}^{\mathrm{sLie}^\rho_{\bar{\mathcal{R}}^!}} \circ \mathrm{Free}^{\mathrm{sLie}^\rho_{\bar{\mathcal{R}}^!}}(\Sigma^k \mathbb{F}_p) \to \mathrm{Free}^{\mathrm{sLie}^\rho_{\bar{\mathcal{R}}^!}}(\Sigma^k \mathbb{F}_p),\ p=2,$$

$$\mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}} \circ \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^k \mathbb{F}_p) \to \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^k \mathbb{F}_p),\ p>2.$$

We need to understand the restriction of the above maps to the additive part, i.e., the horizontal maps of the diagram

$$\begin{tikzcd}
\mathrm{Free}^{(\mathcal{R}')^!} \circ \mathrm{Free}^{(\mathcal{R}')^!}(\Sigma^k \mathbb{F}_p) \arrow[rr] \arrow[d, "\cong"] & & \mathrm{Free}^{(\mathcal{R}')^!}(\Sigma^k \mathbb{F}_p) \arrow[d, "\cong"] \\
\pi_*(\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\mathcal{R}'}, \pi_*(\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\mathcal{R}'}, \Sigma^{-k}\mathbb{F}_p)^\vee)^\vee) \arrow[rr] & & \pi_*(\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\mathcal{R}'}, \Sigma^{-k}\mathbb{F}_p)^\vee)
\end{tikzcd}.$$

Recall that $\mathrm{Free}^{(\mathcal{R}')^!}(\Sigma^k \mathbb{F}_p)$ is by construction isomorphic to the unstable Ext group

$$\mathrm{UnExt}^{*,*}_{\mathcal{R}'}(\mathbb{F}_p, \Sigma^k \mathbb{F}_p) \cong \pi_*(\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\mathcal{R}'}, \Sigma^{-k}\mathbb{F}_p)^\vee).$$

We will make use of a general result of Brantner that follows from [Bra17, Theorem 3.5.1 and 4.3.2]: Suppose that $\mathbf{T}$ is an additive monad on $\mathrm{Mod}_{\mathbb{F}_p}$ associated to the free (unstable) module functor over an algebra R. Then the composition map of the monad $|\mathrm{Bar}_\bullet(\mathrm{id}, \mathbf{T}, -)|^\vee$ is compatible with the Yoneda product on the (unstable) Ext groups over R up to a shearing of the Ext groups.

Here $\mathcal{A}_{\mathcal{R}'}$ is an additive monad associated with the free functor that takes the unstable module over the Koszul algebra $\bar{\mathcal{R}}$. It follows that the top map is a sheared Yoneda product on (unstable) Ext groups. More precisely, given $b \in \mathcal{R}'((0,j),(-v,i+v)) \cong \mathrm{UnExt}^{-v,i+v}_{\mathcal{R}'}(\mathbb{F}_p, \Sigma^j \mathbb{F}_p)$ and $a \in \mathcal{R}'((0,k),(-w,j+w)) \cong \mathrm{UnExt}^{-w,j+w}_{\mathcal{R}'}(\mathbb{F}_p, \Sigma^k \mathbb{F}_p)$, the top map produces an element

$$b \circ a | x_k \mapsto b|a|x_k \in (\mathcal{R}')^!((0,k),(-v-w,i+v+w)) \cong \mathrm{UnExt}^{-v-w,i+v+w}_{\mathcal{R}'}(\mathbb{F}_p, \Sigma^k \mathbb{F}_p)$$

via the composite

$$\begin{tikzcd}
(\mathcal{R}')^!((0,j),(-v,i+v)) \times (\mathcal{R}')^!((0,k),(-w,j+w)) \arrow[d, hook, "\mathrm{susp}^w \times \mathrm{id}"] \\
(\mathcal{R}')^!((0,j+w),(-v,i+v+w)) \times (\mathcal{R}')^!((0,k),(-w,j+w)) \arrow[d, "\cong"] \\
(\mathcal{R}')^!((-w,j+w),(-v-w,i+v+w)) \times (\mathcal{R}')^!((0,k),(-w,j+w)) \arrow[d] \\
(\mathcal{R}')^!((0,k),(-v-w,i+v+w)).
\end{tikzcd}$$

The first map is an injection on the left factor because operations are stable under suspension and here $w \geq 0$, cf. Remark 3.4.2 and 3.4.12. The last map is the composition in the ringoid $(\mathcal{R}')^!$, i.e. juxtaposition corresponding to the Yoneda product on unstable Ext groups. This is exactly the composition product in $\mathcal{P}$.

Therefore the map (3.4) lifts to a map along the E^2-pages of the respective dual bar spectral sequences, given explicitly by $b|x_j \mapsto b \circ (a|x_k) \mapsto b|a|x_k$. Here we use $|$ to denote juxtaposition in $(\mathcal{R}')^!$ and x_k the generator for $\Sigma^k \mathbb{F}_p$. Passing to the E^∞-pages, we deduce that there is a commutative diagram

$$
\begin{tikzcd}
\mathcal{P}^i_j[v] \times \mathcal{P}^j_k[w] \arrow[r, "*"] \arrow[d, "\cong"] & \mathcal{P}^i_k[v+w] \arrow[d, "\cong"] \\
(\mathcal{R}')^!((0,j),(-v,i+v)) \times (\mathcal{R}')^!((0,k),(-w,j+w)) \arrow[r, "\cup"] \arrow[d, hook] & (\mathcal{R}')^!((0,k),(-v-w,i+v+w)) \arrow[d, "\cong"] \\
\begin{array}{c}(\mathcal{R}')^!((-w,j+w),(-v-w,i+v+w)) \\ \times(\mathcal{R}')^!((0,k),(-w,j+w))\end{array} \arrow[r, "|"] \arrow[d, hook] & (\mathcal{R}')^!((0,k),(-v-w,i+v+w)) \arrow[d, hook] \\
\pi_i(\mathcal{L}(\Sigma^j H\mathbb{F}_p)) \times \pi_j(\mathcal{L}(\Sigma^k H\mathbb{F}_p)) \arrow[r, "\circ"] & \pi_{t+l}(\mathcal{L}(\Sigma^k H\mathbb{F}_p))
\end{tikzcd}
$$

as desired. The first two horizontal maps, i.e. the composition product in the power ring $\mathcal{P}$, are given by the sheared Yoneda product, and the bottom horizontal map is given explicitly by the map (3.4).

In particular, the composition product $\mathcal{P}^{j-a-b}_{j-a}[1] \times \mathcal{P}^{j-a}_{j}[1] \to \mathcal{P}^{j-a-b}_{j}[2]$ sends (R^b, R^a) to $R^{(b,a)} = (Q^{b-1})^*(Q^{a-1})^*$ for $p=2$, and $(\beta^{\varepsilon_1} R^{b'-\varepsilon_1}, \beta^{\varepsilon_2} R^{a'-\varepsilon_2})$ to

$$R^{(b'-\varepsilon_1, a'-\varepsilon_2, \varepsilon_1, \varepsilon_2)} = (\beta^{1-\varepsilon_1} Q^{b'-1})^*(\beta^{1-\varepsilon_2} Q^{a'-1})^*$$

for $p > 2$ where $2(p-1)a' = a, 2(p-1)b' = b$. The two classes in the composition are defined whenever $a > -j+1$ and $b > a-j$ for $p=2$, and $2a' > -j$ and $2b' > a+\varepsilon_2 - j$ for $p > 2$. Hence all the Adem relations hold.

When $p=2$, a basis for additive unary operations on a degree j class is given by all monomials $(Q^{i_1})^*(Q^{i_2})^* \cdots (Q^{i_s})^* \in (\mathcal{R}')^!((0,j),(s,m))$ such that $i_s \geq -j+1$ and $i_l > 2i_{l+1}$ for all $1 \leq l < s$, cf. Proposition 3.4.9. Any such monomial is the image of the (well-defined) iterated composition $R^{i_1+1}R^{i_2+1} \cdots R^{i_s+1}$ in $\mathcal{P}^{j-m-s}_j[s]$. Hence every additive unary operation $R^{(i_1,\dots,i_s)}$ can be written as a linear combination of compositions $R^{j_1}\dots R^{j_s}$ of operations in $\mathcal{P}^*_*[1]$. The case $p>2$ is analogous. $\square$

Corollary 3.5.7. *When j gets arbitrarily large, we deduce that the algebra of unary operations on a degree j homotopy class of a spectral partition Lie algebra, or equivalently a degree $-j$ class in mod p TAQ cohomology, is the Koszul dual algebra of the mod p Dyer-Lashof algebra.*

This is because when the degree j of a class x gets arbitrarily large, all unary operations and relations are defined on x, while the shifted Lie brackets and restrictions vanish for degree reasons. For $p>2$ one needs be careful about the precise duality. The Dyer-Lashof operation Q^i is sent to $\beta R^i=\beta P^i$ in cohomological degree $2(p-1)i+1$ and βQ^i to $R^i=P^i$ in cohomological degree $2(p-1)i$ since the Bockstein homomorphism increases cohomological degree by one.

3.6 Shifted restricted Lie algebra structure

Next we examine the shifted restricted Lie algebra structure on the homotopy groups of spectral partition Lie algebras and the reduced mod p TAQ cohomology. Some of the methods in this section are inspired by the thesis works of Antolín-Camarena and Brantner regarding the shifted Lie algebra structure on the mod 2 homology and the Lubin Tate theory of spectral Lie algebras in [AC20, Bra17].

We showed in Proposition 3.4.10 and 3.4.16 that in the case of a free $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$-algebra on a bounded $\mathbb{F}_p$-module A as a trivial $\mathbb{P}$-algebra, there is a shifted (restricted) Lie bracket on the André-Quillen cohomology of the trivial $\mathrm{Poly}_{\mathcal{R}}$-algebra $\pi_*(A)$, which is the E^2-page of the dual bar spectral sequence that converges to $\overline{\mathrm{TAQ}}^{-*}(A)\cong\pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A^\vee))$. Now we show that there is a shifted restricted Lie algebra structure on the homotopy groups of any spectral partition Lie algebra, or the TAQ cohomology of any $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebra A. The shifted Lie algebra structure exists at the level of $H\mathbb{F}_p$-modules and agrees with the shifted Lie algebra structure on the E^2-page.

Recall that the second Goodwillie derivative of the identity functor $\partial_2(\mathrm{Id})\simeq\mathbb{S}^{-1}$ is a naïve Σ_2-spectrum with trivial Σ_2-action. By Proposition 3.1.2, the weight 2 part of the free spectral partition Lie algebra on a bounded $H\mathbb{F}_p$-module A is

$$\big((\partial_2(\mathrm{Id})\otimes H\mathbb{F}_p)\overset{h\Sigma_2}{\otimes} A^{\otimes 2}\big) \xhookrightarrow{\iota_2} \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A) \simeq \bigoplus_n (\partial_n(\mathrm{Id})\otimes H\mathbb{F}_p)\overset{h\Sigma_n}{\otimes}(A)^{\otimes n}.$$

Here ι_n denotes the inclusion of the weight n homogeneous piece.

The binary operation $[-,-]$ representing the shifted Lie bracket is encoded by the weight two part of the structure map

$$\xi_2 : \mathbb{S}^{-1}\otimes (A^{\otimes 2})^{h\Sigma_2} \simeq (\partial_2(\mathrm{Id})\otimes H\mathbb{F}_p)\overset{h\Sigma_2}{\otimes} A^{\otimes 2} \xrightarrow{\iota_2} \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A) \to A,$$

explicitly given as follows.

Construction 3.6.1. For $x : \Sigma^j H\mathbb{F}_p \to A$, $y : \Sigma^k H\mathbb{F}_p \to A$ representing two homotopy classes of a spectral partition Lie algebra A, we have a map of $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$-algebras

$$\theta : \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p \oplus \Sigma^k H\mathbb{F}_p) \to \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A) \to A,$$

where the second map is the $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$-algebra structural map of A. Write $X = \Sigma^j H\mathbb{F}_p$ and $Y = \Sigma^k H\mathbb{F}_p$. There is a binary operation $[-,-]$ on $\pi_*(A)$ represented by the map

$$\mathbb{S}^{-1}\otimes X\otimes Y \simeq \mathbb{S}^{-1}\otimes (X\otimes Y \oplus Y\otimes X)^{h\Sigma_2} \hookrightarrow \mathbb{S}^{-1}\otimes \big((X\oplus Y)^{\otimes 2}\big)^{h\Sigma_2} \simeq \partial_2(\mathrm{Id})\overset{h\Sigma_2}{\otimes}(X\oplus Y)^{\otimes 2} \tag{3.5}$$

followed by the composite

$$\theta\circ\iota_2 : \partial_2(\mathrm{Id})\overset{h\Sigma_2}{\otimes}(X\oplus Y)^{\otimes 2} \hookrightarrow \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p \oplus \Sigma^k H\mathbb{F}_p) \to A,$$

which sends the pair of homotopy classes x and y to a class $[x,y]\in\pi_*(A)$.

At the level of the dual bar spectral sequence, the binary operation $[x,y]\in\pi_*(A)$ is represented uniquely up to a nonzero scalar by the weight 2 part of the composite

$$\begin{aligned}\Sigma^{-1}(\Sigma^j\mathbb{F}_2\oplus\Sigma^k\mathbb{F}_2) \to \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_2}}(\Sigma^j\mathbb{F}_2\oplus\Sigma^k\mathbb{F}_2) \hookrightarrow \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_2}}(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)\\ \cong \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_2\oplus\Sigma^k H\mathbb{F}_2)) \xrightarrow{\theta_*} \pi_*(A),\end{aligned}$$

$$\Sigma^{-1}(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)\to \mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)\hookrightarrow \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)$$
$$\cong \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p\oplus\Sigma^k H\mathbb{F}_p))\xrightarrow{\theta_*}\pi_*(A),$$

for $p>2$.

More precisely, since the dual bar spectral sequence converging to $\pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p\oplus\Sigma^k H\mathbb{F}_p))$ collapses on the E^2-page with no extension problems, there are unique preimages of x,y on the E^2-page, which we again call x,y by abuse of notations. The binary operation $[-,-]$ is then given at all primes p by

$$\begin{aligned}\Sigma^{-1}(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)\to \mathbb{F}_p\{[x,y]\}&\hookrightarrow \mathrm{wt}_2\big(\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)\big)\\ &\cong \pi_*\big(\mathbb{S}^{-1}\overset{h\Sigma_2}{\otimes}(\Sigma^j H\mathbb{F}_p\oplus\Sigma^k H\mathbb{F}_p)^{\otimes 2}\big)\\ &\xrightarrow{(\iota_2)_*}\mathrm{wt}_2[\pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p\oplus\Sigma^k H\mathbb{F}_p))]\xrightarrow{\theta_*}\pi_*(A)\end{aligned}$$

up to a nonzero scalar c. We fix a choice of the generator for the shifted Lie bracket on the E^2-page of the dual bar spectral sequence so that $c=1$, and by abuse of notation we use $[-,-]$ to denote both the binary operation on $\pi_*(A)$ and the shifted Lie bracket on the E^2-page.

Translating to cohomological grading, we constructed a binary operation

$$[-,-]:\overline{\mathrm{TAQ}}^m(A)\otimes\overline{\mathrm{TAQ}}^n(A)\to\overline{\mathrm{TAQ}}^{m+n+1}(A)$$

for any $\mathbb{E}_\infty^{\mathrm{nu}}$-$H\mathbb{F}_p$-algebra A.

First we show that this binary operation is indeed a shifted Lie bracket in a general sense.

Proposition 3.6.2. *The binary operation $[-,-]$ in Construction 3.6.1 satisfies graded commutativity $[x,y]=(-1)^{|x||y|}[y,x]$ and the graded Jacobi identity*

$$(-1)^{|x||z|}[x,[y,z]]+(-1)^{|x||y|}[y,[z,x]]+(-1)^{|y||z|}[z,[x,y]]=0.$$

We shall see in the next proposition that $[x,x]=0$ for all x at $p=2$ and $[x,[x,x]]=0$

for all x at $p=3$. Hence $[-,-]$ equips the homotopy groups of any spectral partition Lie algebra with a shifted Lie algebra structure as is the convention of this paper.

Proof. Graded commutativity $[x,y]=(-1)^{|x||y|}[y,x]$ follows by construction, with the sign coming from the induced action of the transposition $(12)\in\Sigma_2$. To check the graded Jacobi identity of the shifted bracket, we use an argument adapted from [AC20]. Let $A=\Sigma^j H\mathbb{F}_p\oplus\Sigma^k H\mathbb{F}_p\oplus\Sigma^l H\mathbb{F}_p$. The iteration $[[-,-],-]$ of the binary operation $[-,-]$ is given by the weight 3 summand

$$\begin{aligned}
&(\partial_2(\mathrm{Id})\otimes H\mathbb{F}_p)\otimes\Big(\big((\partial_1(\mathrm{Id})\otimes H\mathbb{F}_p)\overset{h\Sigma_1}{\otimes}A\big)\otimes\big((\partial_2(\mathrm{Id})\otimes H\mathbb{F}_p)\overset{h\Sigma_2}{\otimes}A^{\otimes 2}\big)\Big)\\
&\xhookrightarrow{(3.5)}\mathrm{wt}_3\Big[(\partial_2(\mathrm{Id})\otimes H\mathbb{F}_p)\overset{h\Sigma_2}{\otimes}\Big(\big((\partial_1(\mathrm{Id})\otimes H\mathbb{F}_p)\overset{h\Sigma_1}{\otimes}A\big)\oplus\big((\partial_2(\mathrm{Id})\otimes H\mathbb{F}_p)\overset{h\Sigma_2}{\otimes}A^{\otimes 2}\big)\Big)^{\otimes 2}\Big]\\
&\to(\partial_3(\mathrm{Id})\otimes H\mathbb{F}_p)\overset{h\Sigma_3}{\otimes}A^{\otimes 3}
\end{aligned}$$

of the monad composition $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}\circ\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}\to\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$ applied to A. Since $\partial_2(\mathrm{Id})\simeq\mathbb{S}^{-1}$ and $\partial_1(\mathrm{Id})\simeq\mathbb{S}$ both have trivial actions by the symmetric groups, we deduce that the source of the above structure map is equivalent to

$$\partial_2(\mathrm{Id})\otimes(\partial_1(\mathrm{Id})\otimes\partial_2(\mathrm{Id}))\otimes H\mathbb{F}_p\otimes(A^{\otimes 3})^{h\Sigma_3}.$$

Denote by ν the structure map $\partial_2(\mathrm{Id})\otimes(\partial_1(\mathrm{Id})\otimes\partial_2(\mathrm{Id}))\to\partial_3(\mathrm{Id})$. The graded Jacobi identity

$$(-1)^{|x||z|}[x,[y,z]]+(-1)^{|x||y|}[y,[z,x]]+(-1)^{|y||z|}[z,[x,y]]=0$$

is then equivalent to showing that $\nu+(\sigma)_*\nu+(\sigma)_*^2\nu$ is null-homotopic, where $(\sigma)_*$ is the induced action of the cyclic permutation (123). It was proved in [AC20, Proposition 5.2] that $\nu+(\sigma)_*\nu+(\sigma)_*^2\nu$ is null-homotopic. □

Next we investigate the interaction between the shifted Lie bracket $[-,-]$ and the unary operation in Construction 3.5.1.

Proposition 3.6.3. *Given any $x,y\in\pi_*(A)$ and unary operation α of positive weight, we have $[x,\alpha(y)]=0$ unless one of the following condition is satisfied:*

(1) $p=2$ and α is an iteration of bottom unary operations on y. The bottom unary operation on a class y is given by $R^{-|y|+1}$ in Construction 3.5.1, which equips the shifted Lie bracket with a restriction map $x \mapsto x^{[2]}$. In particular $[x,x]=0$ for all x.

(2) $p>2$ and y is in odd degree α is an iteration of bottom unary operations on y. The bottom unary operation on an odd class y is given by $R^{(-|y|+1)/2}$ in Construction 3.5.1. In particular $[x,[x,x]]=0$ for all x when $p=3$.

Proof. We use an argument adapted from [Bra17, Proposition 4.3.15]. Suppose that α is a nonempty sequence of operations with weight w divisible by p, and $x : \Sigma^j H\mathbb{F}_p \to A, y : \Sigma^k H\mathbb{F}_p \to A$ representing two homotopy classes of a spectral partition Lie algebra A. The operation $[x, \alpha(y)]$ is encoded by the weight $w+1$ part of the structure map $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty} \circ \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty} \to \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$, i.e.,

$$\begin{aligned}
\Sigma^{j+k+|\alpha|-1} H\mathbb{F}_p &\to d_2 \otimes \Big(\big(d_1 \overset{h\Sigma_1}{\otimes} \Sigma^j H\mathbb{F}_p\big) \otimes \big(d_w \overset{h\Sigma_w}{\otimes} (\Sigma^k H\mathbb{F}_p)^{\otimes w}\big) \Big) \\
&\xhookrightarrow{(3.5)} \mathrm{wt}_{w+1} \Big[d_2 \overset{h\Sigma_2}{\otimes} \Big(\big(d_1 \overset{h\Sigma_1}{\otimes} \Sigma^j H\mathbb{F}_p\big) \oplus \big(d_w \overset{h\Sigma_w}{\otimes} (\Sigma^k H\mathbb{F}_p)^{\otimes w}\big) \Big)^{\otimes 2} \Big] \\
&\to d_{w+1} \overset{h(\Sigma_1 \times \Sigma_w)}{\otimes} (\Sigma^j H\mathbb{F}_p \otimes \Sigma^{kw} H\mathbb{F}_p) \\
&\hookrightarrow \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p \oplus \Sigma^k H\mathbb{F}_p) \to \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(A) \to A,
\end{aligned} \tag{3.6}$$

where we write d_n for $\partial_n(\mathrm{Id}) \otimes H\mathbb{F}_p$ for ease of notation. Note that the action obtained by restriction to $\Sigma_1 \times \Sigma_w \subset \Sigma_{w+1}$ on $\partial_{w+1}(\mathrm{Id})$ is freely induced from the action of the trivial subgroup on $\mathbb{S}^{-w}$. For any finite group G, we have $(\mathrm{Ind}^G_{\{e\}}(X) \otimes Y)^{hG} \simeq X \otimes Y$ for all G-spectra Y by the Wirthmüller isomorphism. Hence

$$(\partial_{w+1}(\mathrm{Id}) \otimes H\mathbb{F}_p) \overset{h(\Sigma_1 \times \Sigma_w)}{\otimes} \Sigma^{j+kw} H\mathbb{F}_p \simeq \Sigma^{j+(k-1)w} H\mathbb{F}_p.$$

In particular, the $\mathbb{F}_p$-module of weight $w+1$ operations on spectral partition Lie algebras coming from the bracket of one weight one operation and one weight w operation is one-dimensional. Note that $[[\cdots[[x,y],y]\cdots],y]$, where we take the bracket with y exactly w times, is a class of weight $w+1$ obtained by taking the bracket of y with a length w bracket

$[[\cdots[[x,y],y]\cdots],y]$ where we take the bracket with y exactly $w-1$ times in

$$\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_p}} \circ \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_p}}(\Sigma^j\mathbb{F}_p \oplus \Sigma^k\mathbb{F}_p) \subset \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\bar{\mathcal{R}}^!}} \circ \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\bar{\mathcal{R}}^!}}(\Sigma^j\mathbb{F}_p \oplus \Sigma^k\mathbb{F}_p)$$
$$\cong \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty} \circ \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p \oplus \Sigma^k H\mathbb{F}_p)) \xrightarrow{\theta_*} \pi_*(A)$$

when $p=2$ and

$$\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}} \circ \mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j\mathbb{F}_p \oplus \Sigma^k\mathbb{F}_p) \subset \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}} \circ \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^j\mathbb{F}_p \oplus \Sigma^k\mathbb{F}_p)$$
$$\cong \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty} \circ \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p \oplus \Sigma^k H\mathbb{F}_p)) \xrightarrow{\theta_*} \pi_*(A)$$

for $p>2$. In the free case this class is nonzero, so we obtain a generator γ of the $\mathbb{F}_p$-module of weight $w+1$ operations coming from the bracket of one weight one operation and one weight w operation.

If α represents an iteration of the self-bracket, then $[x,\alpha(y)]=0$ by the Jacobi identity when $p\neq 3$. When $p=3$ and $|x|=k$ is even, a degree count shows that $[x,[x,x]]=0$ since the weight 3 part of the E^2-page of the dual bar spectral sequence

$$E^2 \cong \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}} \circ \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^k\mathbb{F}_p) \Rightarrow \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty} \circ \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^k H\mathbb{F}_p))$$

has nothing in total degree $3k-2$.

Suppose that α is not an iteration of the self-bracket. If α is not an iteration of the bottom operation $R^{-|x|+1}$ on x when $p=2$, or an iteration of the bottom operation $R^{(-|x|+1)/2}$ on odd x when $p>2$. Then a comparison of topological degrees shows that $[x,\alpha(y)]$ has strictly smaller topological degree than the generator $\gamma=[[\cdots[[x,y],y]\cdots],y]$ of weight $w+1$ operations coming from the bracket of one weight one operation and one weight w operation. Therefore it has to be zero.

If $p=2$ and α is the bottom operation $R^{-|y|+1}$ on y, then we know that $[x,(Q^{-|y|})^*(y)] = [[x,y],y]$ on the $E^\infty \cong E^2$-page of the dual bar spectral sequence converging to $\pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_2,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_2 \oplus \Sigma^k H\mathbb{F}_2)))$ by Proposition 3.4.10. This is the image of $[x,(Q^{-|y|})^*(y)]$ under the summand

$$\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_2}} \circ \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\bar{\mathcal{R}}^!}}(\Sigma^j\mathbb{F}_p \oplus \Sigma^k\mathbb{F}_p) \to \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\bar{\mathcal{R}}^!}}(\Sigma^j\mathbb{F}_p \oplus \Sigma^k\mathbb{F}_p)$$

of the map of E^2-pages of the dual bar spectral sequences for the monad composition map

$$\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty} \circ \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p \oplus \Sigma^k H\mathbb{F}_p) \to \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p \oplus \Sigma^k H\mathbb{F}_p).$$

Passing to the E^∞-pages of the dual bar spectral sequences of the composition map, we deduce that $[x, R^{-|y|+1}(y)] = \gamma = [[x,y],y]$, i.e., the bottom operation serves as the restriction for the bracket. By induction we conclude that if α is the n-th iteration of the restriction, then $[x,\alpha(y)] = \gamma$ in weight $2n+1$. Note that there is only one unary operation of degree $-|x|+1$ on any class x up to a scalar, and the restriction is such an unary operation. Hence we deduce that $[x,x] = 0$ since taking self-bracket is an additive operation while the restriction is not. $\square$

When $p > 2$, we expect to see a shifted restricted Lie algebra structure on the E^∞-page analogous to the case $p = 2$ with the restriction given by the bottom operation, as was noted in [BCN21, Remark 4.49] and by Basterra and Mandell, cf. [Law20, Example 1.8.8].

Definition 3.6.4. (cf. [Jac41], [Fre00]) A shifted *restricted* Lie algebra over $\mathbb{F}_p$, denoted as a $\mathrm{sLie}^{\rho}_{\mathbb{F}_p}$-algebra, is a graded $\mathbb{F}_p$-module $L = L_\bullet$ with a shifted Lie bracket $L_m \otimes L_n \to L_{m+n-1}$ and a restriction map $x \mapsto x^{[p]}$ with $x^{[p]} \in L_{p|x|-p+1}$ whenever $|x|$ is odd, satisfying the following identities:

1. $(cx)^{[p]} = c^p x^{[p]}$ for all odd degree $x \in L$ and $c \in \mathbb{F}_p$;

2. $\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)$ for all odd degree $x \in L$;

3. For all odd degree $x, y \in L$, $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i}{i}(x,y)$, where s_i is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}(tx+y)^{p-1}(x)$.

Here $\mathrm{ad}(x)$ stands for the self-map $y \mapsto [y,x]$ on L.

Lemma 3.6.5. *If j is odd, then $\overline{\mathrm{TAQ}}^{-*}(\Sigma^{-j} H\mathbb{F}_p) \cong \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p))$ admits a restriction $x \mapsto x^{[p]}$ that coincides with the bottom operation $R^{(-j+1)/2}$ on the generator x in degree j up to a unit λ_j that depends only on j. If j is even, such a map does not exist. In general the shifted Lie bracket on the homotopy groups of spectral partition Lie algebras* *admits a restriction that on any odd degree class x is represented by the bottom operation $R^{(-|x|+1)/2}$ up to a unit $\lambda_{|x|}$.*

Proof. It suffices to check the cases of single generators and two generators. Since the category of $H\mathbb{F}_p$-modules is equivalent to the derived category of chain complexes over $\mathbb{F}_p$ ([Lur17, 7.1.1.16]), we can think of $\pi_{s+t}(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},A)|)$ as the homology of the chain complex

$$C = C_*(\mathbb{D}|\mathrm{Bar}_\bullet(\mathrm{id},\mathbb{P},A)|) \cong \bigoplus_n (\mathrm{sLie}(n)\otimes(A^\vee)^{\otimes n})^{h\Sigma_n}$$

when A is bounded by [BCN21, Remark 4.49], and its homotopy group is a shifted restricted Lie algebra. Here sLie is the shifted Lie operad in the dg-category, with $\mathrm{sLie}(r)$ concentrated in dimension $1-r$, weight r. It remains to show that the restriction is nonzero and identify the restriction on the generator x of $\Sigma^j\mathbb{F}_p$.

There is a homotopy fixed points spectral sequence

$$E^2_{s,t} = H^s\Big(\Sigma_p, \pi_t\big(\mathrm{sLie}(p)\otimes(\Sigma^j\mathbb{F}_p)^{\otimes p}\big)\Big) \Rightarrow \pi_{t-s}\Big(\big(\mathrm{sLie}(p)\otimes(\Sigma^j H\mathbb{F}_p)^{\otimes p}\big)^{h\Sigma_p}\Big),$$

where $\Sigma^j\mathbb{F}_p$ is considered as a one-dimensional chain complex over $\mathbb{F}_p$ concentrated in homological degree j with no differentials. Since $\pi_t(\mathrm{sLie}(p)\otimes(\Sigma^j\mathbb{F}_p)^{\otimes p}) = 0$ unless $t = pj+1-p$, the E^2-page of the homotopy fixed points spectral is concentrated on a single line $t = pj+1-p$. Hence the spectral sequence collapses and there are no extension problems.

Taking $s=0$, we deduce that

$$\pi_{pn+1-p}\big(\mathrm{sLie}(p)\otimes(\Sigma^j\mathbb{F}_p)^{\otimes p}\big) \cong H^0\big(\Sigma_p;\mathrm{sLie}(p)\otimes(\Sigma^j\mathbb{F}_p)^{\otimes p}\big),$$

where the right hand side has coefficients in $\mathrm{Mod}_{\mathbb{F}_p}$. By [Fre00, Theorem 1.2.5], if j is odd then

$$H^0\big(\Sigma_p;\mathrm{sLie}(p)\otimes(\Sigma^j\mathbb{F}_p)^{\otimes p})\big) \cong \big(\mathrm{sLie}(p)\otimes(\Sigma^j\mathbb{F}_p)^{\otimes p}\big)^{\Sigma_p} \cong \big(\Sigma^{-1}\mathrm{Lie}(p)\otimes(\Sigma^{j+1}\mathbb{F}_p)^{\otimes p}\big)^{\Sigma_p}$$

contains an element that serves as the restriction $x^{[p]}$ in the free shifted restricted Lie algebra on $\Sigma^j\mathbb{F}_p$. Since $x^{[p]}$ is in topological degree $pj+1-p$ and the only element in the weight p part of $\pi_{pj+1-p}(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p))$ comes from the bottom operation $(\beta Q^{(-j+1)/2})^*(x)$ up to a unit λ on the E^2-page of the dual bar spectral sequence, we conclude that the restriction on x is given by the image of $\lambda R^{(-j+1)/2}(x)$ in $\pi_{pj+1-p}(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p))$. This unit λ is fixed for any class of a given degree j by functoriality of the restriction map and the bottom Dyer-Lashof operation on an odd class x as the p-fold Massey product on x.

The class x does not admit a restriction when j is even because $\pi_{pj+1-p}(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p)) = 0$, which is as expected for a shifted restricted Lie algebra.

Next we take $A = \Sigma^j H\mathbb{F}_p \oplus \Sigma^k H\mathbb{F}_p$, with j odd. There is a homotopy fixed points spectral sequence

$$E^2_{s,t} = \bigoplus_n H^s\left(\Sigma_n, \pi_t\left(\mathrm{sLie}(n)\otimes(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)^{\otimes n}\right)\right) \Rightarrow \bigoplus_n \pi_{t-s}\left(\left(\mathrm{sLie}(n)\otimes(\Sigma^j H\mathbb{F}_p\oplus\Sigma^k H\mathbb{F}_p)^{\otimes n}\right)^{h\Sigma_n}\right).$$

On the line $s=0$, we have

$$\bigoplus_n H^0\left(\Sigma_n; \mathrm{sLie}(n)\otimes(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)^{\otimes n})\right) \cong \bigoplus_n \left(\mathrm{sLie}(n)\otimes(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)^{\otimes n}\right)^{\Sigma_n},$$

which is the free shifted restricted Lie algebra on the $\mathbb{F}_p$-module $\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p$ by [Fre00, Theorem 1.2.5]. A prior the bracket on the E^2-page of the homotopy fixed points spectral sequence survives to a bracket on the E^∞-page that agrees with the shifted Lie bracket $[-,-]$ in Construction 3.6.1 up to a nonzero scalar c. Hence we choose a generator for $\mathrm{sLie}(2)$ so that $c=1$, and by abuse of notation we also denote the bracket on the E^2-page by $[-,-]$. The $(p+1)$-th summand of the spectral sequence collapses on the E^2-page with no extension problems, since the group cohomology of Σ_n with coefficients in $\mathrm{Mod}_{\mathbb{F}_p}$ is concentrated in degree 0 when n is coprime to p. From the computation on one generator, we deduce that the restriction $x^{[p]}$ on the generator x of $\Sigma^j\mathbb{F}_p$ on the E^2-page of the homotopy fixed points spectral sequence survives to the element $\lambda_j R^{(-j+1)/2}(x)$ with λ_j the unit given in the first part of the proof. Furthermore, the identity $[y,x^{[p]}] = [[\cdots[[y,x],x]\cdots],x]$ on the E^2-page survives to an identity in $\pi_{pj+k-p}(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}(\Sigma^j H\mathbb{F}_p \oplus$

$\Sigma^k H\mathbb{F}_p)$).

Similarly, suppose that j and k are both odd and let x, y represent the geneorator of $\Sigma^j\mathbb{F}_p$ and $\Sigma^k\mathbb{F}_p$ on the E^2-page above. Then the identity $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i}{i}(x,y)$ on the line $s=0$ of the E^2-page survives to on identity $(x+y)^{[p]} = \lambda_j R^{(-j+1)/2}(x) + \lambda_k R^{(-k+1)/2}(y) + \sum_{i=1}^{p-1} \frac{s_i}{i}(x,y)$ on the E^∞-page via the collapse of the homotopy fixed points spectral sequence, where s_i is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}(tx+y)^{p-1}(x)$. In particular, if $j=k$ then

$$\lambda_j R^{(-j+1)/2}(x+y) = \lambda_j R^{(-j+1)/2}(x) + \lambda_j R^{(-j+1)/2}(y) + \sum_{i=1}^{p-1} \frac{s_i}{i}(x,y),$$

so the bottom operation on an odd class is not additive in general even though it lifts to an additive operation on the E^2-page of the dual bar spectral sequence.

Finally, we want to show that the collection $x^{[p]} := \lambda_j R^{(-|x|+1)/2}(x)$ for $|x| = j$ odd and λ_j a unit depending only on j defines a restriction map $(-)^{[p]}$ for the shifted Lie bracket on the homotopy groups of spectral partition Lie algebras by extending to linear sums of classes x, y with $|x| = j \neq |y| = k$ odd via $(x+y)^{[p]} := \lambda_j R^{(-j+1)/2}(x) + \lambda_k R^{(-k+1)/2}(y) + \sum_{i=1}^{p-1} \frac{s_i}{i}(x,y)$. The pth iteration of $[-,x]$ on a class y is encoded by a summand in the weight $p+1$ part of the iterated monad composition

$$(\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty})^{\circ p} \to (\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty})^{\circ p-1} \to \cdots \to \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$$

applied to $H\Sigma^j\mathbb{F}_p \oplus H\Sigma^k\mathbb{F}_p$. Explicitly, this summand is the $(p-1)$-th iteration of $\partial_2(\mathrm{Id}) \otimes (\partial_1(\mathrm{Id}) \otimes (-))$ on $\partial_2(\mathrm{Id})$. Note that the last step $\mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty} \circ \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty} \to \mathrm{Lie}^{\pi}_{\mathbb{F}_p,\mathbb{E}_\infty}$ of the above chain of compositions is

$$\begin{aligned}&(\partial_2(\mathrm{Id}) \otimes H\mathbb{F}_p) \otimes \Big(\big((\partial_1(\mathrm{Id}) \otimes H\mathbb{F}_p) \overset{h\Sigma_1}{\otimes} \Sigma^j H\mathbb{F}_p \big) \otimes \big((\partial_p(\mathrm{Id}) \otimes H\mathbb{F}_p) \overset{h\Sigma_p}{\otimes} (\Sigma^k H\mathbb{F}_p)^{\otimes p} \big) \Big) \\ \to &(\partial_{p+1}(\mathrm{Id}) \otimes H\mathbb{F}_p) \overset{h(\Sigma_1 \times \Sigma_p)}{\otimes} (\Sigma^j H\mathbb{F}_p \otimes \Sigma^{kp} H\mathbb{F}_p)\end{aligned}$$

as in (3.6), which we showed to be one-dimensional in Proposition 3.6.3 with $[[\cdots[[y,x],x]\cdots],x]$

where x appears p times a generator. The monad composition induces a map of homotopy fixed points spectral sequences, both of which collapse on the E^2-page at weight $p+1$ with no extension problems. Hence we get a map that is the weight $p+1$ part of

$$\bigoplus_n \Big(\mathrm{sLie}(n)\otimes\big(\pi_*(\bigoplus_m \mathrm{sLie}(m)\overset{h\Sigma_m}{\otimes}(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)^{\otimes m})\big)^{\otimes n}\Big)^{\Sigma_n}\to\bigoplus_n\Big(\mathrm{sLie}(n)\otimes(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)^{\otimes n}\Big)^{\Sigma_n}$$

along the line $s=0$ on the E^2-pages. This map has as summand

$$\begin{aligned}&\mathrm{sLie}(2)\overset{\Sigma_2}{\otimes}\Big(\big(\mathrm{sLie}(1)\otimes(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)^{\otimes 1}\big)^{\Sigma_1}\oplus\big(\mathrm{sLie}(p)\otimes(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)^{\otimes p}\big)^{\Sigma_p}\Big)^{\otimes 2}\\&\to\big(\mathrm{sLie}(p+1)\otimes(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)^{\otimes p+1}\big)^{\Sigma_{p+1}},\end{aligned}$$

with further summand

$$\phi:\mathrm{sLie}(2)\otimes\Big(\Sigma^j\mathbb{F}_p\otimes\big(\mathrm{sLie}(p)\otimes(\Sigma^k\mathbb{F}_p)^{\otimes p}\big)^{\Sigma_p}\Big)\to\big(\mathrm{sLie}(p+1)\otimes(\Sigma^j\mathbb{F}_p\oplus\Sigma^k\mathbb{F}_p)^{\otimes p+1}\big)^{\Sigma_{p+1}}.$$

The map ϕ agrees with the construction 3.6.1 on the E^∞-page, i.e., it is the evaluation of the free shifted restricted Lie bracket. The image of $[y,R^{(-|x|+1)/2}(x)]$ under ϕ is $[y,x^{[p]}]=[[\cdots[[y,x],x]\cdots],x]$ up to a unit $\lambda_{|x|}$. Hence on the E^∞-page we have $[y,\lambda_{|x|}R^{(-|x|+1)/2}(x)]=[[\cdots[[y,x],x]\cdots],x]$ as desired. □

To sum up, we have the following theorem.

Theorem 3.6.6. *The binary operation $[-,-]$ constructed above equips the homotopy groups of any spectral partition Lie algebra A with a shifted restricted Lie algebra bracket*

$$[-,-]:\pi_j(A)\otimes\pi_k(A)\to\pi_{j+k-1}(A)$$

over $\mathbb{F}_p$. If $p=2$, for all j and $x\in\pi_j(A)$ the restriction $x^{[2]}$ is represented by the bottom operation $R^{-j+1}(x)$. The restriction map on a sum of classes x,y in degrees $j\neq k$ is given by

$$(x+y)^{[2]}=\lambda_jR^{-j+1}(x)+\lambda_kR^{-k+1}(y)+[x,y].$$

If $p>2$, for all odd j and $x\in\pi_j(A)$ the restriction $x^{[p]}$ is the bottom operation $R^{(-j+1)/2}(x)$

up to a unit λ_j. The restriction map on a sum of classes x, y in degrees $j \neq k$ are given by

$$(x+y)^{[p]} = \lambda_j R^{(-j+1)/2}(x) + \lambda_k R^{(-k+1)/2}(y) + \sum_{i=1}^{p-1} \frac{s_i}{i}(x,y),$$

where s_i is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}(tx+y)^{p-1}(x)$.

The bracket is compatible with the unary operations in Theorem 3.5.5 and 3.5.6 in the sense that $[x, \alpha(y)] = 0$ for $x, y \in \pi_*(A)$ and any unary operations α of positive weight that is not an iteration of the restriction map. Equivalently, for any $\mathbb{E}_\infty^{\mathrm{nu}}$-$H\mathbb{F}_p$-algebra A, there is a shifted Lie bracket with restriction

$$[-,-] : \overline{\mathrm{TAQ}}^j(A) \otimes \overline{\mathrm{TAQ}}^k(A) \to \overline{\mathrm{TAQ}}^{j+k+1}(A)$$

satisfying the above conditions.

