Lectures on Pseudodifferential operators

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Abstract. These are the lectures notes, together with additional material, for the course 18.157, Microlocal Analysis, at MIT in Fall of 2005.

Contents

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Introduction

I plan to take a rather direct and geometric approach to microlocal analysis in these lectures. The initial goal is to define the space of pseudodifferential operators

$$\Psi^m(X; E, F) \ni A : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F). \tag{0.1}$$

Here, m is the 'order' of the pseudodifferential operator, X is the compact manifold on which it is defined and E and F are complex vector bundles over X between the sections of which it acts. Thus, the first few lectures are devoted to the definition, and investigation of the elementary properties, of these operators.

In the approach taken here, this space is defined in terms of another, more general, object

$$\Psi^m(X; E, F) = \mathrm{I}^m(X^2, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R). \tag{0.2}$$

Namely, the space on the right is the space of *conormal distributions* on the compact manifold X^2, with respect to the submanifold Diag, the diagonal, as sections of the bundle $\mathrm{Hom}(E, F) \otimes \Omega_R$, where the precise definition of these bundles is discussed later. So we will proceed to define the right side, but in general for any *embedded* compact submanifold of a compact manifold and any complex vector bundle over the latter

$$\mathcal{C}^\infty(X; E) \hookrightarrow \mathrm{I}^m(X, Y; E). \tag{0.3}$$

Here I have included the fact that smooth sections of the bundle are included in the conormal space, for any order. In fact the elements of $\mathrm{I}^m(X, Y; E)$ are arbitrary smooth sections away from Y, they are singular only at Y and then only in a very special way.

To define the space we use the collar neighbourhood theorem to define a 'normal fibration'. This means identifying a neighbourhood of Y in X with a neighbourhood of the zero section of the normal bundle to Y in X. We denote the latter NY (in which the notation for X does not appear, perhaps it should, say as in $N\{Y; X\}$ but that is a bit heavy-handed) and then, by definition,

$$\mathrm{I}^m(X, Y; E)/\mathcal{C}^\infty(X; E) \longleftrightarrow \mathrm{I}^m_{\mathcal{S}}(NY, O_{NY}; E)/\mathcal{S}(NY, E). \tag{0.4}$$

The space on the top on the right hand side here is almost the same as the one on the left, except that the total space of a vector bundle is not compact, so we need to specify the behaviour of things at infinity and in this case they are required to be 'Schwartz', meaning rapidly decaying with all derivatives; that is what the subscript '$\mathcal{S}$' indicates.

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Finally (going backwards) then we free ourselves of the origin of the bundle NY and replace it by a general real vector bundle W over a compact manifold Y, with E a vector bundle over Y (pulled back to W) and we want to define

$$\mathrm{I}^m_{\mathcal{S}}(W, O_W; E) \xleftarrow{\mathcal{F}^{-1}} \rho^{-m}\mathcal{C}^\infty(\overline{W'}; E \otimes \Omega_{\mathrm{fib}}). \tag{0.5}$$

Here we use the *fibrewise* Fourier transform to identify distributions on W with distributions on the dual bundle, W' (they really have to be fibre-densities accounting for the extra factor Ω_{fib}) and $\rho \in \mathcal{C}^\infty(\overline{W'})$ is a defining function for the boundary of the radial compactification, $\overline{W'}$, which is a compact manifold with boundary made up from W'.

Of course we do all this in the opposite order, which corresponds to the first four lectures. Namely, in the remainder of this first lecture I will first describe various compactifications of a vector space and their invariance under linear transformations, so that the fibrewise compactification of a vector bundle makes sense. In particular the meaning of the notation for the compact manifold $\overline{W'}$ on the right in (0.5) is fixed. Once we have that, and the properties of the Fourier transform are recalled, we can define the left side of (0.5) in terms of the right and discuss the main properties of these spaces. Enough information is needed to show that the identification (0.4) makes sense independent of the choice of the normal fibration which underlies it, which is the main content of Lecture 3. Then (0.2) gives a definition of $\Psi^m(X; E, F)$. Of course we need to discuss more of the properties, in particular the way these act as operators and especially the 'symbolic' and the composition properties; I hope most of this will be done by the end of Lecture 5.

A word is in order about why I have chosen to take this rather sophisticated approach to pseudodifferential operators. The idea is that this approach allows easy generalization. As we shall see below, there are many 'variants' of the space $\Psi^m(X; E, F)$. A large class (namely the 'geometric' ones) of these variants can be readily obtained by changing the compactification of the normal bundle to the diagonal to a different one. Then the same procedure gives a class of operators and, under certain conditions, composition properties can be proved the same way.

Now, my *aim* (this of course is written right at the beginning of the semester) in the rest of the course is to cover the following topics.

(1) Pseudodifferential operators on compact manifolds.
(2) Hodge theorem.
(3) Hörmander's theorem.
(4) Spectral asymptotics for the Laplacian.
(5) Dirac operators
(6) Isotropic algebra
(7) K-theory and classifying spaces.
(8) Chern forms.
(9) Fibrations and product-type operators.
(10) Index theorem.
(11) Eta invariant.
(12) Determinant bundle.
(13) Gerbes.

What will I assume? I hope this is at the level of graduate students with a bit of background. By this I mean I will rather freely use the following

- Differential Geometry:

Manifolds and vector bundles
Forms and deRham theorem
Lie groups
Local symplectic geometry

- Differential Analysis:
  Schwartz distributions
  Fourier transform
  Borel's lemma
  Sobolev spaces on $\mathbb{R}^n$
  Operators on Hilbert space

Lack of knowledge of one or two of these things should not be taken as a bar to preceeding!

In these lecture notes I will limit the 'body' of the notes, forming the first section of each chapter, to the material I think is essential for the main line of the course – and this will be pretty much the content of the lectures. On the other hand I will try to include as addenda to each lecture some more background, various extension and refinements, some indications of directions for further reading, exercises and problems (by this I mean I claim to know the answer to the former but not necessarily to the latter!) I hope to persuade participants in the course to write something up for these addenda.

After a few lectures I will be able to indicate how the present treatment is related to some of the many other treatments of this subject which are available elsewhere.

CHAPTER 1

Compactifications of a vector space

Lecture 1: 8 September, 2005

We need to consider the spaces of functions we will insert in the right hand side of (0.5); such functions are often called 'symbols' or (probably better) 'amplitudes.' To start, consider the case in which the submanifold Y is a point, so its normal bundle is just a vector space. There is nothing special about the dual space to a given vector space, so we just consider an arbitrary, real, finite-dimensional vector space W. This is also a $\mathcal{C}^\infty$ manifold so the space $\mathcal{C}^\infty(W)$ of smooth functions is well defined. However these functions are unconstrained near infinity. To introduce appropriate classes of functions we introduce various compactifications of W. Although these compactifications are introduced here for the specific purpose of describing functions with 'good behaviour at infinity' they have many other uses – some of which will be indicated later.

The general idea of compactification is that if U is a smooth manifold which is not compact then we may be able to find a compact manifold, possibly with boundary or with corners, X, and a smooth injection

$$U \hookrightarrow X \tag{L1.1}$$

which is a diffeomorphism of U onto an open dense subset of X. Since the pull-back map is then injective (a smooth function being determined by its values on a dense set), we may identify $\mathcal{C}^\infty(X)$ as a subset of $\mathcal{C}^\infty(U)$; these functions may be thought of as 'controlled at infinity'.

For a vector space we will define several different compactifications. To do so we start with $\mathbb{R}^n$, define a compactification and then check invariance under choice of the basis which leads to the identification $W \longleftrightarrow \mathbb{R}^n$. If invariance under all general linear transformations does not hold then the compactification depends on some additional structure on W.

L1.1. One-point compactification. The first compactification I will discuss is the 1-point compactification. In fact it will turn out that this is not used for quite a while below, for reasons that will become apparent. However, its relation to the compactifications that we will use is worth understanding and it will eventually reappear in the proof of the Atiyah-Singer index theorem.

One way to define the 1-point compactificaton of $\mathbb{R}^n$ is to use a stereographic projection. Thus we first identify $\mathbb{R}^n$, with variable z, with a hyperplane in $\mathbb{R}^{n+1}$,

$$\mathbb{R}^n \ni z \longmapsto (1, z) \in \mathbb{R}^{n+1}_{z_0,z}. \tag{L1.2}$$

Now, consider the sphere of radius 1 centred at the origin in $\mathbb{R}^{n+1}$ and draw the line through $(-1, 0) \in \mathbb{R} \times \mathbb{R}^n$ and $(1, z)$; it meets the sphere at the point $P(z)$ which we can easily find. Namely, the line is

$$\mathbb{R} \ni t \longmapsto (2t - 1, tz) \tag{L1.3}$$

which meets the sphere at the solutions of

$$4t^2 - 4t + 1 + t^2|z|^2 = 1.$$

This has the trivial solution $t = 0$, just the South Pole, and the non-trivial solution $P(z)$ given by

$$t = \frac{4}{4 + |z|^2} \Longrightarrow P(z) = \left(\frac{4 - |z|^2}{4 + |z|^2}, \frac{z}{4 + |z|^2}\right). \tag{L1.4}$$

Thus P is a diffeomorphism from $\mathbb{R}^n$ into the complement of the South Pole in the sphere. Indeed, the inverse is given by

$$(Z_0, Z) \longmapsto z = \frac{8Z}{1 + Z_0},\ |z|^2 = 4\frac{1 - Z_0}{1 + Z_0}, \tag{L1.5}$$

which is smooth in $Z_0 > -1$ on the sphere. This formula also shows that the reflection in the equatorial plane, $Z_0 \longmapsto -Z_0$, on the sphere induces the inversion $z \longmapsto z/|z|^2$. So, a smooth function on $\mathbb{R}^n$ is of the form P^*f for $f \in \mathcal{C}^\infty(\mathbb{S}^n)$, if and only if there exists $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $f(z) = g(z/|z|^2)$ outside the origin.

So, what is wrong with the 1-point compactification? For one thing, it does not have enough invariance. Let me use the notation

$$^1\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\mathrm{SP}\} \tag{L1.6}$$

for this set with the $\mathcal{C}^\infty$ structure coming from P, so it is just the sphere. Then P is the inclusion $P : \mathbb{R}^n \longrightarrow {}^1\overline{\mathbb{R}^n}$. Certainly orthogonal transformations lift to this manifold, so there is a commutative diagramme

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "P"] \arrow[d, "O"'] & {}^1\overline{\mathbb{R}^n} \arrow[d, "\tilde{O}"] \\ \mathbb{R}^n \arrow[r, "P"'] & {}^1\overline{\mathbb{R}^n} \end{tikzcd}, \text{ for } O \in \mathrm{O}(n),\ \tilde{O}(Z_0, Z) = (Z_0, OZ). \tag{L1.7}$$

On the other hand, not all elements of $\mathrm{GL}(n, \mathbb{R})$ lift smoothly in this way. To see this, suppose $G \in \mathrm{GL}(n, \mathbb{R})$ lifts in the sense that there is a commutative diagramme of smooth maps as in (L1.7). Then the smoothness of the inversion means that $|Gz|^{-2}$ must be a smooth function of the variables $z/|z|^2$ near infinity. Inverting again, and using the homogeneity of G this means that

$$\frac{|z|^2}{|Gz|^2} \text{ is a smooth function of } z \text{ near } 0. \tag{L1.8}$$

Now, it is well known that this is only the case if $|Gz|^2 = s^2|z|^2$ for some $s > 0$, i.e. if G is *conformal*[1]. So, for instance the scaling in one variable, $z = (z_1, z') \longmapsto (sz_1, z')$, is not conformal, hence does not extend smoothly to the 1-point compactification of $\mathbb{R}^n$ (if $n > 1$!)

[1]Exercise: Check this!

L1.2. Radial compactification. Next we consider the most important compactification in the sequel, the *radial compactification.* We use the same approach as above for the 1-point compactification. So with the same embedding of $\mathbb{R}^n$ as the hyperplane $Z_0 = 1$ in $\mathbb{R}^{n+1}$ as in (L1.2), consider the modifed sterographic projection based on the line through the origin of the unit sphere, rather than the South Pole. The intersection of $[0,1] \ni t \longrightarrow (t, tz)$ with the unit sphere in $Z_0 > 0$ occurs at $t = (1+|z|^2)^{-\frac{1}{2}}$. Thus the compactifying map is

$$\text{(L1.9)} \quad R : \mathbb{R}^n \ni z \longrightarrow \left(\frac{1}{(1+|z|^2)^{\frac{1}{2}}}, \frac{z}{(1+|z|^2)^{\frac{1}{2}}}\right) \in \mathbb{S}^{n,1} = \{(Z_0, Z); Z_0 \geq 0, Z_0^2 + |Z|^2 = 1\}.$$

It is clearly a diffeomorphism, since the inverse can we written

$$\text{(L1.10)} \qquad z = Z/Z_0 \text{ in } Z_0 > 0.$$

I will denote this radial compactification by $\overline{\mathbb{R}^n} = \mathbb{S}^{n,1}$ with R used to identify the interior with $\mathbb{R}^n$.

Thus the radial compactification embeds $\mathbb{R}^n$ as the open upper half-sphere. This is diffeomorphic to a closed ball and it is tempting to look at the projection on the last n variables in (L1.9) and consider

$$\text{(L1.11)} \qquad Q : \mathbb{R}^n \ni z \longrightarrow \frac{z}{(1+|z|^2)^{\frac{1}{2}}} \in \mathbb{B}^n = \{Z \in \mathbb{R}^n; |Z| \leq 1\}.$$

This is the *quadratic compactification.* It is *not* the same as the radial compactification (L1.9) since the function $Z_0 = (1+|z|^2)^{-\frac{1}{2}}$ is not smooth on it! Rather $(1+|z|^2)^{-1}$ is the pull back of a defining function for the boundary of the ball under Q. This corresponds to the fact that the inverse of the projection of the upper half-sphere to the ball has a square-root singularity. When it comes up, and it will, the quadratic compactification will be denoted ${}^q\overline{\mathbb{R}^n}$.

So, returning to the radial compactification, observe as before that orthogonal transformations lift to $\overline{\mathbb{R}^n}$. Indeed the orthogonal transformation can be extended to act on the $\mathbb{R}^n$ factor of $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ and then intertwines with the standard action on $\mathbb{R}^n$ as in (L1.7).

To examine the lift of a general linear transformation we can proceed directly using homogoneity. Subsequently I will proceed more indirectly, by considering the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$. The indirect approach has certain advantages as we shall see below. However, if $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an invertible linear transformation we can see directly that it lifts to a diffeomorphism of the radial compactification $\tilde{G} : \overline{\mathbb{R}^n} \longrightarrow \overline{\mathbb{R}^n}$. This just means showing that the diffeomorphism $R^{-1}GR$ induced on the interior of the upper half sphere by its identification, through R, with $\mathbb{R}^n$, extends smoothly up to the boundary. Notice that a neighbourhood of the boundary of $\mathbb{S}^{n,1}$ can be identified with the product $[0, \frac{1}{2}) \times \mathbb{S}^{n-1}$ using the variables $\frac{1}{|z|}, \frac{z}{|z|}$. Indeed a smooth function on $\mathbb{R}^n$ extends to be smooth on $\mathbb{S}^{n,1}$ under the identification R if and only if it is a smooth function of $|z|^{-1}$ and $\frac{z}{|z|} \in \mathbb{S}^{n-1}$ outside the origin. To see this, just note that

$$\text{(L1.12)} \qquad (1+|z|^2)^{-\frac{1}{2}} = s(1+s^2)^{\frac{1}{2}}, \ s = |z|^{-1}$$

is a smooth function of $|z|^{-1}$, and conversely. Similarly

$$\frac{z}{(1+|z|^2)^{\frac{1}{2}}} = (1+|z|^{-2})^{-\frac{1}{2}}\frac{z}{|z|}$$

is a smooth function of $z/|z|$ and $1/|z|$ and conversely $z/|z|$ is a smooth function of $z/(1+|z|^2)^{\frac{1}{2}}$ and $(1+|z|^2)^{\frac{1}{2}}$.

Thus the smoothness on the radial compactification is reduced to showing that

$$\frac{1}{|Gz|} \text{ and } \frac{Gz}{|Gz|} \text{ are smooth functions of } \frac{1}{|z|},\ \frac{z}{|z|}$$

up to $1/|z| = 0$. Since G is invertible, $|Gz| > 0$ on the sphere $|z| = 1$, so this the smoothness holds there and by the linearity (hence homogeneity) of G,

$$|Gz| = |z|\left|G(\frac{z}{|z|})\right| \Longrightarrow \frac{1}{|Gz|} = \frac{1}{|z|}\frac{1}{|G(\frac{z}{|z|})|},\ \frac{Gz}{|Gz|} = \frac{G\frac{z}{|z}|}{|G\frac{z}{|z|}|}.$$

This proves the desired smoothness.