Remark 3.6.7. The interaction between the unary operations and the shifted Lie bracket on the homotopy groups of spectral partition Lie algebras differ from that on the homology of spectral Lie algebras. It was shown in [AC20, Kja18] that on the mod p homology groups of spectral Lie algebras, the bracket $[x, \alpha(y)]$ always vanishes if α is a unary operation of positive weight. In comparison, on the homotopy groups of spectral partition Lie algebras the bracket $[x, \alpha(y)]$ does not necessarily vanish when α is an iteration of the restriction map, which is a non-additive unary operation. This phenomenon also shows up in the Lubin-Tate theory of spectral Lie algebras, as was observed in [Bra17, Proposition 4.3.16] that the non-additive unary operation θ interacts nontrivially with the bracket. For instance, when $p = 2$, the bottom non-vanishing operation $\bar{Q}^{|x|}$ on a mod 2 homology class x of a free spectral Lie algebra is identified with the nonzero self-bracket $[x,x]$ by [AC20, Lemma 6.4]. Hence $[y, \bar{Q}^{|x|}(x)] = 0$ by the Jacobi identity for all x, y. In comparison, the bottom operation $R^{-|x|+1}$ on a class in the homotopy group of a free spectral partition Lie algebra over $H\mathbb{F}_2$ represents the restriction on x, so $[y, R^{-|x|+1}(x)] = [[y,x],x]$ is nonzero. Whereas self-brackets always vanish in shifted restricted Lie algebras over $\mathbb{F}_2$, cf. [Fre00, Remark 1.2.9].

3.6.1 Generation

Finally we put all the structures together to obtain the optimal target category for the homotopy group of spectral partition Lie algebras, or equivalently the reduced mod p TAQ cohomology.

Definition 3.6.8. A $\mathcal{P}$-sLie$^{\rho}$*-algebra* L is a module over the power ring $\mathcal{P}$, together with a shifted Lie bracket and a restriction $(-)^{[p]}$, that satisfies the following conditions:

(1) If $p=2$, for all j and $x \in \pi_j(A)$ the restriction $x^{[2]}$ is given by the bottom operation $R^{-j+1}(x)$. The restriction map on a sum of classes x and y is given by

$$(x+y)^{[2]} = R^{-j+1}(x) + R^{-k+1}(y) + [x,y].$$

If $p>2$, for all odd j and $x \in \pi_j(A)$ the restriction $x^{[p]}$ is up to a unit λ_j the bottom operation $R^{(-j+1)/2}(x)$. The restriction map on the sum of classes x, y in degrees $j \neq k$ are given by

$$(x+y)^{[p]} = \lambda_j R^{(-j+1)/2}(x) + \lambda_k R^{(-k+1)/2}(y) + \sum_{i=1}^{p-1} s_i(x,y),$$

where s_i is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}(tx+y)^{p-1}(x)$;

(2) The bracket $[y, \alpha(x)]$ vanishes for any $x, y \in L$ and α a unary operation of positive weight, unless α is an iteration of the restriction map.

Denote by $\mathrm{sLie}^{\rho}_{\mathcal{P}}$ the category of $\mathcal{P}$-sLie$^{\rho}$-algebras.

Hence the homotopy group every spectral partition Lie algebra, or the reduced TAQ cohomology of any $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebra, has the structure of an $\mathrm{sLie}^{\rho}_{\mathcal{P}}$-algebra. The free $\mathcal{P}$-sLie$^{\rho}$-algebra functor $\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathcal{P}}}$ on a $\mathbb{F}_p$-module M can be computed as follows: first we take the free shifted restricted Lie algebra over $\mathbb{F}_p$, then take the free $\mathcal{P}$-module on $\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_p}}(M)$. If $p=2$ then we define the bottom operation $R^{-|x|+1}(x)$ to be the restriction $x^{[2]}$; if $p>2$, we identify the restriction $x \mapsto x^{[p]}$ with the bottom operation $R^{(-|x|+1)/2}(x)$ up to a unit $\lambda_{|x|}$ for any odd degree x. Finally we extend the shifted Lie bracket and the restriction map to the quotient of $\mathrm{Free}^{\mathcal{P}}\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_p}}(M)$ by the above identification, subject to the conditions in Definition 3.6.9.

Note that when $p > 2$, given an $\mathbb{F}_p$-module M with basis $\{x_1, \ldots, x_k\}$, a basis for $\mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathbb{F}_p}}(M)$ is given by

$$\{v\} \cup \{u, u^{[p]}, (u^{[p]})^{[p]}, \ldots\},$$

where u ranges over shifted brackets represented by Lyndon words in letters $x_1, \ldots, x_k$ with odd degree, and v those with even degree. (See, for instance, [BKS05, section 2].) Proposition 3.4.10 and 3.4.16 immediately imply the following.

Corollary 3.6.9. *The canonical map of* $\mathcal{P}$-sLie^{ρ}*-algebras*

$$\alpha : \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathcal{P}}} \pi_*(A) \to \pi_*(\mathrm{Lie}^{\pi}_{\mathbb{F}_p, \mathbb{E}_\infty}(A))$$

is an isomorphism when A is any direct sum of shifts of $H\mathbb{F}_p$'s.

Hence we have identified the target category for the homotopy groups of spectral partition Lie algebras and the reduced TAQ cohomology of any $\mathbb{E}_\infty$-$H\mathbb{F}_p$-algebra.

3.7 Operations on mod p $\mathbb{S}$-linear TAQ cohomology

As an application, we obtain in this section the structure of natural operations on the mod p $\mathbb{S}$-linear TAQ cohomology $\mathrm{TAQ}^*(-, \mathbb{S}; H\mathbb{F}_p)$ of $\mathbb{E}_\infty$-$\mathbb{S}$-algebras, as well as determining their relations. The results in this section are largely inspired by conversations with Tyler Lawson.

Recall from [Law20, 1.8] that the mod p TAQ homology of an $\mathbb{E}_\infty$-$\mathbb{S}$-algebra R can be computed by

$$\mathrm{TAQ}_*(R, \mathbb{S}; H\mathbb{F}_p) \simeq \pi_*(|\mathrm{Bar}_\bullet(H\mathbb{F}_p \otimes \mathrm{id}, \mathbb{E}_\infty, R)|).$$

For any $\mathbb{E}^{\mathrm{nu}}_\infty$-$\mathbb{S}$-algebra A, the reduced mod p TAQ cohomology of an $\mathbb{E}_\infty$-$\mathbb{S}$-algebra $\mathbb{S} \oplus A$ is the same as the mod p TAQ cohomology groups

$$\overline{\mathrm{TAQ}}^n(A, \mathbb{S}; H\mathbb{F}_p) := [\Sigma^{-n}|\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{E}^{\mathrm{nu}}_\infty, A)|, H\mathbb{F}_p]_{\mathrm{Sp}}.$$

When A is of finite type, $\overline{\mathrm{TAQ}}^*(A, \mathbb{S}; H\mathbb{F}_p) \simeq \pi_{-*}(\mathbb{D}|\mathrm{Bar}_\bullet(H\mathbb{F}_p \otimes \mathrm{id}, \mathbb{E}^{\mathrm{nu}}_\infty, A)|)$. Since all operations vanish on the unit of the mod p TAQ cohomology except for multiplication by units, we will again compute operations on $\mathrm{TAQ}^*(-,\mathbb{S};H\mathbb{F}_p)$ by throwing away the base point and computing the dual bar spectral sequence on representing objects.

Corollary 3.7.1. *Unary operations on a degree j cohomology class in the reduced mod p $\mathbb{S}$-linear TAQ cohomology $\overline{\mathrm{TAQ}}^*(A,\mathbb{S};H\mathbb{F}_p)$ of $\mathbb{E}_\infty^{\mathrm{nu}}$-$\mathbb{S}$-algebras A are parametrized by the free $\mathcal{P}$-sLie^ρ-algebra $\mathrm{Free}^{\mathrm{sLie}^\rho_{\mathcal{P}}}(\Sigma^{-j}\mathcal{A})$, where $\mathcal{A}$ is the mod p Steenrod algebra with homological grading.*

In general, for any tuple $(i_1,\dots i_k)$, the k-ary cohomology operations

$$\prod_{i=1}^{k}\mathrm{TAQ}^{i_l}(-,\mathbb{S};H\mathbb{F}_p)\to\mathrm{TAQ}^m(-,\mathbb{S};H\mathbb{F}_p)$$

away from the unit are parametrized by the homological degree $-m$ part of

$$\mathrm{Free}^{\mathrm{sLie}^\rho_{\mathcal{P}}}(\Sigma^{-i_1}\mathcal{A}\oplus\cdots\oplus\Sigma^{-i_k}\mathcal{A}).$$

Proof. The representing objects for the mod p TAQ cohomology functor $\mathrm{TAQ}^*(-,\mathbb{S};H\mathbb{F}_p)$ are the trivial square-zero extensions $\mathbb{S}\oplus\Sigma^{i_1}H\mathbb{F}_p\oplus\cdots\Sigma^{i_n}H\mathbb{F}_p$. To compute the unary operations, we plug in the trivial algebras $\mathrm{S}\oplus\Sigma^{j}H\mathbb{F}_p$. There is a base change formula

$$\mathrm{TAQ}(R,\mathbb{S};H\mathbb{F}_p)\underset{\mathbb{S}}{\otimes}H\mathbb{F}_p\simeq\mathrm{TAQ}(R\underset{\mathbb{S}}{\otimes}H\mathbb{F}_p,H\mathbb{F}_p;H\mathbb{F}_p),$$

so unary operations on a degree j cohomology class are parametrized by the reduced mod p TAQ cohomology $\overline{\mathrm{TAQ}}^*(\Sigma^j H\mathbb{F}_p\otimes H\mathbb{F}_p,H\mathbb{F}_p;H\mathbb{F}_p)$.

It follows from the limiting case of Proposition 3.4.10 and 3.4.16 that the dual bar spectral sequence takes the form

$$E^2_{s,t}=\pi_s(\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Poly}_{\mathcal{R}},\pi_*(\Sigma^j H\mathbb{F}_p\otimes H\mathbb{F}_p))^\vee)_t\Rightarrow\overline{\mathrm{TAQ}}^{-s-t}(\Sigma^j H\mathbb{F}_p\otimes H\mathbb{F}_p,H\mathbb{F}_p;H\mathbb{F}_p)$$

and collapses on the E^2-page. Hence we deduce that

$$E^\infty_{s,t}\cong E^2_{s,t}\cong\mathrm{Free}^{\mathrm{sLie}^\rho_{R^!}}(\Sigma^{-j}\mathcal{A})\cong\mathrm{Free}^{\mathrm{sLie}^\rho_{\mathcal{P}}}(\Sigma^{-j}\mathcal{A}),\ p=2,$$

$$E^{\infty}_{s,t} \cong E^2_{s,t} \cong \mathrm{Free}^{(\mathcal{R}')^!}\mathrm{Free}^{\mathrm{sLie}_{\mathbb{F}_p}}(\Sigma^{-j}\mathcal{A}) \cong \mathrm{Free}^{\mathrm{sLie}^{\rho}_{\mathcal{P}}}(\Sigma^{-j}\mathcal{A}),\ p>2$$

The computation for $k>1$ is similar. □

The Steenrod operations commute with the bracket via the usual Cartan formula

$$Sq^a[x,y] = \sum_i [Sq^i(x), Sq^{a-i}(y)],\ \text{for } p=2,$$

$$P^a[x,y] = \sum_i [P^i(x), P^{a-i}(y)],\ \ \beta P^a[x,y] = \sum_i ([\beta P^i(x), P^{a-i}(y)] + [P^i(x), \beta P^{a-i}(y)])$$

for $p>2$.

Finally we deduce the relations between the Steenrod operations and the unary $\mathbb{F}_p$-linear TAQ cohomology operations.

Proposition 3.7.2. *The Steenrod operations commute with unary $\mathbb{F}_p$-linear TAQ cohomology operations R^i via the Nishida relations on mod p cohomology, i.e.,*

$$Sq^a R^{-|x|+1}(x) = \sum \binom{j-c}{a-2c} R^{a+j+1-c} Sq^c(x) + \sum_{l<k, l+k=a} [Sq^l(x), Sq^k(x)],$$

$$Sq^a R^b(x) = \sum \binom{b-1-c}{a-2c} R^{a+b-c} Sq^c(x),\ \ b > -|x|+1$$

for $p=2$. For $p>2$ we have

$$\begin{aligned} P^n \beta R^j(x) =& (-1)^{n-i} \sum_i \binom{(j-i)(p-1)}{n-pi} \beta R^{n+j-i} P^i(x) \\ &+ (-1)^{n-i} \sum_i \binom{(j-i)(p-1)-1}{n-pi-1} R^{n+j-i} \beta P^i(x), \end{aligned}$$

$$P^n R^j(x) = (-1)^{n-i} \sum_i \binom{(j-i)(p-1)-1}{n-pi} R^{n+j-i} P^i(x)$$

for all $2j > -|x|+1$, *as well as*

$$P^n R^j(x) = (-1)^{n-i}\sum_i \binom{(j-i)(p-1)-1}{n-pi} R^{n+j-i}P^i(x)$$
$$+\frac{1}{\lambda_{|x|}}\sum_{I,\sigma\in\Sigma_p,\sigma(1)=1}[[\cdots[[P^{i_{\sigma(1)}}(x),P^{i_{\sigma(2)}}(x)],P^{i_{\sigma(3)}}(x)]\cdots],P^{i_{\sigma(p)}}(x)]$$

when the degree of x *is odd and* $2j = -|x|+1$, *where the bracket term sums over all nondecreasing sequences* $I = (0 \le i_1 \le i_2 \le \ldots \le i_p)$ *with* $i_1 + i_2 + \cdots + i_p = n$ *for* $p > 2$.

Recall that $\lambda_{|x|}$ is the unit by which bottom operation on an odd degree class x differs from the restriction $x^{[p]}$ on x, cf. Lemma 3.6.5.

Remark 3.7.3. Note that the commuting relations between the Steenrod operations and the TAQ cohomology operations R^i coincide with the Adem relations for Steenrod algebras, thereby reinforcing the heuristics that the operations R^i are extended Steenrod operations.

Proof of Proposition 6.2. Since the operations $R^i = (Q^{i-1})^*$ come from the linear dual of the Dyer-Lashof operations Q^{i-1}, the Steenrod operations commute with R^i via the Nishida relations on cohomology. When $p = 2$ the relations are worked out explicitly, for example, by Miller in [Mil16]. The Nishida relations for applying a Steenrod operation to the bottom operation on x involves an extra bracket term because the bottom operation is the restriction on x.

For $p > 2$, the Nishida relations on cohomology can be read off from Theorem 3 and its corollary in Nishida's original paper [Nis68]:

$$P^n\beta R^j = (-1)^{n-i}\sum_i \binom{(j-i)(p-1)}{n-pi}\beta R^{n+j-i}P^i + (-1)^{n-i}\sum_i \binom{(j-i)(p-1)-1}{n-pi-1} R^{n+j-i}\beta P^i,$$

$$P^n R^j = (-1)^{n-i}\sum_i \binom{(j-i)(p-1)-1}{n-pi} R^{n+j-i}P^i.$$

Analogous to the case $p = 2$, when x is a class in odd degree, the Nishida relations for the steenrod action on the bottom class $P^n R^{(-|x|+1)/2}(x)$ involve extra bracket terms since $\lambda_{|x|}R^{(-|x|+1)/2}(x)$ is the restriction on x.

In order to determine the extra bracket terms, we need an explicit expression for the restriction map on an odd class. In the setting of unshifted graded $\mathbb{F}_p$-modules, this is worked out by Fresse in [Fre00, Remark 1.2.8]. Note that there is an embedding of the Lie operad into the associative operad Assoc. Furthermore, there is an identity

$$\sum_{\sigma\in\Sigma_p} X_{\sigma(1)}\cdots X_{\sigma(p)} = \sum_{\sigma\in\Sigma_p,\sigma(1)=1} \langle\langle\cdots\langle\langle X_{\sigma(1)},X_{\sigma(2)}\rangle,X_{\sigma(3)}\rangle\cdots\rangle,X_{\sigma(p)}\rangle \tag{3.7}$$

in the associative operad, where $\langle x,y\rangle = xy-yx$ is the commutator. For $x\in V$ in even degree, the pth power on x is given by

$$\sum_{\sigma\in\Sigma_p} X_{\sigma(1)}\cdots X_{\sigma(p)}\otimes x^{\otimes p}\in(\mathrm{Assoc}(p)\otimes V^{\otimes p})^{\Sigma_p}\cong(\mathrm{Assoc}(p)\otimes V^{\otimes p})_{\Sigma_p}.$$

Using the identity (3.7), we can pull back the pth power on x along the embedding

$$(\mathrm{Lie}(p)\otimes V^{\otimes p})^{\Sigma_p}\hookrightarrow(\mathrm{Assoc}(p)\otimes V^{\otimes p})^{\Sigma_p}.$$

The resulting element is the restriction on x in the free restricted Lie algebra on V, i.e.,

$$x^{[p]} = \Big(\sum_{\sigma\in\Sigma_p,\sigma(1)=1}[[\cdots[[X_{\sigma(1)},X_{\sigma(2)}],X_{\sigma(3)}]\cdots],X_{\sigma(p)}]\Big)\otimes x^{\otimes p}\in(\mathrm{Lie}(p)\otimes V^{\otimes p})^{\Sigma_p}. \tag{3.8}$$

Since we are working with shifted graded $\mathbb{F}_p$-modules, the commutator in the shifted graded associative algebra is $\langle x,y\rangle = xy-(-1)^{(|y|-1)(|x|-1)}yx$. If x,y are both in odd degrees, then $\langle x,y\rangle = xy-yx$. Hence the identity (3.7) pulls back to the restriction map (3.8) on an odd class x in the free shifted graded restricted Lie algebra over $\mathbb{F}_p$. Now we apply the Steenrod operation P^n to the pth power on x and use the Cartan formula. Note that the Steenrod operations P^a raises degree by an even number, so none of the signs are altered. Pulling back to the free shifted restricted Lie algebra, we deduce that the bracket terms in the Nishida relation for $P^nR^{(-|x|+1)/2}(x)$ consists of terms

$$\frac{1}{\lambda_{|x|}}\sum_{\sigma\in\Sigma_p,\sigma(1)=1}[[\cdots[[P^{i_{\sigma(1)}}(x),P^{i_{\sigma(2)}}(x)],P^{i_{\sigma(3)}}(x)]\cdots],P^{i_{\sigma(p)}}(x)]$$

for all nondecreasing sequences $0\le i_1\le i_2\le\ldots\le i_p$ with $i_1+i_2+\cdots+i_p=n$. $\square$

Chapter 4

The bar spectral sequence for spectral Lie algebras

In this chapter, we examine the bar spectral sequence for spectral Lie algebras in $\mathrm{Mod}_{\mathbb{F}_p}$. As an application, we deduce new information about the mod p homology of labeled configuration spaces.

For L a spectral Lie algebra, its *topological Quillen object* is the bar construction

$$\mathrm{TQ}^{s\mathscr{L}}(L) := |\mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, L)|.$$

We define its mod p *topological Quillen homology* to be

$$\mathrm{TQ}_*^{s\mathscr{L}}(L;\mathbb{F}_p) := \pi_*(|\mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, L)| \otimes \mathbb{F}_p).$$

Then the bar spectral sequence (2.2) for a spectral Lie algebra L converges to the homotopy of the mod p topological Quillen homology of L.

4.1 Operations on the mod 2 homology of spectral Lie algebras

To compute the E^2-page of the bar spectral sequence computing the mod 2 topological Quillen homology, we recall the structure on the mod 2 homology of an algebra L over the spectral Lie operad studied in [AC20, Beh12]. It consists of a $\mathrm{Lie}^s_{\mathbb{F}_2}$-algebra structure along with Dyer-Lashof like unary operations.

The second structure map of a spectral Lie algebra L is given by

$$\xi : \partial_2(\mathrm{Id}) \underset{h\Sigma_2}{\otimes} L^{\otimes 2} \simeq \partial_2(\mathrm{Id}) \otimes L^{\otimes 2}_{h\Sigma_2} \simeq \mathbb{S}^{-1} \otimes L^{\otimes 2}_{h\Sigma_2} \to L.$$

At the level of homology, this gives rise to a shifted Lie bracket

$$[-,-] : H_m(L) \otimes H_n(L) \to H_{m+n-1}(L),$$

making $H_*(L)$ a graded shifted Lie algebra [AC20, Proposition 5.2].

For L a connective spectral Lie algebra, Behrens defined unary operations of weight 2

$$\bar{Q}^j : H_d(L) \to H_{d+j-1}(L)$$

on the mod 2 homology of L via $x \mapsto \xi_* \sigma^{-1} Q^j(x)$, where $Q^j : H_d(L) \to H_{d+j}(L^{\otimes 2}_{h\Sigma_2})$ is an extended Dyer-Lashof operation $x \mapsto e_{j-d} \otimes x \otimes x$, $\sigma^{-1} : H_*(L^{\otimes 2}_{h\Sigma_2}) \to H_{*-1}(\partial_2(\mathrm{Id}) \otimes L^{\otimes 2}_{h\Sigma_2})$ is the desuspension isomorphism, and ξ is the second structure map [Beh12, Section 1.5][AC20, Definition 5.4]. Furthermore, Behrens showed that the quadratic relations

$$\bar{Q}^r \bar{Q}^s = \sum_{l=0}^{r-s-1} \binom{r-2l-1}{s-l} \bar{Q}^{r+s-l} \bar{Q}^l \tag{4.1}$$

for $s < r \leq 2s$ generate all the relations among the unary operations on a class in some positive degree [Beh12, Theorem 1.5.1]. By definition, for x a homogeneous class $\bar{Q}^i(x) = 0$ for all $i < |x|$, hence $\bar{Q}^r \bar{Q}^s(x) = 0$ for $|x| \geq 1$ and $r \leq s$.

Since the extended Dyer-Lashof operations are defined on the mod 2 homology of all nonconnective spectra, the operations $\bar{Q}^i$ for all $i \in \mathbb{Z}$ can be defined on the mod 2 homology of any spectral Lie algebra L with $\bar{Q}^i(x) = 0$ for any homogeneous class $x \in H_*(L)$ and $i < |x|$. Let $\bar{\mathcal{R}}$ be the quotient algebra of the free algebra over $\mathbb{F}_2$ on generators $\{\bar{Q}^j\}_{j\in\mathbb{Z}}$ by the two sided ideal generated by the relations

$$\bar{Q}^r\bar{Q}^s = \sum_{l \leq r-s-1} \binom{r-2l-1}{s-l} \bar{Q}^{r+s-l}\bar{Q}^l \tag{4.2}$$

for all $r \leq 2s$.

The Lie^s-bracket interact with the unary operations in the following way.

Proposition 4.1.1. *[AC20, Lemma 6.4, 6.5] For any $j \in \mathbb{Z}$ and x, y homogeneous classes in the mod 2 homology of a spectral Lie algebra, we have $[\bar{Q}^j(x), y] = 0$ and $\bar{Q}^{|x|}(x) = [x, x]$.*

Remark 4.1.2. It follows that $\bar{Q}^{2|x|-1}\bar{Q}^{|x|}(x) = [[x,x],[x,x]] = 0$. This is guaranteed by the Behrens' relations, since $r = 2|x| - 1 \leq s = 2|x|$ and the right hand side of (4.2) vanished due to instability of the extended Dyer-Lashof operations.

Sometimes it is more convenient to switch to the lower indexing $\bar{Q}_j(x) := \bar{Q}^{|x|+j}(x)$, which automatically takes into account the instability condition.

Definition 4.1.3. The lower indexed $\bar{\mathcal{R}}$-algebra is generated by symbols $\bar{Q}_j$ for $j \geq 0$ and relations

$$\bar{Q}_a\bar{Q}_b = \sum_{0 \leq c < (a+2b-1)/3} \binom{a+b-2c-2}{b-c} \bar{Q}_{a+2b-2c}\bar{Q}_c \tag{4.3}$$

for $0 \leq a \leq b+1$. When $j < 0$ we set $\bar{Q}_j = 0$.

Definition 4.1.4. An $\mathbb{F}_2$-module $M_\bullet$ over $\bar{\mathcal{R}}$ is *allowable* if for any homogeneous element $x \in M_\bullet$ we have $\bar{Q}^{j_1}\bar{Q}^{j_2}\cdots\bar{Q}^{j_m}(x) = 0$ whenever $j_1 < j_2 + \cdots + j_m + |x|$. Alternatively, an allowable $\bar{\mathcal{R}}$-module M is a module over the lower-indexed $\bar{\mathcal{R}}$-algebra.

Now we extend Behrens' results to all spectral Lie algebras.

Proposition 4.1.5. *For L any spectral Lie algebra, its mod 2 homology $H_*(L)$ is an allowable module over $\bar{\mathcal{R}}$. Furthermore, for all $k \geq 0$ and $n \in \mathbb{Z}$ there is an isomorphism of*

$\mathbb{F}_2$*-modules*

$$H_*(\partial_{2^k}(\mathrm{Id}) \underset{h\Sigma_{2^k}}{\otimes} (\mathbb{S}^n)^{\otimes 2^k}) \cong \mathbb{F}_2\{\bar{Q}^{j_1}\cdots\bar{Q}^{j_k}(x_n), j_l > 2j_{l+1}\forall l<k,\ j_k \geq n\}$$
$$\cong \mathbb{F}_2\{\bar{Q}_{i_1}\cdots\bar{Q}_{i_k}(x_n),\ \forall l,\ i_l \geq 0,\ i_l > i_{l+1}+1\}.$$

Proof. The connectedness assumption in Behrens' proof of [Beh12, Theorem 1.5.1] is necessary only because of the connectedness assumption on the following two inputs to the proof. Kuhn [Kuh83, Example 7.6] (see also [Beh12, Lemma 1.4.3]) showed that for Y a connected space, the transfer $\tau : H_*(Y^{\otimes 4}_{h\Sigma_4}) \to H_*(Y^{\otimes 4}_{h\Sigma_2\wr\Sigma_2})$ is given by

$$Q^rQ^s \mapsto Q^r \wr Q^s + \sum_t \left[\binom{s-r+t}{s-t} + \binom{s-r+t}{2t-r}\right] Q^{r+s-t}\wr Q^t. \tag{4.4}$$

On the other hand, Arone and Mahowald's computation [AM99, Theorem 3.16]

$$H_*(\partial_{2^k}(\mathrm{Id}) \underset{h\Sigma_{2^k}}{\otimes} (\mathbb{S}^n)^{\otimes 2^k}) \cong \mathbb{F}_2\{\bar{Q}^{j_1}\cdots\bar{Q}^{j_k}(x_n), j_l > 2j_{l+1}\forall l<k, j_k \geq n\}$$

works for any odd integer n, and extends to positive even integers via the fiber sequence

$$\partial_{2m}(\mathrm{Id}) \underset{h\Sigma_{2m}}{\otimes} (\mathbb{S}^n)^{\otimes 2m} \xrightarrow{E} \Sigma^{-1}\partial_{2m}(\mathrm{Id}) \underset{h\Sigma_{2m}}{\otimes} (\mathbb{S}^{n+1})^{\otimes 2m} \xrightarrow{H} \Sigma^{-1}\partial_m(\mathrm{Id}) \underset{h\Sigma_m}{\otimes} (\mathbb{S}^{2n+1})^{\otimes m},$$

which was obtained by differentiating the EHP sequence [AM99, Proposition 4.7][Beh12, Corollary 2.1.4]. Behrens proved the relations by using the transfer formula and inductively checking that they are compatible with the operadic composition; then he provided a basis by comparing with Arone-Mashowald's answer. Hence we only need to remove the connectedness assumption on both inputs.

Note that Kuhn's transfer formula can be obtained as a consequence of the computation of the transfer map $\tau_0 : H_*(B\Sigma_4) \to H_*(B\Sigma_2 \wr \Sigma_2)$ on group homology by Priddy [Kuh85, section 4]. For any $j,n \in \mathbb{Z}$, the Dyer-Lashof operation Q^j on a class x in degree n is defined via the canonical isomorphism $H_{n+j}((\Sigma^n\mathbb{F}_2)^{\otimes 2}_{h\Sigma_2}) \cong H_{j-n}(B\Sigma_2)[2n]$ [May70], where $[k]$ denotes a shift in homological degree by k. Similarly, the wreath product $Q^r \wr Q^s$ and the weight 4 operation Q^rQ^s are defined in $H_{j-n}(B\Sigma_2\wr\Sigma_2)[4n]$ and $H_{j-n}(B\Sigma_4)[4n]$ respectively, so the transfer map τ on a class in degree n of any spectrum Y is a shift of τ_0 by $4n$.

Next we extend the computation of Arone-Mahowald to nonnegative spheres. We make use of the long exact sequence

$$\cdots \to H_*(\Sigma^{-2}\partial_m(\mathrm{Id}) \underset{h\Sigma_m}{\otimes} (\mathbb{S}^{2n+1})^{\otimes m}) \xrightarrow{P_*} H_*(\partial_{2m}(\mathrm{Id}) \underset{h\Sigma_{2m}}{\otimes} (\mathbb{S}^n)^{\otimes 2m})$$
$$\xrightarrow{E_*} H_*(\Sigma^{-1}\partial_{2m}(\mathrm{Id}) \underset{h\Sigma_{2m}}{\otimes} (\mathbb{S}^{n+1})^{\otimes 2m}) \xrightarrow{H_*} \cdots$$

and isomorphisms

$$H_*\Big(\partial_{2m-1}(\mathrm{Id}) \underset{h\Sigma_{2m-1}}{\otimes} (\mathbb{S}^{2n})^{\otimes(2m-1)}\Big) \cong H_*\Big(\Sigma^{-1}(\partial_{2m-1}(\mathrm{Id}) \underset{h\Sigma_{2m-1}}{\otimes} (\mathbb{S}^{2n+1})^{\otimes(2m-1)})\Big)$$

for all n by Brantner [Bra17, 4.1.3], cf. [Kja18, Lemma 4.4]. There is an equivalence of $H\mathbb{F}_2$-modules with Σ_m-action

$$\partial_m(\mathrm{Id}) \underset{h\Sigma_m}{\otimes} (\Sigma^n H\mathbb{F}_2)^{\otimes m}) \simeq \Sigma^{2mn}\partial_m(\mathrm{Id}) \underset{h\Sigma_m}{\otimes} (\Sigma^{-n} H\mathbb{F}_2)^{\otimes m})$$

for any integers $m,n \geq 0$, where the action on Σ^{2mn} is trivial. Hence we obtain an isomorphism

$$H_*(\partial_m(\mathrm{Id}) \underset{h\Sigma_m}{\otimes} (\mathbb{S}^n)^{\otimes m})) \cong H_*(\Sigma^{2mn}\partial_m(\mathrm{Id}) \underset{h\Sigma_m}{\otimes} (\mathbb{S}^{-n})^{\otimes m}))$$

sending $\bar{Q}_{j_1}\cdots\bar{Q}_{j_k}(\iota_n)$ to $\sigma^{2^{k+1}n}\bar{Q}_{j_1}\cdots\bar{Q}_{j_k}(\iota_{-n})$ when $m = 2^k$, and both vanish when $m \neq 2^k$ for some $k \geq 0$. This addresses the case of the negative spheres.

For $n = 0$, we use the long exact sequence. It follows from the case $n = 1$ that $H_*(\partial_m(\mathrm{Id}) \otimes_{h\Sigma_m} (\mathbb{S}^0)^{\otimes m}) = 0$ when m is not a power of 2. Now suppose that $m = 2^k$. By the [Beh12, Proposition 2.2.5] (cf. remark after [Kja18, Proposition 4.3]), the maps E_* and P_* preserve the $\bar{Q}$ operations, sending the class $\bar{Q}^J(x_n)$ to $\sigma^{-1}\bar{Q}^J(x_{n+1})$ and $\sigma^{-2}\bar{Q}^J x_{2n+1}$ to $\bar{Q}^J\bar{Q}^n(x_n)$ respectively. This addresses the case $n = 0$. □

Denote by $\mathrm{Mod}_{\bar{\mathcal{R}}}$ the category of allowable $\bar{\mathcal{R}}$-modules and $\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}$ the free allowable $\bar{\mathcal{R}}$-module functor, which is left adjoint to the underlying functor $U_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Mod}_{\bar{\mathcal{R}}}} : \mathrm{Mod}_{\bar{\mathcal{R}}} \to \mathrm{Mod}_{\mathbb{F}_2}$. We will suppress the adjective allowable from here on. Then there is an additive monad associated with the free $\bar{\mathcal{R}}$-module functor, which we denote by $\mathcal{A}_{\bar{\mathcal{R}}}$.

Definition 4.1.6. [AC20, Definition 6.1] An $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra is a graded $\mathbb{F}_2$-module $L_\bullet$ with a shifted Lie bracket and an (allowable) $\bar{\mathcal{R}}$-module structure on $L_\bullet$ such that

(1) $\bar{Q}_0(x) = \bar{Q}^k(x) = [x,x]$ if $x \in L_k$, and

(2) $[x, \bar{Q}^k(y)] = 0$ for all $x, y \in L$.

Denote by $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$ the category of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras. To describe the free $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra functor, we recall the construction of *Lyndon words* on a set S, which provides a basis for the free $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra on an $\mathbb{F}_2$-module with $\mathbb{F}_2$-basis S.

Construction 4.1.7. [Hal50] The Lyndon words on a set S is defined recursively as follows: The elements of S are Lyndon words of length one and given an arbitrary fixed total ordering. Suppose that we have defined Lyndon words of length less than k with a total ordering. Then a Lyndon word of length k is a formal bracket $\langle w_1, w_2 \rangle$ such that

1. w_1, w_2 are Lyndon words whose lengths add up to k;
2. $w_1 < w_2$ in the order defined thus far;
3. To take into account the Jacobi identity, if $w_2 = \langle w_3, w_4 \rangle$ for some Lyndon words w_3, w_4, then we require $w_3 \leq w_1$.

To extend the total order to Lyndon words of weight at most k, we first impose an arbitrary total ordering on Lyndon words of length k, and then declare that they are greater than all Lyndon words of lower weights.

The free $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra functor can be computed explicitly as follows:

Proposition 4.1.8. *[AC20, Proposition 7.4] Let $V_\bullet$ be an $\mathbb{F}_2$-module with an ordered basis B of $V_\bullet$. First take the free totally isotropic Lie-algebra with $\langle -,- \rangle$ the free* $\mathrm{Lie}^{s,\mathrm{ti}}$ *bracket. Denote by B' the set of Lyndon words on the letters B, which is an $\mathbb{F}_2$-basis of* $\mathrm{Free}^{\mathrm{Lie}^{s,\mathrm{ti}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(V_\bullet)$*. Then we take the free $\bar{\mathcal{R}}$-module on the underlying $\mathbb{F}_2$-module of* $\mathrm{Free}^{\mathrm{Lie}^{s,\mathrm{ti}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(V_\bullet)$ *and obtain a basis consisting of elements of the form $\bar{Q}^I w$ with $w \in B'$. Equip the free $\bar{\mathcal{R}}$-module* $\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\mathrm{Lie}^{s,\mathrm{ti}}(V_\bullet))$ *with a* Lie^s *bracket $[-,-]$ defined on the induced basis by* *requiring* $[\bar{Q}^I w_1, \bar{Q}^J w_2] = 0$ *if* $I \neq \emptyset$ *or* $J \neq \emptyset$*, and setting recursively along the ordering on* B'

1) If $\langle w_1, w_2 \rangle$ *is a Lyndon word, then* $[w_1, w_2] = \langle w_1, w_2 \rangle$*;*

2) $[w, w] := \bar{Q}^{|w|} w$*;*

3) $[w_1, w_2] := [w_2, w_1]$ *if* $w_1 > w_2$*;*

4) $[w_1, w_2] := [w_3, [w_1, w_4]] + [w_4, [w_1, w_3]]$ *if* $w_1 < w_2$ *and* $w_2 = [w_3, w_4]$ *with* $w_1 < w_3$*.*

Antolín-Camarena showed that the monad $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$ parametrizes the mod 2 homology of connected spectral Lie algebras. The connectivity assumption can be removed in view of Proposition 4.1.5. Denote by $\mathrm{Free}^{s\mathscr{L}}$ the free spectral Lie algebra functor on Spectra

$$X \mapsto \bigoplus_{n \geq 1} \partial_n(\mathrm{Id}) \underset{h\Sigma_n}{\otimes} X^{\otimes n}.$$

Theorem 4.1.9. *[AC20, Theorem 7.1] The canonical map*

$$\mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(H_*(X; \mathbb{F}_2)) \to H_*(\mathrm{Free}^{s\mathscr{L}}(X); \mathbb{F}_2)$$

of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$*-algebras is an isomorphism for any spectrum* X*.*

Proof. Behrens proved the theorem in the case when $X = \mathbb{S}^k$, $k > 0$. Antolín-Camarena proved the isomorphism for X a connected spectrum follows: To extend Behrens' theorem to a finite wedge of spheres, he made use of a result of Arone and Kankaarinta that applies Goodwillie calculus to the Hilton-Milnor Theorem [AK98, Theorem 0.1]. To extend to all connected spectra, note that $X \otimes \mathbb{F}_2$ can be written as a filtered colimit of finite wedges of $\mathbb{S}^m \otimes \mathbb{F}_2$ in the category of $\mathbb{F}_2$-module spectra. The same arguments work to extend the isomorphism in Proposition 4.1.5 to all spectra. □

The category $\mathrm{Mod}_{\bar{\mathcal{R}}}$ is stable under the desuspension functor $\Omega := \Sigma^{-1}$ of $\mathbb{F}_2$-modules since the extended Dyer-Lashof operations are. Namely, for $M \in \mathrm{Mod}_{\bar{\mathcal{R}}}$, the $\mathbb{F}_2$-module ΩM has an $\bar{\mathcal{R}}$-module structrue given by $\bar{Q}^j(\sigma^{-1}x) = \sigma^{-1}\bar{Q}^j(x)$ for any $x \in M$. As a result, for $\Sigma^k \mathbb{F}_2$ the trivial $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra, there is an $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-structure on $\Omega\mathfrak{g}$ such that the bracket is trivial.

Proposition 4.1.10. *There is a natural* $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$*-module structure on* $\Omega^n \mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2)$ *for* $1 \le n \le \infty$*, where the bracket is trivial and* $\bar{Q}^j$ *acts by* $x \mapsto \sigma^{-n}\bar{Q}^j(\sigma^n x)$*. The canonical map*

$$\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^k\mathbb{F}_2) \cong \mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^k\mathbb{F}_2) \to \Omega^n \mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2)$$

is surjective.

Proof. There is a canonical colimit-to-limit comparison map

$$\mathrm{Free}^{s\mathscr{L}}(\Sigma^k H\mathbb{F}_2) \to \Omega \mathrm{Free}^{s\mathscr{L}}(\Sigma^{k+1} H\mathbb{F}_2) \tag{4.5}$$

of spectral Lie algebras over $H\mathbb{F}_2$, which after taking homotopy is the composite of the top and right arrows of the diagram

$$\begin{tikzcd}
\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^k\mathbb{F}_2) \arrow[r] \arrow[d, "ev"] & \mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Omega \mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{k+1}\mathbb{F}_2)) \arrow[d, "ev"] \\
\Sigma^k\mathbb{F}_2 \arrow[r, hook, "i"] & \Omega \mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{k+1}\mathbb{F}_2)
\end{tikzcd}.$$

Let x be the generator of $\Sigma^k\mathbb{F}_2$. By naturality of the $\bar{Q}^j$ operation, the class $\bar{Q}^j(x)$ on the top left corner is mapped to $\bar{Q}^j(i(x))$, which is sent to $\sigma^{-1}\bar{Q}^j\sigma(x)$ under evaluation. In general $\bar{Q}^J(x)$ is mapped to $\sigma^{-1}\bar{Q}^J\sigma(x)$ for any sequence J. Since the Lie^s bracket of operations always vanishes and $[i(x), i(x)] = \bar{Q}^{|x|}(i(x)) = \sigma^{-1}Q^{|x|}(\sigma x)$=0, the Lie^s-bracket is trivial on $\Omega \mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{k+1}\mathbb{F}_2)$. Applying Theorem 4.1.9, we see that the composite is surjective since $|\sigma(x)| = |x| + 1$. Iterating the construction yields the claim. $\square$

Quillen homology

Since the path object of $s\mathrm{Mod}_{\mathbb{F}_2}$ lifts to $s\mathrm{Lie}^s_{\mathcal{R}}$, the discussion in the previous subsection guarantees that any $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra $\mathfrak{g}$ has a free (cofibrant) resolution $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ in $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$. The left derived functor of $Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}$ is thus computed by

$$\mathbb{L}Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\mathfrak{g}) \simeq Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}} \mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g}) \simeq \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g}),$$

where $\mathrm{id} : \mathrm{Mod}_{\mathbb{F}_2} \to \mathrm{Mod}_{\mathbb{F}_2}$ is the identity functor considered as the trivial right module over the monad $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$ with structure map the augmentation.

Definition 4.1.11. The *Quillen homology* of a $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra $\mathfrak{g}$, denoted by $\mathrm{HQ}_*^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}(\mathfrak{g})$, is the total left derived functor

$$\mathrm{HQ}_{*,*}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}(\mathfrak{g}) := H_{*,*}\mathbb{L}Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\mathfrak{g}) \simeq \pi_{*,*}\mathrm{Bar}_\bullet\big(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g}\big).$$

Hence the bar spectral sequence we are interested in takes the form

$$E^2_{s,t} = \pi_{s,t}\mathrm{Bar}_\bullet\big(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \pi_*(L \otimes \mathbb{F}_2)\big) \cong \mathrm{HQ}_{s,t}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}(H_*(L;\mathbb{F}_2)) \Rightarrow \mathrm{TQ}^{s\mathscr{L}}_{s+t}(L;\mathbb{F}_2).$$

4.2 Computing the Quillen homology of spectral Lie algebras

In this section, we study the Quillen homology of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras when $p = 2$ via comparison with two smaller double complexes that are easy to compute via Koszul duality arguments.

4.2.1 May-type spectral sequence and an upper bound

First we find an upper bound for $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ by constructing a May-type spectral sequence. The dimensions of its E^1-page is bounded above by the homotopy groups of the bar construction of the following variant of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras whose unary and binary operations do not intertwine.

Definition 4.2.1. Define a $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra to be an $\mathbb{F}_2$-module L with an allowable $\bar{\mathcal{R}}$-module structure and a $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket $\langle -,- \rangle$ such that $\langle x, \bar{Q}^i(y)\rangle = 0$ for all $x, y \in L$. Denote by $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$ the category of $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebras and the monad associated to the free $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra functor.

The underlying $\mathbb{F}_2$-module of the free $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra on on $\mathbb{F}_2$-module V is given by that of $\mathcal{A}_{\bar{\mathcal{R}}} \circ \mathrm{Lie}^{s,\mathrm{ti}}(V)$. Hence $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$ admits an alternative description as the category of algebras over the composite monad $\mathcal{A}_{\bar{\mathcal{R}}} \circ \mathrm{Lie}^{s,\mathrm{ti}}$, with distributive law the natural transformation $\mathrm{Lie}^{s,\mathrm{ti}} \circ \mathcal{A}_{\bar{\mathcal{R}}} \Rightarrow \mathcal{A}_{\bar{\mathcal{R}}} \circ \mathrm{Lie}^{s,\mathrm{ti}}$ determined by $\langle -, \bar{Q}^i(-)\rangle = 0$ for all i.

Remark 4.2.2. Comparing with Proposition 4.1.8, we see that the underlying $\bar{\mathcal{R}}$-modules of the free $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$ and $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra on any $\mathbb{F}_2$-module agree. The only difference between the two free functors is that in the latter we do not change the $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra to a Lie^s-algebra via the identification $\bar{Q}_0(x) = [x,x]$.

We will see later that any $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra $\mathfrak{g}$ can be equipped with a $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra structure. The bar construction $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ is then levelwise isomorphic to $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, \mathfrak{g})$. The latter has simpler face maps in the sense that the face maps preserve the unary and binary structures respectively, whereas in the former, a Lie bracket that is not a self-bracket can be mapped to a self-bracket. To deal with these face maps, we draw inspiration from the May spectral sequence: suppose that we want to compute the Ext groups over a Hopf algebroid (A,Γ). In good cases, there exists a filtration on Γ such that the associated graded is a Hopf algebra (A,Γ'), i.e. the left and right unit are equal. Then we obtain a May spectral sequence with E^1-page the Ext group over the Hopf algebra Γ', whose cochain complex has differentials simpler than th cobar complex for Γ.

To construct a filtration on $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ so that the associated graded assembles to $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, \mathfrak{g})$, first we need to construct a filtration on any $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra so that the two sides of the identification $\bar{Q}_0(x) = [x,x]$ live in different filtrations.

Construction 4.2.3 (Length filtration)**.** Consider the complete filtration

$$\cdots \to \bar{\mathcal{R}}(n) \to \bar{\mathcal{R}}(n-1) \to \cdots \to \bar{\mathcal{R}}(1) \to \bar{\mathcal{R}}$$

of the homogeneous algebra $\bar{\mathcal{R}}$, where $\bar{\mathcal{R}}(n)$ is the ideal generated by monomials $\bar{Q}^I$ with $|I| = n$. Thus we obtain functors $\mathcal{A}_{\bar{\mathcal{R}}(n)}$ on $\mathrm{Mod}_{\mathbb{F}_2}$, sending M to the submodule of $\mathcal{A}_{\bar{\mathcal{R}}}(M)$ consisting of $\bar{Q}^I(x)$ for $x \in M$ and $|I| \geq n$. Thus we obtain a complete increasing filtration $\mathrm{F}^q_l(M) = \mathrm{coker}(\mathcal{A}_{\bar{\mathcal{R}}(q)}(M) \xrightarrow{\mathrm{ev}} M)$, where ev is the $\bar{\mathcal{R}}$-module struture map. We call this the *length filtration* of M.

Given an arbitrary $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra $\mathfrak{g}$, we would like to equip $\mathfrak{g}$ with the structure of an

$\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra. This boils down to producing a method that equips any Lie^s-algebra with a $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra structure.

Construction 4.2.4. ($\mathrm{Lie}^{s,\mathrm{ti}}$-structure on Lie^s-algebras.) For $\mathfrak{g}$ is Lie^s-algebra with bracket $[-,-]$, let V' be the ideal of self-brackets. Thus we obtain a two-step filtration $V' \to \mathfrak{g}$ of $\mathfrak{g}$. Denote by $\langle -,- \rangle$ the canonical $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket on the quotient $V = \mathfrak{g}/V' = Q^{\mathrm{Lie}^s}_{\mathrm{Lie}^{s,\mathrm{ti}}}(\mathfrak{g})$ and consider V' as a trivial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra. Thus we obtain a $\mathrm{Lie}^{s,\mathrm{ti}}$-structure on the associated graded of $\mathfrak{g}$ as the product of V and V' with the above $\mathrm{Lie}^{s,\mathrm{ti}}$-structures. Denote by $\tilde{\mathfrak{g}}$ the resulting $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra with $\langle -,- \rangle$ the $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket.

Therefore, any $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra $\mathfrak{g}$ admits a $\mathrm{Lie}^{s,\mathrm{ti}}$-structure that is unique up to a choice of splitting of $\mathfrak{g} \to V$. Denote by $\tilde{\mathfrak{g}}$ the resulting $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra, which has the same underlying $\bar{\mathcal{R}}$-module structure as $\mathfrak{g}$, cf. Remark 4.2.2.

Remark 4.2.5. If we fix a choice of splitting for $\mathfrak{g} \to V$, then any Lie^s bracket $[x,y]$ in $\mathfrak{g}$ is equal to a sum of self-brackets and the $\mathrm{Lie}^{s,\mathrm{ti}}$-brackets $\langle x,y \rangle$ in $\tilde{\mathfrak{g}}$.

Now we compute the E^2-page of the bar spectral sequence by constructing a May-type spectral sequence in the sense that the filtration comes from the length filtration of $\bar{\mathcal{R}}$-modules in Construction 4.2.3.

Theorem 4.2.6. *Let $\mathfrak{g}$ be a $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra and $\tilde{\mathfrak{g}}$ the associated $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra via Construction 4.2.4. Then there is a May-type spectral sequence with respect to the $\bar{\mathcal{R}}$-module structure converging to $\pi_{s,t}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^s_{\bar{\mathcal{R}}},\mathfrak{g})$. The E^1-page $\bigoplus_q E^1_{q,s,t}$ of the May-type spectral sequence has dimensions bounded above by $\pi_{s,t}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}},\tilde{\mathfrak{g}})$, in the sense that there is an algebraic γ_1-Bockstein spectral sequence converging to the May E^1-page whose E^1-page is $\pi_{s,t}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}},\tilde{\mathfrak{g}})$.*

Proof. We start by inductively constructing a filtration on $(\mathrm{Lie}^s_{\bar{\mathcal{R}}})^{\circ n}(\mathfrak{g})$ that heuristically count the number of $\bar{Q}$ symbols in a given element.

Since any $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra $\mathfrak{g}$ is an $\bar{\mathcal{R}}$-module, it admits a length filtration. The filtration is compatible with the bracket since brackets of operations always vanish (Definition 4.1.6). Furthermore, since any self-bracket $[x,x] = \bar{Q}_0(x)$ is in $\mathrm{F}^1_l(\mathfrak{g})$ and the right hand side is zero in $\mathrm{F}^0_l(\mathfrak{g})$, we deduce that $\mathrm{Gr}^0_l(\mathfrak{g})$ is a $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra, and the $\mathrm{Lie}^{s,\mathrm{ti}}$-structure can be extended to $\bigoplus \mathrm{Gr}^q(\mathfrak{g})$ via trivial extension to positive q. On the other hand, $\bigoplus \mathrm{Gr}^q(\mathfrak{g})$ is an $\bar{\mathcal{R}}$-module since $\bar{\mathcal{R}}$ is homogeneous. Hence $\tilde{g} = \bigoplus \mathrm{Gr}^q_l(\mathfrak{g})$ equipped with the $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket in Construction 4.2.4 is an algebra over the composite monad $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}} = \mathcal{A}_{\bar{\mathcal{R}}} \circ \mathrm{Lie}^{s,\mathrm{ti}}$.

Now we define a new filtration $F^\bullet$ on $\mathrm{Lie}^s_{\bar{\mathcal{R}}}(\mathfrak{g})$ that combines the length filtration on $\mathfrak{g}$, the length filtration on $\mathrm{Lie}^s_{\bar{\mathcal{R}}}(M)$ for any $\mathbb{F}_2$-module M, and the effect of Lie^s brackets. We extend the length filtration on $\mathfrak{g}$ to $\mathrm{Lie}^{s,\mathrm{ti}}(\mathfrak{g})$ via the Day convolution, i.e. for $x \in \mathrm{F}^q_l(\mathfrak{g}), y \in \mathrm{F}^r_l(\mathfrak{g})$, we have $\langle x,y\rangle \in F^{q+r}(\mathrm{Lie}^{s,\mathrm{ti}}(\mathfrak{g}))$, so on and so forth. Then we extend it to a new filtration on $\mathrm{Lie}^s_{\bar{\mathcal{R}}}(\mathfrak{g})$ by combining with the length filtration on $\mathrm{Lie}^s_{\bar{\mathcal{R}}}(M)$ for M an $\mathbb{F}_2$-module, using the fact that when $\mathfrak{g} = M$ is an $\mathbb{F}_2$-module, $\mathrm{Gr}^0_l(\mathrm{Lie}^s_{\bar{\mathcal{R}}}(M)) = \mathrm{Lie}^{s,\mathrm{ti}}(M)$. In particular, after passing to the associated graded, the evaluation map $\mathrm{Lie}^s_{\bar{\mathcal{R}}}(\mathfrak{g}) \to \mathfrak{g}$ assembles to the $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra structure map $ev : \mathcal{A}_{\bar{\mathcal{R}}} \circ \mathrm{Lie}^{s,\mathrm{ti}}(\tilde{\mathfrak{g}}) \to \tilde{\mathfrak{g}}$: for $x \in \mathfrak{g}$, $[x,x] = \bar{Q}_0|x \in \mathrm{Lie}^s_{\bar{\mathcal{R}}}(\mathfrak{g})$ is mapped to a nonzero element only if $x \in \mathrm{F}^0_l(\mathfrak{g})$, in which case $\bar{Q}_0|x \in F^1\mathrm{Lie}^s_{\bar{\mathcal{R}}}(\mathfrak{g})$ and $\bar{Q}_0(x) \in \mathrm{F}^1_l(\mathfrak{g})$ while $[x,x] \in F^0\mathrm{Lie}^s_{\bar{\mathcal{R}}}(\mathfrak{g})$.

Iterating this process, we obtain a filtration $F^\bullet$ on $\mathrm{Lie}^s_{\bar{\mathcal{R}}} \circ (\mathrm{Lie}^s_{\bar{\mathcal{R}}})^{\circ n}(\mathfrak{g})$ for all $n > 0$ by combining the filtration $F^\bullet$ on $(\mathrm{Lie}^s_{\bar{\mathcal{R}}})^{\circ n}(\mathfrak{g})$ with the length filtration on $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$. This is the nth simplicial level of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$, with associated graded assembling to $(\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}})^{\circ n}(\tilde{\mathfrak{g}})$. Explicitly, $F^q\Big((\mathrm{Lie}^s_{\bar{\mathcal{R}}})^{\circ n}(\mathfrak{g})\Big)$ is the collection of elements $\alpha|x$ in simplicial degree n satisfying the following condition: if we rewrite $\alpha|x$ as an element in $(\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}})^{\circ n}(\mathfrak{g})$ via Remark 4.2.2 and Remark 4.2.5, so any Lie^s bracket in $\alpha|x$ is written as a linear combination of $\mathrm{Lie}^{s,\mathrm{ti}}$ brackets and $\bar{Q}_0$ applies to other elements, then the sum of the filtration degree of $x \in \mathfrak{g}$ times the number of times x appears and the number of symbols $\bar{Q}^j$ in any term of $\alpha|x$ coming from applications of the monad $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$ is at most q.

Since $\bar{\mathcal{R}}$ is a homogeneous algebra and evaluation of brackets do not increase the number of $\bar{Q}^j$'s in the expression, the structure map $\mathrm{Lie}^s_{\bar{\mathcal{R}}}(\mathfrak{g}) \to \mathfrak{g}$ is compatible with this filtration, and so are the face maps and degeneracy maps in $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$. The induced filtration $F^\bullet$ on the normalized complex of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ gives rise to a May-type spectral sequence

$$\bigoplus_q E^1_{q,s,t} = \bigoplus_q \pi_{s,t}\mathrm{Gr}^q\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g}) \Rightarrow \pi_{s,t}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g}).$$

Note that the face maps

$$(\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}})^{\circ n}(\tilde{\mathfrak{g}}) = \bigoplus_q \mathrm{Gr}^q \mathrm{Bar}_n(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g}) \to (\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}})^{\circ(n-1)}(\tilde{\mathfrak{g}}) = \bigoplus_q \mathrm{Gr}^q \mathrm{Bar}_{n-1}(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$$

do not change the associated graded degree unless the differential creates self-brackets – evaluating the $\bar{\mathcal{R}}$-module structure or a $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket is either zero or does not change the number of $\bar{Q}$ symbols. Hence they assembles to the $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra structure maps $(\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}})^{\circ j}(\tilde{\mathfrak{g}}) \to (\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}})^{\circ(j-1)}(\tilde{\mathfrak{g}})$ except in the following situation: for $x \in \mathrm{Gr}^0_l(\mathfrak{g})$, $\gamma_1(\bar{Q}^i|x) := [\bar{Q}^i|1|x, 1|\bar{Q}^i|x]$ and $\bar{Q}_0|\bar{Q}^i|x$ are both in the second associated graded piece. Hence the total differential ∂ of the normalized complex of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ sends $\gamma_1(\bar{Q}^i|x) := [\bar{Q}^i|1|x, 1|\bar{Q}^i|x]$ to $[\bar{Q}^i|x, \bar{Q}^i|x] + [\bar{Q}^i|x, 1|\bar{Q}^i(x)] = \bar{Q}_0|\bar{Q}^i|x + [\bar{Q}^i|x, 1|\bar{Q}^i(x)]$ in $E^0_{2,*,*}$, i.e. the self-bracket has not been filtered away. Similarly, any class containing $\gamma_1(\bar{Q}^i|x)$ with $x \in \mathrm{Gr}^0_l(\mathfrak{g})$ has a face map whose target has at least one self-bracket term. Whereas when $x \in \mathrm{F}^1_l(\mathfrak{g})$, the self-brackets in the target of such differentials are not visible in the associated graded because the number of $\bar{Q}^j$'s in the term decrease after we rewrite the self-brackets in terms of $\bar{Q}_0$.

To further filter away the self-brackets in such differentials, we assign weight 1 to $\gamma_1(\bar{Q}^i|x)$ and $[\bar{Q}^i|x, 1|\bar{Q}^i(x)]$ for all i and $x \in \mathrm{Gr}^0_l(\mathfrak{g})$, weight 0 to everything else in $\mathfrak{g}$, $\mathrm{Lie}^s_{\bar{\mathcal{R}}}(\mathfrak{g})$, and $\mathrm{Lie}^s_{\bar{\mathcal{R}}} \circ \mathrm{Lie}^s_{\bar{\mathcal{R}}}(\mathfrak{g})$, including $\bar{Q}_0|\bar{Q}^i|x$. Then we propagate the weight to $(\mathrm{Lie}^s_{\bar{\mathcal{R}}})^{\circ n}(\mathfrak{g})$ for $n > 2$ by stipulating that applying $\bar{Q}$ does not change weight and brackets add weights. The associated graded of this weight filtration is precisely $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, \tilde{\mathfrak{g}})$, since the only face or degeneracy maps that are altered are the ones involving $\gamma_1(\bar{Q}^i|x)$ for $x \in \mathrm{Gr}^0_l(\mathfrak{g})$, whose target no longer contains the self-bracket term $\bar{Q}_0|\bar{Q}^i|x$. Therefore we obtain an algebraic γ_1-Bockstein spectral sequence converging to the E^1-page of the May-type spectral sequence, whose E^1-page has dimensions those of $\pi_{s,t}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, \tilde{\mathfrak{g}})$. Therefore we obtain an upper bound of the dimension of the E^1-page of the May-type spectral sequence $\bigoplus_q E^1_{q,s,t}$. □

Since differentials preserve weights and the γ_1 operation on $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, L)$ appears in weight at least four, we immediately deduce the following from Theorem 4.2.6.

Corollary 4.2.7. *For any* $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$*-algebra* $\mathfrak{g}$*, the homotopy groups of* $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ *and* $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, \tilde{\mathfrak{g}})$ *are isomorphic in weight less than four.*

Proof. In the algebraic γ_1-Bockstein spectral sequence, the differentials do not appear until weight 4 since $\gamma_1(\bar{Q}_j|x)$ has weight 4. By construction, differentials in the May-type spectral sequence occur when the source and target of a face map in $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ have different number of self-brackets. In weight two and three this cannot happen. Hence both spectral sequences collapse in weight less than four. □

Since $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}} = \mathcal{A}_{\bar{\mathcal{R}}} \circ \mathrm{Lie}^{s,\mathrm{ti}}$ is a composite monad, we apply Construction 2.3.2 and Lemma 2.3.3 to compute the homotopy groups of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, L)$ for L a $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra.

Construction 4.2.8. For L a $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra with $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket $\langle -,- \rangle$, denote by $\mathrm{AR}_\bullet(L)$ the bar construction $\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\bar{\mathcal{R}}}, L)$ equipped with a $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket $\langle -,- \rangle$ given levelwise by

$$\langle \alpha_1|\alpha_2|\ldots|\alpha_n|x, \beta_1|\beta_2|\ldots|\beta_n|y \rangle = \begin{cases} 1|\cdots|1|\langle x,y \rangle & \text{if} \quad \alpha_i = \beta_i = 1, 1 \leq i \leq n \\ 0 & \quad\quad \text{otherwise} \end{cases},$$

where $\alpha_i, \beta_j \in \bar{\mathcal{R}}$ and $x, y \in L$. Here we use L to mean the underlying $\bar{\mathcal{R}}$-module $U^{\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\bar{\mathcal{R}}}}(L)$.

Corollary 4.2.9. *For L a $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra with $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket $\langle -,- \rangle$, there is an isomorphism of bigraded homotopy groups*

$$\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, L) \cong \pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, \mathrm{AR}_\bullet(L)).$$

4.2.2 Homology groups of simplicial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebras.

The homotopy group of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, V_\bullet)$ for $V_\bullet$ a simplicial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra can be computed via a shifted version of the classical Chevalley-Eilenberg complex.

Recall from [CE48], [May66A, Section 5] and [Pri70] that given a $\mathrm{Lie}^{\mathrm{ti}}$-algebra L, i.e., an unshifted totally isotropic Lie algebra over $\mathbb{F}_2$, its $\mathrm{Lie}^{\mathrm{ti}}$-algebra homology is computed by

$$H^{\mathrm{Lie}^{\mathrm{ti}}}(L) := H_*(\mathbb{L}Q^{\mathrm{Lie}^{\mathrm{ti}}_{\mathbb{F}_2}}_{\mathrm{Mod}_{\mathbb{F}_2}}(L)[1] \oplus \mathbb{F}_2) = H_*(CE(L)).$$

Here $CE(L)$ is the standard Chevalley-Eilenberg complex, defined to be the exterior algebra on $L[1]$ with differential δ given by

$$\delta(\sigma x_1 \otimes \cdots \otimes \sigma x_n) = \sum_{1 \le i < j \le n} [\sigma x_i, \sigma x_j] \otimes \sigma x_1 \otimes \cdots \otimes \widehat{\sigma x_i} \otimes \cdots \otimes \widehat{\sigma x_j} \otimes \cdots \otimes \sigma x_n.$$

There is no divided power part at $p = 2$. Since we are working with shifted, graded totally-isotropic Lie algebras, we use a modified version for ease of notation. First we note that given a $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra L, there are weak equivalences

$$N(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, L)) \simeq N(\Sigma \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{\mathrm{ti}}_{\mathbb{F}_2}, \Sigma^{-1} L)) \simeq \Sigma \overline{CE}(\Sigma^{-1} L[1])[-1], \tag{4.6}$$

where $\overline{CE}$ is the reduced complex.

Definition 4.2.10. The *Chevalley-Eilenberg complex* for a $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra L is $\mathrm{CE}(L) = (\Lambda^\bullet(L), \delta)$, where $\Lambda^\bullet(L)$ is the free shifted graded exterior algebra on L (placed in homological degree 0) with a shifted graded exterior product $\Sigma^{-1} \otimes [1]$, which we continue to denote by $\otimes$, that increases homological degree by one and decreases internal degree by one, reflecting the behavior of shifted graded Lie brackets. The differential δ is given by

$$\delta(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \le i < j \le n} [x_i, x_j] \otimes x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_n.$$

Then the $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra homology of L is given by

$$H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(L) := \pi_{*,*}(\mathbb{L}Q^{\mathrm{Lie}^{s,\mathrm{ti}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(L) \oplus \mathbb{F}_2) \cong H_{*,*}(N(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, L) \oplus \mathbb{F}_2) \cong H_{*,*}(\mathrm{CE}(L)),$$

where the last isomorphism follows from rearranging the right hand side in (4.6).

In the case where L is a simplicial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra, its Chevalley-Eilenberg complex $\mathrm{CE}(L)$ is the simplicial chain complex obtained by applying the Chevalley-Eilenberg complex levelwise. Then Dold-Kan correspondence says that the homotopy group of $\mathrm{CE}(L)$ is isomorphic to the homology of its total complex. A simplicial version of May's result is recorded in [BHK19, Section 3]. Here we state the shifted version.

Theorem 4.2.11. *[BHK19, Theorem 3.13] Let L be a simplicial* $\mathrm{Lie}^{s,\mathrm{ti}}$*-algebra. Then there is a natural isomorphism*

$$H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(L) := \pi_{*,*}(\mathbb{L}Q_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Lie}^{s,\mathrm{ti}}}(L) \oplus \mathbb{F}_2) \cong H_{*,*}(\mathrm{CE}(L)).$$

In the total complex of $\mathrm{CE}(L)$, the differential in the homological direction is given by δ in Definition 4.2.10. The differential d in the simplicial direction is obtained by applying the shifted graded exterior algebra functor $\Lambda^{\bullet}$ to each simplicial differential d_i of L and taking the alternating sum, i.e.

$$d = d_0 \otimes d_0 \otimes \cdots \otimes d_0 + \cdots + d_r \otimes d_r \otimes \cdots \otimes d_r.$$

Both differentials preserve weights.

If the $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket on a simplicial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra L is trivial, then the differential δ in the homological direction vanishes and $H_{*,*}(\mathrm{CE}(L)) \cong \pi_{*,*}(\Lambda^{\bullet}(L))$. The natural operations on the homotopy groups of simplicial exterior algebras are well-understood by the work of Cartan, Bousfield, and Dwyer. We only state their results in the case of free algebras, and modify the grading to take into account the fact that we work with shifted, graded exterior algebras.

Theorem 4.2.12. *[Dwy80a, Theorem 2.1, Remark 4.4][Bou68][Car54][HM16, Theorem 3.9] Let* $V_{\bullet}$ *be a simplicial graded* $\mathbb{F}_2$*-module. There are natural operations*

$$\gamma_i : \pi_{h,r,t}(\Lambda^h(V_{\bullet})) \to \pi_{2h+1,r+i,2t-1}(\Lambda^{2h+1}(V_{\bullet})), 1 \leq i \leq r$$

for all $r \geq 1$*, satisfying the relations*

$$\gamma_i\gamma_j(x) = \sum_{(i+1)/2 \leq l \leq (i+j)/3} \binom{j-i+l-1}{j-l} \gamma_{i+j-l}\gamma_l(x) \quad \text{for all } i < 2j.$$

Here in the trigrading (h,r,t) records the number of exterior products h, the simplicial degree r in $V_{\bullet}$, and the internal degree t.

Furthermore, they computed the homotopy group of the free exterior algebra on a simplicial $\mathbb{F}_2$-module.

Definition 4.2.13. A sequence $I = (i_1, \dots, i_m)$ is *γ-admissible* if $i_l \geq 2i_{l+1}$ for $1 \leq l \leq m-1$. The *excess* of I is $e(I) = i_1 - i_2 - \cdots - i_m$.

Theorem 4.2.14. *[Bou68, Theorem 8.6][HM16, Theorem 3.19] Let A be a graded $\mathbb{F}_2$-basis for $\pi_*(V_\bullet)$. Then $\pi_{*,*}(\Lambda^\bullet(V_\bullet))$ is the (shifted graded) exterior algebra on generators $\gamma_I(\alpha)$, where $\alpha \in A$ and $I = (i_1, \dots, i_m)$ is γ-admissible with $e(I) \leq s(\alpha)$, where $s(\alpha)$ is the simplicial degree of the basis element α.*

The following is immediate by combining Theorem 4.2.11 and Theorem 4.2.14.

Corollary 4.2.15. *Suppose that L is a $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra with trivial Lie brackets. Then the homotopy group of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, \mathrm{AR}_\bullet(L))$ is isomorphic as a bigraded vector space to the (shifted graded) exterior algebra on generators $\gamma_I(\alpha)$, where α is a basis element of $\pi_{r,*}(\mathrm{AR}_\bullet(L))$ (cf. Construction 4.2.8) and I is γ-admissible with $e(I) \leq r$.*

Now we can compute the homotopy groups of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, L)$ when the $\mathrm{Lie}^{s,\mathrm{ti}}$ structure on L is trivial. First we recall the following result of Priddy that computes the Ext and Tor groups of a homogeneous Koszul algebra, which we make use of to compute the Tor groups over $\bar{\mathcal{R}}$.

Theorem 4.2.16. *[Pri70, Theorem 2.5] Let R be a homogeneous Koszul algebra over $\mathbb{F}_2$ on generators $a_i, i \in J$ in weight 1 and quadratic relations r_j. Let B be a subset of S, the set of nonempty sequences on J, such that there is a basis of R consisting of monomials $\{a_I\}_{I \in S}$. Then the cohomology algebra $H^*(A) = \mathrm{Ext}^*_R(\mathbb{F}_2, \mathbb{F}_2)$ is isomorphic to the tensor algebra on $a_i^\vee$ subject to relations that are linear dual to the r_j's.*

Remark 4.2.17. Call $a_i a_j$ allowable if $(i, j) \in B$ and unallowable otherwise. We identify $\mathrm{Tor}^R_m(\mathbb{F}_2, \mathbb{F}_2)$ with $\mathrm{Ext}^m_R(\mathbb{F}_2, \mathbb{F}_2)$. A cycle corresponding to the class

$$a_{i_1}^\vee a_{i_2}^\vee \cdots a_{i_m}^\vee \in \mathrm{Tor}^R_m(\mathbb{F}_2, \mathbb{F}_2)$$

with (i_k, i_{k+1}) unallowable for all k in the reduced bar complex over R is a sum

$$\sum_j [a_{j_1}|a_{j_2}|\cdots|a_{j_m}] \in R^{\otimes m}$$

that contains the term $[a_{i_1}|a_{i_2}|\cdots|a_{i_m}]$ with nonzero coefficient. We call this the cycle completion of the monomial $[a_{i_1}|a_{i_2}|\cdots|a_{i_m}]$. To find the cycle explicitly, we start with $\alpha_0 = [a_{i_1}|a_{i_2}|\cdots|a_{i_m}]$. The differential ∂ is a sum of face maps composing adjacent terms $a_{i_k}a_{i_{k+1}}$. We use the relation $a_{i_k}a_{i_{k+1}} = \sum b_{j_k}b_{j_{k+1}}$ to cancel out the the terms $[a_{i_1}|\cdots|a_{i_k}a_{i_{k+1}}|\cdots|a_{i_m}]$ in the differential by adding $\sum[a_{i_1}|\cdots|a_{j_{k-1}}|b_{j_k}|b_{j_{k+1}}|a_{j_{k+2}}|\cdots|a_{i_m}]$ to α_0 for all k and denote the resulting sum α_1. Then we pair off the differential for every term in $\alpha_1 - \alpha_0$, i.e. for each nonzero term in $\partial(\alpha_1 - \alpha_0)$ obtained by composing an unallowable 2-tuple via the kth face map, we use the relations in R to find a sum in $R^{\otimes m}$ whose image under the kth face map cancel out that term. Thus we obtain a new sum α_2 such that all terms in the differential on α_1 are paired off. Now we repeat the process again. It has to terminate since the number of unallowable adjacent pairs is nonincreasing for any term at each step and and there are finitely many monomials with a given number of unallowable adjacent pairs. In other words, $a_{i_1}a_{i_2}\cdots a_{i_m}$ can be written as a unique sum of basis monomials through this iterative process in finite steps.

Lemma 4.2.18. *(1). Suppose that $L = \Sigma^k\mathbb{F}_2$ is a trivial $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra. Then the bigraded homotopy group $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, \mathrm{AR}_\bullet(L))$ is the exterior algebra on generators $\gamma_I\bar{Q}^J(x_k)$, where x_k is the generator of $\pi_*(L)$, $J = (j_1, \ldots, j_r)$ satisfies*

$$j_{l+1} + \cdots + j_r + k - (r-l) \le j_l \le 2j_{l+1}$$

for $1 \le l < r$ and $j_r > k$, and I is γ-admissible with $e(I) \le r$. In lower indexing, the generators are $\gamma_I\bar{Q}_J(x_k)$, where $J = (j_1, \ldots, j_r)$ satisfies $0 \le j_l \le j_{l+1} + 1$ for all l, and I is γ-admissible with $e(I) \le r$.

(2). Let L be the $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra with underlying $\bar{\mathcal{R}}$-module $\Omega^n\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2), n \ge 1$ and trivial Lie brackets. Then $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, \mathrm{AR}_\bullet(L))$ is the exterior algebra on generators $\gamma_I\bar{Q}_J(x_k)$, where $J = (j_1, \ldots, j_r)$ satisfies $0 \le j_l \le j_{l+1} + 1$ for all $l < r$ and

$0 \leq j_r < n$, *and* I *is* γ*-admissible with* $e(I) \leq r$.

Proof. (1). In light of Corollary 4.2.15, it suffices to compute

$$\pi_{*,*}(\mathrm{AR}_\bullet(L)) = \pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\bar{\mathcal{R}}}, \Sigma^k\mathbb{F}_2),$$

where the right hand side is the unstable Tor groups $\mathrm{UnTor}^{\bar{\mathcal{R}}}_{*,*}(\mathbb{F}_2, \Sigma^k\mathbb{F}_2)$. The unstable Tor group is computed by taking the homotopy group of the subcomplex of the bar complex computing the Tor group $\mathrm{Tor}^{\bar{\mathcal{R}}}_{*,*}(\mathbb{F}_2, \Sigma^k\mathbb{F}_2)$ obtained by regarding $\Sigma^k\mathbb{F}_2$ as an unstable trivial module over $\bar{\mathcal{R}}$ and imposing the unstability conditions $[\bar{Q}^j|\alpha] = 0$ for $j \leq |\alpha|$, cf. [BC70, §3].

The quadratic algebra $\bar{\mathcal{R}}$ is a homogeneous Koszul algebra, since the canonical basis $\{\bar{Q}^{j_1}\cdots\bar{Q}^{j_r}, j_i > 2j_{i+1}\forall i\}$ of $\bar{\mathcal{R}}$ is a Poincaré-Birkhoff-Witt basis in the sense of Priddy [Pri70, Theorem 5.3]. In particular, it follows from Priddy's machinery [Pri70, Theorem 2.5, 3.8] that the Tor group $\mathrm{Tor}^{\bar{\mathcal{R}}}_{s,*}(\mathbb{F}_2, \mathbb{F}_2)$ has a basis consisting of cycles indexed by $\bar{Q}^{j_1}\cdots\bar{Q}^{j_s}$, where $j_i \leq 2j_{i+1}$ for all i.

To compute the unstable Tor groups on a class x_k of internal degree k, we need to impose the unstability condition $\bar{Q}^j(x) = 0$ for $j < |x|$, then the basis of $\mathrm{UnTor}^{\bar{\mathcal{R}}}_{r,*}(\mathbb{F}_2, \mathbb{F}_2\{x_k\})$ consists of basis elements of $\mathrm{Tor}^{\bar{\mathcal{R}}}_{r,*}(\mathbb{F}_2, \mathbb{F}_2)$ satisfying $j_i > j_{i-1} - 1 + j_{i-2} - 1 + \cdots + j_r - 1 + |x|$ for all $i < r$ and $j_r \geq k$, or equivalently sequences $\bar{Q}_{j_1}\cdots\bar{Q}_{j_s}(x_k)$, where $0 \leq j_i \leq j_{i+1} + 1$ for all i.

(2). Iterating Proposition 4.1.10 yields a canonical map of $\bar{\mathcal{R}}$-modules

$$L = \Omega^n\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2) \to \Omega^\infty\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^\infty\Sigma^k\mathbb{F}_2) \cong \Sigma^k\mathbb{F}_2,$$

which gives rise to a surjective map of $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebras with trivial brackets. The underlying $\mathbb{F}_2$-module of L has basis $\bar{Q}^J x_k$, where $J = (j_1, \ldots, j_r)$ is a basis element of $\bar{\mathcal{R}}$ satisfying $j_r \geq n+k$. Suppose that $\alpha \in \mathrm{AR}_\bullet(L)$ is the cycle completion of an element $\bar{Q}^{j_1}|\cdots|\bar{Q}^{j_r}|x_k$ with $k \leq j_r < n+k$ and $j_{l+1} - 1 + \cdots + j_r - 1 + k \leq j_l \leq 2j_{l+1}$ for $l < r$. Since cycle completion via Behrens' relations in the sense of Remark 4.2.17 cannot increase the index of the right most operation, the differentials supported by α are the same as those supported by its image in $\mathrm{AR}_\bullet(\mathbb{F}_2\{x_k\})$, so α is a nontrivial cycle. Otherwise, all but the rightmost face maps send α to zero, while the rightmost face map from at least one (distinct) term of α is nonzero, so it is impossible to complete the cycle. Switching to lower-indexing yields the desired answer. $\square$

Combining Theorem 4.2.14 and Lemma 4.2.18, we have the following:

Corollary 4.2.19. *For $\mathfrak{g} = \Omega^n \mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2)$ with $1 \le n \le \infty$, the E^1-page $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, \mathfrak{g})$ of the algebraic γ_1-Bockstein spectral sequence (cf. Theorem 4.2.6) is the (shifted graded) exterior algebra on generators $\gamma_I \bar{Q}_J(x_k)$, where $I = (i_1, \dots, i_m)$ is γ-admissible with $e(I) \le r$ and $J = (j_1, \dots, j_r)$ satisfies $0 \le j_l \le j_{l+1} + 1$ for $l < r$, $0 \le j_r < n$.*

4.2.3 Quillen homology of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras with trivial brackets

Next we want to identify the differentials in the May-type spectral sequence and the γ_1-Bockstein spectral sequence when $\mathfrak{g} = \Omega^n \mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2)$. There is no canonical map from $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ to the E^1-page $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, \mathfrak{g})$ of the γ_1-Bockstein spectral sequence; instead we map $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ into the bar construction of another variant of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras.

Definition 4.2.20. Let $\mathrm{Mod}_{\bar{\mathcal{R}}_{>0}} \subset \mathrm{Mod}_{\bar{\mathcal{R}}}$ be the subcategory of allowable $\bar{\mathcal{R}}$-modules M such that $\bar{Q}_0(x) = 0$ for all $x \in M$. Denote by $\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}_{>0}}}_{\mathrm{Mod}_{\mathbb{F}_2}}$ the free $\bar{\mathcal{R}}_{>0}$-module functor, and $\mathcal{A}_{\bar{\mathcal{R}}_{>0}}$ the additive monad associated to the free functor. Let $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}} = \mathcal{A}_{\bar{\mathcal{R}}_{>0}} \circ \mathrm{Lie}^{s,\mathrm{ti}}$, where the composite monad on the right has distributivity given by $[\bar{Q}^j(-), (-)] = 0$.

By Proposition 4.1.8, there is an equivalence $\mathrm{Lie}^s_{\bar{\mathcal{R}}}(M) = \mathcal{A}_{\bar{\mathcal{R}}} \circ \mathrm{Lie}^s(M)/(\bar{Q}_0(x) = [x,x], x \in M)$, while $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}(M) = \mathcal{A}_{\bar{\mathcal{R}}_{>0}} \circ \mathrm{Lie}^{s,\mathrm{ti}}(M) = \mathcal{A}_{\bar{\mathcal{R}}} \circ \mathrm{Lie}^s(M)/\langle \bar{Q}_0(x), [x,x], x \in M\rangle$, where the quotient is taken with respect to the left $\bar{\mathcal{R}}$-algebra ideal. Hence the category $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}$ of $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}$-algebras is the subcategory of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras L satisfying the condition that $\bar{Q}_0(x) = [x,x] = 0$ for all $x \in L$. The inclusion $T^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}}(\mathfrak{g}) : \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}} \to \mathrm{Lie}^s_{\bar{\mathcal{R}}}$ of subcategory admits a left adjoint $Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}}(\mathfrak{g})$ that takes the quotient by the $\bar{\mathcal{R}}$-algebra ideal of the self-brackets. When $\mathfrak{g}$ is a $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra with trivial Lie^s brackets, $Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}}(\mathfrak{g})$ is given by equipping the $\bar{\mathcal{R}}_{>0}$-module $Q^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\bar{\mathcal{R}}_{>0}}}(\mathfrak{g})$ with trivial $\mathrm{Lie}^{s,\mathrm{ti}}$ brackets.

Lemma 4.2.21. *Let $\mathfrak{g}$ be an $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra. There is a surjective map of simplicial $\mathbb{F}_2$-modules*

$$\varphi : \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g}) \to \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}}(\mathfrak{g})).$$

Proof. There is a map of monads $\mathrm{Lie}^s_{\bar{\mathcal{R}}} \to \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}$ that sends the symbol $\bar{Q}_0$ to 0, and this induces the map of bar constructions in question. □

The bigraded homotopy group of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, L)$ is computed in the same way as that of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, L)$ via Lemma 2.3.3 and Lemma 4.2.18 ($\bar{Q}_0$ operation no longer appears in the generators).

Construction 4.2.22. For L a $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}$-algebra with $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket $\langle -,-\rangle$, denote by $\mathrm{AR}^{>0}_\bullet(L)$ the bar construction $\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\bar{\mathcal{R}}_{>0}}, L)$ equipped with the simplicial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra structure given levelwise by

$$\langle \alpha_1|\alpha_2|\dots|\alpha_n|x, \beta_1|\beta_2|\dots|\beta_n|y\rangle = \begin{cases} 1|\cdots|1|\langle x,y\rangle & \text{if} \quad \alpha_i = \beta_i = 1, 1 \le i \le n \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 4.2.23. *(1). There is an isomorphism*

$$\begin{aligned}\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, \Sigma^k\mathbb{F}_2) &\cong \pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, \mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\bar{\mathcal{R}}_{>0}}, \Sigma^k\mathbb{F}_2)) \\ &\cong \pi_{*,*}\Lambda^\bullet(\mathrm{UnTor}^{\bar{\mathcal{R}}_{>0}}_{*,*}(\mathbb{F}_2, \mathbb{F}_2\{x_k\})).\end{aligned}$$

Hence $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, \Sigma^k\mathbb{F}_2)$ is the exterior algebra on generators $\gamma_I\bar{Q}_J(x_k)$, where $J = (j_1, \dots, j_r)$ satisfies $1 \le j_l \le j_l + 1$ for all l and I is γ-admissible with $e(I) \le r$.

(2). The homotopy group of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, \Omega^n\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}_{>0}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2))$ is the exterior algebra on generators $\gamma_I\bar{Q}_J(x_k)$, where $J = (j_1, \dots, j_r)$ satisfies $1 \le j_r < n$ and $1 \le j_l \le j_l + 1$ for $l < r$, while I is γ-admissible with $e(I) \le r$.

(3). For $L = \Omega^n\mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}_{>0}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2)$ with $1 \le n \le \infty$, the quotient map of monads $\mathcal{A}_{\bar{\mathcal{R}}} \to \mathcal{A}_{\bar{\mathcal{R}}_{>0}}$ induces a surjective map $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, L) \to \pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, L)$

that sends the symbol $\bar{Q}_0$ to 0.

In order to use the comparison map (cf. Lemma 4.2.21)

$$\varphi_* : \pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g}) \to \pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, \mathfrak{g})$$

to detect differentials and permanent cycles, we make use of explicit combinatorial formulae of γ_i by Bökstedt and Ottosen. The grading conventions are modified to suit our context.

For $r,i \in \mathbb{N}$ with $1 \le i \le r$, let $U(r,i)$ be the set of pairs (A,B) of ordered sequences $a_1 < \cdots < a_i, b_1 < \cdots < b_i$ such that $\{a_1,\dots,a_i\}$ and $\{b_1,\dots,b_i\}$ are complementary subsets of $\{r-i, r-i+1, \dots, r+i-1\}$. Let $V(r,i) \subset U(r,i)$ be the subset with $a_1 = r-i$.