Let me show directly that the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$ lifts to the radial compactification, although this could also be shown by checking that the lift $\tilde{G}$ depends smoothly on $G \in \mathrm{GL}(n,\mathbb{R})$. For the standard action on $\mathbb{R}^n$, $\mathfrak{gl}(n,\mathbb{R})$ is represented by 'linear' vector fields with the basis

$$z_i\partial_{z_j},\ i,j = 1,\dots,n. \tag{L1.13}$$

Now we wish to show that $z_i\partial_{z_j}$ lifts to a smooth vector field on $\mathbb{S}^{n,1}$ under the indentification R. Set $s = 1/|z|$ and $\omega = z/|z|$. Then outside the origin

$$z_i\partial_{z_j} = a(s,\omega)\partial_s + V(s,\omega) \tag{L1.14}$$

where $a(s,\omega) \in \mathcal{C}^\infty((0,\infty)\times\mathbb{S}^{n-1})$ and V is a smooth vector field on $\mathbb{S}^{n-1}$ depending smoothly on $s \in (0,1)$. We want to understand what happens as $s \downarrow 0$. However, observe that the linear vector field is constant under the homotheity, $z \to rz$, $0 < r \in \mathbb{R}$. The decomposition (L1.14) is unique and so it must scale in the same way. By the definition of these variables the homotheity becomes $s \to r^{-1}s$, $\omega \to \omega$, so we must have

$$a(s,\omega) = sa(1,\omega),\ V(s,\omega) = V(1,\omega) \Longrightarrow z_i\partial_{z_j} = a(\omega)s\partial_s + V(\omega)$$

where now $a(\omega) \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $V(\omega)$ is a smooth vector field on $\mathbb{S}^{n-1}$. This shows that the linear vector fields lift to be smooth on $\mathbb{S}^{n,1}$ and even that

$$z_i\partial_{z_j} \text{ is tangent to the boundary of } \mathbb{S}^{n,1}. \tag{L1.15}$$

From this we can also deduce that $\mathrm{GL}(n,\mathbb{R})$ lifts to act smoothly under radial compactification. Indeed, in any Lie group a neighbourhood of the identity is given by exponentiation of the Lie algebra. Here exponentiation corresponds to integration of the vector field on $\mathbb{R}^n$, or of its extension to $\mathbb{S}^{n,1}$. So the elements in some neighbourhood of the identity extend smoothly to $\mathbb{S}^{n,1}$. More generally, any element of $\mathrm{GL}(n,\mathbb{R})$ is given by a finite composite of an element of $O(n)$ (an orthogonal transformation, needed if the orientation is reversed) and a finite number of elements of some fixed neighbourhood of the identity. Thus the action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ extends smoothly to $\mathbb{S}^{n,1}$ under R.

It is also the case that translations extend to the radial compactification. Here any translation is obtained by exponentiation of a linear combination of the vector fields ∂_{z_i}. Using the same argument as above, the smoothness near the boundary of $\overline{\mathbb{R}^n}$ can be examined in terms of s, ω and the unique decomposition

$$\partial_{z_i} = a_i(s,\omega)\partial_s + V_i(s,\omega) \tag{L1.16}$$

into a vector field on $(0,\infty)_s \times \mathbb{S}^{n-1}_\omega$. Since ∂_{z_i} is homogeneous of degree -1 under the homotheity $z \to rz$ it follows that

$$a_i(s,\omega) = s^2 a_i(1,\omega),\ V_i(s,\omega) = sV_i(1,\omega)$$

and (L1.16) becomes

$$\partial_{z_i} = s(a_i(\omega)s\partial_s + V_i(\omega)) \tag{L1.17}$$

for a smooth function and a smooth vector field on $\mathbb{S}^{n-1}$. Thus for the translations the generating vector fields actually lift to be Z_0 times a smooth vector field tangent to the boundary of $\mathbb{S}^{n,1}$. This will turn out to be important! In any case the translations also lift to smooth diffeomorphisms of $\mathbb{S}^{n,1}$.

This is our basic compactification of a vector space. Why are we interested in it? One very important reason is that the space $\mathcal{C}^\infty(\overline{W})$ is well-defined and is invariant under the general linear group (and translations). It is given many other names in the literature, typically the 'space of classical symbols of order 0'. More generally we can set

$$S^z_{\mathrm{cl}}(W) = \left\{u \in \mathcal{C}^\infty(W); \rho^{-z}u \in \mathcal{C}^\infty(\overline{W})\right\} \tag{L1.18}$$

where $\rho \in \mathcal{C}^\infty(\overline{W})$ is a boundary defining function. This is the space of 'classical symbols of (possibly complex) order z' on W. I will not use this notation very much because there are all sorts of confusions in the literature.

L1.3. Quadratic compactification. I introduced the quadratic compactification of $\mathbb{R}^n$ in (L1.11) above. Essentially by definition, the canonical map between the interiors (given by identification with $\mathbb{R}^n$) extends to a smooth map from the radial to the quadratic compactification, but not the reverse.

A neighbourhood of infinity in ${}^q\overline{\mathbb{R}^n}$ may be smoothly identified with the product $(0,1) \times \mathbb{S}^{n-1} \ni (t,\omega)$ where $\omega = y/|y| \in \mathbb{S}^{n-1}$ and $t = |y|^{-2}$. Since the generators of the translations satisfy

$$\partial_{y_j} t = -2\frac{y_j}{|y|^4} = -2t^{\frac{3}{2}}\omega_j$$

the translations do *not* lift to be smooth.

The radial vector field is

$$\sum_i y_i\partial_{y_i} = -2t\partial_t,$$

so the homotheity does lift to be smooth, namely it becomes $t \to r^{-2}t$. The homogeniety argument used above for the radial compactification then shows that all general linear transformations lift to be smooth, since

$$z_k\partial_{z_i} = a(\omega)t\partial_t + U_{kj} \tag{L1.19}$$

where the U_{kj} are smooth vector fields on the sphere and a is a smooth function on the sphere.

Thus the quadratic compactification is well-defined for a vector space, since it is preserved under linear transformations, but not for an affine space since it is *not* preserved by translations.

1+. Addenda to Lecture 1

1+.1. Explicit models. It is useful to think of the radial compactification of a vector space, $\overline{W}$, as an explicit set with a $\mathcal{C}^\infty$ structure. By abstract nonsense one can do this from the embedding of $\mathbb{R}^n$ into $\mathbb{S}^{n,1}$, but as I show below there is also a more natural geometric approach.

First let me review in a more sophisticated way the construction of the manifold $\overline{W}$ above. First, for $\mathbb{R}^n$ we have an explicit map into $\mathbb{S}^{n,1}$ such that the action of $\mathrm{GL}(n,\mathbb{R})$ extends smoothly

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "P"] \arrow[d, "G"'] & \mathbb{S}^{n,1} \arrow[d, "\tilde{G}"] \\ \mathbb{R}^n \arrow[r, "P"'] & \mathbb{S}^{n,1} \end{tikzcd}, \ \forall\ G \in \mathrm{GL}(n,\mathbb{R}). \tag{1+.20}$$

Now, to a real vector space, W, of dimension n we can associate the set of all linear ismorphisms to $\mathbb{R}^n$

$$P = \{T : W \longrightarrow \mathbb{R}^n,\ \text{linear and invertible}\}. \tag{1+.21}$$

This is a principal $\mathrm{GL}(n,\mathbb{R})$ space. That is, the action of $\mathrm{GL}(n,\mathbb{R})$

$$\mathrm{GL}(n,\mathbb{R}) \times P \ni (G,T) \longmapsto GT \in P \tag{1+.22}$$

is free and transitive. Then we can 'recover' the original vector space W as the quotient, namely as the vector space associated to the standard action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$

$$\tilde{W} = (P \times \mathbb{R}^n)/\sim,\ (T,v) \sim (GT, Gv)\ \forall\ G \in \mathrm{GL}(n,\mathbb{R}). \tag{1+.23}$$

This is canonically isomorphic to to W with the map being

$$\tilde{W} \ni [(T,v)] \longmapsto T^{-1}v \in W \tag{1+.24}$$

since this does not depend on the representative under (1+.23).

Now, what we have done above is to define the radial compactification $\overline{W}$ as the manifold with boundary associated to P by the action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{S}^{n,1}$

$$\overline{W} = (P \times \mathbb{S}^{n,1})/\sim,\ (T,p) \sim (GT, \tilde{G}p)\ \forall\ G \in \mathrm{GL}(n,\mathbb{R}). \tag{1+.25}$$

This is all very well, but it is a little nicer to have something a little lower-tech in mind. If we consider the action of $G \in \mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ it also induces an action on the associated (projective) sphere. That is, consider the set of half rays through the origin

$$\mathbb{S}^{n-1} = (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+,\ \mathbb{R}^+ \times \mathbb{R}^n \ni (s,z) \longmapsto sz \in \mathbb{R}^n. \tag{1+.26}$$

This is a definition of the manifold $\mathbb{S}^{n-1}$. For a general vector space we can similarly define

$$\mathbb{S}W = (W \setminus \{0\})/\mathbb{R}^+,\ \mathbb{R}^+ \times W \ni (s,w) \longmapsto sw \in W. \tag{1+.27}$$

Here is an exercise for you:-

Lemma 1. *The set $W \sqcup \mathbb{S}W$ (the disjoint union) has a unique $\mathcal{C}^\infty$ structure such that the elements of $\mathcal{C}^\infty(W)$ which are homogeneous (for the $\mathbb{R}^+$ action in (1+.27)), of non-positive integral degree, outside some compact neighbourhood of the origin, lift to be smooth (with their asymptotic values on $\mathbb{S}W$ for homogeneity 0* *and* 0 *there for negative homogeneity) and generate the* $\mathcal{C}^\infty$ *structure (i.e. this set of functions contains a coordinate system at each point).*

Thus we can identify $\overline{W} = W \sqcup \mathbb{S}W$ as a set.

Exercise 1. Show that the quadratic compactification of a vector space can be defined as a space associated to the principal $\mathrm{GL}(n, \mathbb{R})$ space P discussed above and also that it is given in a manner similar to Lemma 1 as a different $\mathcal{C}^\infty$ structure on the same set.

Exercise 2. In the case of the 1-point compactification, formulate precisely the notion of a conformal structure on a real vector space and show that the 1-point compactification only depends on it.

1+.2. Inclusions. All three compactificaitons behave well under inclusion of vector spaces – the inclusion extends to a smooth map of the corresponding compactifications (with metric, or at least conformal, consistency required for the one-point compactification).

Proposition 1. *If* $i : V \subset W$ *is a linear subspace of a vector space over the reals then the inclusion map extends by continuity to a smooth map*

$$\begin{aligned} i &: \overline{V} \hookrightarrow \overline{W} \\ i &: {}^q\overline{V} \hookrightarrow {}^q\overline{W}. \end{aligned} \tag{1+.28}$$

Proof. It suffices to check this in a model case $\square$

Exercise 3. In the case of the 1-point compactification, formulate the notion of the conformal structure induced on a subspace $V \subset W$ by the choice of a conformal structure on W and show that provided the compactifications are compatible in this sense then

$$i : {}^1\overline{V} \hookrightarrow {}^1\overline{W} \tag{1+.29}$$

extends to be smooth.

Exercise 4. Show that an injective linear map between vector spaces always extends to a smooth map between the radial or quadratic compactifications. For non-trivial vector spaces (i.e. of positive dimensions) is there ever a map which is not injective yet which has a smooth extension to (one of) these compactifications? Show that there is always a linear map which does not have a smooth extension between either the radial or quadratic compactifications.

1+.3. Relative compactification. If you have done Exercise 4 you will know that the compactifications discussed above do not behave well with respect to projections of vector spaces. The problem is that the points at infinity 'do not know where to go'. For this reason (and others as it turns out) there is more to be done.

Suppose $V \subset W$ is a subspace and we choose a complementary subspace and hence a product decomposition, $W = V \times U$. Take metrics on V and on U and then consider a map analogous to, but more complicated than, (L1.7)

$$\begin{aligned} &R_V : W \ni w = (v, u) \longmapsto (t, s, v', u') = \\ &\left(\frac{1}{(1+|u|^2)^{\frac12}}, \frac{(1+|u|^2)^{\frac12}}{(1+|u|^2+|v|^2)^{\frac12}}, \frac{v}{(1+|u|^2+|v|^2)^{\frac12}}, \frac{u}{(1+|u|^2)^{\frac12}}\right) \in \mathbb{R}^2 \times W. \end{aligned} \tag{1+.30}$$

On the image, $t^2+|u'|^2=1$, $s^2+|v'|^2=1$. These two 'cylinders' meet transversally (their normals are independent) so the intersection is a smooth manifold. The image of the map lies in

(1+.31)
$$ {}^V\overline{W}=\left\{(t,s,v',u')\in\mathbb{R}^2\times V\times U; t\ge 0,\ s\ge 0,\ t^2+|u'|^2=1,\ s^2+|v'|^2=1\right\}$$

which is a compact manifold with corners in which the image is precisely the dense interior, $s>0$, $t>0$. Of course, in principle this depends on the metrics and the choice of transversal subspace U, but in fact it does not.

Lemma 2. *All translations on W lift to be smooth on ${}^V\overline{W}$ as do all general linear transformations of W which map V into itself.*

Proof. The map

$$(t,s,v',u')\longmapsto (v/st,u'/t),\ s,t>0, \tag{1+.32}$$

is a smooth inverse to R_V in (1+.30) so R_V is a diffeomorphism onto this set. Moroever, orthogonal transformations on U and on V left to diffeomorphisms of ${}^V\overline{W}$ since they just act on the variables u' and v'. A general linear transformation of W leaving V fixed can be factored into $G_1\cdot G_2\cdot G_3$ where $G_1\in\mathrm{GL}(V)$ act as the identity on U, G_2 is of the form

$$G_2(v,u)=(v,u+Sv) \tag{1+.33}$$

for a linear map $S:V\longrightarrow U$ and $G_3\in\mathrm{GL}(U)$ acts as the identity on V. Then G_1 is the product of an orthogonal transformation and a finite number of elements of $\mathrm{GL}(V)$ in any preassigned neighbourhood of the identity and similarly for G_3. On the other hand G_2 is connected to the identity by scaling S to 0. Thus, it is enough to show that the Lie algebra lifts to be smooth on ${}^V\overline{W}$, which is to say the vector fields

$$v_i\partial_{v_j},\ u_k\partial_{v_j},\ u_k\partial_{u_l} \tag{1+.34}$$

(which span the linear vector fields on W tangent to V) lift to be smooth.

In the interior, i.e. where $s>0$ and $t>0$ these are certainly smooth. So we consider the three regions near the boundary separately, where

$$\begin{gathered} s\simeq 0,\ t>\epsilon_0>0\\ t\simeq 0,\ s>\epsilon_0>0 \text{ and}\\ t,s\simeq 0. \end{gathered} \tag{1+.35}$$

Arguing as before that near $x=0$ a smooth function of $x(1+x^2)^{-\frac12}$ is just a smooth function of x, we may use as local generating functions (so including coordinates) in these three regions

$$\begin{gathered} \frac{1}{|v|}(\simeq 0),\ \frac{v}{|v|},\ u\\ \frac{1}{|u|}(\simeq 0),\ \frac{v}{|u|},\ \frac{u}{|u|} \text{ and}\\ \frac{1}{|u|}(\simeq 0),\ \frac{|u|}{|v|}(\simeq 0),\ \frac{v}{|v|},\ \frac{u}{|u|}. \end{gathered} \tag{1+.36}$$

This allows us to apply homogeneity arguments much as above but now for the two homogeneities in u and v separately. Note that each of the vector fields in

(1+.34) is homogeneous of non-positive degree in both senses. It follows that all these vector fields lift to be smooth on ${}^V\overline{W}$ proving the Lemma. □

LEMMA 3. *A short exact sequence of linear maps*

$$0 \longrightarrow V \longrightarrow W \longrightarrow W/V \longrightarrow 0 \tag{1+.37}$$

lifts to a sequence of smooth maps

$$\overline{V} \longrightarrow {}^V\overline{W} \longrightarrow \overline{W/V}. \tag{1+.38}$$

PROOF. We just have to do this for the 'model' as in (L1.8). The inclusion is just $v \to (v, 0)$ and

$$R_V(v,0) = (1, \frac{1}{(1+|v|^2)^{\frac{1}{2}}}, \frac{v}{(1+|v|^2)^{\frac{1}{2}}}, 0) = (1, P(v), 0) \tag{1+.39}$$

in terms of the map (L1.9). Similarly the map from ${}^V\overline{W}$ to $\overline{U} = \overline{W/V}$ extending the projection $(v, u) \longmapsto u$ is just

$${}^V\overline{W} \ni (t, s, v', u') \longmapsto (t, u') \in \overline{U} = \mathbb{S}^{n,1}. \tag{1+.40}$$

□

COROLLARY 1. *If $A : V \longrightarrow W$ is any linear map between real vector spaces V and W, with null space* $\operatorname{null}(A) \subset V$, *then A extends to a smooth map*

$$A : {}^{\operatorname{null}(A)}\overline{V} \longrightarrow \overline{W}. \tag{1+.41}$$

EXERCISE 5. Make sure you can give an elegant proof of this!

EXERCISE 6. Show that the second map in (1+.38) is a fibration. In fact, since the base is contractible (being a ball) it is then necessarily reducible to a product. Thus there exists a diffeomorphism $F : \overline{V} \times \overline{W/V} \longrightarrow {}^V\overline{W}$ such that the composite map

$$\overline{V} \times \overline{W/V} \xrightarrow{F} {}^V\overline{W} \longrightarrow \overline{W/V} \tag{1+.42}$$

is just the projection. However, there is no natural choice of F.