Proposition 4.2.24. *[BO06, Theorem 1.3, Lemma 3.1] Suppose that $V_\bullet$ is a simplicial $\mathbb{F}_2$-module with face maps d_j. Let z be a representative of a class $[z] \in \pi_{s,t}(V_\bullet)$ in the normalized complex $N(V_\bullet)$. For $2 \le i \le s$, define*

$$\gamma_i(z) = \sum_{(A,B)\in V(s,i)} s_{a_i}\cdots s_{a_2}s_{a_1}(z) \otimes s_{b_i}\cdots s_{b_2}s_{b_1}(z) \in \Lambda^2(V_\bullet).$$

Then $d_j(\gamma_i(z)) = 0$ for $0 \le j \le i+s$, and the induced operation $\gamma_i : \pi_{s,t}(V_\bullet) \to \pi_{s+i+1,2t-1}(\Lambda^2(V_\bullet))$ are exactly the Dwyer-Bousfield operations in Theorem 4.2.14.

Remark 4.2.25. If in addition $V_\bullet$ is exterior, then the formula above for $i = 1$ induces the operation γ_1 on $\pi_{*,*}(V_\bullet)$. The operation γ_1 is not well-defined when there is some element a in the simplicial commutative algebra $V_\bullet$ such that $a \otimes a \neq 0$. This is because in $N(V_\bullet)$ the differential sends $\gamma_1(a)$ to $a \otimes a$, cf. [Dwy80a, Remark 4.3, 4.4][BO06, Remark 3.2].

Hence we obtain natural operations γ_i for $1 \le i \le s$ on

$$\pi_{s,*}(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, \mathrm{AR}^{>0}_\bullet(\Sigma^k\mathbb{F}_2))) \cong \pi_{s,*}(\Lambda^\bullet(\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\bar{\mathcal{R}}_{>0}}, \Sigma^k\mathbb{F}_2))),$$

and similarly on

$$\pi_{s,*}(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}, \mathrm{AR}_\bullet(\Sigma^k\mathbb{F}_2))) \cong \pi_{s,*}(\Lambda^\bullet(\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{A}_{\bar{\mathcal{R}}}, \Sigma^k\mathbb{F}_2))).$$

Suppose that ξ is a cycle in $\mathrm{AR}_s^{>0}(\Sigma^k\mathbb{F}_2)$. In the total complex of $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}},\mathrm{AR}_\bullet^{>0}(\Sigma^k\mathbb{F}_2))$, a representative for the homotopy class $\gamma_i([\xi])$ is

$$\gamma_i(\xi) = \sum_{(A,B)\in V(s,i)} \langle s_{a_i}\cdots s_{a_2}s_{a_1}(\xi), s_{b_1}s_{b_2}\cdots s_{b_i}(\xi)\rangle \in \mathrm{Lie}^{s,\mathrm{ti}}\circ(\mathcal{A}_{\bar{\mathcal{R}}_{>0}})^{\circ(s+i)}(\Sigma^k\mathbb{F}_2).$$

When we iterate the γ_i operations, the formula is harder to write down explicitly.

Notation 4.2.26. Suppose that $V_\bullet$ is a simplicial $\mathbb{F}_2$-module as a trivial simplicial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra. For distinct classes $[\xi_1],\ldots,[\xi_n]\in\pi_{*,*}(V_\bullet)$, denote by $B(\xi_1,\ldots,\xi_n)$ the cycle in the normalized complex of $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}},V_\bullet)$ that represents the class $[\xi_1]\otimes\cdots\otimes[\xi_n]\in\pi_{*,*}(\Lambda^{n-1}(V_\bullet))\subset\pi_{*,*}(\mathrm{CE}(V_\bullet))\cong\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}},V_\bullet)$, which is obtained by cycle completion via the Jacobi identity in the sense of Remark 4.2.17.

Therefore a homotopy class $[\xi_1]\otimes\cdots\otimes[\xi_l]$ with $l>1$ in $\pi_{s,*}(\Lambda^\bullet(\mathrm{Bar}_\bullet(\mathrm{id},\mathcal{A}_{\bar{\mathcal{R}}_{>0}},\Sigma^k\mathbb{F}_2)))$ is represented by an element $B(\xi_1,\ldots,\xi_l)$ in the summand $(\mathrm{Lie}^{s,\mathrm{ti}})^{\circ(l-1)}\circ(\mathcal{A}_{\bar{\mathcal{R}}_{>0}})^{\circ(s-l+1)}(\Sigma^k\mathbb{F}_2)$ of the total complex of $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}},\mathrm{AR}_\bullet^{>0}(\Sigma^k\mathbb{F}_2))$. Since a representative for the homotopy class $\gamma_j\gamma_i(\xi)$ in the total complex of $\Lambda^\bullet(\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Mod}_{\bar{\mathcal{R}}_{>0}}},\Sigma^k\mathbb{F}_2))$ is given by

$$\gamma_j\gamma_i(\xi) = \sum_{(C,D)\in V(s+i+1,j)}\ \sum_{(A,B)\in V(s,i)} s_C\big(s_A(\xi)\otimes s_B(\xi)\big)\otimes s_D\big(s_A(\xi)\otimes s_B(\xi)\big),$$

a representative for $\gamma_j\gamma_i(\xi)$ in the total complex of $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}},\mathrm{AR}_\bullet^{>0}(\Sigma^k\mathbb{F}_2))$ is given the sum of over all $(A,B)\in V(s,i),(C,D)\in V(s+i+1,j)$ of $B(s_Cs_A(\xi),s_Cs_B(\xi),s_Ds_A(\xi),s_Ds_B(\xi))$, with the three brackets coming from distinct simplicial filtrations.

Theorem 4.2.27. *The Quillen homology*

$$\mathrm{HQ}_{*,*}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}(\Omega^n\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}(\Sigma^{n+k}\mathbb{F}_2))\cong\pi_{s,t}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^s_{\bar{\mathcal{R}}},\Omega^n\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}(\Sigma^{n+k}\mathbb{F}_2))$$

of the $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$*-algebra* $\Omega^n\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}(\Sigma^{n+k}\mathbb{F}_2), 1\le n\le\infty$ *is isomorphic as a bigraded vector space to the exterior algebra on generators* $\gamma_I\bar{Q}_J(x_k)$*, where* $I=(i_1,\ldots,i_m)$ *is* γ*-admissible with* $e(I)\le r$ *and* $i_m\ge 2$*, whereas* $J=(j_1,\ldots,j_r)$ *satisfies* $0\le j_l\le j_{l+1}+1$ *for* $l<r$*,* $0\le j_r<n$ *and if* $j_1=0$ *then either* $r=1$ *or* $i_m=2$*.*

Recall from Proposition 4.1.10 that in the case $n = \infty$, $\Omega^n \mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2)$ is the trivial $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra $\Sigma^k\mathbb{F}_2$.

Before we proceed to prove the theorem, we provide some intuition about the strategy. From the construction of the May-type spectral sequence and the γ_1-Bockstein spectral sequence in Theorem 4.2.6, we see that there is a differential on a class in $\pi_{*,*}(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, L))$ in either spectral sequences if and only if its representative cycle, considered as an element in $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, L)$, admits a face map that evaluates a non-self-bracket to a self-bracket. Remark 4.2.25 and Corollary 4.2.19 indicate that γ_1 is the only operation that arises in $\pi_{*,*}(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, L)) \cong \Lambda\{\gamma_I \bar{Q}_J(x)\}$ with I γ-admissible precisely because self-brackets are zero in $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebras and thus generates all the differentials in the May-type spectral sequence and the γ_1-Bockstein spectral sequence. Hence we expect that $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, L)$ to be a quotient of $\pi_*(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, L))$ (cf. Corollary 4.2.19) by a suitable ideal generated by $\gamma_1(\alpha)$ for all $\alpha \in \pi_{*,*}(\mathrm{AR}_\bullet(L))$, and we use the induced map on homotopy groups of $\varphi : \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, L) \to \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, L)$ from Lemma 4.2.21 to help detect the differentials and permanent cycles.

Proof of Theorem 4.2.27. We focus on the case $L = \Sigma^k\mathbb{F}_2$, since in the cases $n < \infty$ the only difference is an extra condition on the rightmost operation in basis elements, so the same argument applies with no change.

Consider the map

$$\varphi_* : \pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, L) \to \pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, L)$$

from Lemma 4.2.21. Its cokernel consists of all cycles in $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, \Sigma^k\mathbb{F}_2)$ whose preimage is the source of a differential to an element that is in the kernel of φ. Since φ is surjective by Lemma 4.2.21, this is equivalent to finding all classes α that are cycles in $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, \Sigma^k\mathbb{F}_2)$ precisely because the differential ∂' in the normalized complex of $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}}, \Sigma^k\mathbb{F}_2)$ sends α to a linear combination of elements that contain self-brackets or $\bar{Q}_0$. In other words, via the inclusion to $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, \Sigma^k\mathbb{F}_2)$ in Lemma 4.2.23.(3), all elements in the cokernel of ϕ_* support differentials in the May spectral sequence or the γ_1-Bockstein spectral sequence.

We start with the generators of the exterior algebra, cf. Lemma 4.2.18. Let $[\alpha] = \bar{Q}_{j_1}\bar{Q}_{j_2}\cdots\bar{Q}_{j_r}(x_k)$ be a basis element of $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}},\mathrm{AR}_\bullet^{>0}(\Sigma^k\mathbb{F}_2))$, represented by a cycle $\alpha = \bar{Q}_{j_1}|\cdots|\bar{Q}_{j_r}|x_k + \sum_l \bar{Q}_{j'_1}|\cdots|\bar{Q}_{j'_r}|x_k$ in $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}},\Sigma^k\mathbb{F}_2)$. The terms in the summation comes from cycle completion via Behrens' relations in the sense of Remark 4.2.17, with the condition that any term containing $\bar{Q}_0$ is 0. It has preimage $\tilde{\alpha}$ the cycle completion of $\bar{Q}_{j_1}|\cdots|\bar{Q}_{j_r}|x_k$ in $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)$ via Behrens' relations, which is the sum of α and terms $\bar{Q}_{j'_1}|\cdots|\bar{Q}_{j'_r}|x_k$ such that at least one of the $\bar{Q}_{j'_l}, l>1$ is equal to $\bar{Q}_0$. By [BO06, Lemma 3.1], the differential ∂ in the normalized complex of $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}},\Sigma^k\mathbb{F}_2)$ sends $\gamma_i(\alpha), i\geq 2$ to zero because the terms are either zero or cancel out in pairs due to the simplicial identities of face and degeneracy maps. Hence its preimage $\gamma_i(\tilde{\alpha})$ is also a cycle in the normalized complex of $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)$ and hence a permanent cycle in the May spectral sequence. Similarly, for any γ-admissible sequence $I=(i_1,\dots,i_m)$ with $i_m\geq 2$, $\gamma_I(\alpha)$ lifts to a cycle $\gamma_I(\tilde{\alpha})$ in $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)$ and hence a permanent cycle in the May spectral sequence. By naturality of the γ_i operations and Lemma 4.2.23.(3), the class $\gamma_I(\alpha)$ with $i_m\geq 2$ and $\alpha\in\pi_{*,*}(\mathrm{AR}_\bullet(\Sigma^k\mathbb{F}_2))$ is also a permanent cycle.

On the other hand, the differential ∂ sends $\gamma_1(\alpha)$ to $\langle\alpha,\alpha\rangle = 0$ in $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}},\Sigma^k\mathbb{F}_2)$, whereas its preimage $\gamma_1(\tilde{\alpha}) = [s_0\tilde{\alpha},s_1\tilde{\alpha}]$ maps to $[\tilde{\alpha},\tilde{\alpha}] = \bar{Q}_0|\tilde{\alpha}$ under the differential in $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)$, which is in the kernel of φ. In other words, there is a differential in either the May spectral sequence or the γ_1-Bockstein spectral sequence from $\gamma_1(\alpha)\in\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)$ to $\bar{Q}_0\alpha$. Similarly, for any γ-admissible sequence $I=(i_1,\dots,i_m)$ with $i_m\geq 2$, $\gamma_I\gamma_1(\alpha)$ is a cycle in $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}_{>0}},\Sigma^k\mathbb{F}_2)$ because of the self-bracket in $\partial\gamma_I\gamma_1(\alpha) = \gamma_I(\partial(\gamma_1(\alpha)))$ if the simplicial degree of α is $r>1$ and

$$\partial\gamma_I\gamma_1(\alpha) = \partial(\gamma_1(\alpha))\otimes\gamma_1(\alpha)\otimes\gamma_2\gamma_1(\alpha)\otimes\cdots\otimes\gamma_{2^{m-1}}\cdots\gamma_2\gamma_1(\alpha)$$

if $r=1$, cf. [HM16, 3.9.(i)]. On the other hand, its preimage $\gamma_I\gamma_1(\tilde{\alpha})$ is mapped by the total differential in $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)$ to the cycle completion $B(\bar{Q}_0|\tilde{\alpha},\gamma_1(\tilde{\alpha}),\cdots,\gamma_{2^{m-1}}\cdots\gamma_2\gamma_1(\tilde{\alpha}))$ (cf. Notation 4.2.26) if $r=1$, and to $\gamma_I([\tilde{\alpha},\tilde{\alpha}])$ when $r>1$. Note that in $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)\cong\Lambda\{\gamma_I\bar{Q}_J(x_k)\}$ with I γ-admissible and J satisfying certain conditions, we have $[\gamma_I([\tilde{\alpha},\tilde{\alpha}])] = [\gamma_{I'}(\bar{Q}_0|\tilde{\alpha})]$ with $I'=(i_1+1,\dots,i_m+2^{m-1})$. There is a shift in the indexing of the γ operations because by construction the self-brackets appearing in the same bracket term live in distinct filtrations when more γ's are applied, so replacing each self-bracket by a $\bar{Q}_0$ in a cycle will increase the index of the acting γ_i by one. Hence there is a differential in either the May spectral sequence or the γ_1-Bockstein spectral sequence from $\gamma_I\gamma_1(\alpha)$ to $\gamma_{I'}(\bar{Q}_0|\alpha)$, and all the generators $\gamma_I\gamma_1(\alpha)$ of the exterior algebra $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}},\mathrm{AR}_\bullet^{>0}(\Sigma^k\mathbb{F}_2))$ are in the cokernel of φ_*. Again by naturality of the γ_i operations and Lemma 4.2.23.(3), the class $\gamma_I\gamma_1(\alpha)\in\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)$ supports a differential to $\gamma_{I'}(\bar{Q}_0\alpha)$ in the May spectral sequence or the γ_1-Bockstein spectral sequence.

In general, suppose $[\alpha]$ is a basis element of

$$\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)\cong\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}},\mathrm{AR}_\bullet(\Sigma^k\mathbb{F}_2))$$

that is the exterior product of generators $\gamma_{I_1}([\alpha_1]),\dots,\gamma_{I_n}([\alpha_n])$ with each α_i the cycle completion of a basis element $[\alpha_i]\in\pi_{*,*}\mathrm{AR}_\bullet(\Sigma^k\mathbb{F}_2)$. It is represented by a cycle $\alpha=B(\gamma_{I_1}(\alpha_1),\dots,\gamma_{I_n}(\alpha_n))$ in the total complex of $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}},\mathrm{AR}_\bullet(\Sigma^k\mathbb{F}_2))$, cf. Notation 4.2.26, since $d_j(\gamma_{I_l}(\alpha_l))=0$ for all j and l by Proposition 4.2.24. Then $[\alpha]$ supports a differential in the May spectral sequence or the γ_1-Bockstein spectral sequence if and only if at least one of the γ-admissible sequences I_l is of the form $I_l=(i_{l_1},\dots,i_{l_m},1)$. By Corollary 4.2.19, the above covers all classes in the $\mathbb{F}_2$-basis of the E^1-page of $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)$. $\square$

Remark 4.2.28. Note that $\pi_{*,*}\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{Lie}^s_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2\big)$ is the cofree coalgebra on $\Sigma^k\mathbb{F}_2$ over the comonad $|\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{Lie}^s_{\bar{\mathcal{R}}},-\big)|:=\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{Free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}},-)$ on $\mathrm{Mod}_{\mathbb{F}_2}$. The coalgebra structure map is given by

$$\begin{aligned}|\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{Lie}^s_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2\big)|&\xleftarrow{\simeq}|\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{Lie}^s_{\bar{\mathcal{R}}},|\mathrm{Bar}_\bullet\big(\mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}},\mathrm{Lie}^s_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2\big)|\big)|\\&\to|\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{Lie}^s_{\bar{\mathcal{R}}},|\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{Lie}^s_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2\big)|\big)|,\end{aligned}$$

where the last map makes use of the augmentation $\mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}\to\mathrm{id}$, cf. [Bra17, Appendix D]. In particular, $\pi_{*,*}\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{Lie}^s_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2\big)$ records all natural unary operations on a degree k class in the mod 2 Quillen homology of $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebras, and Theorem 4.2.27 gives us a dimension count.

4.3 Application to mod 2 homology of labeled configuration spaces

The rest of the paper is devoted to studying the mod p homology of labeled configuration spaces using the computation of Quillen homology of spectral Lie algebras. The coefficients for homology is $\mathbb{F}_2$ unless otherwise specified.

Let M be a manifold of dimension n and X a spectrum. The configuration space of k points in M labeled by X is the spectrum

$$B_k(M;X) = \Sigma^\infty_+ \mathrm{Conf}_k(M) \underset{\Sigma_k}{\otimes} X^{\otimes k},$$

considered as a weighted spectra of weight k. Here $\mathrm{Conf}_k(M)$ is the space of k-tuples of pairwise distinct points in M. Denote by $s\mathscr{L}$ the monad associated to the free spectral Lie algebra functor $\mathrm{Free}^{s\mathscr{L}}$. The ∞-category of spectral Lie algebras is cotensored in Spaces, and we write $(-)^{M^+}$ for the cotensor with the one-point compactification of M in this category. In [Knu18], Knudsen established the following equivalence using factorization homology, cf. [BHK19, Theorem 5.1].

Theorem 4.3.1. *[Knu18, Section 3.4] Let M be a parallelizable n-manifold and X a spectrum. Consider X as a weighted spectrum of weight one. Then there is an equivalence of weighted spectra*

$$\bigoplus_{k\geq 1} B_k(M;X) \simeq |\, \mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, \mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)^{M^+})\,| \,.$$

The left hand side is weighted by the index k; the weight filtration on the right hand side is given by propagating the weight on X via the free spectral Lie operad functor.

Applying the bar spectral sequence to the bar construction on the right hand side, we obtain the following:

Proposition 4.3.2. *There is a weighted spectral sequence*

$$E^2_{s,t} = \mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{s,t}(H_*(\mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)^{M^+})) \Rightarrow \bigoplus_{k\geq 1} H_{s+t}(B_k(M;X)). \tag{4.7}$$

The $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra structure on the $\mathbb{F}_2$-module

$$H_*(\mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)^{M^+}) \cong \widetilde{H}^*(M^+) \otimes H_*(\mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)) \cong \widetilde{H}^*(M^+) \otimes \mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(H_*(\Sigma^n X))$$

has an explicit description.

Proposition 4.3.3. *[BHK19, Proposition 5.9] Let $\mathfrak{g}$ be a spectral Lie algebra. Then there is a spectral Lie algebra structure on the cotensor $\mathfrak{g}^{M^+}$ in the category of spectra. The weight two structural map factors as*

$$\partial_2(\mathrm{Id}) \otimes (\mathbb{D}(M^+) \otimes \mathfrak{g})^{\otimes 2}_{h\Sigma_2} \to \mathbb{D}(M^+)^{\otimes 2}_{h\Sigma_2} \otimes (\partial_2(\mathrm{Id}) \otimes \mathfrak{g}^{\otimes 2}_{h\Sigma_2}) \xrightarrow{\mathbb{D}(\delta^*)\otimes \xi_*} \mathbb{D}(M^+) \otimes \mathfrak{g},$$

where $\mathbb{D}$ is the Spanier-Whitehead dual and δ the diagonal embedding.

As a result, the shifted Lie bracket on $\widetilde{H}^*(M^+) \otimes H_*(\mathfrak{g})$ is given by

$$[y_1 \otimes x_1, y_2 \otimes x_2] := (y_1 \cup y_2) \otimes [x_1, x_2].$$

On the other hand, the Steenrod operations on $H^*(M^+)$ induces a twisted $\bar{\mathcal{R}}$-module structure in the cotensor.

Proposition 4.3.4. *The operations $\bar{Q}^j$ act on $\widetilde{H}^*(M^+) \otimes H_*(\mathfrak{g})$ by*

$$\bar{Q}^j(y \otimes x) = \sum_i Sq^{i-j}(y) \otimes \bar{Q}^i(x).$$

Proof. Applying the Cartan formula $Q^j(y \otimes x) = \sum_i Q^{j-i}(y) \otimes Q^i(x)$ for the extended Dyer-Lashof operations $Q^j : x \mapsto e_{j-|x|} \otimes x \otimes x$ and the identification $Q^{-i} = Sq^i$ [May70] to the definition of the $\bar{Q}^j$ operations, we have

$$\bar{Q}^j(y \otimes x) = \xi_* \sigma^{-1}(\sum_i Sq^{i-j}(y) \otimes Q^i(x)) = \sum_i Sq^{i-j}(y) \otimes \xi_* \sigma^{-1} Q^i(x) = \sum_i Sq^{i-j}(y) \otimes \bar{Q}^i(x)$$

Here σ^{-1} is the desuspension isomorphism, and ξ is the second structure map of spectral Lieq algebras. $\square$

4.3.1 The universal case

Now we apply Theorem 4.2.27 to the case where M is the Euclidean space. While the homology for $B_k(\mathbb{R}^n;X)$ is well-understood [BMMS88][CLM76][May72], we observe interesting patterns of higher differentials in the associated Knudsen spectral sequence. Furthermore, the computation of the E^2-page in these cases will be useful in deducing the E^2-page for a general M.

Since $\widetilde{H}^*(S^n) = \mathbb{F}_2\{\iota_n\}$ is concentrated in one dimension, the only nonzero Steenrod operation is $Sq^0 = \mathrm{id}$, so the $\bar{\mathcal{R}}$-module structure on $\widetilde{H}^*(S^n) \otimes H_*(\mathfrak{g})$ is given by

$$\bar{Q}^j(\iota_n \otimes x) = \sigma^{-n}\bar{Q}^j(x) = \bar{Q}^j(\sigma^{-n}x), x \in \mathfrak{g}.$$

In the limiting case $M = \mathbb{R}^\infty = \lim_{n\to\infty} \mathbb{R}^n$, we have the stabilization

$$\lim_{n\to\infty} \Omega^n \mathrm{Free}^{s\mathcal{L}}(\Sigma^n X) \simeq X,$$

and the spectral sequence (4.7) becomes

$$E^2_{s,t} = \mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{s,t}(\Sigma^k \mathbb{F}_2) \Rightarrow H_{s+t}(\mathrm{Free}^{\mathbb{E}_\infty}(\mathbb{S}^k)). \tag{4.8}$$

The E^2-page is computed in Theorem 4.2.27. Namely, it is the exterior algebra generated by one class x_k and two types of operations on coalgebras over the comonad $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, -)$

$$\bar{Q}^j : E^2_{h,s,t} \to E^2_{h,s+1,t+j-1}, \quad j \geq t$$

$$\gamma_i : E^2_{h,s,t} \to E^2_{2h+1,s+i,2t-1}, \quad 2 \leq i \leq s$$

under a further splitting of the filtration degree into a sum of homological degree h counting the number of brackets and simplicial degree s counting the number of $\bar{Q}^j$'s.

Comparing with the computation of $H_*(\mathrm{Free}^{\mathbb{E}_\infty}(\mathbb{S}^k))$ [May72][BMMS88], which is the E^∞-page, we can immediately deduce that the E^2-page is much larger. Using sparsity arguments, we can identify higher differentials in low degrees, which allows us to make the following conjecture.

Conjecture 4.3.5. *Each page of the spectral sequence*

$$E^2_{s,t} = \mathrm{HQ}^{\mathrm{Lie}^s_{\mathcal{R}}}_{s,t}(\Sigma^k\mathbb{F}_2) \Rightarrow \pi_{s+t}\mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, \Sigma^k\mathbb{F}_2) \cong H_{s+t}(\mathrm{Free}^{\mathbb{E}_\infty}(\mathbb{S}^k))$$

is an exterior algebra. The higher differentials on the exterior generators of the E^2-page are given as follows:

1. *For an exterior generator $\alpha = \bar{Q}_{j_1}\cdots\bar{Q}_{j_m}(x_k)$ on the E^2-page, we have*

   $$d^{r+1}\gamma_{r+1}(\alpha) = \bar{Q}_r(\alpha)$$

   for $r < m$ and $r \leq j_1 + 1$.

2. *For an exterior generator $\beta = \gamma_{n+1}\bar{Q}_{j_1}\cdots\bar{Q}_{j_m}(x_k)$ on the E^2-page, we have*

   (a) $d^{n+1}(\beta) = \bar{Q}_n\bar{Q}_{j_1}\cdots\bar{Q}_{j_m}(x_k)$,

   (b) $d^{n+1}\gamma_{m+n+1}(\beta) = d^{n+1}(\beta)\otimes\beta$,

   (c) $\gamma_{l-2}d^{n+1}(\beta) = d^{2n+1}\gamma_{n+l+1}(\beta)$ *for $n+1 < l < m$.*

These generate all higher differentials under further applications of γ_i operations in accordance with (2).(b) and (2).(c), as well as the exterior product.

Figure 4-1 is an illustration of the higher differentials in homological Adams grading $(s+t, s)$ for $\beta = \gamma_{n+1}\bar{Q}_{j_1}\cdots\bar{Q}_{j_m}(x_k)$ and $\alpha = \bar{Q}_n\bar{Q}_{j_1}\cdots\bar{Q}_{j_m}(x_k)$ with internal degree b. Set $a = 2b+m+1$. Along the horizontal line $s = m+1$ we have generators $\bar{Q}_1(\alpha),\ldots,\bar{Q}_{n+1}(\alpha)$, each receiving a blue differential via Conjecture 4.3.5.(1). Along the top slope we have, for each i with $n+1 < i < m$, a cyan arrow $d_{2n+1}(\gamma_{n+i}(\beta)) = \gamma_{i+1}(\alpha)$, which correspond to the differentials in Conjecture 4.3.5.(2).(c). Finally we have a gray arrow $d^{n+1}(\gamma_{m+n+1}(\beta)) = \beta\otimes\alpha$, corresponding to Conjecture 4.3.5.(2).(b).

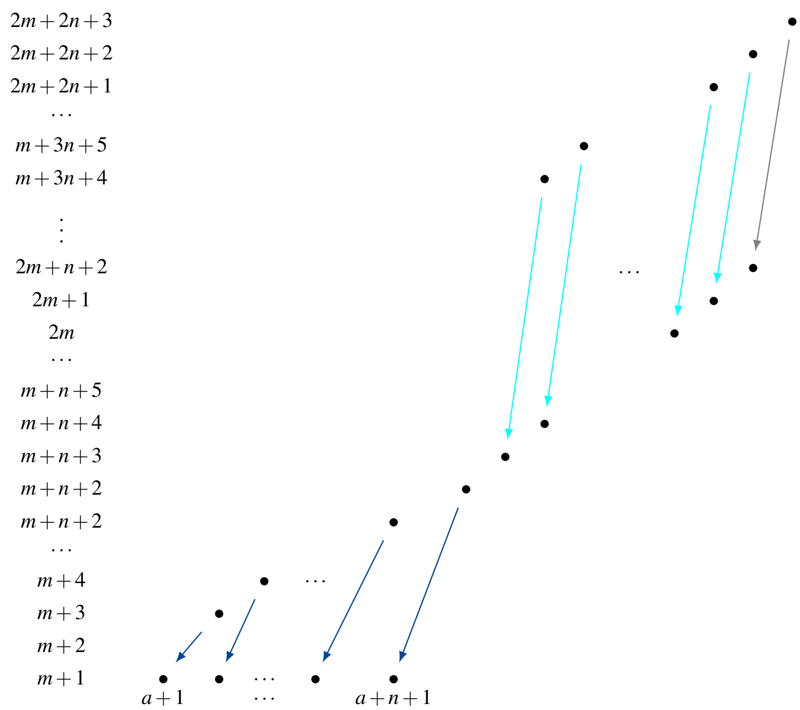


Figure 4-1: Conjectural pattern of universal differentials in the bar spectral sequence.

Remark 4.3.6. The pattern in the universal case is similar to the pattern of universal higher differentials in [Dwy80b, Proposition 2.6] and [Tur98], where divided squares kills off Steenrod operations that are not admissible. Here, the Dyer-Lashof operations $\bar{Q}^j$ on the E^∞-page should be represented by the surviving $\bar{Q}^j$ operations. On the E^2-page, the admissibility condition for $\bar{Q}^j$ allows for more admissible sequences than the Dyer-Lashof algebra. The γ_i operations eliminate the $\bar{Q}^j$ operations that do not satisfy the admissibility condition for Dyer-Lashof operations via higher differentials.

One major difference is that while Steenrod operations can be defined on the spectral sequence filtration-wise in [Dwy80b] and [Tur98], the operations $\bar{Q}^j$ increase filtration by one and hence the classical methods of constructing operations on spectral sequences no longer apply.

In joint work in progress with Andrew Senger, we use a suitable deformation of the comonad associated to the bar construction $|\mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, -)|$ to the ∞-category of Beilinson-connective filtered $\mathbb{F}_2$-modules, which allows us to detect the higher differentials in Conjecture 4.3.5.

Remark 4.3.7. The spectral sequence we study here is analogous to the bar spectral sequence

$$E^2_{s,t} = \pi_s\pi_t \mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{E}^{\mathrm{nu}}_\infty \otimes \mathbb{F}_p, \pi_*(A)) \Rightarrow \pi_{s+t}\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{E}^{\mathrm{nu}}_\infty \otimes \mathbb{F}_p, A)$$

and its dual. The latter was used to identify operations on homotopy groups of spectral partition Lie algebras and mod p TAQ cohomology operations of nonunital $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras in [Zha22], which subsumes unpublished work of Kriz, Basterra and Mandell. The E^2-page of this spectral sequence is the André-Quillen homology of Poly_R-algebras, i.e., graded $\mathbb{F}_2$-modules equipped with Dyer-Lashof operations and a polynomial product that satisfying the Cartan formula. In contrast to Conjecture 4.3.5, this spectral sequence collapses on the E^2-page. Heuristically, the phenomenon here arises from the nonadditivity of the free Lie^s-algebra functor and the order of the factorization $Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}} = Q^{\mathrm{Lie}^s}_{\mathrm{Mod}_{\mathbb{F}_2}} \circ Q^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Lie}^s}$, which results in simplicial homotopy operations. Whereas the Dyer-Lashof operations are additive away from the bottom operations on even degree classes, so the factorization $Q^{\mathrm{Poly}_R}_{\mathrm{Mod}_{\mathbb{F}_p}} = Q^{\mathrm{Mod}_{R_{>0}}}_{\mathrm{Mod}_{\mathbb{F}_p}} \circ Q^{\mathrm{Poly}_R}_{\mathrm{Mod}_{R_{>0}}}$ does not introduce simplicial homotopy operations.

4.3.2 With coefficients

Next, we take up a slightly more complicated case, where $M = \mathbb{R}^n$ with labels in an arbitrary spectrum X. Then $H_*(\mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)^{M^+}) \cong \Omega^n \mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^n H_*(X))$ and the spectral sequence (4.7) becomes

$$E^2_{s,t} = \mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{s,t}(\Omega^n \mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^n H_*(X))) \Rightarrow H_{s+t}(\mathrm{Free}^{\mathbb{E}_n}(X)). \tag{4.9}$$

When $X = \mathbb{S}^k$, the E^2-page $\mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{s,t}(\Omega^n \mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2))$ is computed in Theorem 4.2.27.

Write $H_*(X) \cong \bigoplus_{k,l} \mathbb{F}_2\{x_{k,l}\}$, where $\{x_{k,l}\}_l$ is an $\mathbb{F}_2$-basis of $H_k(X)$ for each k. Then

$$\begin{aligned} \mathfrak{g} = H_*(\Omega^n \mathrm{Free}^{s\mathscr{L}}(\Sigma^n H_*(X))) &\cong \mathbb{F}_2\{\iota_n\} \otimes H_*(\mathrm{Free}^{s\mathscr{L}}(\Sigma^n H_*(X))) \\ &\cong \mathbb{F}_2\{\iota_n\} \otimes \Big(\bigoplus_{w \in W} \mathbb{F}_2\{\bar{Q}^J w, J \in \bar{\mathcal{R}}(d(w))\} \Big) \end{aligned}$$

by [AC20, Proposition 7.3]. Here $\bar{\mathcal{R}}(n)$ is the quotient of $\bar{\mathcal{R}}$ by the relations $\bar{Q}^{j_1} \cdots \bar{Q}^{j_k} = 0$

if $j_1 < j_2 + \cdots j_k + n$, and W is the set of Lyndon words on the set of letters $\{\sigma^n x_{k,l}\}_{k,l}$, which is a basis for the free $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra on generators $\{\sigma^n x_{k,l}\}_{k,l}$.

We define the *degree* of a word $w \in W$ to be $d(w) = 1 + \sum_{k,l} m_{k,l}(w)(n+k-1)$, where $m_{k,l}(w)$ counts the number of times the letter $\sigma^n x_{k,l}$ appears in w. Set

$$\mathfrak{g}_w = \mathbb{F}_2\{\iota_n\} \otimes \mathbb{F}_2\{\bar{Q}^J w, J \in \bar{\mathcal{R}}(n+|w|)\}.$$

Then $\mathfrak{g} \simeq \bigoplus_{w\in W} \mathfrak{g}_w$ with trivial brackets. Note that this splitting is induced by an equivalence of $s\mathscr{L}$-algebras over $\mathbb{F}_2$-module spectra

$$\begin{aligned}\left(\mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)\right)^{(\mathbb{R}^n)^+} \otimes \mathbb{F}_2 \simeq & \mathbb{D}(S^n) \otimes \mathrm{Free}^{s\mathscr{L}}(\Sigma^n X \otimes \mathbb{F}_2) \\ \simeq & \mathbb{D}(S^n) \otimes \mathrm{Free}^{s\mathscr{L}}(\bigvee_{x_{k,l}} \Sigma^{n+k}\mathbb{F}_2) \\ \simeq & \bigvee_{w\in W} \left(\mathrm{Free}^{s\mathscr{L}}(\Sigma^{d(w)}\mathbb{F}_2)\right)^{(\mathbb{R}^n)^+},\end{aligned}$$

where the last step makes use of Corollary 5.13 in [AB21]. The equivalence above would only be that of $\mathbb{F}_2$-module spectra if we did not kill the brackets by cotensoring with $(\mathbb{R}^n)^+$. Therefore we deduce the following:

Proposition 4.3.8. *The spectral sequence* $E^2_{s,t} = \mathrm{HQ}_{s,t}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}(\Omega^n \mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^n H_*(X))) \Rightarrow H_{s+t}(\mathrm{Free}^{\mathbb{E}_n}(X))$ *splits as*

$$E^2_{s,t} \cong \bigoplus_{w\in W} \mathrm{HQ}_{s,t}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}(\mathfrak{g}_w) \Rightarrow \bigoplus_{w\in W} \pi_{s+t}\mathrm{Bar}_\bullet(\mathrm{id}, s\mathscr{L}, \Omega^n \mathrm{Free}^{s\mathscr{L}}(\Sigma^n\Sigma^{d(w)-n}\mathbb{F}_2)).$$

Remark 4.3.9. The canonical map of spectral Lie algebras

$$\Omega^n \mathrm{Free}^{s\mathscr{L}}(\Sigma^n \mathbb{S}^k) \to \Omega^\infty \mathrm{Free}^{s\mathscr{L}}(\Sigma^\infty \mathbb{S}^k)$$

via stabilization induces an embedding of the E^2-pages

$$\mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}(\Omega^n \mathrm{Free}^{\mathrm{Mod}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^{n+k}\mathbb{F}_2)) \to \mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}(\Sigma^k\mathbb{F}_2)$$

by Proposition 4.1.10 and Theorem 4.2.27. We expect that the higher differentials in the target (Conjecture 4.3.5) pull back to higher differentials in the source. Indeed, combinatorially this will yield the computation of the free $\mathbb{E}_n$-algebra on a single generator. If $H_*(X)$ has multiple generators, then the splitting of the spectral sequence above via Lyndon words corresponds precisely the Browder bracket on the free $\mathbb{E}_n$-algebra on those generators, cf. [CLM76, III].

4.4 Upper bounds and low weight computations

For a general parallelizable manifold M of dimension n, the $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra

$$\mathfrak{g} = \widetilde{H}^*(M^+) \otimes \mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^n H_*(X))$$

has non trivial Lie^s-brackets and the precise image of the comparison map φ_* in Lemma 4.2.21 becomes much harder to pin down. Nonetheless, Theorem 4.2.6 and Corollary 4.2.9 allow us to obtain a formula for an upper bound of $\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^s_{\bar{\mathcal{R}}}, \mathfrak{g})$ by

$$\pi_{*,*}\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}, \tilde{\mathfrak{g}}) \cong \pi_{*,*}(\mathrm{CE}(\mathrm{AR}_\bullet(\tilde{\mathfrak{g}})))$$

that is an equivalence in weight less than four. Here $\tilde{\mathfrak{g}} = \widetilde{H}^*(M^+) \otimes \mathrm{Free}^{\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}}(H_*(X))$ is the associated $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra, where $\widetilde{H}^*(M^+)$ is equipped with the $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket coming from the associated $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra of the Lie^s-algebra $H^*(M^+)$ with its usual cup product, cf. Construction 4.2.4. In particular, it follows from Corollary 4.2.7 that in weight less than four, the two homotopy groups are isomorphic.

4.4.1 General upper bounds

We will see that $\pi_{*,*}(\mathrm{CE}(\mathrm{AR}_\bullet(\tilde{\mathfrak{g}})))$ admits a description in terms of the $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra homology of $\tilde{\mathfrak{g}}$. The key observation is that for $\tilde{\mathfrak{g}} = \widetilde{H}^*(M^+) \otimes \mathrm{Free}^{\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}}(H_*(X))$, $\mathrm{AR}_\bullet(\tilde{\mathfrak{g}})$ has trivial $\mathrm{Lie}^{s,\mathrm{ti}}$-structure away from simplicial degree 0 and its degeneracies, cf. Construction 4.2.8, and the $\mathrm{Lie}^{s,\mathrm{ti}}$-bracket on $\tilde{\mathfrak{g}}$ vanishes on elements that involve $\bar{Q}^i$ operations.

Definition 4.4.1. For a $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra $\mathfrak{g}$, we say that its $\mathrm{Lie}^{s,\mathrm{ti}}$-structure is *supported entirely* by a sub-$\mathrm{Lie}^{s,\mathrm{ti}}$-algebra $\mathfrak{g}'$ if the $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra $\mathfrak{g}$ is isomorphic to the product $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra $N \oplus \mathfrak{g}'$, where the $\mathrm{Lie}^{s,\mathrm{ti}}$ bracket vanishes on the complement $N \subset \mathfrak{g}$.

Lemma 4.4.2. *Let* $\tilde{\mathfrak{g}} = L \otimes \mathrm{Free}^{\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(V)$ *be a* $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$*-algebra, where* L *is equipped with a not necessarily unital* $\mathrm{Lie}^{s,\mathrm{ti}}$*-bracket and the* $\mathrm{Lie}^{s,\mathrm{ti}}$*-structure on* $\tilde{\mathfrak{g}}$ *is the usual one on the tensor product. Then*

$$\pi_{*,*}(\mathrm{CE}(\mathrm{AR}_\bullet(\tilde{\mathfrak{g}}))) \cong \Lambda\{\gamma_I(\alpha), \alpha \in A\} \otimes H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}}),$$

where $\alpha \in A$ *is an element of an* $\mathbb{F}_2$*-basis for* $\pi_{\geq 1,*}(\mathrm{AR}_\bullet(\tilde{\mathfrak{g}}))$ *with simplicial degree* $s(\alpha)$*, and* I *is* γ*-admissible with* $e(I) \leq s(\alpha)$*.*

Proof. Since brackets of operations are zero, the $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra $\tilde{\mathfrak{g}}$ is supported entirely by the sub-$\mathrm{Lie}^{s,\mathrm{ti}}$-algebra $\mathfrak{g}'_0 = L \otimes \mathrm{Free}^{\mathrm{Lie}^{s,\mathrm{ti}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(V)$. Furthermore, for all $m \geq 1$, the $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra $\mathrm{AR}_m(\tilde{\mathfrak{g}})$ is supported entirely by the degeneracies coming from $\mathfrak{g}'_0$ by Construction 4.2.8. Hence each simplicial level $\mathrm{AR}_m(\tilde{\mathfrak{g}})$ is isomorphic to the product $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra $T_m \oplus \mathfrak{g}'_m$, where $\mathfrak{g}'_m$ is the sub-$\mathrm{Lie}^{s,\mathrm{ti}}_{\mathbb{F}_2}$-algebra consisting of degeneracies of $\mathfrak{g}'_0$ and T_m a trivial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra. Since the splittings respect the simplicial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra structure of $\mathrm{AR}_\bullet(\tilde{\mathfrak{g}})$, we deduce that $\mathrm{AR}_\bullet(\tilde{\mathfrak{g}}) \cong T_\bullet \oplus \mathfrak{g}'_\bullet$ as simplicial $\mathrm{Lie}^{s,\mathrm{ti}}_{\mathbb{F}_2}$-algebras. This induces a splitting of chain complexes

$$\mathrm{CE}(\mathrm{AR}_\bullet(\tilde{\mathfrak{g}})) \cong \mathrm{CE}(T_\bullet) \otimes \mathrm{CE}(\mathfrak{g}'_\bullet),$$

where $T_\bullet$ is a trivial simplicial $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra and $\mathfrak{g}'_\bullet$ the constant simplicial object on $\mathfrak{g}'_0$. The lemma then follows from Theorem 4.2.14, noting that $H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}}) \cong H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(T_0) \otimes H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\mathfrak{g}'_0)$. □

It remains to compute $\pi_{*,*}(\mathrm{AR}_\bullet(\tilde{\mathfrak{g}}))$ for $\tilde{\mathfrak{g}} = \widetilde{H}^*(M^+) \otimes \mathrm{Free}^{\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}}(H_*(X))$. Since $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are isomorphic as $\bar{\mathcal{R}}$-modules (cf. Remark 4.2.2), we will not distinguish the two. Recall from Proposition 4.3.4 that the $\bar{\mathcal{R}}$-module structure on $\mathfrak{g}$ is twisted by the Steenrod operations in the sense that

$$\bar{Q}^j(y \otimes \alpha) = \sum_{0 \le s \le n} Sq^{j+s}(y) \otimes \bar{Q}^s(\alpha).$$

Notation 4.4.3. Let $H \cup \{z\}$ be an $\mathbb{F}_2$-basis of the cohomology ring $H^*(M^+)$, where z corresponds to the added point in the one-point compactification and H is a basis for $\widetilde{H}^*(M^+)$. For $y \in H$, denote by $|y|$ the cohomological degree of y.

Let $\widetilde{B} = \{x_a\}_a$ be a totally ordered basis for $V = H_*(X)$ and $B = \{\sigma^n x_a\}_a$ with the induced ordering, where n is the dimension of M. Denote by W the set of basic products on the set B. Then

$$\mathfrak{g} = \widetilde{H}^*(M^+) \otimes H_*(\mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)) \cong \bigoplus_{w \in W, y \in H} \mathbb{F}_2\{y\} \otimes \mathbb{F}_2\{\bar{Q}^J w, J \in \bar{\mathcal{R}}(|w|)\}.$$

Proposition 4.4.4. *The bigraded homotopy group* $\pi_{*,*}(\mathrm{AR}_\bullet(\tilde{\mathfrak{g}})) = \pi_{*,*}(\mathrm{AR}_\bullet(\mathfrak{g}))$ *is isomorphic to* $\pi_{*,*}(\mathrm{AR}_\bullet(\mathfrak{g}_{\mathrm{triv}}))$*, where the the untwisted* $\bar{\mathcal{R}}$*-module* $\mathfrak{g}_{\mathrm{triv}}$ *has the same underlying* $\mathbb{F}_2$*-module as* $\mathfrak{g}$ *and the* $\bar{\mathcal{R}}$*-module structure is given by* $\bar{Q}^j(y \otimes x) = y \otimes \bar{Q}^j(x)$ *for all* j*.*

Proof. We make use of a spectral sequence to filter away the twisting by the action of the Steenrod operations. We abuse notation here and denote again by $\mathrm{AR}_\bullet(\mathfrak{g})$ the associated chain complex of $\mathrm{AR}_\bullet(\mathfrak{g})$. Filter $\mathfrak{g}$ in terms of decreasing cohomological degree of $\widetilde{H}^*(M^+)$, so we have

$$F_{-p}(\mathfrak{g}) = \widetilde{H}^{\ge p}(M^+) \otimes \mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(V) \cong \bigoplus_{w \in W, y \in H, |y| \ge p} \mathbb{F}_2\{y \otimes \bar{Q}^J(w), J \in \bar{\mathcal{R}}(|w|)\}$$

with associated graded pieces given by

$$G_{-p}(\mathfrak{g}) = F_{-p}(\mathfrak{g})/F_{-p-1}(\mathfrak{g}) \cong \bigoplus_{w \in W, y \in H, |y| = p} \mathbb{F}_2\{y \otimes \bar{Q}^J(w), J \in \bar{\mathcal{R}}(|w|)\}.$$

Since action by Steenrod operations does not decrease cohomological degree, the induced filtration

$$F_{-p}(\mathrm{AR}_\bullet(\mathfrak{g})) := \mathrm{AR}_\bullet(F_{-p}(\mathfrak{g}))$$

makes $\mathrm{AR}_\bullet(\mathfrak{g})$ a filtered chain complex. The associated graded pieces are

$$G_{-p}(\mathrm{AR}_\bullet(\mathfrak{g})) = \mathrm{AR}_\bullet(G_{-p}(\mathfrak{g})) = \bigoplus_{w\in W, y\in H, |y|=p} \mathrm{AR}_\bullet(\mathbb{F}_2\{y\otimes \bar{Q}^J(w), J\in \bar{\mathcal{R}}(|w|)\})$$

and the induced differential preserves direct summands.

Using the case $M=\mathbb{R}^n$ in Proposition 4.3.8, we deduce that

$$\begin{aligned} E^1_{-p,q} = H_{-p+q}(G_p(\mathrm{AR}_\bullet(\mathfrak{g}))) &\cong \bigoplus_{w\in W, y\in H, |y|=p} \pi_*\Big(\mathrm{AR}_\bullet\big(\mathbb{F}_2\{y\otimes \bar{Q}^J(w), J\in \bar{\mathcal{R}}(|w|)\}\big)\Big) \\ &\cong \bigoplus_{w\in W, y\in H, |y|=p} \mathbb{F}_2\{\bar{Q}^{j_1}\cdots \bar{Q}^{j_m}(y\otimes w), (j_1,\ldots,j_m)\in \bar{\mathcal{R}}(p,|w|)\}, \end{aligned}$$

where $\bar{\mathcal{R}}(p,|w|)$ is the set of sequences $(j_1,\ldots,j_m)$ such that

1. $j_l \le 2j_{l+1}$ for $1\le l<m$ and $|w|-p\le j_m<|w|$;

2. If $m\ge 2$ then $j_l \ge j_{l+1}+\cdots+j_m+|w|-(m-l)$ for $2\le l\le m-1$ and $j_1 > j_2+\cdots+j_m+|w|-(m-1)$.

We claim that every class on the E^1-page survives to a class on the E^∞-page by induction along decreasing cohomological degree on $\widetilde{H}^*(M^+)$.

For $y\in \widetilde{H}^n(M^+)\in F_{-n}(\mathfrak{g})$ a top cohomology class, there are no nonzero Steenrod action on y other than Sq^0, so the differential on β in $\mathrm{AR}_\bullet(\mathfrak{g})$ is the same as the differential in $G_{-n}(\mathrm{AR}_\bullet(\mathfrak{g}))$, i.e. β survives to a nontrivial cycle on the E^∞-page.

Suppose that in $F_{-p-1}(\mathrm{AR}_\bullet(\mathfrak{g}))=\mathrm{AR}_\bullet(F_{-p-1}(\mathfrak{g}))$, any basis element $\beta'=\bar{Q}^{j'_1}\cdots\bar{Q}^{j'_m}(y'\otimes w')$ of the E^1-page is a permanent cycle and they span all permanent cycles in $F_{-p-1}(\mathrm{AR}_\bullet(\mathfrak{g}))$. Let $[\beta]=\bar{Q}^{j_1}\cdots\bar{Q}^{j_m}(y\otimes w)$ be a basis element on the E^1-page, with $y\in\widetilde{H}^p(M^+)$. A cycle representing this class in $\mathrm{AR}_\bullet(G_{-p}(\mathfrak{g}))$ is a finite sum

$$\beta = \bar{Q}^{j_1}|\cdots|\bar{Q}^{j_m}|(y\otimes w)+\sum_l \bar{Q}^{l_1}|\cdots|\bar{Q}^{l_m}|(y\otimes w)$$

obtained by cycle completion via Behrens' relations in the sense of Remark 4.2.17. Note that $l_m \leq j_m < |w|$ for all l. Let d_m be the rightmost face map. Then in $\mathrm{AR}_\bullet(\mathfrak{g})$

$$\begin{aligned}
\partial\beta &= \partial\Big(\bar{Q}^{j_1}|\cdots|\bar{Q}^{j_m}|(y\otimes w)+\sum_l \bar{Q}^{l_1}|\cdots|\bar{Q}^{l_m}|(y\otimes w)\Big)\\
&= 0 + d_m\Big(\bar{Q}^{j_1}|\cdots|\bar{Q}^{j_m}|(y\otimes w)+\sum_l \bar{Q}^{l_1}|\cdots|\bar{Q}^{l_m}|(y\otimes w)\Big)\\
&= \sum_{s\geq 0}\bar{Q}^{j_1}|\cdots|\bar{Q}^{j_{m-1}}|Sq^s(y)\otimes\bar{Q}^{j_m+s}(w)+\sum_l\sum_{s\geq 0}\bar{Q}^{l_1}|\cdots|Q^{l_{m-1}}|Sq^s(y)\otimes\bar{Q}^{l_m+s}(w).
\end{aligned}$$

Note that the sum of these $\theta_l = \bar{Q}^{l_1}|\cdots|Q^{l_{m-1}}|Sq^s(y)\otimes\bar{Q}^{l_m+s}(w)$ or $Q^{j_1}|\cdots|\bar{Q}^{j_{m-1}}|Sq^s(y)\otimes\bar{Q}^{j_m+s}(w)$ over $s\geq 0$ is a boundary in $\mathrm{AR}_\bullet(\mathfrak{g})$: If $l_m+s<|w|$ then $\theta_l=0$. If $l_m+s\geq|w|$ or $j_m+s\geq|w|$, then $s\geq 1$, since $l_m\leq j_m<|w|$, so $\theta_l\in F_{-p-1}(\mathrm{AR}_\bullet(\mathfrak{g}))$. By the inductive hypothesis, the sum of such θ_l is not a nonzero cycle on the E^∞-page and thus the boundary of a finite sum of classes in $F_{-p-1}(\mathrm{AR}_\bullet(\mathfrak{g}))$ of the form $\bar{Q}^{j'_1}|\cdots|\bar{Q}^{j'_m}|(y'\otimes w')$ with $|y'|\geq p+s>p$. Denote by ξ this finite sum, so $\partial(\beta+\xi)=0$ in $\mathrm{AR}_\bullet(\mathfrak{g})$. Note that ξ is not a boundary because it is maximally nondegenerate and $\xi\neq\beta$ since β is not in $F_{-p-1}(\mathrm{AR}_\bullet(\mathfrak{g}))$. Hence $\beta+\xi$ is a cycle in $\mathrm{AR}_\bullet(\mathfrak{g})$ corresponding to the basis element $\beta=\bar{Q}^{j_1}\cdots\bar{Q}^{j_m}(y\otimes w)$ on the E^1-page. Therefore the dimension of the E^1-page is at most that of the E^∞-page, so no differential can happen in the spectral sequence. □

Combing Lemma 4.4.2, Proposition 4.4.4 and Corollary 4.2.7, we deduce the following general upper bound and low weight computation of the E^2-page of the Knudsen spectral sequence.

Theorem 4.4.5. *Let M be a parallelizable manifold of dimension n and X any spectrum. Let $\mathfrak{g}$ denote the $\mathrm{Lie}^s_{\bar{\mathcal{R}}}$-algebra $\widetilde{H}^*(M^+)\otimes\mathrm{Free}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^n H_*(X))$ with $\mathbb{F}_2$-basis B, and $\tilde{\mathfrak{g}}$ the associated $\mathrm{Lie}^{s,\mathrm{ti}}_{\bar{\mathcal{R}}}$-algebra. An upper bound for the E^2-page of the weighted spectral sequence*

$$E^2_{s,t}=\mathrm{HQ}^{\mathrm{Lie}^s_{\bar{\mathcal{R}}}}_{s,t}(\mathfrak{g})\Rightarrow\bigoplus_{k\geq 1}H_{s+t}(B_k(M;X)) \tag{4.10}$$

is given by

$$\pi_{*,*}(\mathrm{CE}(\mathrm{AR}_\bullet(\tilde{\mathfrak{g}})))\cong\Lambda\{\gamma_I\bar{Q}_J(y\otimes w), y\otimes w\in H\otimes B\}\otimes H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}}),$$

where $\gamma_I \bar{Q}_J(y \otimes w)$ satisfies the conditions that

1. $J = (j_1, \dots, j_m)$ *with* $m \geq 1$, $0 \leq j_l \leq j_{l+1} + 1$ *for* $1 \leq l < m$, *and* $0 \leq j_m < |y|$

2. I *is* γ*-admissible with* $e(I) \leq m$.

Furthermore, in weight less than four equality is achieved.

4.4.2 Low weight computations

Theorem 4.4.5 allows us to deduce the degeneration of the spectral sequence at weight two and three using sparsity arguments. Denote by $\mathrm{wt}_k(M)$ the weight k part of a weighted (bi)graded $\mathbb{F}_2$-module M and set $E^r(k) = \mathrm{wt}_k(E^r)$.

Corollary 4.4.6. *Let* $\mathfrak{g}, \tilde{\mathfrak{g}}$ *be the same as in Theorem 4.4.5 and* B, H *bases given in Notation 4.4.3. The weight two part of the spectral sequence (4.10)*

$$E^2_{s,t}(2) = \mathrm{wt}_2(\mathrm{HQ}^{\mathrm{Lie}^s_{\mathcal{R}}}_{s,t}(\mathfrak{g})) \Rightarrow H_{s+t}(B_2(M;X))$$

collapses on the E^2*-page, and hence*

$$E^\infty(2) \cong E^2(2) \cong \mathrm{wt}_2(H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}})) \oplus \bigoplus_{x \in B, y \in H} \{\bar{Q}_j(y \otimes x), 0 \leq j < |y|\}.$$

Proof. Since classes in the tensor factor

$$A = \Lambda\{\gamma_I(\bar{Q}_J(y \otimes w)), y \otimes w \in H \otimes B\}$$

of Theorem 4.4.5 have weight at least two, classes of weight two lie in exactly one of the two tensor components A and $H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}})$. The weight two classes in A are of the form $\bar{Q}_j(y \otimes w)$ where w has weight one, i.e. w is an element of the $\mathbb{F}_2$-basis B of $V = H_*(X)$, cf. Notation 4.4.3. The weight two classes in $H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}})$ are of the form $y \otimes \langle x_a, x_b \rangle$ and $(y \otimes x_a) \otimes (y' \otimes x_b)$. Hence the weight two part of the spectral sequence has E^2-page concentrated in simplicial degrees 0, 1 and thus cannot admit higher differentials. □

In particular, this demonstrates that for a parallelizable M, the $\mathbb{F}_2$-module $H_*(B_2(M;X))$ depends on and only on the cohomology ring $H^*(M^+)$ when $H_*(X)$ has at least two generators.

Remark 4.4.7. This is in contrast to the case where $X = \mathbb{S}^r$ has only one generator in its homology: Bödigheimer-Cohen-Taylor showed that for any n-manifold M,

$$\bigoplus_{k\geq 1} H_*(B_k(M;\mathbb{S}^r)) \cong \bigotimes_{i=0}^{n} H_*(\Omega^{n-i}S^{n+r})^{\otimes \dim H_i(M)}$$

depends only on $H^*(M)$ as an $\mathbb{F}_2$-module [BCT89].

There is a clear bijection from the weight 2 part of their decomposition to the basis above: let x_k denote the generator of the free $\mathbb{E}_n$-algebra $H_*(\Omega^n\Sigma^n S^k)$. For y a basis element of $\widetilde{H}^i(M^+) \cong H_i(M)$, the bijection sends $\bar{Q}_j(y\otimes x_{n+r})$ to the tensor with $Q_j(x_{r+i})$ in the tensor factor $H_*(\Omega^{n-i}S^{n+r})$ corresponding to y and 1 in all other tensor factors. The $\mathrm{Lie}^{s,\mathrm{ti}}$-algebra $\mathrm{Lie}^{s,\mathrm{ti}}\mathfrak{g}$ is trivial, so $\mathrm{wt}_2(H^{\mathrm{Lie}^{s,\mathrm{ti}}}(\mathrm{Lie}^{s,\mathrm{ti}}\mathfrak{g})) \cong \{yy'\}$ where y, y' ranges over distinct basis of $\widetilde{H}^i(M^+)$ and the bijection sends yy' to the tensor with factors y, y' and 1 in all other slots.

On the other hand, the homology of $\mathrm{Conf}_2(M)$, the space of ordered configurations of two points in M, also depends only on the cup product structure of $H^*(M)$ as discussed in [Pet20, Section 1.1].

Corollary 4.4.8. *If in addition M is closed, then the weight three part of the spectral sequence (4.10) collapses on the E^2-page, and a basis for $H_*(B_3(M;X))$ is given by*

$$\begin{aligned} E^\infty(3) \cong E^2(3) \cong & \bigoplus_{x,x'\in B, y,y'\in H} \mathbb{F}_2\{(\bar{Q}_j(y\otimes x))\otimes(y'\otimes x'), 0\leq j<|y|\} \\ & \oplus \mathrm{wt}_3(H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}})), \end{aligned}$$

Proof. Let d denote the generator for $\widetilde{H}^0(M^+) \cong H^0(M)$. Then any element that is a sum of $y\otimes\langle\langle x_1,x_2\rangle,x_3\rangle \in H\otimes B$ is killed by a sum of $(y\otimes\langle x_1,x_2\rangle)\otimes(d\otimes x_3)$. Since classes in A have weights positive powers of two, weight three classes on the E^2-page either live in

$\mathrm{wt}_3(H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}}))$ with simplicial degree one or two, or have the form

$$(\bar{Q}^j(y \otimes x)) \otimes (y' \otimes x') \in \mathrm{wt}_2(A) \otimes \mathrm{wt}_1(H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}}))$$

with simplicial degree two. Hence $E^2(3)$ is concentrated in simplicial degree 1 and 2, so there cannot be any higher differentials. $\square$

At weight four part we can no longer deduce that the spectral sequence (4.10) collapses on the E^2-page using sparsity arguments. An upper bound for the bigraded $\mathbb{F}_2$-module $E^2(4)$ is given by the weight four part of $A \otimes H^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}})$, which consists of:

1. $\bar{Q}_i(y \otimes \langle x, x' \rangle)$ in simplicial degree one,
2. $\bar{Q}_i\bar{Q}_j(y \otimes x)$ and $\bar{Q}_i(y \otimes x) \otimes (y' \otimes \langle x_1, x_2 \rangle)$ in simplicial degree two,
3. $\bar{Q}_i(y \otimes x) \otimes \bar{Q}_j(y' \otimes x')$ and $\bar{Q}_i(y \otimes x) \otimes (y_1 \otimes x_1) \otimes (y_2 \otimes x_2)$ in simplicial degree three,
4. Weight four part of $H^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}})$.

There could well be a d^2-differential from degree considerations.

We close this section by a few example computations: the closed torus, the punctured genus g surfaces with $g \geq 1$ and the (punctured) real projective space $\mathbb{RP}^3$.

4.4.3 Example computations: closed torus and punctured genus g surfaces

Let $\Sigma_{g,1}$ be a once-punctured surface of genus $g \geq 1$ and Σ_1 the closed torus. Let $\widetilde{B} = \{x_i\}_i$ be a totally ordered basis for $H_*(X)$ and $B = \{\sigma^2 x_i\}_i$ with the induced ordering. Then

$$\widetilde{H}^*(\Sigma^+_{g,1}) = \begin{cases} \mathbb{F}_2\{a_i \oplus b_i, i = 1, \dots, g\} & * = 1 \\ \mathbb{F}_2\{c\} & * = 2 \\ 0 & \text{otherwise} \end{cases}$$

with nonzero cup products $a_i \cup b_i = c$ for all i and no nontrivial Steenrod operations.

For the closed surface Σ_1, we further have a class in $\widetilde{H}^2(\Sigma^+_1) \cong H^2(\Sigma_1) = \mathbb{F}_2\{c\}$, with nonzero cup products $a \cup b = c$ and $d \cup y = y$ for all $y \in H^*(\Sigma_1)$.

Weight two

For $M = \Sigma_{g,1}$, the weight two classes supporting nonzero CE differentials are $\partial(a_i \otimes x_1, b_i \otimes x_2) = c \otimes \langle x_1, x_2 \rangle$ for $x_1 \neq x_2 \in B$, since these are only one nonzero cup products. Denote by H^1 the set of generators $\{a_i, b_i, i = 1, \ldots, g\}$ for $\widetilde{H}^1(\Sigma_{g,1}^+)$. Impose a total ordering on $H^1 \cup \{c, d\}$. By Corollary 4.4.6, a basis for $H_*(B_2(\Sigma_{g,1}; X))$ is given by

$$
\begin{aligned}
E^\infty(2) = E^2(2) \cong & \bigoplus_{x \in B} \mathbb{F}_2\{\bar{Q}_0(y \otimes x), y \in H^1; \bar{Q}_0(c \otimes x), \bar{Q}_1(c \otimes x)\} \\
& \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{y \otimes \langle x_1, x_2 \rangle, (y \otimes x_1) \otimes (y \otimes x_2), y \in H^1; (c \otimes x_1) \otimes (c \otimes x_2)\} \\
& \oplus \bigoplus_{x_1, x_2 \in B} \mathbb{F}_2\{(y \otimes x_1) \otimes (c \otimes x_2), y \in H^1\} \oplus \bigoplus_{x \in B} \mathbb{F}_2\{(y \otimes x) \otimes (y' \otimes x), y < y' \in H^1 \cup \{c\}\} \\
& \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y \otimes x_1) \otimes (y' \otimes x_2) + (a_1 \otimes x_1) \otimes (b_1 \otimes x_2), y \neq y' \in H^1, (y, y') \neq (a_i, b_i)\}.
\end{aligned}
$$

For $M = \Sigma_1$, the weight two classes supporting CE differentials are

$$
\delta((a \otimes x_2) \otimes (b \otimes x_2)) = c \otimes \langle x_1, x_2 \rangle \text{ and } \delta((d \otimes x_1) \otimes (y \otimes x_2)) = y \otimes \langle x_1, x_2 \rangle
$$

for $x_1 \neq x_2 \in B$ and $y \in \widetilde{H}^*(\Sigma_1^+)$. By Corollary 4.4.6, a basis for $H_*(B_2(\Sigma_1; X))$ is given by

$$
\begin{aligned}
E^\infty(2) = E^2(2) \cong & \bigoplus_{x \in B} \mathbb{F}_2\{\bar{Q}_0(y \otimes x), y \in H^1; \bar{Q}_0(c \otimes x), \bar{Q}_1(c \otimes x)\} \\
& \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y \otimes x_1) \otimes (y \otimes x_2), y \in H^1; (z \otimes x_1) \otimes (z \otimes x_2)\} \\
& \oplus \bigoplus_{x_1 \neq x_2 \in B} \mathbb{F}_2\{(y \otimes x_1) \otimes (z \otimes x_2), y \in H^1\} \\
& \oplus \bigoplus_{x \in B} \mathbb{F}_2\{(y \otimes x) \otimes (y' \otimes x), \{y < y'\} \in \{a, b, c, d\}\} \\
& \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y \otimes x_1) \otimes (y' \otimes x_2), y, y' \in H^1, \{y, y'\} \neq \{a, b\}\} \\
& \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y \otimes x_1) \otimes (y' \otimes x_2) + (d \otimes x_1) \otimes (c \otimes x_2), \{y, y'\} = \{a, b\} \text{ or } (y, y') = (c, d)\}.
\end{aligned}
$$

Example 4.4.9. When $X = \mathbb{S}^k$ with $k \geq 1$, we have $B = \{x = \sigma_2 \iota_k\}$, so $H_*(B_2(\Sigma_1, S^k))$ has

$\mathbb{F}_2$-basis

$$\{\bar{Q}_0(a\otimes x),\bar{Q}_0(b\otimes x),\bar{Q}_0(c\otimes x),\bar{Q}_1(c\otimes x);(y\otimes x)\otimes(y'\otimes x),\{y<y'\}\subset\{a,b,c,d\}\}.$$

The weight two part of Bödigheimer-Cohen-Taylor's decomposition [BCT89]

$$\bigoplus_{k\geq 1} H_*(B_k(\Sigma_1;\mathbb{S}^k))\cong\bigotimes_{i=0}^{n} H_*(\Omega^{2-i}S^{2+k})^{\otimes\ \dim H_i(M)}\cong H_*(\Omega^2\Sigma^2S^k)\otimes H_*(\Omega\Sigma S^{1+k})^{\otimes 2}\otimes H_*(S^{2+k}) \tag{4.11}$$

is an $\mathbb{F}_2$-module on generators $Q_0(x_k)\otimes 1\otimes 1\otimes 1, Q_1(x_k)\otimes 1\otimes 1\otimes 1, 1\otimes\bar{Q}_0(x_{k+1})\otimes 1\otimes 1, 1\otimes 1\otimes\bar{Q}_0(x_{k+1})\otimes 1$, as well as 6 other elements where we let two of the four tensor factors be 1 and the other two be the weight 1 generators. There is a one-to-one correspondence by sending $y\otimes x$ to $x_{k+2-|y|}$ and $\bar{Q}_i(y\otimes x)$ to $Q_i(x_{k+2-|y|})$ for $y=a,b,c,d$.

Weight three

Classes in $A=\Lambda\big(\gamma_I(\bar{Q}^{j_1}|\cdots|\bar{Q}^{j_m}|(y\otimes w)),m\geq 1\big)$ have weights positive powers of 2. Hence weight three classes in $E^2(3)$ either live in $\mathrm{wt}_3(H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}}))$ or has the form

$$(\bar{Q}_j(y\otimes x))\otimes(y'\otimes x')\in A\otimes H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}}),\ \ x,x'\in B.$$

Let H be the set of generators for $\widetilde{H}^*(\Sigma_{g,1}^+)\cong\widetilde{H}^*(\Sigma_g)$ and H^1 the set of generators for $\widetilde{H}^1(\Sigma_{g,1}^+)$. Recall that $\tilde{\mathfrak{g}}=\widetilde{H}^*(\Sigma_g)\otimes\mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Lie}^{s,\mathrm{ti}}_{\mathcal{R}}}(\Sigma^n H_*(X))$. Then we have

$$\begin{aligned}
E^2(3)\cong&\bigoplus_{x_1,x_2\in B}\mathbb{F}_2\{(\bar{Q}_0(y\otimes x_1))\otimes(y'\otimes x_2),y\in H^1,y'\in H\}\\
&\oplus\bigoplus_{x_1,x_2\in B}\mathbb{F}_2\{(\bar{Q}_0(c\otimes x_1))\otimes(y\otimes x_2),(\bar{Q}_1(c\otimes x))\otimes(y\otimes x_2),y\in H\}\\
&\oplus\mathrm{wt}_3(H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}})).
\end{aligned}$$

A complete list of an $\mathbb{F}_2$-basis of $\mathrm{wt}_3(H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}})$ can be written down in a straight forward way.

The E^2-page is concentrated in simplicial degree $0,1,2$. We need to investigate all classes in $E^2_{2,*}(3)$ to see if they support nontrivial d^2-differentials to $E^2_{0,*+1}(3)$. Note that all classes in $E^2_{0,*}(3)$ are of the form $y \otimes \langle\langle x_1, x_2\rangle, x_3\rangle$ for $y \in H^1$. Since $E^2(3)$ is natural in $H_*(V)$, we can assume $x_1, x_2, x_3 \in B$ have internal degree k respectively. There are two cases:

1. The class $(\bar{Q}_j(y_1 \otimes x_1)) \otimes (y_2 \otimes x_2) \in E^2_{2,*}(3)$ has internal degree at most $3k-5$ for all $y_1, y_2 \in H$, while the class $y \otimes \langle\langle x_1, x_2\rangle, x_1\rangle$ has internal degree $3k-3$ for all $y \in H^1$. Hence they do not support d^2-differentials.

2. The other type of classes in filtration 2 are of the form $(y_1 \otimes x_1) \otimes (y_2 \otimes x_2) \otimes (y_3 \otimes x_3)$ with internal degrees at most $3k-5$, while the class $y \otimes \langle\langle x_1, x_2\rangle, x_3\rangle$ has internal degree $3k-3$. Hence these classes do not support d^2-differentials either.

Therefore the weight three part of the spectral sequence collapses at the E^2-page, and we obtain a basis for $H_*(B_3(\Sigma_{g,1}; X))$.

For the closed surface Σ_1, $\tilde{\mathfrak{g}} = H^*(\Sigma_1) \otimes \mathrm{Free}^{\mathrm{Lie}^{s,\mathrm{ti}}_{\mathcal{R}}}_{\mathrm{Mod}_{\mathbb{F}_2}}(\Sigma^n H_*(X))$ and Corollary 4.4.8 says that

$$
\begin{aligned}
E^\infty(3) = E^2(3) \cong & \bigoplus_{x_1, x_2 \in B} \mathbb{F}_2\{(\bar{Q}_0(y \otimes x_1)) \otimes (y' \otimes x_2), y \in H^1, y' \in H \cup \{d\}\} \\
& \oplus \bigoplus_{x_1, x_2 \in B} \mathbb{F}_2\{(\bar{Q}_0(c \otimes x_1)) \otimes (y \otimes x_2), (\bar{Q}_1(c \otimes x)) \otimes (y \otimes x_2), y \in H \cup \{d\}\} \\
& \oplus \mathrm{wt}_3(H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}})).
\end{aligned}
$$

We do not list the $\mathbb{F}_2$-basis of $\mathrm{wt}_3((H^{\mathrm{Lie}^{s,\mathrm{ti}}}_{*,*}(\tilde{\mathfrak{g}}))$ for simplicity.

Example 4.4.10. As in the weight two case, our basis for $H_*(B_3(\Sigma_1, \mathbb{S}^k)), k \geq 1$ is in bijection with the weight 3 part of Equation (4.11) by sending $y \otimes x$ to $x_{k+2-|y|}$ and $\bar{Q}_i(y \otimes x)$ to $Q_i(x_{k+2-|y|})$ for $y = a, b, c, d$.

4.4.4 Example computations: (punctured) real projective space

The simplest examples of parallelizable manifolds admitting nontrivial Steenrod actions other than Sq^0 are the real projective space $\mathbb{RP}^3$ and the once-punctured real projective space $\dot{\mathbb{RP}}^3$. Let y be a generator for $H^1(\mathbb{RP}^3)$. Then

$$\widetilde{H}^*((\mathbb{RP}^3)^+) \cong H^*(\mathbb{RP}^3) = \mathbb{F}_2[y]/(y^4), \widetilde{H}^*((\dot{\mathbb{RP}}^3)^+) = \widetilde{H}^*(\mathbb{RP}^3) = \mathbb{F}_2\{y, y^2, y^3\}$$

with the obvious cup products and one nontrivial Steenrod operation $Sq^1(y) = y^2$.

Weight two

We deduce $H_*(B_2(\dot{\mathbb{RP}}^3; X))$ and $H_*(B_2(\mathbb{RP}^3; X))$ from Corollary 4.4.6. For $M = \dot{\mathbb{RP}}^3$, there is only one nontrivial cup product $y \cup y^2 = y^3$, so

$$\begin{aligned} E^\infty(2) = E^2(2) = & \bigoplus_{x \in B, a=1,2,3} \mathbb{F}_2\{\bar{Q}_j(y^a \otimes x), 0 \le j < a\} \oplus \bigoplus_{x \in B} \mathbb{F}_2\{(y^a \otimes x) \otimes (y^b \otimes x), 1 \le a < b \le 3\} \\ & \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{y \otimes \langle x_1, x_2 \rangle; (y^a \otimes x_1) \otimes (y^a \otimes x_2), a = 2,3\} \\ & \oplus \bigoplus_{x_1 \neq x_2 \in B} \mathbb{F}_2\{(y^a \otimes x_1) \otimes (y^3 \otimes x_2), a = 1,2\} \\ & \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y^1 \otimes x_1) \otimes (y^2 \otimes x_2) + (y^2 \otimes x_1) \otimes (y^1 \otimes x_2)\}. \end{aligned}$$

For $M = \mathbb{RP}^3$, the nonzero cup products are $y \cup y = y^2, y \cup y^2 = y^3$ and $1 \cup y^a = y^a$ for $0 \le a \le 3$, so

$$\begin{aligned} E^\infty(2) = E^2(2) = & \bigoplus_{x \in B, a=1,2,3} \mathbb{F}_2\{\bar{Q}_j(y^a \otimes x), 0 \le j < a\} \\ & \oplus \bigoplus_{x \in B} \mathbb{F}_2\{(y^a \otimes x) \otimes (y^b \otimes x), 0 \le a < b \le 3\} \\ & \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y^a \otimes x_1) \otimes (y^a \otimes x_2), a = 2,3\} \\ & \oplus \bigoplus_{x_1 \neq x_2 \in B} \mathbb{F}_2\{(y^a \otimes x_1) \otimes (y^3 \otimes x_2), a = 1,2\} \\ & \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y^a \otimes x_1) \otimes (y^b \otimes x_2) + (y^3 \otimes x_1) \otimes (1 \otimes x_2), (a,b) \neq (3,0)\} \\ & \oplus \bigoplus_{x_1 < x_2 \in B} \mathbb{F}_2\{(y^a \otimes x_1) \otimes (1 \otimes x_2) + (1 \otimes x_1) \otimes (y^a \otimes x_2), a = 1,2,3\}. \end{aligned}$$

Example 4.4.11. When $X = \mathbb{S}^k$ with $k \geq 1$, we have $B = \{x = \sigma_2 \iota_k\}$, so $H_*(B_2(\mathbb{RP}^3, S^k))$ has $\mathbb{F}_2$-basis

$$\{\bar{Q}_j(y^a \otimes x), 0 \leq j < a, a = 1,2,3; (y^a \otimes x) \otimes (y^b \otimes x), 0 \leq a < b \leq 3\}.$$

A bijection with weight 3 part of Bödigheimer-Cohen-Taylor's decomposition [BCT89]

$$\begin{aligned}\bigoplus_{k\geq 1} H_*(B_k(\mathbb{RP}^3;\mathbb{S}^k)) \cong & \bigotimes_{i=0}^{n} H_*(\Omega^{3-i} S^{3+k})^{\otimes \dim H_i(M)} \\ \cong & H_*(\Omega^3\Sigma^3 S^k) \otimes H_*(\Omega^2\Sigma^2 S^{k+1}) \otimes H_*(\Omega\Sigma S^{k+2}) \otimes H_*(S^{k+3}) \\ \cong & \mathrm{Free}^{E_3}(\mathbb{F}_2\{x_k\}) \otimes \mathrm{Free}^{E_2}(\mathbb{F}_2\{\{x_{k+1}\}) \otimes \mathrm{Free}^{E_1}(\mathbb{F}_2\{x_{k+2}\}) \otimes \mathbb{F}_2\{x_k\}\end{aligned}$$

is given by sending $y^a \otimes x$ to x_{k+3-a} and $\bar{Q}_i(y^a \otimes x)$ to $Q_i(x_{k+3-a})$ for $0 \leq a \leq 3$.

Weight three

For the closed manifold $\mathbb{RP}^3$ and $\tilde{\mathfrak{g}} = H^*(\mathbb{RP}^3) \otimes \mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Lie}_{\mathcal{R}}^{s,\mathrm{ti}}}(\Sigma^n H_*(X))$, it follows from Corollary 4.4.8 that

$$E^\infty(3) = E^2(3) \cong \mathrm{wt}_3(H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}})) \oplus \bigoplus_{x_1,x_2 \in B, 1\leq a\leq 3, 0\leq b\leq 3} \mathbb{F}_2\{(\bar{Q}_j(y^a \otimes x_1)) \otimes (y^b \otimes x_2), 0 \leq j < a\}.$$

For the punctured real projective space $\dot{\mathbb{RP}}^3$ and $\tilde{\mathfrak{g}} = \widetilde{H}^*(\mathbb{RP}^3) \otimes \mathrm{Free}_{\mathrm{Mod}_{\mathbb{F}_2}}^{\mathrm{Lie}_{\mathcal{R}}^{s,\mathrm{ti}}}(\Sigma^n H_*(X))$, weight three classes in $E^2(3)$ either live in $\mathrm{wt}_3(H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}}))$ or has the form

$$(\bar{Q}_j(y^a \otimes x)) \otimes (y^b \otimes x') \in A \otimes H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}})$$

with $x, x' \in B$ and $1 \leq a, b \leq 3$. Therefore

$$E^2(3) = \mathrm{wt}_3(H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}})) \oplus \bigoplus_{x_1,x_2 \in B, 1\leq a,b\leq 3} \mathbb{F}_2\{(\bar{Q}_j|(y^a \otimes x_1)) \otimes (y^b \otimes x_2), 0 \leq j < a\}.$$

A complete list of an $\mathbb{F}_2$-basis for $\mathrm{wt}_3(H_{*,*}^{\mathrm{Lie}^{s,\mathrm{ti}}}(\tilde{\mathfrak{g}}))$ is given by

1. $y \otimes \langle\langle x_1, x_2\rangle, x_3\rangle$ for $x_1, x_2, x_3 \in B, x_1 < x_2, x_1 < x_3$ in simplicial degree 0;

2. $(y^3 \otimes \langle x_1, x_2\rangle) \otimes (y^b \otimes x_3) + (y^3 \otimes \langle x_1, x_3\rangle) \otimes (y^b \otimes x_2) + (y^3 \otimes \langle x_2, x_3\rangle) \otimes (y^b \otimes x_1)$ for $b = 1, 2$

   and $(y \otimes \langle x_1, x_2\rangle) \otimes (y^2 \otimes x_3) + (y \otimes \langle x_1, x_3\rangle) \otimes (y^2 \otimes x_2) + (y \otimes \langle x_2, x_3\rangle) \otimes (y^2 \otimes x_1)$ for distinct $x_i \in B$ in simplicial degree 1;

3. $(y^a \otimes x_1) \otimes (y^b \otimes x_2) \otimes (y^c \otimes x_3)$ for $\{1,2\}, \{1,1\} \not\subseteq \{a,b,c\}$ and $x_i \in B$;

   $\sum_{\{i,j,k\}=\{1,2,3\}, i<j} (y \otimes x_i) \otimes (y \otimes x_j) \otimes (y^2 \otimes x_k)$,

   $\sum_{\{i,j,k\}=\{1,2,3\}, j<k} (y \otimes x_i) \otimes (y^2 \otimes x_j) \otimes (y^2 \otimes x_k)$ for distinct $x_1, x_2, x_3 \in B$ in simplicial degree 2.

Again the E^2-page is concentrated in simplicial degrees $0, 1, 2$, and we use sparsity to rule out higher differentials. Suppose that x_1, x_2, x_3 have internal degree k. We examine the two cases that could potentially support a d^2-differential.

1. The class $(\bar{Q}_j(y^a \otimes x_1)) \otimes (y^b \otimes x_2) \in E^2_{2,*}(3)$ has internal degree at most $3k-5$ for all $1 \leq a, b \leq 3$, while the class $y \otimes \langle\langle x_1, x_2\rangle, x_1\rangle$ has internal degree $3k-3$. Hence they do not support d^2-differentials.

2. The other type of classes in simplicial degree 2 are of the form $(y^a \otimes x_1) \otimes (y^b \otimes x_2) \otimes (y^c \otimes x_3)$ with internal degrees at most $3k-5$, while the class $y \otimes \langle\langle x_1, x_2\rangle, x_3\rangle$ has internal degree $3k-3$. Hence these classes do not support d^2-differentials either.

Therefore the weight three part of the spectral sequence collapses on the E^2-page, and we obtain a basis for $H_*(B_3(\dot{\mathbb{RP}}^3; X))$.

4.5 Odd primary homology

In this last section, we apply the same methods to study the mod p homology of $B_k(M; X)$ for $p > 2$ via the Knudsen spectral sequence with $\mathbb{F}_p$ coefficient.

4.5.1 Odd primary Knudsen spectral sequence

We start by recalling partial progress in understanding the unary operations on the mod p homology of spectral Lie algebras by Kjaer [Kja18]. He constructed weight p Dyer-Lashof-type operations in analogy to Behrens' construction of $\bar{Q}^j$, which was further clarified by the work of Konovalov.

Proposition 4.5.1. *[Kja18, Definition 3.2][Kon23, Definition 2.5.17] Let L be a spectral Lie algebra. Then $H_*(L;\mathbb{F}_p)$ admits unary operations*

$$\overline{\beta^\varepsilon Q^j} : H_*(L;\mathbb{F}_p) \to H_{*+2(p-1)i-\varepsilon-1}(L;\mathbb{F}_p),\ \varepsilon \in \{0,1\}, j \in \mathbb{Z}.$$

On a class $x \in H_*(L;\mathbb{F}_p)$ such that if $|x|$ is even then $2j \neq x$, the class $\overline{\beta^\varepsilon Q^j}(x)$ is given by by $\xi_*(\sigma^{-1}\beta^\varepsilon Q^j(x))$, where $\beta^\varepsilon Q^j$ is a mod p Dyer-Lashof operation, σ^{-1} the desuspension isomorphism, and $\xi : \partial_p(\mathrm{Id}) \otimes_{h\Sigma_p} L^{\otimes p} \to L$ the pth structure map of the spectral Lie algebra L. When $|x| = 2l$, define $\overline{\beta Q^l}(x)$ via the isomorphism $H_*(\partial_p(\mathrm{id}) \otimes_{h\Sigma_p} (S^{2l})^{\otimes p}) \cong H_*(\Sigma^{-1}(\partial_p(\mathrm{id}) \otimes_{h\Sigma_p} (S^{2l-1})^{\otimes p}))$.

It follows from the instability condition of Dyer-Lashof operations that the allowability condition for the operations $\overline{\beta^\varepsilon}$ are given by $\overline{\beta^\varepsilon Q^j}(x) = 0$ if $j < \frac{|x|}{2}$. Analogous to the case $p = 2$, brackets of unary operations always vanish.

Proposition 4.5.2. *[Kja18, Proposition 3.7] For L a spectral Lie algebra, $[\overline{\beta^\varepsilon Q^j}(x), y] = 0$ for any ε, j and $x, y \in H_*(L;\mathbb{F}_p)$.*

The relations among the unary operations were obtained by Konovalov.