1+.4. Products. One thing we can certainly do is take the product of two vector spaces, $W = V \times U$. Then we can consider the compactification of W given by $\overline{V} \times \overline{U}$. The projection from W to U certainly extends to a smooth map from $\overline{V} \times \overline{U}$ to $\overline{U}$, namely the projection. However we still have the problem of the relationship of $\overline{W}$ to $\overline{V} \times \overline{U}$. The natural map between the interiors, both of which are identified with W, does not extend to a smooth map either way. We are part of the way to overcoming this difficulty with ${}^V\overline{W}$, but this is certainly not 'symmetric' in how it treats V and U so cannot be the full answer.

EXERCISE 7. Define the doubly-relative radial compactification of the product of two vector spaces. Do so by choosing metrics on U and V and then taking the map

$$
\begin{aligned}
R_{U,V} : W = (v,u) \longmapsto \Big(&\frac{1}{(1+|u|^2)^{\frac12}}, \frac{1}{(1+|v|^2)^{\frac12}}, \frac{(1+|u|^2)^{\frac12}}{(1+|u|^2+|v|^2)^{\frac12}}, \\
&\frac{(1+|v|^2)^{\frac12}}{(1+|u|^2+|v|^2)^{\frac12}}, \frac{v}{(1+|v|^2)^{\frac12}}, \frac{(v,u)}{(1+|u|^2+|v|^2)^{\frac12}}, \frac{u}{(1+|u|^2)^{\frac12}}\Big) \\
&\in \mathbb{R}^4 \times V \times W \times U
\end{aligned}
\tag{1+.43}
$$

and showing it to be a diffeomorphism onto its image. Then check that the closure of the image is a manifold with corners (it has three boundary faces provided U and V have dimension at least 2, more if one of them is one-dimensional). Show that all translations on W lift to be smooth as do all linear transformations of W mapping U and V into themselves (i.e. direct products of linear transformations of U and of V.) Denoting the resulting compactifications by ${}^{U,V}\overline{U \times V}$ show that both the inclusions of U and V extend to be smooth

$$
i : \overline{U} \longrightarrow {}^{U,V}\overline{U \times V},\ i : \overline{V} \longrightarrow {}^{U,V}\overline{U \times V}. \tag{1+.44}
$$

Show that the identity map extends to be smooth in two different ways

$$
\begin{tikzcd}
 & {}^{U,V}\overline{U \times V} \arrow[dl] \arrow[dr] & \\
\overline{U \times V} & & \overline{U} \times \overline{V}.
\end{tikzcd}
\tag{1+.45}
$$

The non-invertibility of these maps goes some way to explaining the difference between the radial compactification of the product and the product of the radial compactifications. Draw a picture!

1+.5. Blow up. If you go so far as to actually do Exercise 7 you will come to look for a better way of doing such things. Fortunately there is – and it is discussed in more detail starting in the addenda to the second lecture. For the moment, consider the relationship between the 1-point compactfication of $\mathbb{R}^n$ and its radial compactification. We know (or if you prefer have defined things so) that a function is smooth near the point at infinity of the one-point compactification if it is a smooth function of $z/|z|^2$. On the other hand, a function is smooth near the sphere at infinity of $\overline{\mathbb{R}^n}$ if it is a smooth function of $x = 1/|z|$ and $\omega = z/|z| \in \mathbb{S}^{n-1}$ near $x = 0$. Since $z/|z|^2 = x\omega$ we see that smoothness on ${}^1\overline{\mathbb{R}^n}$ implies smoothness on $\overline{\mathbb{R}^n}$. This of course means that the map sending the whole of infinity to the one point is smooth

$$
\beta : \overline{\mathbb{R}^n} \longrightarrow {}^1\overline{\mathbb{R}^n},\ \overline{\mathbb{R}^n} = [{}^1\overline{\mathbb{R}^n}, \{\infty\}]. \tag{1+.46}
$$

In fact we can see more. Namely, in the coordinates discussed above, the map β is nothing other than the introduction of polar coordinates,

$$
z/|z|^2 = Z = x\omega \tag{1+.47}
$$

and this is what the final notation in (1+.46) indicates.

Definition 1. A manifold with boundary X (denoted subsequently by $[M, \{p\}]$ is the blow-up at $p \in M$ of a manifold M if there is a smooth map

$$
\beta : X \longrightarrow M, \tag{1+.48}
$$

which is a diffeomorphism of $X \setminus \partial X$ to $M \setminus \{p\}$, maps ∂X to p and is such that polar coordinates for some local coordinates aound p lift to a diffeomorphism of a neighbourhood of ∂X to $[0,1) \times \mathbb{S}^{n-1}$, $n = \dim M$.

This definition just means that the blow-up of a point is the introduction of polar coordinates. It would not make much sense if it depended on the choice of local coordinates based at (i.e. vanishing at) p in which polar coordinates where introduced.

Exercise 8. Confirm that change of local coordinates based at $0 \in \mathbb{R}^n$ induces a diffeomorphism on $[0,\epsilon)_r \times \mathbb{S}^{n-1}$ for some $\epsilon > 0$. Hint: First do the linear case; for which one can either use the linear invariance of radial compactification above, or model the argument. Then check the case that the Jacobian is the identity directly.

1+.6. Radial and relative compactification. I will show below that the blow up of a closed embedded submanifold of a manifold is always well-defined and reduces locally to the introduction of polar coordinates in the normal variables. The same notation as above, $[M,Y]$ is used for the blown-up manifold in this more general case; it comes equipped with a smooth blow-down map $\beta : [M,Y] \longrightarrow M$. The reason I bring this up here is that the relative compactification introduced above can also be defined through blow-up from the radial compactification. In this case we are blowing up an embedded submanifold of the boundary of a manifold with boundary.

Proposition 2. *If $V \subset W$ is a non-trivial subspace of a real vector space then there is a natural diffeomorphism*

$$
{}^{V}\overline{W} \equiv [\overline{W}, \mathbb{S}V]. \tag{1+.49}
$$

Exercise 9. See if you can check this in local coordinates – of course it is a bit tricky since I have not explained what the blow-up map really is.

Exercise 10. See what happens in the 'trivial cases' excluded from Proposition 2, meaning $V = \{0\}$ or $V = W$. Namely show that

$$
{}^{\{0\}}\overline{W} \equiv \overline{W},\ {}^{W}\overline{W} \equiv {}^{q}\overline{W}. \tag{1+.50}
$$

A similar discussion applies to the double-relative compactification of a product. Namely, in $\overline{U \times V}$ the two bounding spheres, $\mathbb{S}U$ and $\mathbb{S}V$, of the subspaces are disjoint embedded submanifolds of the boundary. Since they are disjoint the blow-ups of $\mathbb{S}U$ and $\mathbb{S}V$ are completely independent.

Proposition 3. *For any real vector spaces, there is a canonical diffeomorphism*

$$
{}^{U,V}\overline{U \times V} \longrightarrow [\overline{U \times V}, \mathbb{S}U, \mathbb{S}V] \tag{1+.51}
$$

We may also blow up embedded submanifolds of boundary faces of manifolds with corners, provided they meet the other boundary faces in a 'product manner'. In particular we can blow up any boundary face.

Proposition 4. *For any real vector spaces, there is a canonical diffeomorphism*

$$
{}^{U,V}\overline{U \times V} \longrightarrow [\overline{U} \times \overline{V}, \mathbb{S}U \times \mathbb{S}V]. \tag{1+.52}
$$

Notice that $\mathbb{S}U \times \mathbb{S}V$ is indeed the corner of $\overline{U} \times \overline{V}$, since it is the product of the boundaries.

1+.7. Parabolic compactifications. If that wasn't enough there are actually other compactifications, which are not obtained by blow up of the ones I have already considered. What's more they really do show up in analysis, in particular in complex analysis – about which I will say nothing much in this course (but see [**2**].)

CHAPTER 2

Conormal distributions at the origin

Lecture 2: 13 September, 2005

L2.1. Classical symbols. As indicated above, I will define the conormal distributions at the origin of a vector space, starting with $\mathbb{R}^n$, as the inverse Fourier transform of the spaces $\rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$. Here $\rho \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$ is any boundary defining function. For a compact manifold with boundary X, a boundary defining function $\rho \in \mathcal{C}^\infty(X)$ is any function such that $\rho \geq 0$ everywhere, $\partial X = \{\rho = 0\}$ and $d\rho \neq 0$ on ∂X. Such a function always exists and any two are related by

$$\rho' = a\rho,\ 0 < a \in \mathcal{C}^\infty(X). \tag{L2.1}$$

For the radial compactification we know we can take as boundary defining function

$$Z_0 = \rho = \frac{1}{(1 + |\xi|^2)^{\frac{1}{2}}} \tag{L2.2}$$

for any metric.

Then the space $\rho^{-m}\mathcal{C}^\infty(\overline{W})$ for any real vector space W is defined by

$$u \in \rho^{-m}\mathcal{C}^\infty(\overline{W}) \Longleftrightarrow \rho^m u \in \mathcal{C}^\infty(\overline{W}). \tag{L2.3}$$

Traditionally this is called the space of 'classical symbols or order m on W' and denoted $S^m_{\mathrm{cl}}(W)$. I will not use this notation (at least not much) because it is redundant and also there is some confusion in the literature between closely related, but different, spaces.

Now, for a little exercise in abstract nonsense, note that $\rho^{-m}\mathcal{C}^\infty(\overline{W})$ is the space of all global sections of a trivial line bundle, which we denote N_{-m}, so

$$\rho^{-m}\mathcal{C}^\infty(\overline{W}) = \mathcal{C}^\infty(\overline{W}; N_{-m}). \tag{L2.4}$$

Indeed, this is a direct consequence of the relation (L2.1) between any two defining functions. Thus, ρ^{-m} is a global section of this bundle for any boundary defining function ρ. If you want to be pedantic, the fibre at any point $q \in \overline{W}$ (including of course boundary points) may be defined to be

$$(N_{-m})_p = \rho^{-m}\mathcal{C}^\infty(\overline{W})/\mathcal{I}_p \cdot \rho^{-m}\mathcal{C}^\infty(\overline{W}) \tag{L2.5}$$

where $\mathcal{I}_p \subset \mathcal{C}^\infty(\overline{W})$ is the space of smooth functions which vanish at p. It is handy to have the notation (L2.4), and it is little more than notation, since it lets us push the 'weight' function ρ^{-m} into a bundle and 'hide' it.

There are some rather obvious properties of these symbol spaces. Namely they multiply

$$\mathcal{C}^\infty(\overline{W}; N_{-m}) \cdot \mathcal{C}^\infty(\overline{W}; N_{-m'}) = \mathcal{C}^\infty(\overline{W}; N_{-m-m'}),\ \forall\ m, m' \in \mathbb{R}. \tag{L2.6}$$

In particular they are all $\mathcal{C}^\infty(\overline{W})$-modules, corresponding to the case $m = 0$ when N_{-m} is canonically trivial.

We also know that the action of $\mathrm{GL}(W)$ on W, and of W acting by translations, extends smoothly to $\overline{W}$ and necessarily maps the boundary onto itself. It follows that these actions extend to $\mathcal{C}^\infty(\overline{W})$, so G^* acts on $\mathcal{C}^\infty(\overline{W}; N_{-m})$ for any m.

The name 'symbols' is related to the 'symbol estimates' that these functions satisfy. In the case of $\mathbb{R}^n$ we know that $(1+|\xi|^2)^{-\frac{1}{2}}$ is a boundary defining function on $\overline{\mathbb{R}^n}$. Thus if $a \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}; N_{-m})$ then $\rho^m a$ is a bounded function and this reduces to

$$|a(\xi)| \le C(1+|\xi|^2)^{\frac{m}{2}} \Longleftrightarrow |a(\xi)| \le C'(1+|\xi|)^m\ \forall\ \xi \in \mathbb{R}^n. \tag{L2.7}$$

The second, simpler looking, form follows from the fact that $(1+|\xi|^2)^{\frac{1}{2}}$ and $1+|\xi|$ are of the 'same size', meaning each is bounded above and below by some positive multiple of the other. The disadvantage of $1+|\xi|$ is that it is singular at the origin, but it is easier to write. Anyway we also know that $\xi^\alpha \partial_\xi^\beta a \in \rho^{-m+|\alpha|-|\beta|}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ and hence

$$|\xi^\alpha \partial_\xi^\beta a| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|+|\alpha|} \text{ if } a \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}; N_{-m}). \tag{L2.8}$$

This is an explicit form of the statement that differentiation by ξ lowers the order by 1 and multiplication by a polynomial raises the order by the order of the polynomial, i.e.

$$\begin{aligned} \partial_{\xi_i} : \mathcal{C}^\infty(\overline{\mathbb{R}^n}; N_{-m}) &\longrightarrow \mathcal{C}^\infty(\overline{\mathbb{R}^n}; N_{-m+1}) \\ \xi_i \times : \mathcal{C}^\infty(\overline{\mathbb{R}^n}; N_{-m}) &\longrightarrow \mathcal{C}^\infty(\overline{\mathbb{R}^n}; N_{-m-1}). \end{aligned} \tag{L2.9}$$

The symbol estimates (L2.8), even if valid for all α and β, do not imply that $a \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}; N_{-m})$. Some discussion of the extent to which they are weaker and why they usually appear centrally in a treatment of microlocal analysis can be found in the addenda below. The present treatment avoids the use of these larger spaces of symbols 'with bounds', although they still have their place.

One thing that does follow easily from (L2.8) is that symbols of arbitrarily low order are Schwartz functions

Lemma 4. *On any real vector space and for any $m \in \mathbb{R}$,*

$$\bigcap_{N \in \mathbb{N}} \rho^{-m+N}\mathcal{C}^\infty(\overline{W}) = \mathcal{S}(W). \tag{L2.10}$$

Proof. Since we can always replce m in (L2.8) by $m - N$ it follows that if a is in the intersection in (L2.10) then

$$\sup_\xi |\xi^\alpha \partial_\xi^\beta a| < \infty\ \forall\ \alpha, \beta \Longrightarrow a \in \mathcal{S}(W). \tag{L2.11}$$

The converse statement also follows, namely it suffices to show that $\mathcal{S}(W) \subset \mathcal{C}^\infty(\overline{W})$. □

Returning to the general properties of the classical symbol spaces, there is a short exact sequence which will turn out to be of fundamental importance later. Namely, for any m

$$0 \longrightarrow \mathcal{C}^\infty(\overline{W}; N_{-m+1}) \longrightarrow \mathcal{C}^\infty(\overline{W}; N_{-m}) \longrightarrow \mathcal{C}^\infty(\mathbb{S}W; N_{-m}) \longrightarrow 0. \tag{L2.12}$$

In future I will often leave out the zeros at the ends of such short exact sequences. The claim of exactness is just that the second map is injective, the third is surjective and the range of the second is exactly the null space of the third. If we use ρ^{-m} to trivialize the bundle N_{-m} then this just reduces to the short exactness[1] of

$$\rho\mathcal{C}^\infty(\overline{W}) \longrightarrow \mathcal{C}^\infty(\overline{W}) \longrightarrow \mathcal{C}^\infty(\mathbb{S}W). \tag{L2.13}$$

This in turn means the restriction map to the bounding sphere is surjective and that a smooth function is of the form ρf for another smooth f if and only if it vanishes at the boundary; this is a form of Taylor's theorem.

There is an equally important but more complicated version of this called 'asymptotic completeness' of the spaces or 'asymototic summability' of series of symbols.

Proposition 5. *[Asymptotic Completeness] If $a_k \in \mathcal{C}^\infty(\overline{W}; N_{-m+k})$, is* any *sequence then there exists an element $a \in \mathcal{C}^\infty(\overline{W}; N_{-m})$ such that*

$$a - \sum_{k=0}^{N} a_k \in \mathcal{C}^\infty(\overline{W}; N_{-m+N+1}) \ \forall\ N \in \mathbb{N}. \tag{L2.14}$$

Proof. We can multiply everything by ρ^m to reduce to the case $m = 0$. Then it is a form of Borel's Lemma. Namely it follows from the fact that for any (compact) manifold with boundary X and any sequence $b_k \in \rho^k\mathcal{C}^\infty(X)$, $k \in \mathbb{N}_0$, there exists an element $b \in \mathcal{C}^\infty(X)$ such that

$$b - \sum_{k=0}^{N} b_k \in \rho^{N+1}\mathcal{C}^\infty(X) \ \forall\ N \in \mathbb{N}. \tag{L2.15}$$

This in turn can be reduced to the corresponding local statement for a hypersurface $z_1 = 0$ in $\mathbb{R}^n$ and then to the 1-dimensional case, with smoothness in parameters – this is the setting for the original Lemma of É. Borel. Namely that the sequence of derivatives of a smooth function at a fixed point is unconstrained, i.e. if c_k is any sequence of complex numbers then there exists a smooth function $u \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\frac{d^k u}{dx^k}(0) = c_k \ \forall\ k. \tag{L2.16}$$

Let me at least remind you of how this is proved – an extension of this argument leads to (L2.15). Namely one forces the Taylor series to converge, of course without constraints on the c_k's in (L2.16) it will not converge of its own volition! So, choose a cut-off function $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ which is 1 in $|x| < \frac{1}{2}$ and vanishes in $|x| > 1$. Then consider the series of smooth functions

$$b(x) = \sum_k \frac{c_k x^k}{k!}\chi(\frac{x}{\epsilon_k}), \tag{L2.17}$$

[1] Meaning supply your own zeroes at the ends.

where $\epsilon_k > 0$ is a sequence which tends to 0. It follows that the series is finite in any region $x \ge x_0 > 0$ so converges to a smooth function in $x > 0$. In fact it is easy to make it converge uniformily in $x \ge 0$. Indeed, the size of the kth term is

$$|c_k|\epsilon_k^k/k! \tag{L2.18}$$

since the cutoff vanishes when $x > \epsilon_k$. Now the ϵ_k just need to be chosen to vanish rapidly enough and the series will converge uniformly and absolutely. A similar choice allows the series for the derivatives of any order to be made to converge and then a diagonalization arument gives convergence in $\mathcal{C}^\infty(\mathbb{R})$. It follows that the sum satisfies (L2.16). □

The relationship (L2.14) is usually written

$$a \sim \sum_{k=0}^{\infty} a_k. \tag{L2.19}$$

Notice that $a \in \mathcal{C}^\infty(\overline{W}; N_{-m})$ is not uniquely determined by this condition. Any other element a' satisfying (L2.14) in place of a is such that $a'-a \in \mathcal{C}^\infty(\overline{W}; N_{-m+N})$ for all N which is to say $a' - a \in \mathcal{S}(W)$, so the 'asymptotic sum' is determined up to a rapidly decreasing 'error.'