Proposition 4.5.3. *[Kon23, Theorem 8.2.14] Let $\bar{\mathcal{R}}$ be the free algebra over $\mathbb{F}_p$ on generators $\overline{\beta^\varepsilon Q^j}, \varepsilon \in \{0,1\}$, subject to the relations*

$$\begin{aligned}\overline{\beta^\varepsilon Q^j} \cdot \overline{\beta Q^i} =&(-1)^{\varepsilon+1} \sum_{m=pi}^{i+j-1} \binom{p(m-i)-(p-1)j+\varepsilon-1}{m-pi} \overline{\beta Q^m} \cdot \overline{\beta^\varepsilon Q^{j+i-m}} \\ &+(1-\varepsilon) \sum_{m=pi+1}^{i+j-1} \binom{p(m-i)-(p-1)j}{m-pi} \overline{Q^m} \cdot \overline{\beta Q^{j+i-m}}\end{aligned}$$

for $j < pi$, and

$$\overline{\beta^{\varepsilon}Q^{j}}\cdot\overline{Q^{i}} = \sum_{m=pi+1}^{i+j-1}\binom{p(m-i)-(p-1)j-1}{m-pi-1}\overline{\beta^{\varepsilon}Q^{m}}\cdot\overline{Q^{j+i-m}}$$

for $j \le pi$. Then the mod p homology of a spectral Lie algebra is an allowable module over $\bar{\mathcal{R}}$.

Denote by $\mathcal{A}_{\bar{\mathcal{R}}}$ the free allowable $\bar{\mathcal{R}}$-module monad. Let $\mathrm{Lie}^{s}_{\bar{\mathcal{R}}} : \mathrm{Mod}_{\mathbb{F}_p} \to \mathrm{Mod}_{\mathbb{F}_p}$ be the composite monad $\mathcal{A}_{\bar{\mathcal{R}}}\circ\mathrm{Lie}^{s}_{\mathbb{F}_p}$ subject to the commuting relations Proposition 4.5.2 when $p > 3$, and the monad given by $\mathrm{Lie}^{s}_{\bar{\mathcal{R}}}(M) = \mathcal{A}_{\bar{\mathcal{R}}}\circ\mathrm{Lie}^{s}_{\mathbb{F}_3}(M)/\langle\overline{\beta^{\varepsilon}Q^{|x|/2}}(x) = [[x,x],x]\rangle$, where we take the quotient by the $\bar{\mathcal{R}}$-module ideal ranging over $x \in M$ in even degree. For $M \in \mathrm{Mod}_{\mathbb{F}_p}$, let A be an $\mathbb{F}_p$-basis for the free shifted Lie algebra $\mathrm{Free}^{\mathrm{Lie}^{s}_{\mathbb{F}_p}}_{\mathrm{Mod}_{\mathbb{F}_p}}(M)$. The graded $\mathbb{F}_p$-module $\mathrm{Lie}^{s}_{\mathrm{Poly}}(M)$ has basis

$$\{\overline{\beta_1^{\varepsilon_1}Q^{j_1}}\cdots\overline{\beta_k^{\varepsilon_k}Q^{j_k}}|x,\ \ x\in A, j_k \ge \frac{|x|}{2}, j_i \ge pj_{i+1}-\varepsilon_{i+1}\forall i\}.$$

Theorem 4.5.4. *[Kja18, Theorem 5.2][Kon23, Theorem 8.2.17] For X a spectrum. there is an isomorphism of $\mathrm{Lie}^{s}_{\bar{\mathcal{R}}}$-algebras*

$$\mathrm{Lie}^{s}_{\bar{\mathcal{R}}}(H_*(X;\mathbb{F}_p)) \to H_*(\mathrm{Free}^{s\mathscr{L}}(X);\mathbb{F}_p).$$

Remark 4.5.5. For $p = 3$, Kjaer claimed in [Kja18, Corollary 4.7] that the triple bracket on an even degree homology class ι_{2l} of a spectral Lie algebra is zero by showing that

$$[[\iota_{2l},\iota_{2l}],\iota_{2l}] \in H_*(\partial_3(\mathrm{id}) \underset{h\Sigma_3}{\otimes} (S^{2l})^{\otimes 3})$$

vanishes. The claim is incorrect in light of Proposition 4.5.8 below, and was independently observed by Nikolai Konovolav. Specifically, Kjaer argued that in the long exact sequence

$$\cdots \to H_{6l-2}(\Sigma^{-2}(S^{2l})^{\otimes 3}_{h\Sigma_3}) \to H_{6l-2}(\partial_3(\mathrm{id}) \underset{h\Sigma_3}{\otimes} (S^{2l})^{\otimes 3}) \to H_{6l-2}((\Sigma^{-1}(S^{2l})^{\otimes 3}_{h\Sigma 3}) \to \cdots,$$

the middle group is generated as an $\mathbb{F}_3$-module by the bottom operation $\overline{\beta Q^{l}}\iota_{2l}$, which is mapped isomorphically onto $\sigma^{-1}\beta Q^l\iota_{2l}$ by definition of the bottom operation in Definition 3.2. However, $\sigma^{-1}\beta Q^l\iota_{2l} \in H_{6l-2}(\Sigma^{-1}(S^{2l})^{\otimes 3}_{h\Sigma 3}) = 0$. In fact, one can see that the confusion was caused by incorrect placement of parentheses. Since the left term is one-dimensional on $[[\iota_{2l},\iota_{2l}],\iota_{2l}]$, we see that $[[\iota_{2l},\iota_{2l}],\iota_{2l}] = \mu_l\overline{\beta Q^l}\iota_{2l}$, where $\mu_l = \pm 1$. This also motivates the modification of the definition of the bottom operation on an even class in Proposition 4.5.1.

Now we turn to the odd primary Knudsen spectral sequence

$$E^2_{s,t}(k) = \pi_{s,t}\big(\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{Lie}^s_{\mathcal{R}},\mathfrak{g}\big)\otimes\mathbb{F}_p\big)(k) \Rightarrow H_{s+t}(B_k(M;X);\mathbb{F}_p), \tag{4.12}$$

where $\mathfrak{g} = H_*(\mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)^{M^+};\mathbb{F}_p) \cong \widetilde{H}^*(M^+;\mathbb{F}_p)\otimes\mathrm{Lie}^s_{\mathrm{Poly}}(\Sigma^n H_*(X;\mathbb{F}_p))$. Furthermore, $\mathfrak{g}$ has a $\mathrm{Lie}^s_{\mathbb{F}_p}$-structure given by Proposition 4.3.3, i.e.,

$$[y_1\otimes x_1, y_2\otimes x_2] := (y_1\cup y_2)\otimes[x_1,x_2].$$

We proceed to compute the E^2-page of the spectral sequence (4.12) in small weight in terms of $\mathrm{Lie}^s_{\mathbb{F}_p}$-algebra homology.

Definition 4.5.6. [CE48][May66A] For a shifted Lie algebra L over $\mathbb{F}_p$, let L_{even} and L_{odd} denote the elements in L with even and odd degree, respectively. The *Chevalley-Eilenberg* complex of L is the chain complex

$$\mathrm{CE}(L) = (\Gamma^\bullet(L_{\mathrm{even}})\otimes\Lambda^\bullet(L_{\mathrm{odd}}),\partial),$$

where $\Gamma^\bullet$ and $\Lambda^\bullet$ are respectively the graded, shifted divided power and exterior algebra functor over $\mathbb{F}_p$, and the differential ∂ on a general element

$$\gamma_{k_1}(x_1)\gamma_{k_2}(x_2)\cdots\gamma_{k_m}(x_m)\langle y_1,y_2,\ldots,y_n\rangle \in \Gamma^\bullet(L_{\mathrm{even}})\otimes\Lambda^\bullet(L_{\mathrm{odd}})$$

is given by

$$\begin{aligned}
&\sum_{1\le i<j\le m} \gamma_{k_1}(x_1)\cdots\gamma_{k_i-1}(x_i)\cdots\gamma_{k_j-1}(x_j)\cdots\gamma_{k_m}(x_m)\langle[x_i,x_j],y_1,\dots y_n\rangle \\
&+\sum_{1\le i<j\le n}(-1)^{i+j-1}\gamma_{k_1}(x_1)\cdots\gamma_{k_m}(x_m)\langle[y_i,y_j],y_1,\dots,\widehat{y_i},\dots\widehat{y_j},\dots y_n\rangle \\
&+\frac{1}{2}\sum_{i=1}^{m}\gamma_{k_1}(x_1)\cdots\gamma_{k_i-2}(x_i)\cdots\gamma_{k_m}(x_m)\langle[x_i,x_i],y_1,\dots,y_n\rangle \\
&+\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{j-1}\gamma_1([x_i,y_j])\gamma_{k_1}(x_1)\cdots\gamma_{k_i-1}(x_i)\cdots\gamma_{k_m}(x_m)\langle y_1,\dots,\widehat{y_j},\dots,y_n\rangle.
\end{aligned}$$

Proposition 4.5.7. *Let M^n be a parallelizable manifold and X any spectrum.*

1. *For $k<p$, the weight k part of the spectral sequence*

$$E^2_{s,t}(k)=\pi_s\pi_t\big(\mathrm{Bar}_\bullet\big(\mathrm{id},s\mathscr{L},\mathrm{Free}^{s\mathscr{L}}(\Sigma^n X)^{M^+}\big)\otimes\mathbb{F}_p\big)(k)\Rightarrow H_{s+t}(B_k(M;X);\mathbb{F}_p)$$

   has E^2-page given by $\mathrm{wt}_k(H_{*,*}(\mathrm{CE}(\mathfrak{g}))$, where $\mathfrak{g}=\widetilde{H}^*(M^+;\mathbb{F}_p)\otimes\mathrm{Lie}^s_{\mathbb{F}_p}(\Sigma^n H_*(X;\mathbb{F}_p))$.

2. *For $p\ge 5$, the weight p part of the spectral sequence has E^2-page given by*

$$E^2_{*,*}(k)\cong \mathrm{wt}_p(H_{*,*}(\mathrm{CE}(\mathfrak{g})))\oplus\bigoplus_{y\in H,x\in B}\mathbb{F}_p\Big\{\overline{\beta^\varepsilon Q^j}|y\otimes x,\ \frac{|x|-|y|}{2}\le j<\frac{|x|}{2}\Big\},$$

   where H is an $\mathbb{F}_p$-basis of $\widetilde{H}^*(M^+;\mathbb{F}_p)$ and B an $\mathbb{F}_p$-basis of $H_*(X;\mathbb{F}_p)$.

Proof. For $k<p$, all elements in the weight k part of the E^2-page of the spectral sequence do not contain unary operations $\overline{\beta^\varepsilon Q^j}$. When $k=p$, nondegenerate elements of weight p on the E^2-page are either of the form $\overline{\beta^\varepsilon Q^j}|y\otimes x\in\mathrm{Lie}^s_{\overline{\mathcal{R}}}(\mathfrak{g})$, $\overline{\beta^\varepsilon Q^j}(y\otimes x)\in\mathfrak{g}$, or a bracket of weight p. When $p\ge 5$, the unary operation $\beta^\varepsilon Q^j$ cannot be an iteration of brackets on a single element, since $[[x,x],x]=0$ for any x by the Jacobi identity. Hence there is no d_1-differential from a weight p bracket to $\overline{\beta^\varepsilon Q^j}|y\otimes x$ or $y\otimes\overline{\beta^\varepsilon Q^j}(x)$. The same argument in Proposition 4.4.4 implies that the twisting of the action of $\overline{\beta^\varepsilon Q^j}$ by Steenrod operations can be ignored when computing a basis for the E^2-page. □

The condition $p\ge 5$ in part (2) is necessary in light of the following computation for

Euclidean spaces.

Proposition 4.5.8. *For $p \geq 5$, the only higher differential in the weight p part of the spectral sequence (4.12) for $M = \mathbb{R}^n, 2 \leq n \leq \infty$, which converges to $H_*(B_p(\mathbb{R}^n;\mathbb{S}^{2l});\mathbb{F}_p)$, is a d_{p-2}-differential $\gamma_p(x) \mapsto \overline{\beta^\varepsilon Q^l}|y_n \otimes \sigma^n(x)$.*

When $p = 3$, the above spectral sequence has a d^1-differential $\gamma_3(x) \mapsto \overline{\beta^\varepsilon Q^l}|x$.

Heuristically, this is because the bottom non-vanishing mod p Dyer-Lashof operation on a class x of degree $2l$ in the mod p homology of an $\mathbb{E}_n$-algebra is given by $\bar{Q}^l(x) = x^{\otimes p}$, so $\gamma_p(x)$ is redundant.

Proof. Consider the spectral sequence (4.12) when $M = \mathbb{R}^n$ and $X = \mathbb{S}^{2l}$ with $n > 2$, so

$$\mathfrak{g} = \mathbb{F}_p\{y_n\} \otimes \mathrm{Lie}^s_{\mathrm{Poly}}(\mathbb{F}_p\{\sigma^n(x_{2l})\})$$

with y in internal degree $-n$ and x_{2l} in degree $2l$. Set $x = y_n \otimes \sigma^n(x_{2l})$. Then the weight p part of the E^2-page has basis

$$\left\{\overline{\beta^\varepsilon Q^j}|x,\ l \leq j < \frac{2l+n}{2};\ \gamma_p(x)\right\}.$$

Comparing with the weight p part of the E^∞-page, which is the weight p part of the mod p homology of the free $\mathbb{E}_n$-algebra on the $\mathbb{S}^{2l}$, we see that there are two classes that do not survive to the E^∞-page, i.e., $\gamma_p(x)$ in bidegree $(p-1, 2pl-(p-1))$ and $\overline{\beta Q^l}|x$ in bidegree $(1, 2pl-2)$ (cf. [CLM76, III]). Hence there has to be a d_{p-2}-differential from $\gamma_p(x)$ to $\overline{\beta^\varepsilon Q^l}|x$.

When $p = 3$, $\gamma_3(x)$ is represented by the element $[[x,x],x] \in \mathrm{Lie}^s_{\mathbb{F}_p} \circ \mathrm{Lie}^s_{\mathbb{F}_p}(\mathfrak{g}) \subset \mathrm{Lie}^s_{\mathrm{Poly}} \circ \mathrm{Lie}^s_{\mathrm{Poly}}(\mathfrak{g})$. It is mapped by the differential to $[[x,x],x] \in \mathrm{Lie}^s_{\mathbb{F}_p}(\mathfrak{g})$, which by Remark 4.5.5 is indeed $\overline{\beta^\varepsilon Q^l}|x$. $\square$

As an immediate corollary to Proposition 4.5.7, we see that the weight two part of the spectral sequence (4.12) collapses on the E^2-page, since the E^2-page is concentrated in simplicial degree 0 and 1. When $p > 3$, weight three elements on the E^2-page are in simplicial degree 1 or 2 since $[[x,x],x] = 0$ by the Jacobi identity. Hence the weight three part of the spectral sequence (4.12) also collapses on the E^2-page.

Corollary 4.5.9. *Let M^n be a parallelizable manifold and X any spectrum. Let $\mathfrak{g}$ be the* $\mathrm{Lie}^s_{\mathbb{F}_p}$*-algebra* $\widetilde{H}^*(M^+;\mathbb{F}_p)\otimes \mathrm{Lie}^s_{\mathbb{F}_p}(\Sigma^n H_*(X;\mathbb{F}_p))$

1. *For all i, there is an isomorphism of $\mathbb{F}_p$-modules*

$$H_i(B_2(M;X);\mathbb{F}_p)\cong \bigoplus_{s+t=i} \mathrm{wt}_2(H_{s,t}(\mathrm{CE}(\mathfrak{g})).$$

2. *If $p\geq 5$, then $H_i(B_3(M;X);\mathbb{F}_p)\cong \bigoplus_{s+t=i}\mathrm{wt}_3(H_{s,t}(\mathrm{CE}(\mathfrak{g}))$ for all i.*

Remark 4.5.10. For M a connected n-manifold, Bödigheimer-Cohen-Taylor showed that

$$\bigoplus_{k\geq 1} H_*(B_k(M;S^r);\mathbb{F}_p)\cong \bigotimes_{i=0}^{n} H_*(\Omega^{n-i}S^{n+r};\mathbb{F}_p)^{\otimes \dim H_i(M;\mathbb{F}_p)}$$

for $r+n$ odd and $r\geq 0$ [BCT89]. Their proof does not work in the case where $r+n$ is even due to the existence of nontrivial self-brackets in $H_*(\Omega^m\Sigma^m S^l);\mathbb{F}_p)$ when l is even. Roughly speaking, their inductive proof relies on the canonical map $H_*(\Omega^m\Sigma^m S^l;\mathbb{F}_p)\to H_*(\Omega^\infty\Sigma^\infty S^l;\mathbb{F}_p)$ being an injection, which is only true when l is odd. Corollary 4.5.9 shows that when l is even, the mod p homology of $B_k(M;\mathbb{S}^r), k=2,3$ depends on the cup product structure on $H^*(M^+;\mathbb{F}_p)$: if $a\cup b=c$ in $\widetilde{H}^*(M^+;\mathbb{F}_p)$, then the d_1-differential sends $(a\otimes x)\otimes(b\otimes x)$ to $c\otimes[x,x]\in\mathfrak{g}=\widetilde{H}^*(M^+;\mathbb{F}_p)\otimes\mathrm{Lie}^s_{\mathrm{Poly}}(\mathbb{F}_p\{x\})$, which is not zero since x has internal degree l.

At higher weights, there generally will be higher differentials in the odd primary Knudsen spectral sequence (4.12). In recent work with Matthew Chen [CZ22], we make use of Proposition 4.5.7 and Drummond-Cole-Knudsen's computation of the rational homology of the unordered configurations space $B_k(M)$ where $M=\Sigma_1$ or $\Sigma_{g,1}$ [DCK17] to identify the differentials in the Knudsen spectral sequence for $B_k(\Sigma_g;\mathbb{S})$. As a result, we show that the integral homology of $B_k(\Sigma_1)$ is p-torsion-free for $k\leq p$. The same argument works for the punctured surface $\Sigma_{g,1}$ with $g\geq 1$, thereby providing a more elementary proof for [BHK19, Theorem 1.10]

Chapter 5

The structure of the bar spectral sequence

The philosophy of viewing spectral sequences as one-parameter deformations of homotopy theories has recently proven to be useful in a number of situations. For instance, Pstrągowski [Pst] constructed the ∞-category $\mathrm{Syn}_{\mathbb{F}_p}$ of $\mathbb{F}_p$-synthetic spectra as a deformation of the ∞-category Sp of spectra with formal parameter τ. Informally, inverting the parameter τ recovers Sp, whereas modding out by τ yields the derived category of comodules over the dual of the Steenrod algebra. On the other hand, building on the insights of [HL17, Lur11a], Brantner [Bra17] computed the additive operations and their relations on the Morava E-theory of spectral Lie algebras using a deformation of the comonad associated to the bar construction against Rezk's monad of additive power operations on the Morava E-theory of $\mathbb{E}_\infty$-algebras [Rez12].

This chapter records joint work in progress with Andrew Senger on examining the bar spectral sequence as a coalgebra over a filtered comonad. The motivation is to use the comonadic structure map to detect the higher differentials in the bar spectral sequence in Conjecture 4.3.5.

5.1 Categorical setup

We start by starting up the necessary framework to carry out the computation. The main references for this section are [Lur09] and [Pst].

5.1.1 Product-preserving presheaves

Let $\mathcal{C}$ be a small ∞-category and $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ the ∞-category of presheaves on $\mathcal{C}$ with value in Spaces. Denote by ν the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ that sends $X \in \mathcal{C}$ to the presheaf $\mathrm{Map}_{\mathcal{C}}(-, X)$.

Theorem 5.1.1. *[Lur09, Proposition 5.3.6.2, Proposition 5.5.8.10, Lemma 5.5.8.14] Let $\mathcal{C}$ be a small ∞-category with finite coproducts. Denote by $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ the full subcategory generated by the image of the Yoneda embedding under sifted colimits. Then the following statements hold:*

1. *$\mathcal{P}_\Sigma(\mathcal{C})$ is the full subcategory $\mathrm{Fun}^\times(\mathcal{C}, \mathcal{S})$ of product-preserving presheaves, i.e., presheaves $f \in \mathcal{P}(\mathcal{C})$ satisfying $f(X \sqcup X') \xrightarrow{\simeq} f(X) \times f(X')$ for all $X, X' \in \mathcal{C}$;*

2. *The Yoneda embedding $\nu : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ preserves finite coproducts;*

3. *For any ∞-category $\mathcal{D}$ that admits all sifted colimits, precomposing with the Yoneda embedding ν induces an equivalence $\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with inverse left Kan extension along ν. Here $\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D})$ denotes the ∞-category of functors $\mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{D}$ that preserve sifted colimits.*

For example, suppose that $\mathcal{C}$ is an ∞-category with all colimits and $i : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ the finite coproduct-preserving inclusion of a full subcategory $\mathcal{C}_0$ that is closed under finite coproducts. Then there is a natural colimit-preserving functor $\mathrm{Re} : \mathcal{P}_\Sigma(\mathcal{C}_0) \to \mathcal{C}$, called the *realization map*, obtained by left Kan extension of i along the Yoneda embedding $\nu : \mathcal{C}_0 \to \mathcal{P}_\Sigma(\mathcal{C}_0)$.

Proposition 5.1.2. *The right adjoint to $\mathrm{Re} : \mathcal{P}_\Sigma(\mathcal{C}_0) \to \mathcal{C}$ is given by the restricted Yoneda embedding $\nu_0 : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C}_0)$ that sends $X \in \mathcal{C}$ to the product-preserving presheaf $\mathrm{Map}_{\mathcal{C}}(-, X)$.*

Proof. We want to show that there is a natural equivalence

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{Re}(X),Y) \simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C}_0)}(X,\nu_0(Y))$$

for $X \in \mathcal{P}_\Sigma(\mathcal{C}_0)$ and $Y \in \mathcal{C}$. Since $\mathcal{P}_\Sigma(\mathcal{C}_0)$ is generated by the image of $\nu : \mathcal{C}_0 \to \mathcal{P}_\Sigma(\mathcal{C}_0)$ under sifted colimits and Re preserves sifted colimits, it suffices to check the equivalence of the image of ν. By construction Re sends $\nu(X)$ to $X \in \mathcal{C}$ for any $X \in \mathcal{C}_0$. Hence there are natural equivalences

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{Re}\circ\nu(X),Y) \simeq \mathrm{Map}_{\mathcal{C}}(X,Y) \simeq \nu_0(Y)(X) \simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C}_0)}(\nu(X),\nu_0(Y))$$

as desired. □

Proposition 5.1.3. *(1) Suppose that the full sucategory $\mathcal{C}_0$ of $\mathcal{C}$ consists of compact objects. Then the restricted Yoneda embedding $\nu_0 : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C}_0)$ preserves filtered colimits.*

(2) If in addition $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$, then ν_0 is fully faithful and the counit map $\mathrm{Re}\circ\nu_0 \to \mathrm{id}_{\mathcal{C}}$ is an equivalence.

Proof. (1) Suppose that $X \in \mathcal{C}$ is a filtered colimit $X = \operatorname*{colim}_\alpha X_\alpha$ of compact object $X_\alpha \in \mathcal{C}$. For any $Y \in \mathcal{C}_0$, there are natural equivalences

$$\nu_0(X)(Y) \simeq \mathrm{Map}_{\mathcal{C}}(Y,X) \simeq \mathrm{Map}_{\mathcal{C}}(Y,\operatorname*{colim}_\alpha X_\alpha) \simeq \operatorname*{colim}_\alpha \mathrm{Map}_{\mathcal{C}}(Y,X_\alpha) \simeq \operatorname*{colim}_\alpha \nu(X_\alpha)(Y).$$

The second to last equivalence is due to the compactness of Y.

(2) If $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$, then for any $X,Y \in \mathcal{C}$ expressed as filtered colimits $X = \operatorname*{colim}_\alpha X_\alpha, Y = \operatorname*{colim}_\beta Y_\beta$ of compact objects X_α, Y_β, we have

$$\mathrm{Map}_{\mathcal{C}}(X,Y) = \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_\alpha X_\alpha, \operatorname*{colim}_\beta Y_\beta) \simeq \lim_\alpha \operatorname*{colim}_\beta \mathrm{Map}_{\mathcal{C}}(X_\alpha,Y_\beta) \simeq \lim_\alpha \operatorname*{colim}_\beta \mathrm{Map}_{\mathcal{C}}(X_\alpha,Y_\beta).$$

On the other hand, by part (1) we have

$$\begin{aligned}
\mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C}_0)}(\nu_0(X),\nu_0(Y)) &= \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_\alpha \nu(X_\alpha), \operatorname*{colim}_\beta \nu(Y_\beta)) \\
&\simeq \lim_\alpha \mathrm{Map}_{\mathcal{C}}(\nu(X_\alpha), \operatorname*{colim}_\beta \nu(Y_\beta)) \\
&\simeq \lim_\alpha \operatorname*{colim}_\beta \nu(Y_\beta)(X_\alpha) \\
&\simeq \lim_\alpha \operatorname*{colim}_\beta \mathrm{Map}_{\mathcal{C}}(X_\alpha, Y_\beta)
\end{aligned}$$

as desired, so ν_0 is fully faithful. Hence there are natural equivalences

$$\mathrm{Map}_{\mathcal{C}}(X,Y) \simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C}_0)}(\nu_0(X),\nu_0(Y)) \simeq \mathrm{Map}_{\mathcal{C}}(\mathrm{Re}\circ\nu_0(X),Y).$$

It follows from the Yoneda lemma that the counit $\mathrm{Re}\circ\nu_0 \to \mathrm{id}$ is an equivalence. □

Proposition 5.1.4. *[Pst, Section 2.3] Suppose that $\mathcal{C}$ has a symmetric monoidal structure. Then $\mathcal{P}_\Sigma(\mathcal{C})$ admits a unique symmetric monoidal structure such that the symmetric monodial tensor product preserves colimit in both variables and the Yoneda embedding ν is symmetric monoidal.*

5.1.2 Filtered and graded objects

Let $\mathcal{C}$ be a stable, presentable ∞-category. Denote by $\mathbb{Z}$ the symmetric monoidal ∞-category with underlying category the discrete set $\mathbb{Z}$ and the symmetric monoidal product given by addition. Denote by $\mathbb{Z}^{\mathrm{Fil}}$ the symmetric monoidal ∞-category with underlying category the discrete poset $\mathbb{Z}$ under $\leq$ and the symmetric monoidal product given by addition.

Definition 5.1.5. Let $\mathcal{C}^{\mathrm{Fil}}$ denote the presentable ∞-category $\mathrm{Fun}((\mathbb{Z}^{\mathrm{Fil}})^{\mathrm{op}},\mathcal{C})$ of filtered objects in $\mathcal{C}$, whose objects are diagrams in $\mathcal{C}$ of the form

$$C_\bullet = \cdots \to C_2 \to C_1 \to C_0 \to C_{-1} \to \cdots.$$

Let $\mathcal{C}^{\mathrm{Gr}}$ denote the presentable ∞-category $\mathrm{Fun}((\mathbb{Z}^{\mathrm{op}},\mathcal{C})$ of graded objects, whose objects are collections $\{C_n\}_n$ of objects in $\mathcal{C}$.

If $\mathcal{C}$ is equipped with a symmetric monoidal structure, then so do $\mathcal{C}^{\mathrm{Fil}}$ and $\mathcal{C}^{\mathrm{Gr}}$ under the Day convolution.

We have the following adjunctions from standard considerations:

Proposition 5.1.6. 1. *There is a natural left adjoint $Y : \mathcal{C} \to \mathcal{C}^{\mathrm{Fil}}$ sending $C \in \mathcal{C}$ to the constant diagram*

$$\cdots \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} \cdots.$$

Its right adjoint Re *sends a filtered object $C_\bullet$ to* $\operatorname*{colim}_n C_{-n}$.

2. *For each $n \in \mathbb{Z}$, there is a natural left adjoint $Y_n : \mathcal{C} \to \mathcal{C}^{\mathrm{Fil}}$ sending $C \in \mathcal{C}$ to the diagram*

$$\cdots \to 0 \to 0 \to C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} C \xrightarrow{\mathrm{id}} \cdots,$$

where $Y_n(C)_k = 0$ for $k < n$ and $Y_n(C)_k = C$ for $k \geq n$. Its right adjoint $(-)_n$ sends a filtered object $C_\bullet$ to C_n and preserves sifted colimits.

3. *There is a natural left adjoint* $\mathrm{Gr} : \mathcal{C}^{\mathrm{Fil}} \to \mathcal{C}^{\mathrm{Gr}}$ *sending a filtered object $C_\bullet$ to its associated graded object $\{C_n/C_{n+1}\}_n$. Its right adjoint sends the graded object $\{X_n\}_n$ to the filtered object $X_\bullet$ with X_n in the nth place and all the maps are zero.*

Suppose that $\mathcal{C}$ is a presentable stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$.

Definition 5.1.7. The *Postnikov t-structure* on $\mathcal{C}^{\mathrm{Fil}}$ is a pair of full subcategories $((\mathcal{C}^{\mathrm{Fil}})_{\geq 0}, (\mathcal{C}^{\mathrm{Fil}})_{\leq 0})$, where $(\mathcal{C}^{\mathrm{Fil}})_{\geq 0}$ consists of objects $C_\bullet \in \mathcal{C}^{\mathrm{Fil}}$ such that $C_n \in \mathcal{C}_{\geq n}$ for all $n \in \mathbb{Z}$, and $(\mathcal{C}^{\mathrm{Fil}})_{\geq 0}$ consists of objects $C_\bullet \in \mathcal{C}^{\mathrm{Fil}}$ such that $C_n \in \mathcal{C}_{\leq n}$ for all $n \in \mathbb{Z}$.

The natural left adjoint $\tau_{\geq 0} : \mathcal{C}^{\mathrm{Fil}} \to (\mathcal{C}^{\mathrm{Fil}})_{\geq 0}$ to the inclusion $(\mathcal{C}^{\mathrm{Fil}})_{\geq 0} \to \mathcal{C}^{\mathrm{Fil}}$ takes the connective cover with respect to the Postnikov t-structure. Explicitly, for $C_\bullet \in \mathcal{C}$ we have $(\tau_{\geq 0}\, C_\bullet)_n = \tau_{\geq n}\, C_n$, i.e., it takes the n-connective cover of filtration n with respect to the t-structure on $\mathcal{C}$ for all $n \in \mathbb{Z}$.

Since $\mathcal{C}^{\mathrm{Fil}}$ is presentable and $\tau_{\geq 0}$ preserves sifted colimits, the subcategory $(\mathcal{C}^{\mathrm{Fil}})_{\geq 0}$ is closed under sifted colimits. If $\mathcal{C}$ has a symmetric monoidal structure, then $(\mathcal{C}^{\mathrm{Fil}})_{\geq 0}$ is closed under the induced symmetric monoidal structure in $\mathcal{C}^{\mathrm{Fil}}$.

5.1.3 Product-preserving presheaves as connective filtered objects

Let k be a field and Mod_k the ∞-category of k-module spectra. Denote by $\mathrm{Mod}_k^{\mathrm{ff}}$ the full subcategory of Mod_k consisting of finite free objects. In this section, we identify the category $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$ with the category $(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}$ of Postnikov connective filtered k-modules.

Let c be the composite

$$\mathrm{Mod}_k^{\mathrm{ff}} \hookrightarrow \mathrm{Mod}_k \xrightarrow{Y} \mathrm{Mod}_k^{\mathrm{Fil}} \xrightarrow{\tau_{\geq 0}} (\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0},$$

where Y is the constant embedding in Proposition 5.1.6 and $\tau_{\geq 0}$ the connective cover with respect to the Postnikov t-structure. Then c sends $\Sigma^n k$ to $k^{n,n}$ for all n, where $k^{n,n}$ is the filtered object

$$\cdots \to 0 \to 0 \to \Sigma^n k \to \Sigma^n k \to \cdots,$$

where $(k^{n,n})_i = \Sigma^n k$ for $i \leq n$ and $(k^{n,n})_i = 0$ for $i > n$.

Since the Yoneda embedding $\nu : \mathrm{Mod}_k^{\mathrm{ff}} \to \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$ preserves finite coproducts, by Theorem 5.1.1.(3) there is a unique colimit-preserving functor Φ that serves as the left Kan extension of c along ν.

$$\begin{tikzcd}
\mathrm{Mod}_k^{\mathrm{ff}} \arrow[r, "c"] \arrow[d, "\nu"] & (\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0} \\
\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

Theorem 5.1.8. *The functor $\Phi : \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \to (\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}$ induces an equivalence of ∞-categories.*

The theorem will follow from a general criterion, which we recall below.

Definition 5.1.9. [Lur09, Definition 5.5.8.18] Let $\mathcal{C}$ be an ∞-category with all sifted colimits. An object $X \in \mathcal{C}$ is *projective* if the functor $\mathrm{Map}_{\mathcal{C}}(X, -)$ preserves geometric realizations.

If in addition $\mathcal{C}$ admits filtered colimits, then $X \in \mathcal{C}$ is compact projective if $\mathrm{Map}_{\mathcal{C}}(X, -)$ preserves filtered colimits and geometric realization.

Proposition 5.1.10. *[Lur09, Proposition 5.5.8.22] Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts, $\mathcal{D}$ an ∞-category which admits sifted colimits, and $F : \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{D}$ the left Kan extension of $f : \mathcal{C} \to \mathcal{D}$ along the Yoneda embedding $\nu : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$. Then F is an equivalence if and only if the following conditions are satisfied:*

1. *The functor f is fully faithful;*

2. *The essential image of f consists of compact projective objects of $\mathcal{D}$;*

3. *$\mathcal{D}$ is generated by the essential image of f under sifted colimits.*

Proof of Theorem 5.1.8. We check that the functor

$$c : \mathrm{Mod}_k^{\mathrm{ff}} \hookrightarrow \mathrm{Mod}_k \xrightarrow{Y} \mathrm{Mod}_k^{\mathrm{Fil}} \xrightarrow{\tau_{\geq 0}} (\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}$$

satisfies the three conditions in Proposition 5.1.10.

(1). The category $\mathrm{Mod}_k^{\mathrm{ff}}$ is generated under finite coproducts by shifts of k. Since c preserves finite coproducts, it suffices to check that there are natural equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mod}_k}(\Sigma^i k, \Sigma^j k) = \mathrm{Map}_{\mathrm{Mod}_k^{\mathrm{ff}}}(\Sigma^i k, \Sigma^j k) &\simeq \mathrm{Map}_{(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}}(c(\Sigma^i k), c(\Sigma^j k)) \\ &= \mathrm{Map}_{(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}}(k^{i,i}, k^{j,j})\end{aligned}$$

for all i, j. When $j \geq i$, this follows from the adjunction $Y_n \dashv (-)_n$ Proposition 5.1.6.(2). When $j < i$, both sides are trivial since $\pi_n(\mathrm{Map}_{\mathrm{Mod}_k^{\mathrm{ff}}}(\Sigma^i k, \Sigma^j k)) \cong \mathrm{Ext}_k^{-n}(\Sigma^i k, \Sigma^j k))$. Hence the functor c is fully faithful.

(2). Next we want to show that the essential image of c consists of compact projective objects of $(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}$, i.e., $\mathrm{Map}_{(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}}(c(X), -)$ preserves sifted colimits for all $X \in \mathrm{Mod}_k^{\mathrm{ff}}$. It suffices to consider $X = \Sigma^n k$. Let $C_\bullet = \operatorname{colim}_\alpha C_\bullet^\alpha$ be any sifted colimit in $(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}$. Since $c(X) = k^{n,n} = Y_n(k)$ and Y_n is left adjoint to the sifted-colimit-preserving functor $(-)_n$ (Proposition 5.1.6.(2)), there are natural equivalences

$$\begin{aligned}\mathrm{Map}_{(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}}(k^{n,n}, \operatorname{colim}_\alpha C_\bullet^\alpha) &\simeq \mathrm{Map}_{\mathrm{Mod}_{k,\geq n}}\Big(\Sigma^n k, \big(\operatorname{colim}_\alpha C_\bullet^\alpha\big)_n\Big) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{k,\geq n}}\big(\Sigma^n k, \operatorname{colim}_\alpha (C_\bullet^\alpha)_n\big).\end{aligned}$$

Since $(C_\bullet^\alpha)_n$ is n-connective by assumption, there are further natural equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mod}_{k,\geq n}}\big(\Sigma^n k, \operatorname*{colim}_\alpha (C_\bullet^\alpha)_n\big) \simeq \mathrm{Map}_{\mathrm{Mod}_{k,\geq 0}}\big(k, \operatorname*{colim}_\alpha \Sigma^{-n}(C_\bullet^\alpha)_n\big) &\simeq \Omega^\infty\big(\operatorname*{colim}_\alpha \Sigma^{-n}(C_\bullet^\alpha)_n\big)\big) \\ &\simeq \operatorname*{colim}_\alpha \Omega^\infty\Sigma^{-n}(C_\bullet^\alpha)_n.\end{aligned}$$

The last equivalence is because $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathrm{Spaces}$ preserves sifted colimits [Lur17, Proposition 1.4.3.9] and so does the forgetful functor $\mathrm{Mod}_{k,\geq 0} \to \mathrm{Sp}_{\geq 0}$. Similarly, there are natural equivalences

$$\begin{aligned}\operatorname*{colim}_\alpha \mathrm{Map}_{(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}}(k^{n,n}, C_\bullet^\alpha) \simeq \operatorname*{colim}_\alpha \mathrm{Map}_{\mathrm{Mod}_{k,\geq n}}(\Sigma^n k, C_\bullet^\alpha) &\simeq \operatorname*{colim}_\alpha \mathrm{Map}_{\mathrm{Mod}_{k,\geq 0}}(k, \Sigma^{-n} C_\bullet^\alpha) \\ &\simeq \operatorname*{colim}_\alpha \Omega^\infty\Sigma^{-n}(C_\bullet^\alpha)_n.\end{aligned}$$

Therefore $k^{n,n}$ is compact projective in $(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}$ for all n.

(3). Since $(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}$ has all finite coproducts, we will show instead that $\{k^{n,n}\}_{n\in\mathbb{Z}}$ generates $(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}$ under colimits. This is equivalent to showing that $\mathrm{Map}_{(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}}(k^{n,n}, C_\bullet) \simeq 0$ if and only if $C_\bullet$ is the constant object on 0. As in part (2), we have

$$\mathrm{Map}_{(\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}}(k^{n,n}, C_\bullet) \simeq \mathrm{Map}_{\mathrm{Mod}_{k,\geq n}}(\Sigma^n k, C_n) \simeq \mathrm{Map}_{\mathrm{Mod}_{k,\geq 0}}(k, \Sigma^{-n} C_n) \simeq \Omega^\infty\Sigma^{-n} C_n,$$

which is zero if and only if $C_n \simeq 0$ as desired. □

5.1.4 Grading conventions

Denote by $\Sigma^n = \Sigma^{n,n}$ the nth categorical suspension in $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$ for $n \geq 0$. Since $\mathrm{Re} : \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \to \mathrm{Mod}_k$ is a left adjoint, it commutes with categorical suspension, so $\mathrm{Re}(\Sigma\nu(X)) \simeq \Sigma X$.

Fro all $n \in \mathbb{Z}$, define the nth internal suspension $\Sigma^{n,0}$ in $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$ by $\Sigma^{n,0}X(Y) := X(\Sigma^n Y)$ where $X \in \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}), Y \in \mathrm{Mod}_k^{\mathrm{ff}}$. In particular, there is a natural equivalence $\Sigma^{1,0}\nu(Y) \simeq \nu(\Sigma Y)$ for $Y \in \mathrm{Mod}_k^{\mathrm{ff}}$.

Therefore we obtain a bigrading on $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$ with respect to the set of compact generators $\{\Sigma^{a,b}\nu(k), a \in \mathbb{Z}, b \in \mathbb{Z}_{\geq 0}\}$.

Definition 5.1.11. The bigraded homotopy groups of $X \in \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$ are given by

$$\pi_{a,b}(X) := \pi_0(\mathrm{Map}_{\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})}(\Sigma^{a,b}\nu(k), X))$$

for all $a \in \mathbb{Z}, b \in \mathbb{Z}_{\geq 0}$.

Therefore the realization map Re induces a map of homotopy groups $\pi_{a,b}(X) \to \pi_a(\mathrm{Re}(X))$.

5.1.5 The spectral sequence of a filtered object

Given a filtered object $\mathcal{C}_\bullet \in \mathrm{Mod}_k^{\mathrm{Fil}}$, there is an associated spectral sequence recovering the usual spectral sequence associated to a filtered chain complex over k, which we recall in this section.

For each $i \in \mathbb{N}$, the category $\mathrm{Mod}_k^{\mathrm{Fil}}$ admits natural automorphims $(i) : \mathrm{Mod}_k^{\mathrm{Fil}} \to \mathrm{Mod}_k^{\mathrm{Fil}}$ sending $C_\bullet$ to $C_{\bullet+i}$ by precomposing with the automorphism $(\mathbb{Z}^{\mathrm{Fil}})^{\mathrm{op}} \to (\mathbb{Z}^{\mathrm{Fil}})^{\mathrm{op}}$ sending n to $n-i$ for all $n \in \mathbb{Z}$.

Definition 5.1.12. Denote by $\tau : (1) \to (0) = \mathrm{id}$ the natural transformation that encodes the shift map in filtration, and by $C\tau$ the cofiber of $\mathbb{1}(1) \to \mathbb{1}$. Here $\mathbb{1}$ is the monoidal unit of $\mathrm{Mod}_k^{\mathrm{Fil}}$.

For example, on the constant object $\cdots \to 0 \to 0 \to X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} \cdots$, the natural transformation τ encodes the following diagram in Mod_k:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[r, "\mathrm{id}"] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X \arrow[r, "\mathrm{id}"] & X \arrow[r, "\mathrm{id}"] & X \arrow[r, "\mathrm{id}"] & \cdots
\end{tikzcd}$$

Recall that there is a realization functor $\mathrm{Re} : \mathrm{Mod}_k^{\mathrm{Fil}} \to \mathrm{Mod}_k$ sending $C_\bullet$ to $\operatorname{colim}_n C_{-n}$. We say that a filtered object is *complete* if $\lim_n C_n = 0$.

Proposition 5.1.13. *[Lur17, Section 1.2.2] To each complete filtered object $C_\bullet \in \mathrm{Mod}_k^{\mathrm{Fil}}$, there is an associated spectral sequence*

$$E^2_{p+q,q} = \pi_{p+q}\mathrm{Gr}(C_\bullet)_p \simeq \pi_{p+q}(C_\bullet \otimes C\tau)_p \Rightarrow \pi_{p+q}(\mathrm{Re}(C_\bullet)),$$

which coincides with the usual spectral sequence of a filtered object.