L2.2. Classical conormal distibutions. Now, we are finally in a position to define the 'classical conormal distributions' on $\mathbb{R}^n$ with respect to the origin

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) = \mathcal{F}^{-1}\left(\rho^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^n})\right),\ m = m' + \frac{n}{4}. \tag{L2.20}$$

As promised these are just the inverse Fourier transforms of our symbol spaces. Notice however that I have shifted the 'order' on the left by a constant that depends on the dimension only. This 'normalization' is for reasons related to the 'principle of stationary phase' that will not show up for quite a long time, but leaving it out will cause more confusion than putting it in.

The simplest nontrivial example of a conormal distribution with respect to the origin of $\mathbb{R}^n$ is the Dirac delta 'function', the inverse Fourier transform of the constant function 1. According to (L2.20) it has 'order $n/4$' (however this is just a choice of normalization and doesn't correspond to a meaningful regularity statement)

$$\delta_0 \in I^{\frac{n}{4}}_{\mathcal{S}}(\mathbb{R}^n, \{0\}). \tag{L2.21}$$

However this is almost enough to allow one to remember the normalization (which I have a hard time doing)!

So, what are the basic properties. Certainly (L2.20) defines a space of tempered distributions

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \subset \mathcal{S}'(\mathbb{R}^n). \tag{L2.22}$$

Since $\rho\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \subset \mathcal{C}^\infty(\overline{\mathbb{R}^n})$ it follows that

$$I^{m-k}_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \subset I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \text{ if } k \in \mathbb{N}. \tag{L2.23}$$

Now the estimate (L2.7) shows that for the Fourier transform

$$\text{if } u \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \text{ then } a = \mathcal{F}(u) \in \rho^{-m+\frac{n}{4}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}),$$
$$\text{so } -m + \frac{n}{4} > n \Longrightarrow a \in L^1(\mathbb{R}^n)$$

and then the inverse Fourier transform

$$u(z) = (2\pi)^{-1} \int_{\mathbb{R}^n} a(\zeta) d\zeta, \ |u(z)| \le (2\pi)^{-n} \int_{\mathbb{R}^n} |a(\zeta)| d\zeta$$

is bounded. It is in fact also continuous (by the continuity-in-the-mean of L^1 functions) and vanishes at infinity so

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \subset \mathcal{C}^0_0(\mathbb{R}^n) \text{ if } m < -\frac{3n}{4}, \tag{L2.24}$$

the space of continuous functions which vanish at infinity.

Since the right hand side in (L2.20) is a space of smooth functions with some growth it is reasonable to expect the elements of the space on the left to be smooth with some localized singularities. That is indeed the case and we will show that

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\})\big|_{\mathbb{R}^n \setminus \{0\}} \subset \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}), \tag{L2.25}$$

so the only singularities in an element of $I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\})$ are at the origin, i.e. $\operatorname{sing\,supp}(u) \subset \{0\}$. We will actually prove something even stronger.

Lemma 5. *If $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ is equal to 1 in a neighbourhood of the origin then*

$$u \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \Longrightarrow (1 - \chi)u \in \mathcal{S}(\mathbb{R}^n). \tag{L2.26}$$

Proof. The Fourier transform has the property that

$$\mathcal{F}(z^\alpha D^\beta_z u) = (-D_\zeta)^\alpha (\zeta^\beta \mathcal{F}(u)) \tag{L2.27}$$

where $D_{z_k} = \frac{1}{i}\partial_{z_k}$ takes care of the factors of i. Recalling (L2.9) for the symbol spaces (and of course (L2.20)) we see that

$$z^\alpha D^\beta_z : I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \longrightarrow I^{m+|\beta|-|\alpha|}_{\mathcal{S}}(\mathbb{R}^n, \{0\}), \tag{L2.28}$$

just the opposite of the symbols spaces, so that differentiation raises the order but multiplication by a monomial lowers the order by the degree. Combining this with (L2.24) we conclude that

$$u \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \Longrightarrow z^\alpha D^\beta_z u \in \mathcal{C}^0_0(\mathbb{R}^n) \text{ if } |\alpha| > m + |\beta| + \frac{3n}{4}. \tag{L2.29}$$

So, adding a large number of terms we see that

$$|z|^{2N} u \in \mathcal{C}^p_0(\mathbb{R}^n) \text{ is bounded with its first } p \text{ derivatives if } 2N > m + p + \frac{3n}{4}. \tag{L2.30}$$

Now, multiplying by the cutoff $(1 - \chi)$ the same is true of $(1 - \chi)u$, However, $|x|^{2N}$ then does not vanish on the support, so we conclude that

$$|D^\beta_z\, ((1 - \chi)u)\,| \le C_{N,p}(1 + |z|)^{-2N}, \ 2N > m + p + \frac{3n}{4}, \ |\beta| \le p. \tag{L2.31}$$

Since m is fixed, we can simply take N very large and hence conclude that $(1-\chi)u \in \mathcal{S}(\mathbb{R}^n)$ which was the claim. □

This is the reason for the suffix $\mathcal{S}$ in the definition (L2.20); these distributions are rapidly decaying at infinity with all derivatives, it is just that they may be singular in a very specific way at the origin.

Next time I will talk more about invariance, showing that for a vector space there is an invariant version of the Fourier transform giving an isomorphism

$$\mathcal{F} : \mathcal{S}(W) \longrightarrow \mathcal{S}(W'; \Omega W') \tag{L2.32}$$

onto the Schwartz space of densities. In any case it is pretty clear that $I^m(\mathbb{R}^n, \{0\})$ is invariant under the action of $\mathrm{GL}(n, \mathbb{R})$ since

$$\mathcal{F}(G^*u) = ((G^{-1})^t)^*\mathcal{F}u \cdot |\det G|^{-1},\ G \in \mathrm{GL}(n, \mathbb{R}). \tag{L2.33}$$

We will eventually need more invariance than this, namely that the nature of the singularity at the origin is the same in any coordinates based at the origin.

2+. Addenda to Lecture 2

2+.1. Borel's lemma. Let me go a little further with the proof of Borel's lemma. As noted above, the series (L2.17) converges uniformly, with all derivatives, on compact subsets of in $|x| > 0$ if we simply require $\epsilon_k \to 0$. The estimates (L2.18) can be extended to the derivatives. Namely for any $j \geq k$ (only to avoid complications with indices)

$$D_x^j\left(\frac{c_k x^k}{k!}\chi(\frac{x}{\epsilon_k})\right) = \sum_{p=0}^{j}\binom{j}{p}\frac{c_k x^{k-j+p}}{(k-j+p)!}\epsilon_k^{-p}\chi^{(p)}(\frac{x}{\epsilon_k}) \Longrightarrow \tag{2+.34}$$
$$|D_x^j\left(\frac{c_k x^k}{k!}\chi(\frac{x}{\epsilon_k})\right)| \leq C_{k,j}\epsilon_k^{k-j}$$

where $C_{k,j}$ is a constant that does not depend on ϵ_k. It follows that if we choose

$$\epsilon_k < 2^{-k}/(1 + C_{k,j})\ \forall\ k > j,\ \forall\ j \tag{2+.35}$$

then the series of jth derivatives converges absolutely and uniformly for all x. The important point here is that making (2+.35) hold for all j represents only a finite number of conditions on each ϵ_k, namely there are conditions only for $0 \leq j \leq k$. Thus choosing each ϵ_k to be small enough the series (L2.17) converges uniformly, will all its derivatives. The sum is therefore a smooth function and it satisfies (L2.16).

A similar argument applies in more variables. If $u_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is a sequence with each element supported in a fixed compact set K then choosing $\epsilon_k > 0$ small enough ensures that

$$u(x, y) = \sum_k \frac{u_k(y)x^k}{k!}\chi(\frac{x}{\epsilon_k}) \in \mathcal{C}_c^\infty(\mathbb{R}^n) \tag{2+.36}$$

converges absolutely and uniformly with all its partial derivatives and satisfies

$$\partial_x^k u(0, y) = u_k(y)\ \forall\ k. \tag{2+.37}$$

Indeed, we simply have to arrange that all the differentiated series, with both x and y derivatives, converge absolutely and uniformly. The x derivatives behave exactly as before and the y derivatives fall on the u_k only. Thus we can arrange that the series for $\partial_x^k\partial_y^\alpha u$ converges by choosing

$$\epsilon_k < \epsilon_{k,j,\alpha}\ \forall\ k > j + |\alpha|. \tag{2+.38}$$

Here $\epsilon_{k,j,\alpha}$ is the same constant as in (2+.35) except that the $|c_k|$'s leading to the bound are replaced by the supremums of the $\partial_y^\alpha u_k$. Again the important point is that the convergence of each of the series is determined by what happens from some (any) finite point onwards. Thus we only need impose the bound on ϵ_k for $k > j + |\alpha|$ as in (2+.38). So again this is only a finite number of conditions on each ϵ_k but implies the uniform convergence of the series for all partial derivatives, so (2+.37) follows.

The general case now follows by use of a partition of unity to reduce the problem to a finite number applications of the construction above on $\mathbb{R}^n$.

2+.2. Symbols with bounds. As remarked above, the 'symbol estimates' (L2.8) do not imply that $a \in \rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$. To understand a little better what they do mean, first observe that the case $m = 0$ is fundamental since

$$a \text{ satifies (L2.8)} \iff (1+|\xi|^2)^{-m/2}a \text{ satifies (L2.8) with } m = 0. \tag{2+.39}$$

In fact the estimates with $\alpha \neq 0$ in (L2.8) are redundant, since they follow from those with $\alpha = 0$. It is also possible to reorganize these estimates as follows.

Exercise 11. Show (probably using induction) that the estimates (L2.8) for $m = 0$ are equivalent to the statements

$$\left(\prod_{j=1}^{N} V_{k_j l_j}\right) a \in L^\infty(\mathbb{R}^n),\ V_{kl} = \xi_k \partial_{\xi_l} \tag{2+.40}$$

for all N and all integer sequences k_j, l_j (including implicitly the case of no factors at all).

The operators V_{kl} are the linear vector fields on $\mathbb{R}^n$ and we know from § L1.2 that these lift to $\overline{\mathbb{R}^n}$ to span, near infinity, all vector fields tangent to the boundary.

Definition 2. On any compact manifold with boundary X let $\mathcal{V}_{\mathrm{b}}(X)$ denote the Lie algebra of all those smooth vector fields on X which are tangent to the boundary and define

$$\mathcal{A}(X) = \{a \in \mathcal{C}^\infty(\operatorname{int} X) \cap L^\infty(X); V_1 \cdots V_N a \in L^\infty(X),\ \forall\ V_i \in \mathcal{V}_{\mathrm{b}}(X),\ \forall\ N\}\,. \tag{2+.41}$$

Using the discussion of compactification last time, try your hand at a proof of

Proposition 6. *The symbol estimates* (L2.8) *are equivalent to requiring* $a \in \rho^{-m}\mathcal{A}(\overline{\mathbb{R}^n})$.

2+.3. Density and approximation. It is quite usual to replace the classical spaces by the larger spaces (with weaker topology) introduced above

$$\rho^{-m}\mathcal{C}^\infty(\overline{W}) \subset \rho^{-m}\mathcal{A}(\overline{W}). \tag{2+.42}$$

One reason for this is that it allows density arguments to be used.

Lemma 6. *For any $a \in \rho^{-m}\mathcal{C}^\infty(\overline{W})$ there exists a sequence $a_k \in \mathcal{S}(W)$ such that*

$$\begin{gathered} a_k \text{ is bounded in } \rho^{-m}\mathcal{A}(\overline{W}) \text{ and} \\ a_k \longrightarrow a \text{ in the topology of } \rho^{-m'}\mathcal{A}(\overline{W})\ \forall\ m' > m. \end{gathered} \tag{2+.43}$$

Proof. In fact we can take the sequence to be in $\mathcal{C}_c^\infty(W) \subset \mathcal{S}(W)$. Namely, if $\rho \in \mathcal{C}^\infty(\overline{W})$ is a defining function for 'infinity' and $\phi \in \mathcal{C}_c^\infty(0,\infty)$ has $\rho(x) = 1$ in $x > 1$ then

$$a_k = \phi(k\rho)a \in \mathcal{C}_c^\infty(W) \tag{2+.44}$$

has the desired properties. Indeed the result is equivalent to the special case $m = 0$ applied to $\rho^m a$. Thus we may assume that $a \in \mathcal{C}^\infty(\overline{W})$ in which case it follows directly from the definition, (2+.44), that a_k is bounded in $L^\infty(W)$ and that for any $\epsilon > 0$, $\rho^\epsilon a_k \longrightarrow \rho^\epsilon a$ in $L^\infty(\overline{W})$. These are the first estimates corresponding to (2+.43), which is the same statement after applying any number of smooth vector fields V_i tangent to the boundary of $\overline{W}$. Thus it is enough to check that for such vector fields and any $\epsilon > 0$,

(2+.45)
$$V_1 \dots V_N a_k \text{ is bounded in } L^\infty(W) \text{ and } \rho^\epsilon V_1 \dots V_N a_k \longrightarrow \rho^\epsilon a \text{ in } L^\infty(W).$$

This in turn follows by observing the boundedness of all the terms arising from differentiating the cut-off $\phi(k\rho)$ and the fact that they are supported arbitrarily close to the boundary (so when multiplied by ρ^ϵ each of them tends to zero). □

Note that you cannot do much better than this, namely $\mathcal{S}(W)$ is certainly not dense in $\rho^{-m}\mathcal{C}^\infty(\overline{W})$ in our 'classical symbol topology' (just the topology of $\mathcal{C}^\infty(\overline{W})$ on $\rho^m a$) – in fact it is a closed subspace!

2+.4. Asymptotic summation. If one wishes to use these larger symbol spaces, $\rho^{-m}\mathcal{A}(\overline{W})$ (which by the way would normally be denoted $S^m_{1,0}(W)$, with the $1,0$ suffix being a special case of a more general ρ, δ notation) then one needs to check various properties of it. Essentially by definition $\xi^\alpha \partial_\xi^\beta$ maps $\rho^{-m}\mathcal{A}(\overline{\mathbb{R}^n})$ into $\rho^{-(m+|\alpha|-|\beta|)}$ with $|\alpha| = \alpha_1 + \dots + \alpha_n$. Slightly more serious is the analogue of Borel's lemma, which is

Proposition 7. *[Asymptotic summability]. If $a_j \in \rho^{-m_j}\mathcal{A}(\overline{W})$ is a sequence with $m_j \to -\infty$ then there exists $a \in \rho^{-M}\mathcal{A}(\overline{W})$ where $M = \max_j m_j$ such that*

$$a - \sum_{j \le N} \in \rho^{-M(N)}\mathcal{A}(\overline{W}),\ M(N) = \max_{j>N} m_j,\ \forall\ N. \tag{2+.46}$$

Sketch only. The same method as for Borel's lemma, based on (2+.36), works. □

2+.5. Homogeneity and conormality. It is natural to ask exactly what these conormal distributions, both 'classical' and corresponding to symbols with bounds, are like. In the classical case it is possible to see quite explicitly the local behaviour of the singularity at the origin.

Lemma 7. *If $a \in \rho^{-m}(\overline{\mathbb{R}^n})$ with $m \notin \mathbb{Z}$ then there exists a sequence of functions $u_k \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $k \in \mathbb{N}_0$, such that the inverse Fourier transform*

(2+.47)
$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(\xi) d\xi \in \mathcal{S}'(\mathbb{R}^n) \text{ satisfies}$$
$$u - \sum_{0 \le k \le N} |x|^{-m-n+k} u_k(\frac{x}{|x|}) = U_N\big|_{x\neq 0},\ U_N \in \mathcal{C}^{N-n-[m]}(\mathbb{R}^n),\ N > n + [m]$$

where $[m]$ is the integral part of m.

In fact the coefficients u_k in the expansion (2+.47) are completely determined by a (remember that m is not an integer here) and conversely they completely determine the singularity of a in the sense that two classical symbols a and a' giving the same expansions differ by an element of $\mathcal{S}(\mathbb{R}^n)$. There is in fact no mystery about the u_k, they can be computed by formally substituting the Taylor series expansion of a at infinity, so

$$(2+.48)\quad a - \sum_{0\le k<N} |\xi|^{m-k} a_k(\frac{\xi}{|\xi|})| \le C|\xi|^{m-N} \text{ in } |\xi|>1 \Longrightarrow$$
$$u_k(\mu) = (2\pi)^{-n}\gamma_{m-k}\int_{\mathbb{S}^{n-1}} e^{i\mu\cdot\omega} a_k(\omega)d\omega,\ \mu\in\mathbb{S}^{n-1}$$

for certain constants γ_{m-k} which I leave you to evaluate.

In the case of integral m the result is almost the same, but a little more complicated. The expansion of a, in (2+.48) is always the same. However the expansion of u depends a little on how big the integer m is. If $m \le -n$, so $m = -n-p$ for some non-negative integer p, then we need to replace (2+.47) by

$$(2+.49)\quad u - p_N(x)\log|x| - \sum_{0\le k\le N} |x|^{-m-n+k} v_{-m-n+k}(\frac{x}{|x|}) = U_N\big|_{x\ne 0},$$
$$U_N \in \mathcal{C}^{N-n-[m]}(\mathbb{R}^n),\ N > n+[m]$$

where p_N is a fixed formal power series starting with terms of homogeneity at least $-m-n$ in x truncated at level N,

$$(2+.50)\qquad p_N(x) = \sum_{-m-n\le|\alpha|<N} p_\alpha x^\alpha$$

where the p_α are constants independent of N, and the u_k are smooth functions on the sphere which satisfy the constraints

$$(2+.51)\qquad \int_{\mathbb{S}^{n-1}} v_{-q}(\omega)\omega^\alpha d\omega = 0,\ |\alpha|\le q.$$

All such functions occur in, and are determined by, these expansions and again the singularity of u is determined by then expansion. The normalization (2+.51) means that there are no polynomials in the expansion in (2+.49), which is naturel since these do not correspond to singularities at the origin for u. The corresponding singular terms occur with the logarithmic coefficient.

When $-n < m < 0$ the expansion is the same, except there are additional terms of homogeneity between 0 and $-m-n$ which are subject to no constraints. When m is a non-negative integer the are terms which do not appear in the expansion (which is in $x \ne 0$ where u is smooth) but correspond to the delta functions at the origin. Thus, the expansion of a has a unique polynomial part with inverse Fourier transform a sum of derivatives of the delta function. So one can consider $a \in \rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ without polynomial part. Then there is an expansion just as in (2+.49) except that the terms now of non-negative integral homogeneity must satisfy the same integral constraints as in (2+.51).

One way to make the relationship between homogeneity and conormality explicit is to check

LEMMA 8. *Any distribution on $\mathbb{R}^n$ which is smooth outside the origin and 'homogeneous modulo $\mathcal{C}^\infty$' of some degree, i.e.*

$$(2+.52)\qquad u(tx) = t^h u(x) + F(t,x),\ t>0, x\in\mathbb{R}^n \text{ with } F\in\mathcal{C}^\infty((0,\infty)\times\mathbb{R}^n)$$

is equal to a (classical) conormal distribution in a neighbourhood of 0. *Conversely, finite sums $\psi_k u_k + \psi$ with the u_k of this form and 'homogeneous' of degree $m-k$*

with the ψ_k, $\psi \in \mathcal{S}(\mathbb{R}^n)$ *are dense in the space of classical conormal distributions of order* $-m + \frac{3n}{4}$.

2+.6. Blow up of the origin. The operation of 'blowing up a submanifold' is in many senses dual to the process of compactification discussed last time. For one thing it is related to maps *into* the space in question, rather than maps from the space into a compactification. Thus for a vector space W, the space '$[W, \{0\}]$', which is 'W blown up at the origin' is associated to a map, namely polar coordinates

$$\beta : [0, \infty) \times \mathbb{S}^{n-1} \ni (r, \omega) \longmapsto r\omega \in \mathbb{R}^n. \tag{2+.53}$$

Here we can think of the sphere as the usual 'Euclidean sphere of radius 1'

$$\mathbb{S}^{n-1} = \{z \in \mathbb{R}^n; |z| = 1\}. \tag{2+.54}$$

At any point of $\mathbb{S}^{n-1}$ there are 'projective coordinates'. Namely at each point there can be at most one component z_j with $z_j^2 = 1$ and any $n-1$ components, not including one with $z_j^2 = 1$, give local coordinates. This is just the implicit function theorem since

$$\sum_j z_j dz_j = 0 \tag{2+.55}$$

is the only constraint on the differentials, so any $n-1$ of them are independent unless they include a dz_j with $z_i = 0$ for all $i \neq j$ (which means $z_j^2 = 1$ and $dz_j = 0$ on $\mathbb{S}^{n-1}$).

Thus the smoothness of (2+.53) follows from the smoothness of the components, as functions on $\mathbb{S}^{n-1}$. It is surjective, since $0 \in \mathbb{R}^n$ is the image of $\{0\} \times \mathbb{S}^{n-1}$ and any other point $0 \neq z \in \mathbb{R}^n$ is the image of $(|z|, z/|z|)$. In fact this shows that β is a diffeomorphism of $(0, \infty) \times \mathbb{S}^{n-1}$ onto $\mathbb{R}^n \setminus \{0\}$, with the inverse being $r = |z|$, $\omega = z/|z|$.

Now, the standard action of the orthogonal group on the sphere, which is induced from the action on $\mathbb{R}^n$, commutes with β

$$\beta(r, O\omega) = O\beta(r, \omega) \ \forall \ r \in [0, \infty), \ \omega \in \mathbb{S}^{n-1}.$$

Just as for the map defining radial compactification, it is important to know that a general element of $\mathrm{GL}(n, \mathbb{R})$ lifts under β.

Lemma 9. *There is a smooth action of* $\mathrm{GL}(n, \mathbb{R})$ *on* $[0, \infty) \times \mathbb{S}^{n-1}$ *which is intertwined with the standard action on* $\mathbb{R}^n$ *by* β :

$$\begin{tikzcd} {[0,\infty) \times \mathbb{S}^{n-1}} \arrow[r, "\beta"] \arrow[d, "\tilde{A}"'] & \mathbb{R}^n \arrow[d, "A"] \\ {[0,\infty) \times \mathbb{S}^{n-1}} \arrow[r, "\beta"'] & \mathbb{R}^n. \end{tikzcd} \tag{2+.56}$$

Proof. See the discussion in the case of radial compactification. The Lie algebra of $\mathrm{GL}(N, \mathbb{R})$ consists of the linear vector fields $z_i \partial_j$. Each of these is homogeneous of degree 0 under the homotheity $z \longmapsto sz$, $s \in (0, \infty)$. Since β is a diffeomorphism, there is a unique smooth vector field V_{ij} on $(0, \infty) \times \mathbb{S}^{n-1}$ such that $\beta_*(V_{ij}) = z_i \partial_j$ at each point. Thus $V_{ij} = a(r, \omega)\partial_r + V'_{ij}(r)$ where $V'_{ij}(r)$ is a smooth vector field on the sphere, depending smoothly on $r \in (0, r)$. By the homogeneity $a(r, \omega) = ra(1, \omega)$ and V'_{ij} is independent of r. Thus

$$V_{ij} = a(\omega) r \partial_r + V'_{ij}(\omega) \tag{2+.57}$$

extends to be smooth down to $r = 0$ (and tangent to $r = 0$). As in the case of compactification this shows that any element $A \in \mathrm{GL}(n, \mathbb{R})$ lifts to a smooth diffeomorphism $\tilde{A}$ of $[0, \infty) \times \mathbb{S}^{n-1}$ and in fact gives a smooth action

$$\mathrm{GL}(n, \mathbb{R}) \times [0, \infty) \times \mathbb{S}^{n-1} \longrightarrow [0, \infty) \times \mathbb{S}^{n-1}. \tag{2+.58}$$

$\square$

Continuing to follow the discussion of the radial compactification, this shows that we may define $[W, \{0\}]$ as a manifold associated to the principal $\mathrm{GL}(n, \mathbb{R})$-space, $P(W)$, of bases of W. Thus $\mathrm{GL}(n, \mathbb{R})$ acts on $P(W)$ by replacing a basis by the corresponding linear combination of its elements and $\mathrm{GL}(W)$ acts on it by acting on the elements of the basis. From this abstract point of view we may set

$$[W, \{0\}] = \left(P(W) \times [0, \infty) \times \mathbb{S}^{n-1}\right) / \, \mathrm{GL}(n, \mathbb{R}). \tag{2+.59}$$

Exercise 12. Show that $[W, \{0\}]$ is a manifold with boundary, diffeomorphic to $[0, \infty) \times \mathbb{S}^{n-1}$, that $\mathrm{GL}(W)$ acts smoothly on it and that there is a smooth map (the blow-down map)

$$\beta : [W, \{0\}] \longrightarrow W \tag{2+.60}$$

which intertwines the actions, maps the boundary to $\{0\}$ and is a diffeomorphism of the interior to $W \setminus \{0\}$.

As with conormal distributions, I will show later how to extend this notion to blowing up an embedded submanifold of a given manifold, by passing through the special case of blowing up the zero section of a vector bundle. It is also convenient to have a concrete realization of the blown-up space.

Exercise 13. Define the sphere of W to be

$$\mathbb{S}W = (W \setminus \{0\}) / \mathbb{R}^+ \tag{2+.61}$$

where $\mathbb{R}^+$ acts by multiplication. Show that $\mathbb{S}W$ is a smooth compact manifold diffeomorphic to $\mathbb{S}^{n-1}$, $n = \dim W$, that there is is a natural diffeomorphism $\mathbb{S}W \longrightarrow \partial[W, \{0\}]$ and a unique $\mathcal{C}^\infty$ structure on the disjoint union so that

$$[W, \{0\}] = (W \setminus \{0\}) \sqcup \mathbb{S}W. \tag{2+.62}$$

CHAPTER 3

Conormality at the zero section

Lecture 3: 15 September, 2005

Next we turn to the case of a real vector bundle $W \longrightarrow Y$ over a compact manifold Y and define the space of conormal distributions on the total space W of the vector bundle with respect to (i.e. only singular at) the zero section 0_W. The latter is a compact embedded submanifold canonically isomorphic to Y.

Before I do this, I want to point out some further properties in the case of the conormal distributions with respect to the origin of a vector space. In particular there is another important invariance property, the proof of which I want to go through. I will also indicate in a simple example how these spaces can be used.

First, these distibutions can be integrated

$$\int_{\mathbb{R}^n} : I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \longrightarrow \mathbb{C}. \tag{L3.1}$$

The integral is well-defined on both distributions of compact support and on $\mathcal{S}(\mathbb{R}^n)$ and we know, using (L2.26), that any $u \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\})$ can be written as a sum $u = \chi u + (1 - \chi)u$ of one term of each type. The value of integral is independent of the particular splitting since the definitions agree on the intersection, namely $\mathcal{C}^\infty_c(\mathbb{R}^n)$. In terms of the Fourier transform $a = \mathcal{F}(u)$ the integral can be written explictly:-

$$\int_{\mathbb{R}^n} u(x)dx = a(0). \tag{L3.2}$$

As we shall see this rather trivial observation is decidedly useful later.

For a general vector space we will only get a well-defined map analogous to (L3.1) if we have chosen a volume form, which could be the Lebesgue form for some identification with $\mathbb{R}^n$. I will discuss densities, a better way to do this, later.

So, returning to the case of $\mathbb{R}^n$ recall that we have already shown that

$$z^\alpha D^\beta_z : I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \longrightarrow I^{m+|\beta|-|\alpha|}_{\mathcal{S}}(\mathbb{R}^n, \{0\}). \tag{L3.3}$$

This follows directly from the properties of the Fourier transform. It is also clear that convolution behaves well

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) * I^{m'}_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \subset I^{m+m'+\frac{n}{4}}_{\mathcal{S}}(\mathbb{R}^n, \{0\}). \tag{L3.4}$$

Indeed, the Fourier transform of the convolution is the product of the Fourier transforms so

(L3.5)

$$\widehat{u * v} = \widehat{u}\widehat{v} \in \rho^{-m-m'-\frac{n}{2}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \text{ if } \widehat{u} \in \rho^{-m-\frac{n}{4}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \text{ and } \widehat{v} \in \rho^{-m'-\frac{n}{4}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}).$$

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It is also the case that $I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$ is a $\mathcal{S}(\mathbb{R}^n)$-module, that is multiplication by a Schwartz function maps this space into itself

$$\mathcal{S}(\mathbb{R}^n)\cdot I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})\subset I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\}). \tag{L3.6}$$

Perhaps the obvious way to approach this is the opposite to (L3.5). That is, take the Fourier tranform and then show that

$$\mathcal{S}(\mathbb{R}^n)*\rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n})\subset\rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n}). \tag{L3.7}$$

This is not so hard, and may well be informative. However I will prove it in a slightly different way, using an asymptotic completeness argument.

So we wish to show that if $\phi\in\mathcal{S}(\mathbb{R}^n)$ and $u\in I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$ then $\phi u\in I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$. We can simplify this a little by choosing a cutoff function $\chi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ which is identically equal to 1 near the origin and splitting $u=\chi u+(1-\chi)u$ into a compactly supported term and a term in $\mathcal{S}(\mathbb{R}^n)$; then we can ignore the latter since it is in an algebra contained in $I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$. Now, we can similarly split ϕ into a part supported near, and a part supported away from, the origin. If the latter is supported in the complement of the support of u (now compact) then the product is zero. Thus we are reduced to the special case

$$\mathcal{C}^\infty_c(\mathbb{R}^n)\cdot I^m_{\mathrm{c}}(\mathbb{R}^n,\{0\})\subset I^m_{\mathrm{c}}(\mathbb{R}^n,\{0\}) \tag{L3.8}$$

where the suffix 'c' indicates that supports are compact, as opposed to the Schwartz property at infinity.

Now, let us replace ϕ by its Taylor series expansion, to high order and with remainder, about the origin

$$\phi(z)=\sum_{|\alpha|\le N}c_\alpha z^\alpha+\sum_{|\alpha|=N+1}\phi_\alpha(z)z^\alpha,\ \phi_\alpha(z)\in\mathcal{C}^\infty(\mathbb{R}^n). \tag{L3.9}$$

If you recall, this is proved by radial integration. Now, multiplying $\phi(z)$ by another cutoff $\chi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ which is equal to 1 in a neighbourhood of the support of u (so $\chi u=u$) we find that

$$\phi(z)u=\sum_{|\alpha|\le N}c_\alpha z^\alpha u+\sum_{|\alpha|=N+1}\phi^{(N)}_\alpha(z)z^\alpha u,\ \phi^{(N)}_\alpha\in\mathcal{C}^\infty_c(\mathbb{R}^n). \tag{L3.10}$$

The advantage of doing this is that we know about all the terms in the first sum, namely $z^\alpha u\in I^{m-|\alpha|}_{\mathcal{S}}(\mathbb{R}^n,\{0\})$. Similarly the remainder terms are of the form

$$\sum_{|\alpha|=N+1}\phi^{(N)}_\alpha(z)u^{(N)}_\alpha,\ u^{(N)}_\alpha\in I^{m-N-1}_{\mathrm{c}}(\mathbb{R}^n,0). \tag{L3.11}$$

On the other hand from the estimates I did last time, we know that if $N>m+n+p$ then

$$u^{(N)}_\alpha\in\mathcal{C}^p_{\mathrm{c}}(\mathbb{R}^n),\ \forall\ |\alpha|=N+1. \tag{L3.12}$$

After multiplying by a smooth function of compact support this remains true. Note that in the first sum in (L3.10) the term of order α is fixed once $N\ge|\alpha|$. Thus, by asymptotic completeness we can find one element $v\in I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$ such that

$$v-\sum_{|\alpha|\le N}c_\alpha z^\alpha u\in I^{m-N-1}_{\mathcal{S}}(\mathbb{R}^n,\{0\})\ \forall\ N. \tag{L3.13}$$

Combining this with (L3.10) and (L3.11), with the same estimate on the regularity at the origin for the difference in (L3.13) we conclude that

$$\phi u - v \in \mathcal{S}(\mathbb{R}^n) + \mathcal{C}^p_c(\mathbb{R}^n)\ \forall\ p \tag{L3.14}$$

and hence $\phi u - v \in \mathcal{S}(\mathbb{R}^n)$ which proves (L3.8) and hence (L3.6).

Let me make an immediate application of this to a 'baby' problem which is intended to illustrate how we can use these conormal distributions. Observe that the constants are in the space $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ so

$$\delta \in I_{\mathcal{S}}^{-\frac{n}{4}}(\mathbb{R}^n, \{0\}) \tag{L3.15}$$

just to make you think of an example.

Now, combining (L3.3) and (L3.6) we see that if P is a differential operator with Schwartz coefficients

$$P = \sum_{|\alpha|\le k} p_\alpha(z) D_z^\alpha,\ p_\alpha(z) \in \mathcal{S}(\mathbb{R}^n) \tag{L3.16}$$

then

$$P : I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \longrightarrow I^{m+k}_{\mathcal{S}}(\mathbb{R}^n, \{0\}). \tag{L3.17}$$

The project is to try to partially invert this map by showing that

$$\begin{gathered}\text{Given } f \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\})\ \exists\ u \in I^{m-k}_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \\ \text{s.t. } Pu = f + g,\ g \in \mathcal{S}(\mathbb{R}^n) \text{ provided } P \text{ is elliptic at } 0\end{gathered} \tag{L3.18}$$

Of course I have not said what the condition of ellipticity means, but we will find out in the proof. We only 'partially' solve the problem in the sense that there is a Schwartz error, but at least we can remove the singularity.