Remark 5.1.14. We use the homological Adams grading in this paper.

Under the equivalence $\Phi : \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \to (\mathrm{Mod}_k^{\mathrm{Fil}}))_{\geq 0}$, the generators $\Sigma^{b,0}\nu(k)$ are sent to $k^{b,b}$. The categorical suspension Σ in $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$ is sent to the pointwise suspension in $(\mathrm{Mod}_k^{\mathrm{Fil}}))\geq 0$ and the internal suspension $\Sigma^{b,0}$ is sent to the composition of the shift $(-b)$ with the bth pointwise suspension. Hence the natural transformation $\tau : (1) \to \mathrm{id}$ in $(\mathrm{Mod}_k^{\mathrm{Fil}}))_{\geq 0}$ corresponds to the natural transformation defined by $\Sigma^{0,1}X(Y) \simeq \Sigma X(\Sigma^{-1}Y) \to X(Y)$ in $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$, which we will also denote by τ.

Proposition 5.1.15. *The restricted realization functor* $\mathrm{Re} : (\mathrm{Mod}_k^{\mathrm{Fil}}))_{\geq 0} \to \mathrm{Mod}_k$ *and the realization functor* $\mathrm{Re} : \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \to \mathrm{Mod}_k$ *are compatible under* Φ*.*

Proof. The functor Re preserves sifted colimits and there is a natural equivalence $\mathrm{Re} \circ \Phi(\nu(X)) \simeq X \simeq \mathrm{Re} \circ \nu(X)$ for $X \in \mathrm{Mod}_k^{\mathrm{ff}}$. □

5.2 Bar spectral sequences via deformed comonads

Let k be a field and Mod_k the ∞-category of k-module spectra. Let $\mathcal{O}$ be a nonunital ∞-operad in Mod_k and $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_k)$ the ∞-category of $\mathcal{O}$-algebras in Mod_k. There is a commonad $\mathrm{sqz} \circ \mathrm{cot}$ on Mod_k associated to the operad $\mathcal{O}$ that classically comes from the Quillen adjunction between the indecomposable functor and the square-zero extension. The goal of this section is to construct a lift the commonad $\mathrm{sqz} \circ \mathrm{cot}$ to a commonad on product-preserving presheaves.

5.2.1 Comonads and the weight grading

We start by recalling the adjunction

$$\mathrm{cot} \dashv \mathrm{sqz} : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_k) \to \mathrm{Mod}_k$$

that gives raise to the commonad of interest. Furthermore, this commonad is compatible with a natural weight-grading on $\mathcal{O}$-algebras that reflects the decomposition of the free

$\mathcal{O}$-algebra into homogeneous pieces. The main references for this subsection are [Bra17, Appendix] and [BM19, Section 4, 5].

Given a nonunital ∞-operad $\mathcal{O}$ in $\mathcal{C}$, i.e., an object in $\mathrm{Alg}(\mathrm{Sseq}(\mathcal{C}))$, there is a free-forgetful adjunction $\mathrm{free}_{\mathcal{O}} \dashv \mathrm{forget} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ that gives rise to an augmented monad on $\mathcal{C}$. The free $\mathcal{O}$-algebra on $X \in \mathcal{C}$ is given by $\mathrm{free}_{\mathcal{O}}(X) \simeq \bigoplus_{i\geq 1} O(i) \otimes_{h\Sigma_i} X^{\otimes i}$. On the other hand, there is a square-zero functor $\mathrm{sqz} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, sending an object $X \in \mathcal{C}$ to $X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ with the trivial $\mathcal{O}$-algebra structure. It admits a left adjoint cot, called the cotangent fiber.

Definition 5.2.1. Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a symmetric monoidal structure. Denote by $\mathcal{C}^{\mathrm{Fil},\geq 1} := \mathrm{Fun}((\mathbb{Z}^{\mathrm{Fil}}_{\geq 1})^{\mathrm{op}}, \mathcal{C})$ the ∞-category of *positively-filtered* objects. An object $C_\bullet$ in $\mathcal{C}^{\mathrm{Fil},\geq 1}$ is a diagram $\cdots \to C_3 \to C_2 \to C_1$ in $\mathcal{C}$.

Let $\mathcal{C}^{\mathrm{wt}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}_{\geq 1}, \mathcal{C})$ be the ∞-category of *weight-graded* objects.

Here $\mathbb{Z}^{\mathrm{Fil}}_{\geq 1} \subset \mathbb{Z}^{\mathrm{Fil}}$ is the subcategory with underlying category the discrete set $\mathbb{Z}_{\geq 1}$, which is closed under the symmetric monoidal product given by addition. Hence we obtain analogous adjunctions as in proposition 5.1.6.

The underlying functors $\mathrm{und} : \mathcal{C}^{\mathrm{wt}} \to \mathcal{C}$ sending $\{X_n\}_{n\geq 1}$ to $\bigoplus_{n\geq 1} X_n$, and $\mathrm{und} : \mathcal{C}^{\mathrm{Fil},\geq 1} \to \mathcal{C}$ sending $C_\bullet$ to C_1 are symmetric monoidal, and so is the associated graded functor $\mathcal{C}^{\mathrm{Fil},\geq 0} \to \mathcal{C}^{\mathrm{wt}}$. Hence we obtain underlying functors $\mathrm{und} : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathrm{Fil},\geq 1}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $\mathrm{und} : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathrm{wt}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. When the context is clear, we write $\mathrm{Alg}_{\mathcal{O}}$ for $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $\mathrm{Alg}^{\mathrm{wt}}_{\mathcal{O}}$ for $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathrm{wt}})$.

Definition 5.2.2. [BM19, Construction 5.2] The *adic filtration* functor $\mathrm{adic} : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathrm{Fil},\geq 1})$ is the left adjoint to $\mathrm{und} : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathrm{Fil},\geq 1}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.

On a free $\mathcal{O}$-algebra $A = \mathrm{free}_{\mathcal{O}}(X) \in \mathrm{Alg}_{\mathcal{O}}$, the adic filtration sends A to the positively-filtered object

$$\mathrm{adic}(\mathrm{free}_{\mathcal{O}}(X))_n = \bigoplus_{i\geq n} \mathcal{O}(i) \underset{h\Sigma i}{\otimes} X^{\otimes i}$$

as expected. The composition $\mathrm{Gr} \circ \mathrm{adic}$ sends $\mathrm{free}_{\mathcal{O}}(X)$ to the weight-graded object $\{\mathcal{O}(i) \otimes_{h\Sigma i} X^{\otimes i}\}_{i\geq 1}$, which is left ajoint to the underlying functor $\mathrm{und} : \mathrm{Alg}^{\mathrm{wt}}_{\mathcal{O}} \to \mathrm{Alg}_{\mathcal{O}}$.

Proposition 5.2.3. *[BM19, Remark 4.24, Proposition 5.8] Let $\mathcal{C} = \mathrm{Mod}_k$. There is a commutative diagram of sifted-colimit-preserving functors*

$$\begin{tikzcd}
\mathrm{Mod}_k^{\mathrm{wt}} \arrow[r, shift left, "\mathrm{free}_{\mathcal{O}}"] \arrow[d, "\mathrm{und}"] & \mathrm{Alg}_{\mathcal{O}}^{\mathrm{wt}} \arrow[l, shift left, "\mathrm{forget}"] \arrow[r, shift left, "\mathrm{cot}"] \arrow[d, "\mathrm{und}"] & \mathrm{Mod}_k^{\mathrm{wt}} \arrow[l, shift left, "\mathrm{sqz}"] \arrow[d, "\mathrm{und}"] \\
\mathrm{Mod}_k \arrow[r, shift left, "\mathrm{free}_{\mathcal{O}}"] & \mathrm{Alg}_{\mathcal{O}} \arrow[l, shift left, "\mathrm{forget}"] \arrow[r, shift left, "\mathrm{cot}"] & \mathrm{Mod}_k \arrow[l, shift left, "\mathrm{sqz}"]
\end{tikzcd}$$

where horizontal composites are the identity in both directions, and all horizontal pairs are natural adjunctions. The functor cot *is computed by the geometric realization of the bar construction $|\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, -)|$ for both $\mathrm{Alg}_{\mathcal{O}}$ and $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{wt}}$.*

Remark 5.2.4. Proposition 5.8 in [BM19] is concerned only with the case $\mathcal{O} = \mathbb{E}_\infty^{\mathrm{nu}}$ and restricts to connective objects. Since we do not require the horizontal rows in the diagram to be comonadic adjunctions, both conditions can be dropped and the proof is standard.

The adjunctions in the right square give rise to sifted-colimit-preserving comonads $\mathrm{cot} \circ \mathrm{sqz}$ on Mod_k and $\mathrm{Mod}_k^{\mathrm{wt}}$ compatible with $\mathrm{und} : \mathrm{Mod}_k^{\mathrm{wt}} \to \mathrm{Mod}_k$.

5.2.2 Functoriality of product-preserving presheaves

In order to construct lifts of the commonad $\mathrm{sqz} \circ \mathrm{cot}$, we need a few functoriality results that will be used repeatedly in the sections to follow.

Proposition 5.2.5. *[Pst, Proposition 2.10, A.13] Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then the induced adjunction $F^* \dashv F_* : \mathcal{P}(\mathcal{C}) \rightleftarrows \mathcal{P}(\mathcal{D})$ restricts to an adjunction $F^* \dashv F_* : \mathcal{P}_\Sigma(\mathcal{C}) \rightleftarrows \mathcal{P}_\Sigma(\mathcal{D})$. Here F_* is given by precomposition and F^* the left Kan extension of $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\nu} \mathcal{P}_\Sigma(\mathcal{D})$ along the Yoneda embedding $\nu : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$. Furthermore, the right adjoint F_* preseves sifted-colimits.*

In particular, there is an equivalence $F^* \circ \nu \simeq \nu \circ F$ by construction.

Suppose that $\mathcal{C}, \mathcal{D}$ are ∞-categories with all colimits. Furthermore $i : \mathcal{C}_0 \hookrightarrow \mathcal{C}$, $i : \mathcal{D}_0 \hookrightarrow \mathcal{D}$ are finite coproduct-preserving inclusions of full subcategories $\mathcal{C}_0$, $\mathcal{D}_0$ that are closed under finite coproducts.

Proposition 5.2.6. *Suppose that we have an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ such that F restricts to a finite coproduct-preserving functor $f : \mathcal{C}_0 \to \mathcal{D}_0$.*

1. *There is a natural equivalence $f_* \circ \nu_0(X) \simeq \nu_0 \circ G(X) \in \mathcal{P}_\Sigma(\mathcal{C}_0)$ for $X \in \mathcal{D}$;*

2. *There is a natural equivalence $f_* \circ \nu(X) \simeq \nu_0 \circ G \circ i(X) \in \mathcal{P}_\Sigma(\mathcal{C}_0)$ for $X \in \mathcal{D}_0$;*

3. *If in addition G preserves sifted colimits, then there is a natural equivalence $\mathrm{Re} \circ f_*(X) \simeq G \circ \mathrm{Re}(X) \in \mathcal{D}$ for $X \in \mathcal{P}_\Sigma(\mathcal{C}_0)$.*

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "f"] \arrow[d, "\nu"] \arrow[dd, bend right=60, "i"'] & \mathcal{D}_0 \arrow[d, "\nu"] \arrow[dd, bend left=60, "i"] \\
\mathcal{P}_\Sigma(\mathcal{C}_0) \arrow[r, shift left, "f^*"] \arrow[d, shift right, "\mathrm{Re}"'] & \mathcal{P}_\Sigma(\mathcal{D}_0) \arrow[l, shift left, "f_*"] \arrow[d, shift right, "\mathrm{Re}"'] \\
\mathcal{C} \arrow[r, shift left, "F"] \arrow[u, shift right, "\nu_0"'] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[u, shift right, "\nu_0"']
\end{tikzcd}$$

Proof. (1). For a given $X \in \mathcal{D}$ and any $Y \in \mathcal{C}_0$, there are natural equivalences

$$f_*(\nu_0(X))(Y) \simeq \nu_0(X)(f(Y)) \simeq \mathrm{Map}_{\mathcal{D}}(F(Y), X) \simeq \mathrm{Map}_{\mathcal{C}}(Y, G(X)) \simeq \nu_0(G(X))(Y).$$

(2). For $X \in \mathcal{D}_0$, there are natural equivalences

$$f_* \circ \nu(X) \simeq f_* \circ \nu_0(i(X)) \simeq \nu_0 \circ G(i(X)).$$

(3). Note that Re and f_* both preserves sifted colimits by Proposition 5.2.5. Hence it suffices to check on the image of ν. For a given $X \in \mathcal{C}_0$, $G \circ \mathrm{Re}(\nu(X)) \simeq G(i(X))$. For any $Y \in \mathcal{D}_0$,

$$\begin{aligned} f_*\nu(X)(Y) \simeq \nu(X)(f(Y)) \simeq \mathrm{Map}_{\mathcal{D}_0}(f(Y), X) &\simeq \mathrm{Map}_{\mathcal{D}}(F(Y), i(X)) \\ &\simeq \mathrm{Map}_{\mathcal{C}}(Y, G(i(X))) \simeq \nu_0(G(i(X)))(Y). \end{aligned}$$

Hence for any $Z \in \mathcal{D}$, there are natural equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{D}_0}(\mathrm{Re} \circ f_*(\nu(X)), Z) \simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{D}_0)}(f_*(\nu(X)), \nu_0(Z)) &\simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{D}_0)}(\nu_0(G(i(X))), \nu_0(Z)) \\ &\simeq \mathrm{Map}_{\mathcal{D}_0}(G(i(X)), Z). \end{aligned}$$

The first equivalence follows from the adjunction $\mathrm{Re} \dashv \nu_0$ in Proposition 5.1.2. The second equivalence follows from part (2). By the Yoneda lemma, there is a natural equivalence $\mathrm{Re} \circ f_*(\nu(X)) \simeq G(i(X))$ as desired. $\square$

5.2.3 Deforming adjunctions

The goal of this section is to construct compatible deformations of the comonad $\mathrm{cot} \circ \mathrm{sqz}$ associated to a nonunital operad $\mathcal{O}$ in Mod_k.

Recall that $\mathrm{Mod}_k^{\mathrm{ff}}$ is the full subcategory of Mod_k consisting of finite free objects. Denote by $\mathrm{Mod}_k^{\mathrm{ff,wt}}$ the full subcategory of $\mathrm{Mod}_k^{\mathrm{ff,wt}}$ consisting of objects $X_\bullet$ whose underlying object $\bigoplus_i X_i$ is finite free. Let $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ff}}$ (resp $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ff,wt}}$) be the full subcategory of $\mathrm{Alg}_{\mathcal{O}}$ (resp. $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{wt}}$) consisting of the essential image of $\mathrm{Mod}_k^{\mathrm{ff}}$ (resp. $\mathrm{Mod}_k^{\mathrm{ff,wt}}$) under $\mathrm{free}_{\mathcal{O}}$.

By Proposition 5.2.5, the restriction of the cotangent functor $\mathrm{cot} : \mathrm{Alg}_{\mathcal{O}}^{\mathrm{ff}} \to \mathrm{Mod}_k^{\mathrm{ff}}$ gives rise to a sifted-colimit-preserving comonad $\mathrm{cot}^* \circ \mathrm{cot}_*$ on $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$. We want to show that this comonad is a lift of the comonad $\mathrm{cot} \circ \mathrm{sqz}$ on Mod_k via the realization map Re.

Proposition 5.2.7. *There is a commutative diagram*

$$\begin{array}{ccccc} \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) & \xrightarrow{\mathrm{cot}_*} & \mathcal{P}_\Sigma(\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ff}}) & \xrightarrow{\mathrm{cot}^*} & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \\ \downarrow{\scriptstyle \mathrm{Re}} & & \downarrow{\scriptstyle \mathrm{Re}} & & \downarrow{\scriptstyle \mathrm{Re}} \\ \mathrm{Mod}_k & \xrightarrow{\mathrm{sqz}} & \mathrm{Alg}_{\mathcal{O}} & \xrightarrow{\mathrm{cot}} & \mathrm{Mod}_k \end{array}.$$

Proof. Since every arrow in the diagram preserves sifted-colimits, it suffices to check on the image of ν. It follows from Proposition 5.2.6.(1) and the adjunction $\mathrm{cot} \dashv \mathrm{sqz}$ that the left square commutes. On the other hand, $\mathrm{cot}^* \nu(Y) \simeq \nu(\mathrm{cot}(Y))$ by definition, so the right square commutes. $\square$

Note that the exact same arguments goes through in the weight-graded setting, which yields the weight-graded version of Proposition 5.2.7.

Proposition 5.2.8. *The restriction of the cotangent functor* $\mathrm{cot} : \mathrm{Alg}_{\mathcal{O}}^{\mathrm{ff,wt}} \to \mathrm{Mod}_k^{\mathrm{ff,wt}}$ *gives rise to a sifted-colimit-preserving comonad* $\mathrm{cot}^* \circ \mathrm{cot}_*$ *on* $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff,wt}})$ *and there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff,wt}}) \arrow[r, "\cot_*"] \arrow[d, "\mathrm{Re}"] & \mathcal{P}_\Sigma(\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ff,wt}}) \arrow[r, "\cot^*"] \arrow[d, "\mathrm{Re}"] & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff,wt}}) \arrow[d, "\mathrm{Re}"] \\
\mathrm{Mod}_k^{\mathrm{wt}} \arrow[r, "\mathrm{sqz}"] & \mathrm{Alg}_{\mathcal{O}}^{\mathrm{wt}} \arrow[r, "\cot"] & \mathrm{Mod}_k^{\mathrm{wt}}
\end{tikzcd}.$$

The weight-graded version is compatible with the ungraded version via the underlying functor $\mathrm{und} : \mathrm{Mod}_k^{\mathrm{wt}} \to \mathrm{Mod}_k$ and its restriction $\mathrm{und} : \mathrm{Mod}_k^{\mathrm{wt,ff}} \to \mathrm{Mod}_k^{\mathrm{ff}}$.

Proposition 5.2.9. *There is a commutative cube of sifted-colimit-preserving functors*

$$\begin{tikzcd}
\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff,wt}}) \arrow[rrr, "\cot^*\circ\cot_*"] \arrow[dr, "\mathrm{und}^*"'] \arrow[dd, "\mathrm{Re}"'] & & & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff,wt}}) \arrow[dr, "\mathrm{und}^*"] \arrow[dd, "\mathrm{Re}"' near start] & \\
& \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \arrow[rrr, "\cot^*\circ\cot_*" near start] \arrow[dd, "\mathrm{Re}"' near start] & & & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \arrow[dd, "\mathrm{Re}"] \\
\mathrm{Mod}_k^{\mathrm{wt}} \arrow[rrr, "\cot\circ\mathrm{sqz}" near end] \arrow[dr, "\mathrm{und}"'] & & & \mathrm{Mod}_k^{\mathrm{wt}} \arrow[dr, "\mathrm{und}"] & \\
& \mathrm{Mod}_k \arrow[rrr, "\cot\circ\mathrm{sqz}"] & & & \mathrm{Mod}_k
\end{tikzcd}.$$

Proof. The two squares on the side of the cube are commutative by naturality of the realization map Re. The front and back squares are commutative by Proposition 5.2.7 and 5.2.8. The bottom square is commutative by Proposition 5.2.3. It remains to check the commutativity of the top square, i.e., the composition of squares

$$\begin{tikzcd}
\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff,wt}}) \arrow[r, "\cot_*"] \arrow[d, "\mathrm{und}^*"] & \mathcal{P}_\Sigma(\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ff,wt}}) \arrow[r, "\cot^*"] \arrow[d, "\mathrm{und}^*"] & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff,wt}}) \arrow[d, "\mathrm{und}^*"] \\
\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \arrow[r, "\cot_*"] & \mathcal{P}_\Sigma(\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ff}}) \arrow[r, "\cot^*"] & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})
\end{tikzcd}.$$

The right square of left adjoints is commutative by functoriality. All arrows in the left square preserve sifted colimits, so it suffices to check on the image of ν. It follows from Proposition 5.2.6.(2) and the adjunction $\cot \dashv \mathrm{sqz}$ that for any $X \in \mathrm{Mod}_k^{\mathrm{ff,wt}}$ or $\mathrm{Mod}_k^{\mathrm{ff}}$, there are natural equivalences

$$\mathrm{und}^* \circ \cot_*(\nu(X)) \simeq \mathrm{und}^* \circ \nu_0(\mathrm{sqz}(i(X))) \simeq \nu_0(\mathrm{und} \circ \mathrm{sqz}(i(X))).$$

Similarly, we have $\cot_* \circ \mathrm{und}^*(\nu(X)) \simeq \cot_*(\nu(\mathrm{und}(X))) \simeq \nu_0(\mathrm{sqz}(i(\mathrm{und}(X))))$. Hence the left square commutes by Proposition 5.2.3. $\square$

5.2.4 Bar spectral sequence as décalage

For any $\mathcal{O}$-algebra A in Mod_k, there is a bar spectral sequence obtained by the skeletal filtration of the geometric realization of the bar construction

$$E^2_{s,t} = \pi_s(\pi_t(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A)) \Rightarrow \pi_{s+t}(|\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A)|).$$

When $A = \mathrm{sqz}(X)$ is a trivial $\mathcal{O}$-algebra, we have $\mathrm{cot} \circ \mathrm{sqz}(X) \simeq |\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, \mathrm{sqz}(X))|$. The comonad $\mathrm{cot}^* \circ \mathrm{cot}_*$ on $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$ is a deformation of the comonad $\mathrm{cot} \circ \mathrm{sqz}$ on Mod_k in light of Proposition 5.2.7.

In order to identify the bar spectral sequence for an $\mathcal{O}$-algebra A with the décalage of the filtered object $\Phi(\mathrm{cot}^* \circ \nu(A)) \in (\mathrm{Mod}_k^{\mathrm{Fil}})_{\geq 0}$, we need a more explicit formula for $\mathrm{cot}^* \circ \nu(A) \in \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$.

Lemma 5.2.10. *There is a natural equivalence $\nu_0(A) \simeq |\nu_0 \mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, A)|$ for $A \in \mathrm{Alg}_{\mathcal{O}}$.*

Proof. For any $M \in \mathrm{Mod}_k^{\mathrm{ff}}$, $\nu_0(A)(\mathrm{free}_{\mathcal{O}}(M)) = \mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}}(\mathrm{free}_{\mathcal{O}}(M), A) \simeq \mathrm{Map}_{\mathrm{Mod}_k}(M, A)$, and

$$\begin{aligned} |\nu_0 \mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, A)|(\mathrm{free}_{\mathcal{O}}(M)) &= |\mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}}(\mathrm{free}_{\mathcal{O}}(M), \mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, A))| \\ &\simeq |\mathrm{Map}_{\mathrm{Mod}_k}(M, \mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, A)))|. \end{aligned}$$

The augmented simplicial object $\mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, A) \to A$ in $\mathrm{Alg}_{\mathcal{O}}$ admits a splitting after forgetting to Mod_k. Hence the equivalence $\mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, A) \xrightarrow{\simeq} A$ in Mod_k is preserved by any functor. This completes the proof. □

Denote by $\mathrm{Alg}^{\mathrm{f}}_{\mathcal{O}}$ the full subcategory of $\mathrm{Alg}_{\mathcal{O}}$ consisting of free $\mathcal{O}$-algebras.

Lemma 5.2.11. *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{P}_\Sigma(\mathrm{Alg}^{\mathrm{ff}}_{\mathcal{O}}) & \xrightarrow{\mathrm{cot}^*} & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \\ {\scriptstyle \nu_0}\uparrow & & {\scriptstyle \nu_0}\uparrow \\ \mathrm{Alg}^{\mathrm{f}}_{\mathcal{O}} & \xrightarrow{\mathrm{cot}} & \mathrm{Mod}_k \end{array}.$$

Proof. Recall that $\cot^*$ preserves all colimits by Proposition 5.2.5. Applying Lemma 5.2.10 for $A \in \mathrm{Alg}^{\mathrm{f}}_{\mathcal{O}}$ yields natural equivalences $\cot^* \circ \nu_0(A) \simeq |\cot^* \nu_0 \mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, A)|$. Express $\mathrm{free}^{\circ i}_{\mathcal{O}}(A)$ as filtered colimits $\mathrm{free}^{\circ i}_{\mathcal{O}}(A) = \operatorname{colim}_{\alpha_i} X_{\alpha_i}$ of $X_{\alpha_i} \in \mathrm{Alg}^{\mathrm{ff}}_{\mathcal{O}}$ inductively along i, i.e. there is an inclusion of filtered systems $\{X_{\alpha_i}\}_{\alpha_i} \subset \{X_{\alpha_{i+1}}\}_{\alpha_{i+1}}$ induced by the map $\mathrm{free}^{\circ i}_{\mathcal{O}}(A) \to \mathrm{free}^{\circ (i+1)}_{\mathcal{O}}(A)$ of free $\mathcal{O}$-algebras for all i. By Proposition 5.1.3 ν_0 preserve filtered colimits, so

$$\begin{aligned}\cot^* \circ \nu_0(A) \simeq |\cot^* \circ \nu_0(\mathrm{free}^{\circ i}_{\mathcal{O}}(A))| \simeq |\cot^* \circ \nu_0(\operatorname*{colim}_{\alpha_i} X_{\alpha_i})| &\simeq |\operatorname*{colim}_{\alpha_i} \cot^* \circ \nu(X_{\alpha_i})| \\ &\simeq |\operatorname*{colim}_{\alpha_i} \nu(\cot(X_{\alpha_i}))|.\end{aligned}$$

Appealing again to the splitting of the augmented simplicial object $\mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, A) \to A$ in Mod_k, for any $Y \in \mathrm{Mod}^{\mathrm{ff}}_k$ there are natural equivalences

$$\begin{aligned}\nu_0(\cot(A))(Y) = \mathrm{Map}_{\mathrm{Mod}_k}(Y, \cot(A)) &\simeq |\mathrm{Map}_{\mathrm{Mod}_k}(Y, \cot \mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, A))| \\ &\simeq |\mathrm{Map}_{\mathrm{Mod}_k}(Y, \operatorname*{colim}_{\alpha_i} \cot(X_{\alpha_i}))| \\ &\simeq |\nu_0(\operatorname*{colim}_{\alpha_i} \cot(X_{\alpha_i}))(Y)| \simeq |\operatorname*{colim}_{\alpha_i} \nu(\cot(X_{\alpha_i}))|(Y)\end{aligned}$$

as desired. In the second equivalence we used the fact cot is a left adjoint. □

Now we are ready to prove the main theorem of this section.

Theorem 5.2.12. *For $A \in \mathrm{Alg}_{\mathcal{O}}$, the bar spectral sequence*

$$E^2_{s,t} = \pi_s(\pi_t(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A) \Rightarrow \pi_{s+t}(|\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A)|)$$

is naturally isomorphic to the spectral sequence for the realization of $(\Phi(\cot^* \circ \nu_0(A)) \in (\mathrm{Mod}^{\mathrm{Fil}}_k)_{\geq 0}$. If $A = \mathrm{sqz}(X)$ with $X \in \mathrm{Mod}^{\mathrm{ff}}_k$, then the bar spectral sequence is isomorphic to the spectral sequence for the realization $\Phi(\cot^* \circ \cot_*(\nu(X)))$.

Proof. Since the simplicial maps in the bar construction preserve internal degrees, we can consider the functor $F : \mathrm{Alg}_{\mathcal{O}} \to (\mathrm{Mod}^{\mathrm{Fil}}_k)_{\geq 0}$ given by the level-wise whitehead tower of the bar construction $F(A)_t = |\tau_{\geq t} \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A)|$, with associated graded $F(A)_t / F(A)_{t-1} \simeq$

$|\pi_t(\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A))|$. Hence the spectral sequence associated to the filtered object $F(A)$ is precisely the bar spectral sequence for A by Proposition 5.1.13.

It remains to show that there is a natural equivalence $F(A) \simeq \Phi(\cot^* \circ v_0(A))$. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}} \arrow[r, "v_0"] \arrow[ddr, "\mathrm{Bar}_\bullet(-)"'] & \mathcal{P}_\Sigma(\mathrm{Alg}^{\mathrm{ff}}_{\mathcal{O}}) \arrow[r, "\cot^*"] & \mathcal{P}_\Sigma(\mathrm{Mod}^{\mathrm{ff}}_k) \arrow[r, "\Phi"] & (\mathrm{Mod}^{\mathrm{Fil}}_k)_{\geq 0} \\
& \mathcal{P}_\Sigma(\mathrm{Alg}^{\mathrm{ff}}_{\mathcal{O}})^{\Delta^{\mathrm{op}}} \arrow[r, "\cot^*"] \arrow[u, "|-|"] & \mathcal{P}_\Sigma(\mathrm{Mod}^{\mathrm{ff}}_k)^{\Delta^{\mathrm{op}}} \arrow[r, "\Phi"] \arrow[u, "|-|"] & (\mathrm{Mod}^{\mathrm{Fil}}_k)^{\Delta^{\mathrm{op}}}_{\geq 0} \arrow[u, "|-|"] \\
& (\mathrm{Alg}^{\mathrm{f}}_{\mathcal{O}})^{\Delta^{\mathrm{op}}} \arrow[r, "\cot"] \arrow[u, "v_0"] & (\mathrm{Mod}_k)^{\Delta^{\mathrm{op}}} \arrow[u, "v_0"] &
\end{tikzcd},$$

where $\mathrm{Bar}_\bullet(-) = \mathrm{Bar}_\bullet(\mathrm{free}_{\mathcal{O}}, \mathrm{free}_{\mathcal{O}}, -)$ is the free resolution in $\mathrm{Alg}_{\mathcal{O}}$. The left square commutes by Lemma 5.2.10, the upper middle square commutes because $\cot^*$ preserves geometric realization, the bottom square commutes by Lemma 5.2.11, and the right square commutes because Φ preserves geometric realization.

Hence there is a natural equivalence $\Phi(\cot^* \circ v_0(A)) \simeq |\Phi(v_0 \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A))|$. Since v_0 preserves filtered colimits by Proposition 5.1.3 and $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A)$ is levelwise given by filtered colimits of finite free k-modules, $\Phi \circ v_0 \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A)$ is precisely the simplicial filtered object $F_\bullet$ with

$$(F_\bullet)_n = \tau_{\geq n} \mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, A).$$

Taking geometric realization completes the proof.

When $A = \mathrm{sqz}(X)$ with $X \in \mathrm{Mod}^{\mathrm{ff}}_k$, there is a natural equivalence $\cot^* \circ \cot_* v(X) \simeq \cot^* v_0(\mathrm{sqz}(X))$ for $X \in \mathrm{Mod}^{\mathrm{ff}}_k$ by Proposition 5.2.6.(2). $\square$

5.2.5 Weight decomposition

The bar spectral sequence has a natural weight decomposition induced by the weight decomposition of the comonad $\cot \circ \mathrm{sqz}$, which we recall below.

The comonad $\cot \circ \mathrm{sqz}$ associated to a nonunital operad $\mathcal{O}$ on Mod_k admits a natural decomposition into homogeneous pieces: for $X \in \mathrm{Mod}_k$,

$$\cot \circ \mathrm{sqz}(X) = |\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}_{\mathcal{O}}, \mathrm{sqz}(X))| \simeq \mathrm{Bar}(\mathcal{O}) \circ V \simeq \bigoplus_{i \geq 1} \mathrm{Bar}(\mathcal{O})(i) \underset{h\Sigma_i}{\otimes} X^{\otimes n},$$

where $\mathrm{Bar}(\mathcal{O}) \simeq |\mathrm{Bar}_\bullet(1, \mathcal{O}, 1)|$ is the bar construction in the ∞-category of symmetric sequences in Mod_k and $\mathrm{Bar}(\mathcal{O}) \circ X$ denotes the composition of symmetric sequences with X regarded as a symmetric sequence concentrated in arity 1. (Br Appendix D)

The weight i part of the comonad $\cot \circ \mathrm{sqz}$ can be extracted as follows. There are natural left adjoints $c : \mathrm{Mod}_k \to \mathrm{Mod}_k^{\mathrm{wt}}$ sending X to the weight-graded object $X_\bullet$ with $X_1 = X$ and $X_i = 0$ for $i > 1$ and $(-)_i : \mathrm{Mod}_k^{\mathrm{wt}} \to \mathrm{Mod}_k$ sending $X_\bullet$ to X_i. Hence the composite

$$D_i := (\cot \circ \mathrm{sqz})_i : \mathrm{Mod}_k \xrightarrow{c} \mathrm{Mod}_k^{\mathrm{wt}} \xrightarrow{\cot \circ \mathrm{sqz}} \mathrm{Mod}_k^{\mathrm{wt}} \xrightarrow{(-)_i} \mathrm{Mod}_k$$

sends X to $\mathrm{Bar}(\mathcal{O})(i) \otimes_{h\Sigma_i} X^{\otimes i}$.

Similarly, we would like to extract the weight i piece of the deformed comonad $\cot^* \circ \cot_*$. Note that the functor c restricts to $c : \mathrm{Mod}_k^{\mathrm{ff}} \to \mathrm{Mod}_k^{\mathrm{ff,wt}}$ and $(-)_i$ restricts to $(-)_i : \mathrm{Mod}_k^{\mathrm{ff,wt}} \to \mathrm{Mod}_k^{\mathrm{ff}}$.

Proposition 5.2.13. *There is a commutative diagram of sifted-colimit-preserving functors*

$$\begin{tikzcd}
\mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \arrow[r, "c^*"] \arrow[d, "\mathrm{Re}"] & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff,wt}}) \arrow[r, "\cot^* \circ \cot_*"] \arrow[d, "\mathrm{Re}"] & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff,wt}}) \arrow[r, "(-)_i^*"] \arrow[d, "\mathrm{Re}"] & \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}) \arrow[d, "\mathrm{Re}"] \\
\mathrm{Mod}_k \arrow[r, "c"] & \mathrm{Mod}_k^{\mathrm{wt}} \arrow[r, "\cot \circ \mathrm{sqz}"] & \mathrm{Mod}_k^{\mathrm{wt}} \arrow[r, "(-)_i"] & \mathrm{Mod}_k
\end{tikzcd}.$$

Denote by $\mathbb{D}_i$ the composite along the top horizontal line, which preserves sifted colimits. Hence there is a natural equivalence $\mathrm{Re} \circ \mathbb{D}_i(\nu(X)) \simeq D_i(X)$.

Proof. The leftmost and rightmost squares are commutative by naturality of the realization map Re and the middle square is commutative by Proposition 5.2.8. □

5.3 Universal differentials in the bar spectral sequence

From here on, we specialize to the case where $k = \mathbb{F}_2$ and $\mathcal{O} = s\mathscr{L}$ is the spectral Lie operad in $\mathrm{Mod}_{\mathbb{F}_2}$. Then $\cot \circ \mathrm{sqz}(X) \simeq \mathrm{Bar}(s\mathscr{L}) \circ X \simeq \bigoplus_{i \geq 1} (X^{\otimes i})_{h\Sigma_i}$, with $D_2(X) = (X^{\otimes 2})_{h\Sigma_2}$.

We start by computing the differential on any element in the weight 2 part of the bar spectral sequence in the universal class, which is encoded by $\mathbb{D}_2$.

5.3.1 Computing $\mathbb{D}_2$

Lemma 5.3.1. *For $X \in \mathrm{Mod}_k^{\mathrm{ff}}$, there is a natural equivalence $\mathbb{D}_2(\nu(X)) \simeq \Sigma\nu_0(\Sigma^{-1}D_2(X))$.*

Proof. Recall that the weight 2 part of the bar spectral sequence for $\mathrm{sqz}(X)$ collapses on the E^2-page with no extension problems. Furthermore, all permanent cycles are in filtration 1. Hence $\mathrm{Gr}(\Phi(\mathbb{D}_2(X))) \simeq G_1(D_2(X))$ by theorem 5.2.12, where $G_1(D_2(X))$ is the graded object with $D_2(X)$ in degree 1 and 0 otherwise. The adjunction in Proposition 5.1.6.(2) yields a natural transformation $\Phi(\mathbb{D}_2(X)) \to Y_1(D_2(X))$, where $Y_1(D_2(X))$ is the filtered object with $Y_1(D_2(X))_i = 0$ for $i \leq 0$ and $Y_1(D_2(X)) = D_2(X)$ for $i \geq 1$. Hence we obtain a natural transformation $\mathbb{D}_2(X)\nu \to \Sigma\nu_0(\Sigma^{-1}D_2(X))$, which is an equivalence. □

Lemma 5.3.2. *There are natural equivalences $\mathbb{D}_2(X \oplus Y) \simeq \mathbb{D}_2(X) \oplus \mathbb{D}_2(Y) \oplus \Sigma^{1,0}X \otimes Y$, symmetric in $X, Y \in \mathcal{P}_\Sigma(\mathrm{Mod}_{\mathbb{F}_2}^{\mathrm{ff}})$. In general*

$$\mathbb{D}_2(X_0 \oplus \cdots \oplus X_n) \simeq \mathbb{D}_2(X_0) \oplus \cdots \oplus \mathbb{D}_2(X_n) \oplus \bigoplus_{0 \leq i < j \leq n} \Sigma^{1,0}X_i \otimes X_j,$$

for any $n \geq 1$.

Proof. Since finite products are finite coproducts, for $X, Y \in \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}})$ there are natural comparison maps symmetric in X, Y

$$\begin{tikzcd} \mathbb{D}_2(X) \oplus \mathbb{D}_2(Y) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \mathbb{D}_2(X \oplus Y) \arrow[r] & \mathbb{D}_2(X) \oplus \mathbb{D}_2(Y) \end{tikzcd},$$

exhibiting $\mathbb{D}_2(X) \oplus \mathbb{D}_2(Y)$ functorially as a direct summand of $\mathbb{D}_2(X \oplus Y)$. In other words, there is a functor $\mathbb{G} : \mathcal{P}_\Sigma(\mathrm{Mod}_{\mathbb{F}_2}^{\mathrm{ff}})^{\times 2} \to \mathcal{P}_\Sigma(\mathrm{Mod}_{\mathbb{F}_2}^{\mathrm{ff}})$ symmetric and sifted-colimit-preserving in both arguments such that $\mathbb{D}_2(X \oplus Y) \simeq \mathbb{D}_2(X) \oplus \mathbb{D}_2(Y) \oplus \mathbb{G}(X, Y)$. By Lemma 5.3.1, there are natural equivalences

$$\begin{aligned}
\mathbb{D}_2(\nu(X)\oplus\nu(Y)) \simeq \mathbb{D}_2(\nu(X\oplus Y)) &\simeq \Sigma\nu_0(\Sigma^{-1}D_2(X\oplus Y)) \\
&\simeq \Sigma\nu_0(\Sigma^{-1}(D_2(X)\oplus D_2(Y)\oplus X\otimes Y)) \\
&\simeq \Sigma\nu_0(\Sigma^{-1}D_2(X))\oplus\Sigma\nu_0(\Sigma^{-1}D_2(Y))\oplus\Sigma\nu(\Sigma^{-1}X\otimes Y).
\end{aligned}$$

Therefore, there is a natural equivalence $\mathbb{G}(\nu(X),\nu(Y))\simeq\Sigma\nu_0(\Sigma^{-1}X\otimes Y)\simeq\Sigma^{1,0}\nu(X\otimes Y)$. The first part of the lemma follows from the fact that $\mathbb{G}$ preserves sifted colimits and ν is symmetric monoidal (Proposition 5.1.4). The second part follows from standard induction on $\bigoplus_{1\le i\le n}X_i\simeq(\bigoplus_{1\le i\le n-1}X_i)\oplus X_n$. □

Proposition 5.3.3. *For $X\in\mathcal{P}_\Sigma(\mathrm{Mod}^{\mathrm{ff}}_{\mathbb{F}_2})$, there is a natural cofiber sequence*

$$\Sigma\mathbb{D}_2(X)\to\mathbb{D}_2(\Sigma X)\to\Sigma^{3,2}X^{\otimes 2}.$$

Proof. Consider the resolution of ΣX obtained by smash product with the simplcial circle

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & * \sqcup s_1(c)\sqcup s_0(c) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & * \sqcup c \arrow[r, shift left=1] \arrow[r, shift right=1] & *
\end{tikzcd}$$

given by

$$*\sqcup(\bigsqcup_{0\le i\le n} s_n\circ s_{n-1}\circ\cdots\circ\hat{s}_i\circ\cdots\circ s_0(c))$$

on simplicial level $n+1$, cf. [Lod11, 1.2]. We omit the degeneracy maps in the diagrams for simplicity. Denote by X_i the degeneracy $s_n\circ s_{n-1}\circ\cdots\circ\hat{s}_i\circ\cdots\circ s_0(c)\otimes X$ in simplicial level n for all $i\le n$. Recall that $\mathbb{D}_2$ commutes with geometric realizations. Applying the direct sum decomposition in Lemma 5.3.2 levelwise to the simplicial resolutions of $\mathbb{D}_2(\Sigma X)\to\Sigma\mathbb{D}_2(X)$ yields a cofiber sequence of simplicial resolutions

$$\begin{tikzcd}
\cdots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & \mathbb{D}_2(X)^{\oplus 3} \arrow[d] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbb{D}_2(X)^{\oplus 2} \arrow[d] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{D}_2(X) \arrow[d] \arrow[r, shift left=1] \arrow[r, shift right=1] & * \arrow[d] \arrow[r] & \Sigma\mathbb{D}_2(X) \arrow[d] \\
\cdots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & \mathbb{D}_2(X^{\oplus 3}) \arrow[d] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbb{D}_2(X^{\oplus 2}) \arrow[d] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{D}_2(X) \arrow[d] \arrow[r, shift left=1] \arrow[r, shift right=1] & * \arrow[d] \arrow[r] & \mathbb{D}_2(\Sigma X) \arrow[d] \\
\cdots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & (\Sigma^{1,0}X^{\otimes 2})^{\oplus 3} \arrow[u, dashed, bend right] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \Sigma^{1,0}X^{\otimes 2} \arrow[u, dashed, bend right] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & * \arrow[u, dashed, bend right] \arrow[r, shift left=1] \arrow[r, shift right=1] & * \arrow[u, dashed, bend right] \arrow[r] & Y
\end{tikzcd}.$$

Since the resolution of Y is a sub simplicial object of the resolution of $\mathbb{D}_2(\Sigma X)$, the degeneracy maps are the restriction of those in $\mathbb{D}_2(\Sigma X)$. Appealing again to the natural decomposition in Lemma 5.3.2, the restricted degeneracy maps are given by $s_i(x \otimes y) \simeq s_i(x) \otimes s_i(y) \simeq s_i(y) \otimes s_i(x)$. It follows from standard induction that the simplicial level $n+1$ of Y is explicitly given by $(\Sigma^{1,0} X^{\otimes 2})^{\oplus \binom{n}{2}} = \bigoplus_{0 \le i < j \le n} X_i \otimes X_j$. It remains to show that Y is equivalent to $\Sigma^2 \Sigma^{1,0} X^{\otimes 2}$, which has a resolution obtained by tensoring $\Sigma^{1,0} X^{\otimes 2}$ with the simplicial 2-sphere given by

$$* \sqcup \Big(\coprod_{0 \le i < j \le n} s_n \circ s_{n-1} \circ \cdots \circ \hat{s}_j \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_0(c) \Big)$$

on simplicial level $n+1$ for all $n \ge 1$, i.e., all possible degeneracies of a two-cell c and the basepoint. We abbreviate the degeneracy $s_n \circ s_{n-1} \circ \cdots \circ \hat{s}_j \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_0(c)) \otimes (\Sigma^{1,0} X^{\otimes 2})$ as $\Sigma^{1,0} X_{i,j}$ for all $i < j$.

Consider the partial simplicial diagram $Y_{\le 2} \in \mathrm{Fun}(\Delta^{\mathrm{op}}_{\le 2}, \mathcal{P}_\Sigma(\mathrm{Mod}_k^{\mathrm{ff}}))$ of Y given by

$$X^{\otimes 2} \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; * \;\substack{\longrightarrow\\[-2pt]\longrightarrow}\; *$$

Left Kan extension along the inclusion $\Delta^{\mathrm{op}}_{\le 2} \to \Delta^{\mathrm{op}}$ yields a simplicial object that is precisely the resolution of $\Sigma^2 X^{\otimes 2}$, Hence we obtain a natural map $\phi : \Sigma^{1,0} \Sigma^2 X^{\otimes 2} \to Y$ coming from

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] \arrow[r, shift right=2] & (\Sigma^{1,0} X^{\otimes 2})^{\oplus 3} \arrow[r, shift left=1.5] \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] \arrow[r, shift right=1.5] \arrow[d, "\phi_3"] & \Sigma^{1,0} X^{\otimes 2} \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] \arrow[d, "\phi_2"] & * \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] \arrow[d, "\phi_1"] & * \arrow[r] \arrow[d, "\phi_0"] & \Sigma^2 \Sigma^{1,0} X^{\otimes 2} \arrow[d, "\phi"] \\
\cdots \arrow[r, shift left=2] \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] \arrow[r, shift right=2] & (\Sigma^{1,0} X^{\otimes 2})^{\oplus 3} \arrow[r, shift left=1.5] \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] \arrow[r, shift right=1.5] & \Sigma^{1,0} X^{\otimes 2} \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] & * \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] & * \arrow[r] & Y
\end{tikzcd}.$$

It remains to show that ϕ is an equivalence.

We will show by induction that ϕ_n is an equivalence for all n. By construction $\phi_2 : X_{0,1} \to X_0 \otimes X_1$ is an equivalence. Suppose that ϕ_n is an equivalence $X_{i,j}$ to $X_i \otimes X_j$ for all $0 \le i < j \le n-1$. Consider the degeneracy map $s_n : \bigoplus_{0 \le i < j \le n-1} X_{i,j} \to \bigoplus_{0 \le i < j \le n-1} X_{i,j}$, along with the restriction of the degeneracy maps $s_{n-1} : \bigoplus_{0 \le i \le n-2} \Sigma^{1,0} X_{i,n-1} \to \bigoplus_{0 \le i \le n-2} \Sigma^{1,0} X_{i,n}$ and $s_{n-2} : \Sigma^{1,0} X_{n-2,n-1} \to \Sigma^{1,0} X_{n-1,n}$ from simplicial level n to $n+1$ in $\Sigma^2 \Sigma^{1,0} X^{\otimes 2}$. All three maps are equivalences, so we obtain an equivalence

$$s_n \oplus s_{n-1} \oplus s_{n-2} : \Big(\bigoplus_{0 \le i < j \le n-1} \Sigma^{1,0} X_{i,j} \Big) \oplus \Big(\bigoplus_{0 \le i \le n-2} \Sigma^{1,0} X_{i,n-1} \Big) \oplus X_{n-2,n-1} \to \bigoplus_{0 \le i < j \le n} \Sigma^{1,0} X_{i,j}.$$

Similarly, the degeneracy map $s_n : \bigoplus_{0\le i<j\le n-1} \Sigma^{1,0} X_i \otimes X_j \to \bigoplus_{0\le i<j\le n-1} \Sigma^{1,0} X_i \otimes X_j$ and the restriction of the degeneracy map $s_{n-1} : \bigoplus_{0\le i\le n-2} \Sigma^{1,0} X_i \otimes X_{n-1} \to \bigoplus_{0\le i\le n-2} \Sigma^{1,0} X_i \otimes X_n$, $s_{n-2} : \Sigma^{1,0} X_{n-2} \otimes X_{n-1} \to \Sigma^{1,0} X_{n-1} \otimes X_n$ from simplicial level n to $n+1$ in Y are both equivalences, so we obtain an equivalence $s_n \oplus s_{n-1} \oplus s_{n-2}$:

$$\Big(\bigoplus_{0\le i<j\le n-1} \Sigma^{1,0} X_i \otimes X_j\Big) \oplus \Big(\bigoplus_{0\le i\le n-2} \Sigma^{1,0} X_i \otimes X_{n-1}\Big) \oplus \Sigma^{1,0} X_{n-2} \otimes X_{n-1} \to \bigoplus_{0\le i<j\le n} \Sigma^{1,0} X_i \otimes X_j.$$

Thus we obtain a commutative diagram

$$\begin{tikzcd}
\Big(\bigoplus\limits_{0\le i<j\le n-1} \Sigma^{1,0} X_{i,j}\Big) \oplus \Big(\bigoplus\limits_{0\le i\le n-2} \Sigma^{1,0} X_{i,n-1}\Big) \oplus \Sigma^{1,0} X_{n-2,n-1} \arrow[r, "s_n\oplus s_{n-1}\oplus s_{n-2}"] \arrow[d, "\phi_n"] & \bigoplus\limits_{0\le i<j\le n} \Sigma^{1,0} X_{i,j} \arrow[d, "\phi_{n+1}"] \\
\Big(\bigoplus\limits_{0\le i<j\le n-1} \Sigma^{1,0} X_i \otimes X_j\Big) \oplus \Big(\bigoplus\limits_{0\le i\le n-2} \Sigma^{1,0} X_i \otimes X_{n-1}\Big) \oplus \Sigma^{1,0} X_{n-2} \otimes X_{n-1} \arrow[r, "s_n\oplus s_{n-1}\oplus s_{n-2}"] & \bigoplus\limits_{0\le i<j\le n} \Sigma^{1,0} X_i \otimes X_j.
\end{tikzcd}$$

Since the top, left, and bottom arrows are all equivalences, so is ϕ_{n+1}. This completes the proof. $\square$

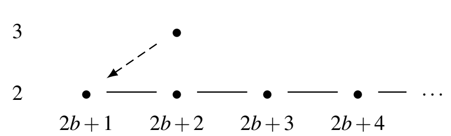
For simplicity, we use $\mathbb{F}_2^{a,b}$ to denote $\Sigma^{a,b} \nu(\mathbb{F}_2)$.

Lemma 5.3.4. *For all $a, n \ge 1$ and $b \in \mathbb{Z}$,*

$$\mathbb{D}_2(\mathbb{F}_2^{a+b,a}) \simeq \Big(\bigoplus_{i\ge 0} \mathbb{F}_2^{2a+2b+i,a+1}\Big) \oplus \bigoplus_{n=1}^{a-1} \Sigma^{2b+a+n,a+1} C\tau^n.$$

Proof. Recall that when $a = 0$, then $\mathbb{D}_2(\nu(\Sigma^b \mathbb{F}_2)) \simeq \Sigma \nu_0(\Sigma^{-1} D_2(\mathbb{F}_2)) \simeq \bigoplus_{i\ge 0} \mathbb{F}_2^{2b+i,1}$ by Proposition 5.1.3 and Lemma 5.3.1.

If $a = 1$, Proposition 5.3.3 yields a cofiber sequence $\mathbb{F}_2^{2b+1,2} \to \Sigma \mathbb{D}_2(\nu(\Sigma^b \mathbb{F}_2)) \to \mathbb{D}_2(\Sigma \nu(\Sigma^b \mathbb{F}_2)) \to \mathbb{F}_2^{2b+2,3}$, so the spectral sequence for $\pi_* \mathrm{Re}(\mathbb{D}_2(\Sigma \nu(\Sigma^b \mathbb{F}_2)))$ is



The only possible attaching map goes from $\Sigma^{2+2b,3}\nu(\mathbb{F}_2)$ to the bottom cell of $\Sigma\mathbb{D}_2(\nu(\Sigma^b\mathbb{F}_2))$. This map is indeed nonzero since we know the realization is $\mathrm{Re}\circ\mathbb{D}_2(\Sigma\nu(\Sigma^b\mathbb{F}_2)) \simeq D_2(\Sigma^{b+1}\mathbb{F}_2)$ by Proposition 5.2.13. Hence in the cofiber sequence $\mathbb{F}_2^{2b+1,2}$ hits the bottom cell and $\mathbb{D}_2(\Sigma\nu(\Sigma^b\mathbb{F}_2)) \simeq \bigoplus_{i\geq 2}\mathbb{F}_2^{2b+i,2}$.

If $a=2$, setting $X=\mathbb{F}_2^{b+2,2}$ in Proposition 5.3.3 yields a cofiber sequence

$$\Sigma\mathbb{D}_2(\Sigma\nu(\Sigma^b\mathbb{F}_2)) \to \mathbb{D}_2(\Sigma^2\nu(\Sigma^b\mathbb{F}_2)) \to \mathbb{F}_2^{2b+4,5}.$$

Hence the spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma^2\nu(\Sigma^b\mathbb{F}_2)))$ is given by Figure 5-1.

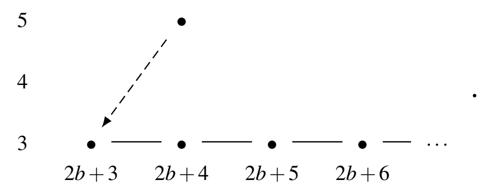


Figure 5-1: The spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma^2\nu(\Sigma^b\mathbb{F}_2)))$.

The only possible differential goes from $\mathbb{F}_2^{2b+4,5}$ to the bottom cell of $\Sigma\mathbb{D}_2(\Sigma\nu(\Sigma^b\mathbb{F}_2))$. The differential indeed happens since the realization $\mathrm{Re}\circ\mathbb{D}_2(\Sigma^2\nu(\Sigma^b\mathbb{F}_2)) \simeq D_2(\Sigma^{b+2}\mathbb{F}_2)$ (Proposition 5.1.15, Proposition 5.2.13) has only one class in degree $s+t=2b+4$. Hence

$$\mathbb{D}_2(\Sigma^2\nu(\Sigma^b\mathbb{F}_2)) \simeq \mathbb{D}_2(\mathbb{F}_2^{2+b,2}) \simeq \left(\bigoplus_{i\geq 0}\mathbb{F}_2^{2b+4+i,3}\right)\oplus\Sigma^{2b+3,3}C\tau.$$

Now we induct on $a\geq 2$. Suppose that

$$\mathbb{D}_2(\Sigma^a\nu(\Sigma^b\mathbb{F}_2)) \simeq \mathbb{D}_2(\Sigma^{a+b,a}\nu(\mathbb{F}_2)) \simeq \left(\bigoplus_{i\geq 0}\mathbb{F}_2^{2a+2b+i,a+1}\right)\oplus\bigoplus_{n=1}^{a-1}\Sigma^{2b+a+n,a+1}C\tau^n.$$

Setting $X=\mathbb{F}_2^{a+b,a}$ in Proposition 5.3.3 yields a cofiber sequence

$$\Sigma\mathbb{D}_2(\Sigma^a\nu(\Sigma^b\mathbb{F}_2)) \to \mathbb{D}_2(\Sigma^{a+1}\nu(\Sigma^b\mathbb{F}_2)) \to \mathbb{F}_2^{2a+2b+2,2a+3}.$$

Hence the spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma^{a+1}\nu(\Sigma^b\mathbb{F}_2)))$ is given by Figure 5-2.

The only possible differential has source $\mathbb{F}_2^{2a+2b+2,2a+3}$, as indicated by the two dashed

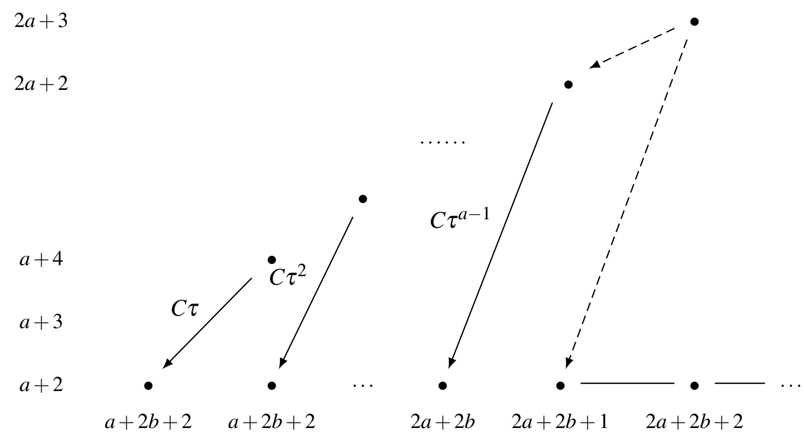


Figure 5-2: The spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma^{a+1}\nu(\Sigma^b\mathbb{F}_2)))$.

arrows. Suppose that the short arrow is nontrivial. Then the τ^{a-1}-torsion class in bidegree $(s+t,s)=(2a+2b,a+2)$ cannot receive another differential and thus is a permanent cycle, a contradiction. On the other hand, the class $\mathbb{F}_2^{2a+2b+2,2a+3}$ must support a differential since $\mathrm{Re}\circ\mathbb{D}_2(\Sigma^{a+1}\nu(\Sigma^b\mathbb{F}_2))\simeq D_2(\Sigma^{b+a+1}\mathbb{F}_2)$ has only one class in degree $s+t=2a+2b+2$. Therefore

$$\mathbb{D}_2(\Sigma^{a+1}\nu(\Sigma^b\mathbb{F}_2))\simeq\mathbb{D}_2(\mathbb{F}_2^{a+b+1,a+1})\simeq\left(\bigoplus_{i\geq 0}\mathbb{F}_2^{2a+2b+2+i,a+2}\right)\oplus\bigoplus_{n=1}^{a}\Sigma^{a+2b+1+n,a+2}C\tau^n.$$

$\square$

Notation 5.3.5. We name the elements on the E^2-page of the spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\mathbb{F}_2^{a+b,a}))$ as follows. When $a=0,1$, let $\bar{Q}_i(x_{a+b,a})=\bar{Q}^{i+b}(x_{b,a})$ denote the unique element in bidegree $(a+2b+i,a+1)$ for $i\geq a$. When $a\geq 2$, let $\bar{Q}_i(x_{a+b,a})=\bar{Q}^{i+b}(x_{a+b,a})$ denote the unique element in bidegree $(a+2b+i,a+1)$ for $i\geq 1$ and $\gamma_j(x_{a+b,a})$ the unique element in bidegree $(a+2b+j,a+j+1)$ for $2\leq j\leq a$.

By Lemma 5.3.4, the cofiber sequence $\Sigma\mathbb{D}_2(\mathbb{F}_2^{a+b,a})\to\mathbb{D}_2(\mathbb{F}_2^{a+b+1,a+1})\to\mathbb{F}_2^{2a+2b+2,2a+3}$ splits after tensoring with $C\tau$. Hence $\sigma\bar{Q}_i x_{a+b,a}=\bar{Q}_i(\sigma x_{a+b,a})=\bar{Q}_i(x_{a+b+1,a+1})$, and similarly $\sigma\gamma_i(x_{a+b,a})=\gamma_i(\sigma x_{a+b,a})=\gamma_i(x_{a+b+1,a+1})$.

Next we study the application $\mathbb{D}_2$ to a shift of $C\tau^n$, which encodes how differentials are propagated via the comonad structure.

Lemma 5.3.6. *For $a-1 \geq n \geq 1$ and any $b \in \mathbb{Z}$, there is an equivalence of filtered objects*

$$\mathbb{D}_2(\Sigma^{a+b,a}C\tau^n) \simeq \Big(\bigoplus_{i=1}^{2n} \Sigma^{a+2b-n+i+1,a+n+2}C\tau^i\Big) \oplus \Big(\bigoplus_{i=1}^{n} \Sigma^{a+2b+i,a+1}C\tau^i\Big)$$
$$\oplus \Big(\bigoplus_{j\geq 0} \Sigma^{a+2b+n+j+1,a+1}C\tau^n\Big) \oplus \Big(\bigoplus_{j=0}^{a-n-2} \Sigma^{a+2b+n+j+2,a+n+j+3}C\tau^{2n}\Big) \oplus \Sigma^{2a+2b+1,2a+n+2}C\tau^n.$$

Remark 5.3.7. In general, the spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma^{a+b,a}C\tau^n))$ looks like Figure 5-3 with the $s+t$ degree shifted to the right by $2b$.

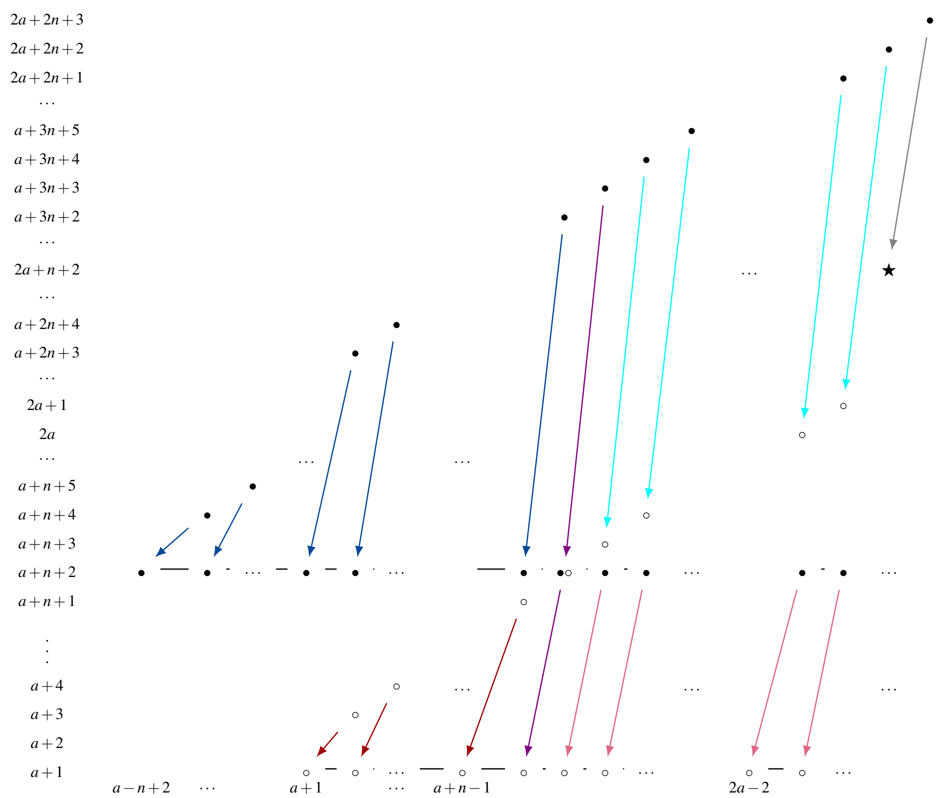


Figure 5-3: The spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma^{a+b,a}C\tau^n))$.

In light of Notation 5.3.5, the lemma says the following. Suppose that we have a d_{n+1}-differential from $\gamma_{n+1}(x)$ to $\bar{Q}_n(x)$ represented by $\Sigma^{a+b,a}C\tau^n$ in the spectral sequence for $\pi_*(\mathrm{Re}(\mathbb{D}_2(\mathbb{F}_2^{a-1+c,a-1})))$, where $a-1 \geq n+1 \geq 2$ and $b = 2c+n-1$. Applying $\mathbb{D}_2$ to this differential, we deduce that the differentials in the spectral sequence for $\pi_*(\mathrm{Re}(\mathbb{D}_2(\Sigma^{a+b,a}C\tau)))$ are the following:

1. For $2 \leq i \leq n$, a d_i-differential $\gamma_i \circ \bar{Q}_n(x) \mapsto \bar{Q}_{i-1} \circ \bar{Q}_n(x)$, represented by a red arrow

connecting white dots coming from $\mathbb{D}_2(\mathbb{F}_2^{a+b,a})$;

2. For $1 \le j \le 2n-1$, a d_j-differential $\gamma_j \circ \gamma_{n+1}(x) \mapsto \bar{Q}_j \circ \gamma_{n+1}(x)$, represented by a blue arrow connecting black dots coming from $\mathbb{D}_2(\mathbb{F}_2^{a+b+1,a+n+1})$;

3. A d_{n+1}-differential $\gamma_{n+1} \circ \bar{Q}_n(x) + \bar{Q}_{2n} \circ \gamma_{n+1}(x) \mapsto \bar{Q}_n \circ \bar{Q}_n(x)$, and a d_{2n+1}-differential from $\gamma_{2n} \circ \gamma_{n+1}(x)$ to the remaining $\mathbb{F}_2^{a+2b+n+1,a+n+2}$, both arrows are in purple;

4. For each $i > n$, a d_{n+1}-differential $\bar{Q}_{n+i} \circ \gamma_{n+1}(x) \mapsto \bar{Q}_i \circ \bar{Q}_n(x)$, represented by a pink arrow connecting a black dot and a white dot;

5. For each i with $n+1 \le i \le a-1$, a d_{2n+1}-differential $\gamma_{n+i} \circ \gamma_{n+1}(x) \mapsto \gamma_{i+1} \circ \bar{Q}_n(x)$, represented by a cyan arrow connecting a black dot and a white dot;

6. A d_n-differential $\gamma_{a+n} \circ \gamma_{n+1}(x) \mapsto \Sigma^{0,1}\gamma_{n+1}(x) \cdot \bar{Q}_n(x)$, connecting the top black dot to the cross term represented by the star.

Proof of Lemma 5.3.6. We assume $b = 0$, since changing b by 1 simply shifts the $s+t$ degree by 2.

Using the defining cofiber sequence $\mathbb{F}_2^{0,0} \to C\tau \to \mathbb{F}_2^{1,n+1}$, we deduce that $\mathrm{Gr}(\mathbb{D}_2(C\tau)) \simeq \mathbb{D}_2(\mathbb{F}_2^{0,0}) \oplus \mathbb{D}_2(\mathbb{F}_2^{1,n+1}) \oplus \Sigma^{0,1}\mathbb{F}_2^{0,0} \otimes \mathbb{F}_2^{1,n+1}$, so the E^2-page and higher differentials of the spectral sequence for $\pi_*(\mathrm{Re}(\mathbb{D}_2(C\tau)))$ is Figure 5-4.

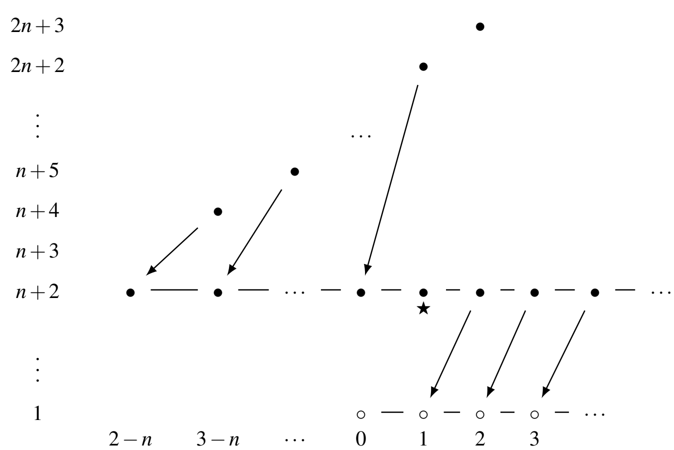


Figure 5-4: The spectral sequence for $\pi_*(\mathrm{Re}(\mathbb{D}_2(C\tau)))$.

Classes from the three summands labeled respectively by white dots, black dots, and star. Since $\mathrm{Re}(\mathbb{D}_2(C\tau)) \simeq D_2(\mathrm{Re}(C\tau)) \simeq 0$, we deduce that the two parallel lines are killed by d_2-differentials. Starting from $s+t=2$ to the right, there is only one way for each class on the top line to be killed, as indicated by the solid arrows.

Now consider the cofiber sequence $\mathbb{F}_2^{1,1} \to \Sigma^n C\tau \to \mathbb{F}_2^{2,n+2}$. The E^2-page of the spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma^a C\tau))$ is given by $\mathbb{D}_2(\mathbb{F}_2^{1,1}) \oplus \mathbb{D}_2(\mathbb{F}_2^{2,n+2}) \oplus \Sigma^{0,1}\mathbb{F}_2^{1,1} \otimes \mathbb{F}_2^{2,n+2}$ as depicted below on the left. On the other hand, it follows from the cofiber sequence $\Sigma\mathbb{D}_2(C\tau^n) \to \mathbb{D}_2(\Sigma C\tau^n) \to \Sigma^{2,3}C\tau^n \otimes C\tau^n$ (Proposition 5.3.3) that the spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma C\tau))$ is depicted in Figure 5-5 on the right, where the classes represented by asterisks come from $\Sigma^{2,3}C\tau^n \otimes C\tau^n$.

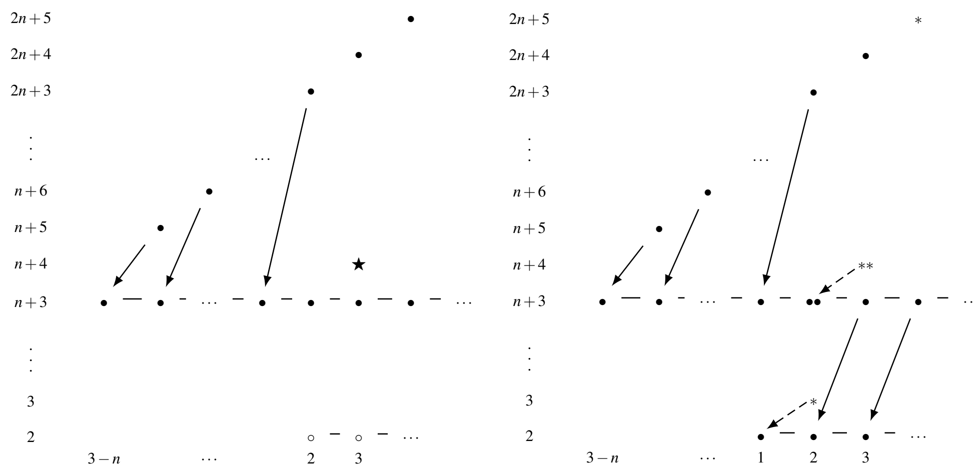


Figure 5-5: The spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma C\tau))$.

Hence there have to be two d_1-differentials as indicated by the dashed arrows that preempt two d_n-differentials in $\Sigma\mathbb{D}_2(C\tau)$. The rest of the differentials are then forced by a sparsity argument, and we conclude that

$$\mathbb{D}_2(\Sigma C\tau^n) \simeq \Big(\bigoplus_{i=1}^{n} \Sigma^{2-n+i,n+3}C\tau^i\Big) \oplus \Big(\bigoplus_{i\geq 1} \Sigma^{i+1,2}C\tau^n\Big) \oplus \Sigma^{2a+2b+1,n+4}C\tau^n.$$

Now we induct along $s=a$. We will explain the case $n=1$ in full details and the cases $n>2$ is analogous. Suppose that

$$\mathbb{D}_2(\Sigma^a C\tau) \simeq \Sigma^{a+1,a+3}C\tau \oplus \Big(\bigoplus_{i\geq 1} \Sigma^{a+i,a+1}C\tau\Big) \oplus \Big(\bigoplus_{j=2}^{a} \Sigma^{a+j,a+1+j}C\tau^2\Big) \oplus \Sigma^{2a+1,2a+3}C\tau.$$

There is a cofiber sequence $\mathbb{F}_2^{a+1,a+1} \to \Sigma^{a+1}C\tau \to \mathbb{F}_2^{a+2,a+3}$. The E^2-page of the spectral sequence for $\pi_* \mathrm{Re}(\mathbb{D}_2(\Sigma^{a+1}C\tau))$ is given by

$$\mathbb{D}_2(\mathrm{Gr}(\Sigma^{a+1}C\tau)) \simeq \mathbb{D}_2(\mathbb{F}_2^{a+1,a+1}) \oplus \mathbb{D}_2(\mathbb{F}_2^{a+2,a+3}) \oplus \mathbb{F}_2^{2a+3,2a+5}$$

as depicted in Figure 5-6.

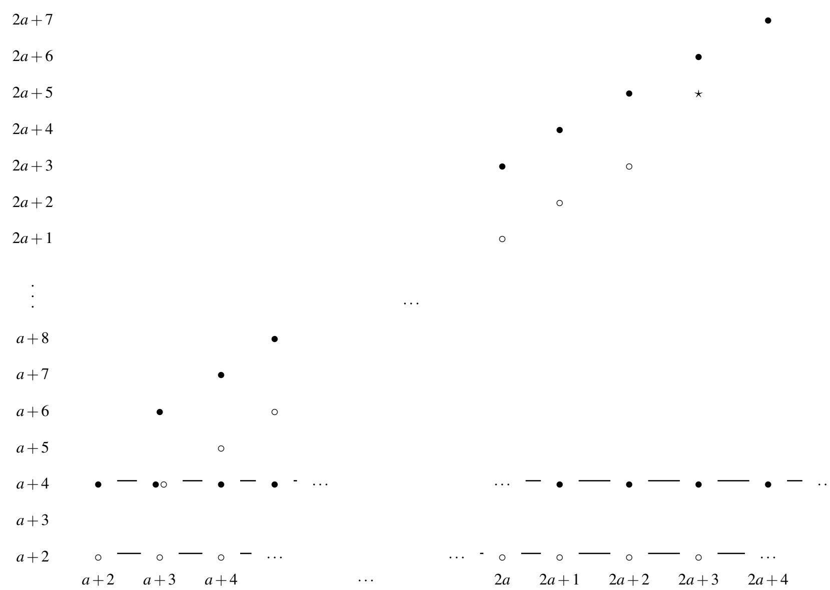


Figure 5-6: The E^2-page of the spectral sequence for $\pi_* \mathrm{Re}(\mathbb{D}_2(\Sigma^{a+1}C\tau))$.

On the other hand, it follows from the cofiber sequence

$$\Sigma\mathbb{D}_2(\Sigma^a C\tau) \to \mathbb{D}_2(\Sigma^{a+1}C\tau) \to \Sigma^{2a+2,2a+3}C\tau \otimes C\tau$$

(Proposition 5.3.3) that the spectral sequence for $\pi_* \mathrm{Re}(\mathbb{D}_2(\Sigma^{a+1}C\tau))$ is depicted in Figure 5-7, where the classes represented by asterisks come from $\Sigma^{2a+2,2a+3}C\tau \otimes C\tau$. Hence there is a d_1-differential from $(2a+3, 2a+5)$ to $(2a+2, 2a+4)$, as indicated by the dashed arrow on the right in Figure 5-7, which preempts the d_2-differential from $(2a+3, 2a+6)$ to $(2a+2, 2a+4)$ in $\Sigma\mathbb{D}_2(\Sigma^a C\tau^n)$. The rest of the differentials are therefore forced upon us for degree reasons after the d_2-differentials kill the parallel lines at the bottom. Therefore

$$\mathbb{D}_2(\Sigma^{a+1}C\tau) \simeq \Sigma^{a+2,a+4}C\tau \oplus \left(\bigoplus_{i\geq 1} \Sigma^{a+1+i,a+2}C\tau\right) \oplus \left(\bigoplus_{j=2}^{a} \Sigma^{a+1+j,a+2+j}C\tau^2\right) \oplus \Sigma^{2a+3,2a+5}C\tau.$$

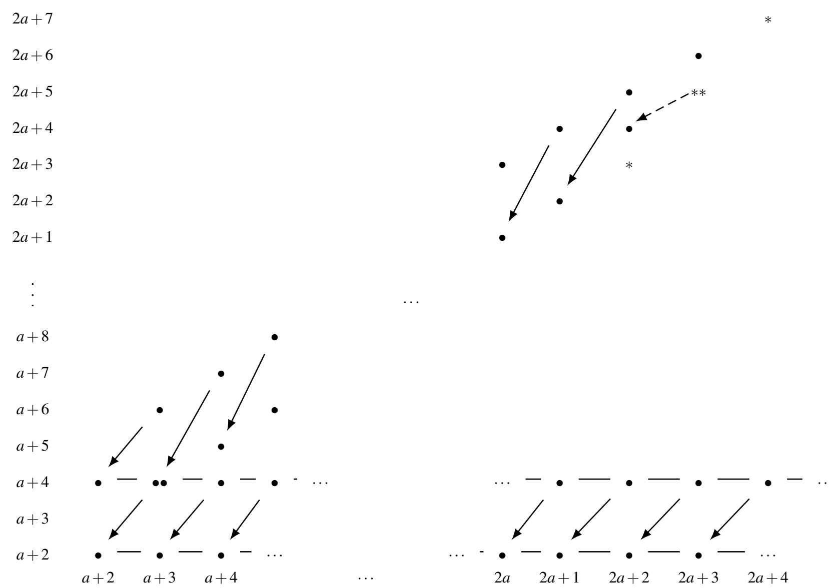


Figure 5-7: The spectral sequence for $\pi_*\mathrm{Re}(\mathbb{D}_2(\Sigma^{a+1}C\tau))$.

For $n > 1$, the inductive step is the same: comparing the E^2-page $\mathbb{D}_2(\bigoplus_i \mathrm{Gr}(\Sigma^{a+1}C\tau)_i) \simeq \mathbb{D}_2(\mathbb{F}_2^{a+1,a+1}) \oplus \mathbb{D}_2(\mathbb{F}_2^{a+2,a+2+n}) \oplus \mathbb{F}_2^{2a+3,2a+4+n}$ of the spectral sequence for $\pi_*\mathrm{Re}(\Sigma^{a+1}C\tau^n)$ with the E^2-page coming from the cofiber sequence

$$\Sigma\mathbb{D}_2(\Sigma^a C\tau^n) \to \mathbb{D}_2(\Sigma^{a+1}C\tau^n) \to \Sigma^{2,3}(\Sigma^a C\tau^n)^{\otimes 2}$$

forces a d_1-differential from the class $\mathbb{F}_2^{2a+3,2a+n+4}$ in $\Sigma^{2,3}(\Sigma^a C\tau^n)^{\otimes 2}$ to the class $\mathbb{F}_2^{2a+2,2a+n+3}$ in $\mathbb{D}_2(\Sigma^{a+1}C\tau^n)$. After that all the remaining differentials are determined uniquely for degree reasons, noting that the first $n-1$ d_{n+1}-differentials between the two parallel lines are preempted by shorter differentials in $\mathbb{D}_2(\mathbb{F}_2^{a+1,a+1})$. $\square$

Therefore we have computed the differential on any element in the weight 2 part of the bar spectral sequence in the universal class.

5.3.2 Looking ahead: Differentials from the comonad structure

Now that we have a good understanding of $\mathbb{D}_2$, we want to use the command structure map

$$\mathbb{D}_{2n} \to \mathbb{D}_2 \circ \mathbb{D}_n \oplus \bigoplus_{ij=2n, i>2, j>1} \mathbb{D}_i \circ \mathbb{D}_j$$

on $v(X)$ to inductively deduce the higher differentials in the bar spectral sequence for $\Sigma^k\mathbb{F}_2$ as a trivial spectral Lie algebra.

The bar construction on the E^2-page $\pi_*\Big(\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}},\Sigma^k\mathbb{F}_2\big)\Big)$ is a coalgebra over the comonad $\pi_*\big(\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}},\mathrm{sqz}(-))\big)$ on $\mathrm{Mod}_{\mathbb{F}_2}$. The coalgebra structure map is given by

$$|\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}},\Sigma^k\mathbb{F}_2\big)| \xleftarrow{\simeq} |\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}},|\mathrm{Bar}_\bullet\big(\mathrm{free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}},\mathrm{free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}},\Sigma^k\mathbb{F}_2\big)|\big)|$$
$$\to |\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}},|\mathrm{Bar}_\bullet\big(\mathrm{id},\mathrm{free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}},\Sigma^k\mathbb{F}_2\big)|\big)|,$$

where the last map makes use of the augmentation $\mathrm{free}_{\mathrm{Lie}^s_{\bar{\mathcal{R}}}} \to \mathrm{id}$, cf. [Bra17, Appendix D].

First we give an explicit description of the composite

$$\xi_2 : \mathbb{D}_{2n} \to \bigoplus_{ij=2n,i,j>1} \mathbb{D}_i \circ \mathbb{D}_j \twoheadrightarrow \mathbb{D}_2 \circ \mathbb{D}_n$$

of the comonad structure map followed by projection on to a summand on the E^2-page, i.e., after tensoring with $C\tau$.

Lemma 5.3.8. *The map $\xi_2 \otimes C\tau$ is given by the following on the basis elements in Theorem 4.2.27:*

1. $\bar{Q}^i\bar{Q}^j\bar{Q}^{j_1}\cdots\bar{Q}^{j_m}(x_k) \mapsto \bar{Q}^i\circ\bar{Q}^j\bar{Q}^{j_1}\cdots\bar{Q}^{j_m}(x_k)+\sum_l \alpha_l\bar{Q}^{i+j-l}\circ\bar{Q}^l\bar{Q}^{j_1}\cdots\bar{Q}^{j_m}(x_k)$, *where the sum ranges over the nonzero terms on the right hand side of Behrens' relation* $\bar{Q}^i\bar{Q}^j = \sum_l \alpha_l\bar{Q}^{i+j-l}\bar{Q}^l$.
2. $\gamma_i Q^j\bar{Q}^{j_1}\cdots\bar{Q}^{j_m}(x_k) \mapsto \gamma_i\circ Q^j\bar{Q}^{j_1}\cdots\bar{Q}^{j_m}(x_k)$.

Furthermore, for any cycle α on the E^1-page, the chain $\bar{Q}^j|\gamma_i(\alpha)$ does not survive to the E^2-page.

Proof. For $i \le 2j$, the comonad structure map sends $\bar{Q}^i\bar{Q}^j$ to $\bar{Q}^i\circ\bar{Q}^j+\sum_{l,i+j-l>l}\alpha_l\bar{Q}^{i+j-l}\circ\bar{Q}^l$, where the sum ranges over the nonzero terms on the right hand side of the relation $\bar{Q}^i\bar{Q}^j = \sum_{l,i+j-l>l}\alpha_l\bar{Q}^{i+j-l}\bar{Q}^l$. This can be deduced either from Priddy's machinery of Koszul duality [Pri70] on the subcomplex $\mathrm{Bar}_\bullet(\mathrm{id},\mathrm{free}_{\bar{\mathcal{R}}},\Sigma^k\mathbb{F}_2)$, or directly from the fact that the class $\bar{Q}^i\bar{Q}^j(x) \in E^2$ is represented by the cycle $\bar{Q}^i|\bar{Q}^j|x + \sum_{l,i+j-l>l} \alpha_l \bar{Q}^{i+j-l}|\bar{Q}^l|x$ on the E^1-page. Since any l in the sum is less than j, the term $\bar{Q}^l\bar{Q}^{j_1}\cdots\bar{Q}^{j_m}(x_k)$ is either zero or a basis element of the E^2-page and hence always well defined. This proves (1).

Item (2) follow from the explicit chain-level construction of γ_i, see [Dwy80a, Section 4].

For any cycle z on the E^1-page, the differential d_1 sends $\bar{Q}^j|\gamma_i(z)$ to $\bar{Q}^j\gamma_i(z)$ since $\gamma_i(z)$ is a cycle. In order for $\bar{Q}^j|\gamma_i(z)$ to survive to the E^2-page, we need to complete the cycle by finding a different chain α such that

$$d_1(\alpha) = \bar{Q}^j\gamma_i(z) = \sum_{a,b\in V_{r,i}} \bar{Q}^j[s_{a_1}s_{a_2}\cdots s_{a_i}(z), s_{b_1}s_{b_2}\cdots s_{b_i}(z)]$$

for some r, i, cf. Proposition 4.2.24. Note that the outmost $\bar{Q}^j$ and Lie bracket come from the same simplicial level, and hence cannot be rewritten as a sum of other compositions of operations in a free spectral Lie algebra. Furthermore, there is no chain β such that $d_1(\beta) = \gamma_i(z)$, i.e., there is no chain such that $d_1(\bar{Q}^j\beta) = \bar{Q}^j\gamma_i(z)$. Therefore such an α does not exist. $\square$

The comonadic composition on $\gamma_i\gamma_j$ turns out to be rather tricky, and we intend to investigate it in some future endeavor.

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