As in the discussion above, we do not get to our goal immediately, but we proceed by steps. Suppose $u_0 \in I^{m-k}(\mathbb{R}^n, \{0\})$ then we know from (L3.17) that $Pu \in I^m(\mathbb{R}^n, \{0\})$ but we can get more information about the 'leading singularity'. Namely, the part of the sum in (L3.16) over $|\alpha| < k$ maps u into $I^{m-1}(\mathbb{R}^n, \{0\})$. Similarly, any part of the coefficients which vanishes at the origin has a factor of z_j in it and so, even after k differentiations, this part maps into $I^{m-1}(\mathbb{R}^n, \{0\})$ as well. Thus

$$Pu_0 = \sum_{|\alpha|=k} p_\alpha(0) D_z^\alpha u_0 + f',\ f' \in I^{m-1}(\mathbb{R}^n, \{0\}). \tag{L3.19}$$

Taking the Fourier transform of this 'leading term' we get

$$p_k(0,\xi)\widehat{u_0}(\xi) \in \rho^{-m-k+\frac{n}{4}} \mathcal{C}^\infty(\overline{\mathbb{R}^n}),\ p_k(0,\xi) = \sum_{|\alpha|=k} p_a\alpha(0)\xi^\alpha. \tag{L3.20}$$

So we want to solve

$$p_k(0,\xi)\widehat{u_0}(\xi) = \widehat{f}(\xi) \tag{L3.21}$$

just as though we were solving a constant coefficient operator. Of course in general this does not have a smooth solution because of the zeros of $p_k(0,\xi)$. We say that

$$P \text{ is elliptic at } 0 \text{ if } p_k(0,\xi) \neq 0 \text{ in } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{L3.22}$$

Even assuming this we cannot quite solve (L3.21) since (unless we are in the completely trivial case where $k = 0$) $p_k(0, \xi)$ does vanish at the origin. However we can choose

$$\widehat{u_0}(\xi) = \frac{(1 - \chi(\xi))\widehat{f}(\xi)}{p_k(0, \xi)} \in \rho^{-m-k+\frac{n}{4}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}). \tag{L3.23}$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 near zero. Since p_k is homogeneous of degree k (and non-vanishing) $(1 - \chi)/p_k \in \rho^k\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ from which (L3.23) follows. Moreover then we get

$$p_k(0, \xi)\widehat{u_0} = \widehat{f} + g, \ g \in \mathcal{S}(\mathbb{R}^n). \tag{L3.24}$$

Inserting this in (L3.19) we have made progress, namely we have shown that

$$\text{Given } f \in I^m(\mathbb{R}^n, \{0\}) \ \exists \ u_0 \in I^{m-k}(\mathbb{R}^n, \{0\}) \text{ s.t.} \quad Pu = f + f'', \ f'' \in I^{m-1}(\mathbb{R}^n, \{0\}) \tag{L3.25}$$

provided P satisfies the ellipticity condition (L3.22).

Now we can proceed by induction. Namely the order m in (L3.25) is arbitrary and the inductive statement is that we have constructed

$$u_j \in I^{m-k-j}(\mathbb{R}^n, \{0\}), \ j = 0, \dots, l \quad \text{s.t. } P(\sum_{j=0}^{l} u_j) = f - f_{l+1} \text{ where } f_{l+1} \in I^{m-l-1}(\mathbb{R}^n, \{0\}). \tag{L3.26}$$

Then we use (L3.25) with m replaced by $m - l - 1$ and f replaced by f_{l+1} to construct u_{l+1} and then define $f_{l+2} = f_{l+1} - Pu_{l+1} \in I_{\mathcal{S}}^{m-l-2}(\mathbb{R}^n, \{0\})$. This proves the inductive statement for all l.

Finally we use asymtotic completeness, which shows that there exists one fixed $u \in I_{\mathcal{S}}^{m-k}(\mathbb{R}^n, \{0\})$ such that

$$u - \sum_{j=0}^{l} u_j \in I_{\mathcal{S}}^{m-k-l-1}(\mathbb{R}^n, \{0\}) \ \forall \ l \tag{L3.27}$$

and from this (L3.18) follows.

This argument is a model for quite a few arguments below.

Now, what I really want to do today is to define conormal distributions on a vector bundle. What we need here is the invariance under linear transformations, which we have already checked. However we also want to be able to write things in an invariant form and to do so it is convenient to use the language of densities.

Recall that given a vector space over the reals there are many 'associated' vector spaces. The dual W', tensor powers and in particular exterior powers – the totally antisymmetric parts of the tensor powers. If $\dim W = n$ then the maximal (non-trivial) exterior power is $\Lambda^n W$. Its elements are n-multilinear and totally antisymmetric forms

$$\mu : (W')^{\times n} = W' \times W' \times \dots W' \longrightarrow \mathbb{R}$$

where multilinearity is linearity in each of the n variables separately and antisymmetry reduces to oddness under the exchange of any neighbouring pair of variables. This is a 1-dimensional vector space and by standard linear algebra its dual is canonically isomorphic to $\Lambda^n(W')$. That is, μ can be identified (canonically so we use the same name) with a linear map

$$\mu : \Lambda^n(W') \longrightarrow \mathbb{R}. \tag{L3.28}$$

The fundamental property of these forms is that on $\mathbb{R}^n$, for the action of $\mathrm{GL}(n, \mathbb{R})$,

$$(G^*\mu)(w_1, \dots, w_n) = \mu(Gw_1, \dots, Gw_n) = \det G \cdot \mu(w_1, \dots, w_n) \tag{L3.29}$$

in terms of (3). Of course it has to be a multiple since the space is 1-dimensional.

Now, in place of (L3.28) we can consider more general maps

$$\Omega^t W = \{\nu : \Lambda^n(W') \setminus \{0\} \longrightarrow \mathbb{R},\ \nu(cv) = |c|^t \nu(v)\ \forall\ c \in \mathbb{R} \setminus \{0\},\ v \in \Lambda^n(W')\}. \tag{L3.30}$$

Instead of being linear these are absolutely homogeneous of degree t. If $t = 0$ we just have constants but in all cases, for each $t \in \mathbb{R}$, these are linear spaces of dimension 1. In the special case $t = 1$ we just use the notation ΩW. Observe that if $\mu \in \Lambda^n W$ then $|\mu| \in \Omega W$ and any element is equal to $\pm|\mu|$ for some such μ. Thus the only real difference between $\Lambda^n W$ and ΩW is to do with orientation. Anyway, it follows from this observation that in the case of $\mathbb{R}^n$,

$$G^*\nu = |\det G|\nu \tag{L3.31}$$

in terms of the same action of $\mathrm{GL}(n, \mathbb{R})$ on $\Lambda^n \mathbb{R}^n$. This is the reason densities are important, because they transform in such a way that integration becomes invariant (for the moment under linear transformations).

Let us apply this discussion directly to the Fourier transform. For Schwartz functions on $\mathbb{R}^n$

$$\widehat{G^*u}(\zeta) = \int e^{-iz\cdot\zeta} u(Gz) dz,\ G \in \mathrm{GL}(n, \mathbb{R}). \tag{L3.32}$$

Changing the variable of integration from z to $y = Gz$ we 'know' (from integration theory) that

$$\widehat{G^*u}(\zeta) = \int e^{-iy\cdot(G^{-1})^t\zeta} u(y) dy |\det G|^{-1} \tag{L3.33}$$

where I have used the definition of the transpose to write $G^{-1}y \cdot \zeta = y \cdot (G^{-1})^t \zeta$. Thus

$$\widehat{G^*u}(\zeta) = |\det G|^{-1}.((G^{-1})^t)^* \hat{u} \tag{L3.34}$$

The action via the transpose of the inverse is exactly what we expect on the dual space but there is an extra factor of the determinant, admittedly just a constant but annoying nevertheless! We can remove this and get complete invariance by redefining the Fourier transform as a density

$$\mathcal{F}u(\zeta) = \hat{u}(\zeta)|d\zeta| \in \mathcal{S}(\mathbb{R}^n; \Omega). \tag{L3.35}$$

Now, from the discussion above this transforms in precisely the correct way so that we have a map which is independent of the choice of linear coordiantes

$$\mathcal{F} : \mathcal{S}(W) \longrightarrow \mathcal{S}(W'; \Omega W') \tag{L3.36}$$

where the image space is just the space of Schwartz functions valued in $\Omega W'$ (which is just a 1-dimensional vector space.)

As a consequence of this we can now identify, independent of the choice of basis

$$I^m(W, O_W) = \mathcal{F}^{-1}\left(\rho^{-m-\frac{n}{4}} \mathcal{C}^\infty(\overline{W'}; \Omega W')\right). \tag{L3.37}$$

Then

$$I^{m-1}(W,O_W)\longrightarrow I^m(W,O_W)\xrightarrow{\sigma_m}\mathcal{C}^\infty(\mathbb{S}W';N_{-m-\frac{n}{4}}\otimes\Omega W') \tag{L3.38}$$

is a short exact sequence.

One advantage of this definition, or the coordinate version for $\mathbb{R}^n$, is that we can immediately see what it means for such a distribution to 'depend smoothly on parameters.' Said another way, these spaces come with topologies, since the Fourier transform is used as an isomorphism, we can use the topology (uniform convergence of all derivatives on compact sets) on $\mathcal{C}^\infty(\overline{W'})$ to give a toplogy on

$$\rho^{-m}\mathcal{C}^\infty(\overline{W'};\Omega W')=\mathcal{C}^\infty(\overline{W'};N_{-m}\otimes\Omega W')$$

for any m and hence we have a topology on $I^m(W,0_W)$.

So, if Y is a compact manifold, what is $\mathcal{C}^\infty(Y;I^m(W,O_W))$ for a vector space W? It is a space of distributions on $Y\times W$ which is identified by Fourier transform with $\mathcal{C}^\infty(Y;\rho^{-m-\frac{n}{4}}\mathcal{C}^\infty(\overline{W'};\Omega W'))$. Again the definition of the topology just given means that we remove the weight factor and take a basis of the vector space and so identify this with $\mathcal{C}^\infty(Y;\mathcal{C}^\infty(\mathbb{S}^{n,1}))$. Now, it is a standard analytic fact that (for any manifolds)

$$\mathcal{C}^\infty(Y;\mathcal{C}^\infty(\mathbb{S}^{n,1}))=\mathcal{C}^\infty(Y\times\mathbb{S}^{n,1})=\mathcal{C}^\infty(Y\times\mathbb{S}^n)\big|_{Y\times\mathbb{S}^{n,1}} \tag{L3.39}$$

is just the space of smooth functions on the product manifold, itself a compact manifold with boundary. Or, backing up a little with the identifications it is the same thing as

$$\rho^{-m-\frac{n}{4}}\mathcal{C}^\infty(Y\times\overline{W'};\Omega W'). \tag{L3.40}$$

For reasons that might seem trivial compared to the resulting annoyance, we identify this space, as a space of distributions on $Y\times W$ with

$$I^m_{\mathcal{S}}(Y\times W,Y\times\{0\})=\mathcal{F}^{-1}\left(\rho^{-m+\frac{d}{4}-\frac{n}{4}}\mathcal{C}^\infty(Y\times\overline{W'};\Omega W')\right),\quad d=\dim Y,\ n=\dim W. \tag{L3.41}$$

Here $\mathcal{F}$ is to be interpreted as in (L3.36).

Now, suppose that rather than a product with a vector space, W is a smooth real vector bundle over the compact manifold Y. We want to define $I^m_{\mathcal{S}}(W,O_W)$ so that it reduces to (L3.41) in case the bundle is trivial.

Let me start with the radial compactification of the real vector bundle W. I will, for just this once, take the 'high road' of associated bundles, but then give a transition-map description.

A real vector bundle over Y is a manifold W with a smooth surjective map $\pi:W\longrightarrow Y$ which is a submersion (has surjective differential at each point), is such that, for each $y\in Y$, $\pi^{-1}(y)=W_y$ has a linear structure (over $\mathbb{R}$) and which is also locally trivial in the sense that Y has a covering by open sets $\mathcal{U}$ such that for each $U\in\mathcal{U}$, there is a diffeomorphism $T_U:\pi^{-1}(U)\longrightarrow U\times\mathbb{R}^n$ giving a commutative diagramme with the projections

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "F_U"] \arrow[dr, "\pi"'] & & U\times\mathbb{R}^n \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd} \tag{L3.42}$$

and which is linear on the fibres.

From a vector bundle we can construct a principal bundle. Namely for each $y \in Y$ set

$$P_y = \{T : W_y \longrightarrow \mathbb{R}^n \text{ a linear isomorphism.}\} \tag{L3.43}$$

This is a principal $\mathrm{GL}(n, \mathbb{R})$-space since if $T \in P_y$ then $GT \in P_y$ for each $G \in \mathrm{GL}(n, \mathbb{R})$ and this action of $\mathrm{GL}(n, \mathbb{R})$ is simple and transitive. Putting these spaces together set

$$\begin{tikzcd} P = \bigcup_{y \in Y} P_y \arrow[d, "\pi_P"] \\ Y. \end{tikzcd} \tag{L3.44}$$

The local trivializations (L3.42) of W provide sections over the sets $U \in \mathcal{U}$ of P giving corresponding maps

$$\begin{tikzcd} \pi_P^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times \mathrm{GL}(n, \mathbb{R}) \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd} \tag{L3.45}$$

which fix a consistent $\mathcal{C}^\infty$ structure on P.

The vector bundle W can be recovered from the principal bundle P as

$$W = P \times \mathbb{R}^n / \mathrm{GL}(n, \mathbb{R}),\ G(p, v) = (Gp, Gv). \tag{L3.46}$$

In this way we can easily define the radial compactification of W by taking the extension of the $\mathrm{GL}(n, \mathbb{R})$ action to $\mathbb{S}^{n,1}$ and so setting

$$\overline{W} = P \times \mathbb{S}^{n,1} / \mathrm{GL}(n, \mathbb{R}),\ G(p, q) = (Gp, Gq),\ W \hookrightarrow \overline{W} \tag{L3.47}$$

embeds W as the interior of a compact manifold with boundary. Thus we have defined the corresponding 'symbol spaces'

$$\rho^{-m} \mathcal{C}^\infty(\overline{W}; \Omega_{\mathrm{fib}}),\ \Omega_{\mathrm{fib}} = \Omega W. \tag{L3.48}$$

where ρ is as before a defining function for the boundary (which always exists globally).

Thinking in terms of transition maps for local trivializations suppose that U_i, U_j are elements of $\mathcal{U}$ (and a finite number of its elements must cover by the compactness of Y) the two maps (L3.42) combine over $U_{ij} = U_i \cap U_j$, assuming this is non-empty, to give a smooth map

$$h_{ij} : U_{ij} \longrightarrow \mathrm{GL}(n, \mathbb{R}),\ h_{ij}(y) = F_{U_i} \circ F_{U_j}^{-1}. \tag{L3.49}$$

Then the vector bundle can be thought of as the union of the $U_i \times \mathbb{R}^n$ with these identifications over U_{ij}. The fact that the spaces (L3.48) are well defined reduces to the $\mathrm{GL}(n, \mathbb{R})$-invariance of the radial compactification, which of course we used in the 'high road' definition above.

Now we can extend the definition (L3.41) from the product case to the general bundle case by setting

$$I^m_{\mathcal{S}}(W, 0_W) = \mathcal{F}_{\mathrm{fib}}^{-1}\left(\rho^{-m+\frac{d}{4}-\frac{n}{4}} \mathcal{C}^\infty(\overline{W'}; \Omega_{\mathrm{fib}})\right), \quad d = \dim Y,\ n = \dim W. \tag{L3.50}$$

I am being a little casual about the fibrewise Fourier transform but we can see that it all makes sense by the local trivialization approach. In fact the global behaviour in the base is not a big issue. I have done it this way so that the bookkeeping is fairly straightforward.

What bookkeeping? Well, the important property here, that we used repeatedly in the construction at the beginning of the lecture, is the short exact sequence which became (L3.38) in the invariant notation for a vector space. Now we get

$$I^{m-1}_{\mathcal{S}}(W, O_W) \longrightarrow I^{m}_{\mathcal{S}}(W, O_W) \xrightarrow{\sigma_m} \mathcal{C}^\infty(\mathbb{S}W'; N_{-m'} \otimes \Omega W'),\\ m' = m - \frac{d}{4} + \frac{n}{4},\ d = \dim Y,\ n = \dim W_y \tag{L3.51}$$

is exact.

Exercise 14. Check that you do understand what (L3.51) means and how to prove it. In a nutshell, the space $\mathbb{S}W'$ is the boundary of the radial compactification of W' and the surjectivity of the second map corresponds to the fact that every element of $\rho^{-m+\frac{d}{4}-\frac{n}{4}}\mathcal{C}^\infty(Y \times \overline{W'}; \Omega_{\text{fib}})$ corresponds to a (unique) conormal distribution by (L3.50). The injectivity on the right is almost by definition and the exactness in the middle is precisely the fact that an element of $\mathcal{C}^\infty(\overline{W'})$ which vanishes on $\mathbb{S}W'$ is an element of $\rho\mathcal{C}^\infty(\overline{W'})$ and conversely.

Finally let me review what we need for the next step, to define $I^m(X, Y)$ where Y is a compact embedded submanifold of a compact manifold X using the Collar Neighbourhood theorem. From the definition above it is immediate, or rather built into the definition that if $g : W \longrightarrow W$ is a bundle isomorphism then

$$g^* : I^m(W, 0_W) \longrightarrow I^m(W, 0_W). \tag{L3.52}$$

This is true whether g projects to the identity on the base (the usual meaning of a bundle isomorphism) or projects to a non-trivial diffeomorphism of the base.

The point is that this is by no means strong enough for what we want. Indeed we will need to consider a *diffeomorphism* between neighbourhoods of the zero section, N and N', of W but which need not preserve the fibres and even if it does, need not be linear. Of course it will be assumed to map the zero section into itself, otherwise (L3.52) could not possibly hold. Moreover, because of the known invariance under bundle isomorphisms we can assume a bit more. First we shall require that

$$g : 0_W \longrightarrow 0_W \text{ is the identity.} \tag{L3.53}$$

Now, this means that at each point $y \in 0_W$ (which is just Y) the tangent space to 0_W is mapped into itself as the identity too. The quotient

$$T_yW/T_y0_W = T_yW_y = W_y \tag{L3.54}$$

is naturally identified with the fibre of W through the point. So it makes sense to add a second condition to (L3.53) on the differential (i.e. the Jacobian) of g at each point of 0_W :

$$g_* : W_y \longrightarrow W_y \text{ is the identity.} \tag{L3.55}$$

Lemma 10. *Any diffeomorphism of a neighbourhood of 0_W onto its image in W which maps 0_W onto itself pointwise can be factorized as $g \cdot h$ where h is a bundle isomorphism and g satisfies* (L3.53) *and* (L3.55).

So, we want to show that (L3.52) holds for g as in (L3.53), (L3.55). Of course it only makes sense to apply g^* to functions or distributions with support in the image set N' but this is no problem since outside any given neighbourhood of 0_W we already know that our conormal distributions are smooth.

Let me check what (L3.53) and (L3.55) mean in local coordinates. If we take a local trivialization of W over some open set $U \subset Y$ and use coordinates y in U and fibre coordinates z in $W_U = U \times \mathbb{R}^n$ then

$$g(y,z) = (y + \sum_j z_j m_j(y,z), z + \sum_{ij} z_i z_j a_{ij}(y,z)) \tag{L3.56}$$

where the m_j and a_{ij} are just some smooth functions. This follows by writing $g(y,z) = (Y(y,z), Z(y,z))$. The fact that $0_W = \{z = 0\}$ is mapped into itself means $Z(y,0) = 0$, the fact that this map on Y is the identity means $Y(y,0) = y$ which gives the first part of (L3.56) and then the part of the Jacobian in (L3.55) is just $\partial Z/\partial z(y,0) = \mathrm{Id}$ which gives the second part of (L3.56).

Now, to show (L3.52) we will use 'Moser's method'. This is based first on the fact that the map in (L3.56) is connected to the identity by a curve of diffeomorphisms (possibly in a smaller neighbourhood of the zero section) of the same type. Locally (and that is all that really matters) this is clear since we can consider

$$g_s(y,z) = (y + s\sum_j z_j m_j(y,z), z + s\sum_{ij} z_i z_j a_{ij}(y,z)). \tag{L3.57}$$

So we want to show that $g_1^* u = v \in I^m_{\mathcal{S}}(W, 0_W)$ if $u \in I^m_{\mathcal{S}}(W, 0_W)$ (and has support sufficiently close to 0_W). Now, the clever idea of Moser (not in this context) is to try to construct a smooth curve

$$u_s \in \mathcal{C}^\infty([0,1]_s; I^m_{\mathcal{S}}(W, 0_W)) \text{ s.t. } u_1 = u \text{ and } \frac{d}{ds} g_s^* u_s = 0. \tag{L3.58}$$

If we could do this (and actually we can) then we conclude that $g_s^* u_s$ is constant, so

$$g^* u = g_1^* u_1 = g_0^* u_0 = u_0 \in I^m_{\mathcal{S}}(W, 0_W). \tag{L3.59}$$

So, why might we expect to be able to do this? Well, the 'trick' here is the identity

$$\frac{d}{ds} g_s^* u_s = g_s^* (\frac{du}{ds} + V_s u_s) \tag{L3.60}$$

where V_s is a vector field determined by g_s. Once we work out what this vector field is, we need to choose u_s to satify, in addition to (L3.58),

$$\frac{du}{ds} + V_s u_s = 0. \tag{L3.61}$$

The remarkable thing is that g_s has disappeared, we only need to consider its 'infinitesmal generator' V_s.

3+. Addenda to Lecture 3

3+.1. Densities. If U is any finite dimensional (complex) vector space we set

$$I^m_{\mathcal{S}}(W, \{0\}; U) = I^m_{\mathcal{S}}(W, \{0\}) \otimes_{\mathbb{C}} U \tag{3+.62}$$

and identify it as the 'space of conormal distributions with values in U.' (Of course you can do this with all distributions, etc).

EXERCISE 15. Check that the Fourier transform gives an isomorphism

$$cF : I^m_{\mathcal{S}}(W, \{0\}; U) \longrightarrow \rho^{-m'}\mathcal{C}^\infty(\overline{W'}; U \otimes \Omega W'),\ m' = m + \frac{n}{4}. \tag{3+.63}$$

Show further that there is a canonical isomorphism $\Omega U' = (\Omega U)'$ for any vector space, and hence that $\Omega U' \otimes \Omega U \equiv \mathbb{C}$ (or $\mathbb{R}$ if U is real) is canonically trivial. Hence (or directly) show that the integration map (L3.1) gives a linearly-invariant map

$$\int_{\mathbb{R}^n} : I^m_{\mathcal{S}}(W, \{0\}; \Omega W) \longrightarrow \mathbb{C} \tag{3+.64}$$

(as it should).

3+.2. Properties of conormal distributions.

3+.3. The Thom class.

3+.4. Submanifolds and restriction.

CHAPTER 4

Conormality at a submanifold

Lecture 4: 20 September, 2005

Last time I defined the space of conormal distributions at the zero section of a real vector bundle and checked the basic properties. These include invariance under bundle transformations and diffeomorphism of the base. The next step is to transfer the defintion to a general embedded submanifold. As noted at the end of last lecture, to do this we need a more general invariance result. To make a change of pace I will do this locally rather than globally. There is no particularly compelling reason for this, I just felt it was time to make sure we could 'see' what is happening.

Thus consider a trivial vector bundle over $\mathbb{R}^n$, $W = \mathbb{R}^n \times \mathbb{R}^k$. We have not really defined the conormal distibutions with respect to $\mathbb{R}^n \times \{0\}$ 'globally' on $\mathbb{R}^n$, although we could easily do so – and indeed I will need them later. Let me instead consider the space of conormal distributions on $\mathbb{R}^n \times \mathbb{R}^k$ with compact support and in fact supported in some bounded open set $N \subset \mathbb{R}^n \times \mathbb{R}^k$ which meets $\mathbb{R}^n \times \{0\}$ (so that we are not just looking at smooth functions). Since N is bounded we can choose a large constant so that $N \subset [-\pi, \pi]^n \times \mathbb{R}^k$ and then we may think of it as a subset of a trivial bundle over the torus

$$N \subset \mathbb{T}^n \times \mathbb{R}^n,\ \mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n. \tag{L4.1}$$

So, by definition the conormal distributions supported in N are just the fibre inverse Fourier transforms of classical symbols, the elements of

(L4.2)

$$\mathrm{I}^m_{\mathcal{S}}(W, O_W) = \mathcal{F}^{-1}_{\text{fib}}\left(\mathcal{C}^\infty(W'; \Omega_{\text{fib}}(W') \otimes N_{-m'})\right),\ W = \mathbb{T}^n \times \mathbb{R}^n,\ m' = m - \frac{n}{4} + \frac{k}{4},$$

to which we simply add the condition that

$$\operatorname{supp}(u) \subset N. \tag{L4.3}$$

The main invariance result I will prove is

Proposition 8. *If $F : N' \longrightarrow N$ is a diffeomorphism, between open subsets of $\mathbb{R}^n \times \mathbb{R}^k$ both satisfying* (L4.1)*, and which satisfies*

$$\begin{cases} F(p) = p \\ F_* = \mathrm{Id} \ \text{ on } N'_p(\mathbb{R}^n \times \{0\}) \end{cases} \quad \forall\ p \in N' \cap (\mathbb{R}^n \times \{0\}) \tag{L4.4}$$

then

$$u \in \mathrm{I}^m_{\mathcal{S}}(W, O_W) \text{ and } \operatorname{supp}(u) \Subset N \Longrightarrow F^*u \in \mathrm{I}^m_{\mathcal{S}}(W, O_W). \tag{L4.5}$$

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PROOF. As discussed last time, we will use Moser's method which depends on the construction of a 1-parameter family of such diffeomorphism.

LEMMA 11. *If* $F : N' \longrightarrow N$ *is as in Proposition 8 then for some open* $N'' \subset N'$ *with* $N'' \cap (\mathbb{R}^n \times \{0\}) = N' \cap (\mathbb{R}^n \times \{0\})$ *there is a smooth 1-parameter family of maps* $F_s : N'' \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$, $s \in [0,1]$, *which are diffeomorphisms onto their ranges and satisfy* $F_0 = \mathrm{Id}$, $F_1 = F\big|_{N''}$ *and* (L4.4) *for each* t.

PROOF. The assumptions on the diffeomorphism F imply that

$$F(x,z) = (x + \sum_{z_j} G_j(x,z), z + \sum_{jk} z_j z_k H_{jk}(x,z)),\ (x,z) \in N'. \tag{L4.6}$$

Indeed, the first restriction on the components realizes the condition $F(x,0) = (x,0)$ and the second correspond to the requirement that the Jacobian $\partial_z \partial_z F(x,0) = \mathrm{Id}$. Then we can simply set

$$\begin{aligned} F_s(x,z) = (X(s), Z(s)),\ X_i(s) &= x_i + s \sum_{z_j} G_{ij}(x,z), \\ Z_p(s) = z_p + s \sum_{jk} z_j z_k H_{pjk}(x,z)),\ &(x,z) \in N'' = N' \cap |z| < \epsilon \end{aligned} \tag{L4.7}$$

where choosing $\epsilon > 0$ small enough ensures, by the inverse function theorem, that all the maps are diffeomorphisms onto their images. $\square$

Recall that for any smooth function (and hence by continuity also for distributions) the chain rule becomes

$$\frac{d}{ds} F_s^* v_s = F_s^* (\frac{d}{ds} v_s + V(s) v_s) \tag{L4.8}$$

for a smooth vector field V_s. Indeed the vector field is just

$$\frac{dX_i}{ds} \partial_{X_i} + \frac{dZ_p}{ds} \partial_{Z_p} \tag{L4.9}$$

where the coefficients should be treated as functions of $(X(s), Z(s))$. It follows from (L4.7) that

$$V(s) = \sum_k Z_k V_k,\ V_k \text{ smooth and tangent to } Z = 0, \tag{L4.10}$$

which is to say the zero section.

To prove the proposition, consider u as in (L4.5). We will choose a curve of distributions supported very close to $N'' \cap (\mathbb{R}^n \times \{0\})$ and such that

$$\frac{d}{ds} u(s) + V(s) u(s) \in \mathcal{C}^\infty,\ u(1) = u. \tag{L4.11}$$

Recall that we have shown above that the action of any smooth vector field tangent to the zero section leaves the order of a conormal distribution unchanged and multiplying by any Z_k lowers it. Thus

$$V(s) : \{u \in \mathrm{I}^m_{\mathcal{S}}(W, O_W); \mathrm{supp}(u) \Subset F_s(N'')\} \longrightarrow \mathrm{I}^{m-1}_{\mathcal{S}}(W, O_W). \tag{L4.12}$$

So in fact it is easy to solve (L4.11) iteratively. Just make a first choice of $u_0 = u$ which is constant. This means that we have the initial step for the inductive hypothesis

$$\frac{d}{ds}u_{(N)}(s)+V(s)u_{(N)}(s)=f_{N+1}(s)\in \mathrm{I}_{\mathcal{S}}^{m-N-1}(W,O_W),\\ u_{(N)}(s)=u_0(s)+\cdots+u_N(s),\ u_j(1)=0,\ j>1. \tag{L4.13}$$

Supposing we have solved it to level N, setting

$$u_{N+1}(s)=\int_s^1 f_{N+1}(s')ds' \Longrightarrow \frac{d}{ds}u_{N+1}(s)=-f_{N+1}s+f_{N+2} \tag{L4.14}$$

gives the inductive hypothesis at the next level. Taking an asymptotic sum

$$u(s)\sim\sum_j u_j(s) \text{ gives (L4.11).} \tag{L4.15}$$

Notice that I have not bothered talking about the supports here, but they can be arranged to be arbitrarily close to the compact set $\operatorname{supp}(u)\cap(\mathbb{R}^n\times\{0\})$ by making additional smooth errors.

This completes the proof of Proposition 8 since $\frac{d}{ds}F_s^*u(s)$ is smooth in all variables and hence

$$F^*u=F_1^*u(1)=F_0^*u(0)+v=u(0)+v\in \mathrm{I}_{\mathcal{S}}^m(W,O_W) \text{ since } v\in\mathcal{C}_c^\infty(N''). \tag{L4.16}$$

□

We can easily apply this local result to obtain a more global one along the lines that I mentioned last time.

Proposition 9. *Let W be a real vector bundle over a compact manifold Y and suppose that $f : N \longrightarrow N'$ is a diffeomorphism between open neighbourhoods of the zero section 0_W with the properties* (L3.53) *and* (L3.54) *(so it fixes each point of the zero section and has differential projecting to the identity on the normal space to the zero section at each point) then*

$$u\in I_{\mathcal{S}}^m(W,0_W) \text{ with } supp(u)\Subset N' \Longrightarrow f^*u-u\in I_{\mathcal{S}}^{m-1}(W,0_W). \tag{L4.17}$$

and in particular

$$\sigma_m(f^*u)=\sigma_m(u)\in\mathcal{C}^\infty(\mathbb{S}W';N_{-m'}\otimes\Omega W'),\ m'=m-\frac{d}{4}+\frac{n}{4}. \tag{L4.18}$$

Proof. Each point of Y has a neighbourhood in Y over which W is trivial and Proposition 8. Thus, taking a partition of unity ϕ_j of a neighbourhood of $0_W=Y$ in W with each element supported in such a set we may apply Proposition 8 to f and $\phi_j u$ on each set. Since $u-\sum_j(\phi_j u)$ is smooth and $f^*(\phi_j u)-\phi_j u$ is conormal, and of order $m-1$, for each j we deduce the global form (L4.17).

The invariance of the symbol, (L4.18), follows immediately from (L4.17). □

This result in turn allows us to define the space $I^m(X,Y)$ of conormal distributions associated with (only singular at) an embedded closed submanifold of a compact manifold. To do so we need an appropriate form of

Theorem 1. *[Collar Neighbhourhood Theorem] Let $Y\subset X$ be a closed embedded submanifold of a compact manifold (so Y is a closed subset and for each point $y\in Y$ there exist local coordinates on X based at y in which Y meets the coordinate patch in the set given by the vanishing of the last $d-k$ coordinates) then there* *are an open neighbourhood D of Y in X and D' of the zero section of the normal bundle, NY, to Y in X and a diffeomorphism $f : D \longrightarrow D'$ such that*

$$\begin{gathered} f\big|_Y \text{ is the natural identification of } Y \text{ with } 0_{NY} \\ f_* \text{ induces the natural identification of } N_yY \text{ with } N_yY \ \forall\ y \in Y. \end{gathered} \tag{L4.19}$$

Perhaps in this form the theorem requires a little more explanation. First the normal bundle has, as I said early, fibre at a point $y \in Y$ the quotient

$$N_yY = T_yX/Y_yY. \tag{L4.20}$$

If X is a given a Riemannian structure then we may identify this quotient witht he metric normal space and write

$$T_yX = T_yY \oplus N_yY \tag{L4.21}$$

but in general there is no natural way of embedding NY as a subbundle of T_YX. However, once we have a smooth map $f : D \longrightarrow D'$ which maps a neighbourhood of Y in X to a neighbourhood of the zero section of NY, and maps each $y \in Y$ to its image point in 0_{NY} then

$$f_* : T_yX \longrightarrow T_y(NY).$$

Since we are assuming that f maps Y onto 0_{NY} as 'the identity' it must map T_yY to $Y_y(0_{NY}) = T_yY$ as the identity and hence projects to a map on the quotients

$$f_* : N_yY \longrightarrow T_y(NY)/T_y0_{NY} = N_yY \tag{L4.22}$$

where we can identify the normal space to the zero section unambiguously with the fibre for any vector bundle. Thus the second condition is that this map should also be the identity.

Proof. I will not give a complete proof of the Collar Neighbourhood Theorem in this form. Suffice it to say that the standard approach is to use geodesic flow map for a Riemann metric on X. Using the embedding of NY in T_YX coming from (L4.21) one can check that the restriction of the exponential map to a small neighbourhood of the zero section of the normal bundle gives a diffeomorphism onto a neighbourhood of Y and the inverse of this satisfies the two conditions. $\square$

For our application, the uniqueness part is also important. Namely given two local diffeomorphism f_i, $i = 1, 2$, both as in the theorem, the composite $f = f_2 \circ f_1^{-1}$ is a diffeomorphism of one neighbourhood of the zero section of NY to another and it necessarily satisfies both (L3.53) and (L3.54). This means that the definition we have been working towards makes good sense.

Definition 3. If $Y \subset X$ is a closed embedded submanifold of a compact manofold then

$$\begin{gathered} I^m(X,Y) = \big\{u \in \mathcal{C}^{-\infty}(X); u = u_1 + u_2,\ u_2 \in \mathcal{C}^\infty(X) \text{ and} \\ u_1 = f^*v,\ v \in I^m_{\mathcal{S}}(NY, 0_{NY}),\ \operatorname{supp}(v) \subset D' \text{ for some diffeomorphism as in (L4.19)}\big\}. \end{gathered} \tag{L4.23}$$

Now, many properties of the $I^m(X,Y)$ now follow directly from the properties already stablished for the $I^m_{\mathcal{S}}(W, 0_W)$. First the inclusion for these spaces gives immediately

$$I^{m-1}(X,Y) \subset I^m(X,Y). \tag{L4.24}$$

This inclusion is important because it is captured by the symbol. Since this is rather important in the sequel, let me state this formally.

LEMMA 12. *The symbol map on $I^m(NY, 0_{NY})$ induces a symbol map on $I^m(X, Y)$ and this gives a short exact sequence*

$$I^{m-1}(X,Y) \hookrightarrow I^m(X,Y) \xrightarrow{\sigma_m} \mathcal{C}^\infty(SN^*Y; N_{-m'} \otimes \Omega_{\mathrm{fib}}),\\ m' = m - \frac{d}{4} + \frac{n}{4},\ d = \dim Y,\ n = \operatorname{codim} Y. \tag{L4.25}$$

So what are the important properties of these distributions?

(1) Each element of $I^m(X, Y)$ is smooth outside Y and

$$\bigcap_k I^{m-k}(X,Y) = \mathcal{C}^\infty(X). \tag{L4.26}$$

(2) Invariance:- If $F : X' \longrightarrow X$ is a diffeomorphism then

$$F^* : I^m(X,Y) \longrightarrow I^m(X', F^{-1}(Y)),\ \sigma_m(F^*u) = F^*\sigma_m(u) \tag{L4.27}$$

where you need to check the sense in which F^* induces an isomorphism of the conormal bundles N^*Y in X and $N^*(F^{-1}(Y))$ in X'.

(3) Action of differential operators. If $P \in \operatorname{Diff}^k(X)$ (which I have not really defined) then

$$P : I^m(X,Y) \longrightarrow I^{m+k}(X,Y),\ \sigma_{m+k}(Pu) = \sigma_k(P)\big|_{N^*Y}\sigma_m(u). \tag{L4.28}$$

(4) Asymptotic completeness. If $u_k \in I^{m-k}(X, Y)$ then there exists $u \in I^m(X, Y)$ such that

$$u - \sum_{k<N} u_k \in I^{m-N}(X,Y),\ \forall\ N. \tag{L4.29}$$

4+. Addenda to Lecture 4

4+.1. Listing the properties. Let me briefly summarize, again, the properties of the conormal distributions as I have defined them above and outline proofs. For the momemnt we only have 'generalized functions'. For each $m \in \mathbb{C}$ (I have mostly been treating m as real but this is not usd anywhere) and any embedded closed submanifold of a compact manifold, $Y \subset X$, we have defined

$$I^m(X,Y) \subset \mathcal{C}^{-\infty}(X) = (\mathcal{C}^\infty(X;\Omega))'. \tag{4+.30}$$

This is Definition 3 in terms of conormal distributions with respect to the zero section of a vector bundle (in this case the normal bundle to Y in X). The definition in that case is (L3.50) as the inverse fibre Fourier transform of 'symbols' on the radial compactification of the dual bundle. It follows from the inclusion for the symbol spaces that if $k \in \mathbb{N}$ then

$$I^{m-k}(X,Y) \subset I^m(X,Y),\ \bigcap_k I^{m-k}(X,Y) = \mathcal{C}^\infty(X). \tag{4+.31}$$

Asymptpotic completeness of the symbol spaces shows that if $u_k \in I^{m-k}(X, Y)$ then there exists $u \in I^m(X, Y)$ such that

$$u - \sum_{k\le N} u_k \in I^{m-N}(X,Y)\ \forall\ N. \tag{4+.32}$$

The main thing that distinguishes conormal distributions is that their leading singularities are describeable by the principal symbol map which gives a short exact sequence for each m
(4+.33)
$$I^{m-1}(X,Y) \longrightarrow I^m(X,Y) \xrightarrow{\sigma_m} \mathcal{C}^\infty(SN^*Y; N_{m'}\otimes\Omega_{\text{fib}}),\ m' = m - \frac{1}{4}\dim X + \frac{1}{4}\dim X,$$
N_{-m} is the bundle of functions homogeneous of degree m' on N^*Y (the normal bundle to Y in X) and Ω_{fib} is the bundle of densities on the fibres of N^*Y.

Exercise 16. Show that the density bundle on X, restricted to Y, can be decomposed

$$\Omega_Y X = \Omega Y \otimes \Omega_{\text{fib}} NY \tag{4+.34}$$

where $\Omega_{\text{fib}} NY$ is the 'normal density bundle to Y, so is the 'absolute value' of the maximal exterior power of the conormal bundle to Y. (The notation is to indicate that this is the usual normal bundle on the fibres of NY made into a bundle over Y.) So if $0 < \mu \in \mathcal{C}^\infty(Y; \Omega NY)$ is a positive smooth 'normal density' on Y (and such always exists) then

$$u_\mu : \mathcal{C}^\infty(X; \Omega X) \ni \nu \longmapsto \int_Y (\nu/\mu) \in \mathbb{C} \text{ (or } \mathbb{R}) \tag{4+.35}$$

is a well-defined distibution. Show that this 'delta' section is an element of $I^{-\frac{1}{4}\operatorname{codim} Y}(X,Y)$ and compute its symbol (in terms of μ.)

For any differential operator $P \in \operatorname{Diff}^q(X)$ (so $P : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X)$ is a continuous linear operator which is *local*) its symbol $\sigma_q(P)$ is a smooth function on T^*X which is a homogeneous polynomial of degree q on the fibres (defined by the condition

$$P(e^{itf(x)}v(x)) = e^{itf(x)}\left(\sigma_q(tdf)v(x) + O(t^{q-1})\right)\ \forall\ f,\ v \in \mathcal{C}^\infty(X),\ t \in \mathbb{R}) \tag{4+.36}$$

$$P : I^m(X,Y) \longrightarrow I^{m+q}(X,Y),\ \sigma_{m+q}(Pu) = \sigma_q(P)\sigma_m(u). \tag{4+.37}$$

In particular the $I^m(X,Y)$ are $\mathcal{C}^\infty(X)$ modules and they are invariant under diffeomorphisms, so if $f : O \longrightarrow O'$ is a diffeomorphism between open subsets of X, Y and Y' are embedded submanifolds of X and $f(O\cap Y) = O' \cap Y'$ then
(4+.38)
$$f^* : \{u \in I^m(X,Y'); \operatorname{supp}(u) \subset O'\} \longrightarrow I^m(X,Y,\ \sigma_m(f^*u) = (f^*)^*\sigma_m(u)$$
where $f^* : N^*_{O;\cap Y;}Y' \longrightarrow N^*Y$ is the induced map.

Exercise 17. Show that any element of $I^m(X,Y)$ which has support in Y is of the form Pu_μ where u_μ is as in (4+.35) and $P \in \operatorname{Diff}^q(X)$ for some q. What values of m can occur this way?

4+.2. Poincaré forms. Although I have only defined conormal distributions, there is no problem in defining conormal sections of any complex vector bundle E over X (and I will do this next time) giving a space $I^m(X,Y;E)$ with similar properties. In fact I will discuss this in more detail next time. Informally an element of $\mathcal{C}^{-\infty}(X;E)$ is given in terms of any local trivialization of E by a sum over the local basis with distributional coefficients. If these coefficients are in $I^m(X,Y)$ then the distributional section is in $I^m(X,Y;E)$. This tensor-product definition can readily be made rigourous.

Anyway, suppose we have made sense of this already. The 'simplest' sort of conormal distributions are again the 'Dirac delta sections'. One particularly nice example is given by the Poincaré duals of embedded submanifolds. Since this is an opportunity to discuss a little homology, let me do so.

First recall deRham theory in which the spaces of sections of the exterior bundles (exterior powers of the cotangent bundle) over a manifold X are the chain spaces for a (co)homology theory. Namely d gives a complex of differential operators, $d \in \operatorname{Diff}^1(X; \Lambda^k X, \Lambda^{k+1} X)$, $d^2 = 0$

$$\dots \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^{k-1}) \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^k) \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^{k+1}) \xrightarrow{d} \dots. \tag{4+.39}$$

The deRham cohomology groups

$$H^k_{\mathrm{dR}}(X) = \{u \in \mathcal{C}^\infty(X; \Lambda^k); du = 0\}/d\mathcal{C}^\infty(X; \Lambda^{k-1}) \tag{4+.40}$$

are naturally isomorphic (for a compact manifold) to the other 'obvious' cohomology groups – singular, smooth singular or Čech (and as I will discuss later, Hodge).

There are other forms of the deRham groups too. In particular the 'distributional deRham cohomology' is canonically isomorphic to the smooth

$$\{u \in \mathcal{C}^{-\infty}(X; \Lambda^k); du = 0\}/d\mathcal{C}^{-\infty}(X; \Lambda^{k-1}) \equiv H^k_{\mathrm{dR}}(X). \tag{4+.41}$$

Here there is an obvious map from smooth deRham to distributional deRham and this is always an isomorphism. That is, any element of $\mathcal{C}^{-\infty}(X; \Lambda^k)$ which satisfies $du = 0$ is of the form $dv + u'$ with $v \in \mathcal{C}^{-\infty}(X; \Lambda^{k-1})$ and $u' \in \mathcal{C}^\infty(X; \Lambda^k)$ (so of course $du = 0$). This by the way is a consequence of the Hodge theorem proved later (but can be proved more crudely but more directly if you want).

Why care about distributional deRham at all? One reason is the existence of Poincaré dual forms (also sometimes called Leray forms).

Proposition 10. *If $Y \subset X$ is a closed embedded submanifold with an oriented normal bundle then the form given in local coordinates near any point of Y, in which $Y = \{x_{d+1} = \dots = x_n = 0\}$ locally with the correct orientation, by*

$$p_Y = \delta(x_{d+1}) \cdots \delta(x_n) dx_{d+1} \wedge \dots \wedge dx_n \in I^-(X, Y; \Lambda^{n-d}), \ \dim Y = d, \tag{4+.42}$$

is independent of choices, closed and fixes the Poincaré dual class to Y in $H^{n-d}(X)$.

CHAPTER 5

Pseudodifferential operators

Lecture 5: 22 September, 2005

Since it may be a while before I write up the notes from this fifth lecture, I include here my pre-lecture notes

L5.1. Conormal sections of bundles. I had planned to go through the definition of $I^m(X,Y)$ again from the beginning to define instead $I^m(X,Y;E)$ where E is a complex vector bundle over X. I will do this in the addenda and instead give a direct definition which has the virtue of brevity. Namely

$$I^m(X,Y;E) = I^m(X,Y) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;E). \tag{L5.1}$$

Here we use the fact that $I^m(X,Y)$ is a module over $\mathcal{C}^\infty(X)$ – we can multiply by arbitrary $\mathcal{C}^\infty$ functions – and so is $\mathcal{C}^\infty(X;E)$, the space of smooth sections of the vector bundle E. What *precisely* does (L5.1) mean? It means that we define an element of $I^m(X,Y;E)$ as an equivalence class of finite sums of pairs (written multiplicatively)

$$v = [\sum_i u_i e_i],\ e_i \in \mathcal{C}^\infty(X;E),\ u_i \in I^m(X,Y) \tag{L5.2}$$

where the equivalence relation is generated by $\mathcal{C}^\infty(X)$-linearity, i.e.

$$\sum_i u_i e_i \sim \sum_j u'_j e'_j \text{ if } e_i = \sum_j a_{ij} e'_j \text{ and } u'_j = \sum_i a_{ij} u_i,\ a_{ij} \in \mathcal{C}^\infty(X). \tag{L5.3}$$

Then $I^m(X,Y;E)$ is itself a $\mathcal{C}^\infty(X)$-module and if an element, u, has support in an open set over which E is trivial then for any smooth local basis, e_i of E, [1]

$$u = \sum_i u_i e_i,\ u_i \in I^m(X,Y). \tag{L5.4}$$

The definition above can be used for the space of distributional sections, that is

$$\mathcal{C}^{-\infty}(X;E) = \mathcal{C}^{-\infty}(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;E) \tag{L5.5}$$

so $I^m(X,Y;E) \subset \mathcal{C}^{-\infty}(X;E)$ and this tensor product definition is equivalent to the duality definition

$$\mathcal{C}^{-\infty}(X;E) = (\mathcal{C}^\infty(X;E^* \otimes \Omega_X))'. \tag{L5.6}$$

It follows that there are natural injections, as there should be

$$\mathcal{C}^\infty(X;E) \hookrightarrow I^m(X,Y;E) \hookrightarrow \mathcal{C}^{-\infty}(X;E). \tag{L5.7}$$

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[1] Instead this can also be used as the basis of the definition.

L5.2. Integration. Suppose that Z is compact then integration of distibutions is well-defined provided they are valued in the density bundle of Z, for any *vector space* E (not a vector bundle, it has to be globally trivialized)

$$\int_Z : \mathcal{C}^{-\infty}(Z; E \otimes \Omega_Z) \longrightarrow E. \tag{L5.8}$$

Of course this means we can integrate $I^m(Z, Y; E \otimes \Omega_Z)$ under the same conditions.

L5.3. Restriction. Now suppose that $Z \subset X$ is an embedded submanifold which is transversal to Y, meaning that

$$\forall\ p \in Y,\ T_pY + T_pZ = T_pX. \tag{L5.9}$$

Then, the restriction map for smooth sections $\mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(Z; E\big|_Z)$ extends to conormal sections

(L5.10)

$$\big|_Z : I^m(X, Y; E) \longrightarrow I^{m+\frac14 \operatorname{codim} Z}(Z, Y \cap Z; E\big|_Z),\ \sigma_{m+\frac14 \operatorname{codim} Z}(u\big|_Z) = \sigma_m(u)\big|_Z.$$

To prove this, we can use the restriction map from $\mathcal{C}^\infty(X; E)$ to handle any element supported away from Y. So to define $u\big|_Z$ for $u \in I^m(X, Y; E)$ we can suppose that u is supported in any preassigned neighbourhood of Y. In particular we can assume it is supported in the range of some normal fibration of Y.

Now, what does the transversality mean? Fix a point $p \in Y$ then let $Z = \{y_1, \dots, y_k = 0\}$ be given by the vanishing of local defining functions and let $Y = \{t_1, \dots, t_p = 0\}$ be similarly given in terms of local defining b functions. Then (L5.9) means that the differentials of these functions are independent at p, so they can be completed to a local coordinate system based at p, by adding $s_1, \dots, s_{n-p-k}$ where necessarily $k \le n - p$. Thus the y and s together give local coordinates on Y near p. These coordinates give a normal fibration of Y near p – we may identify the normal bundle with the fibres (y, s) =const near Y (and near p.) Now, cover Y by such local coordinate systems and normal fibrations and take a finite partition of unity subordinate to this cover. Using this to decompose $u \in I^m(X, Y)$ we see that each piece, u_i, is of the form

$$u_i \in \mathcal{C}^\infty\left(\mathbb{R}^k; I^{m+\frac{n-p-k}{4}}(\mathbb{R}^{n-p-k} \times \mathbb{R}^p, Y \cap Z; E)\right) \tag{L5.11}$$

with compact support near the origin in all variables. The first variables here are the y's and $Y \cap Z = \mathbb{R}^p \times \{0\}$. Thus, restriction to $y = 0$, which is to say Z, gives a map as in (L5.10) locally. It is clearly consistent[2] under changes of coordinates and so we get (L5.10) with the computation of the symbol also immediate from (L5.11).

L5.4. Push-forward. Let $\phi : X \longrightarrow B$ be a fibration (or if you prefer, for present purposes it is enough to take the projection off a product, i.e. $X = B \times Z$). Suppose that this fibration is transversal to the embedded submanifold $Y \subset X$, meaning that for all $p \in Y$,

$$T_pY + T_p(\phi^{-1}(\phi(p))) = T_pX, \tag{L5.12}$$

which is just the condition that each fibre is transversal to Y. Then fibre integration gives a linear map

$$\phi_* : I^m(X, Y; \phi^*E \otimes \Omega_X) \longrightarrow \mathcal{C}^\infty(B; E \otimes \Omega_B) \tag{L5.13}$$

[2] See problem X

for any smooth vector bundle E over B.

First recall that this is true in the case $m = -\infty$, i.e.

$$\phi_* : \mathcal{C}^\infty(X; \phi^* E \otimes \Omega_X) \longrightarrow \mathcal{C}^\infty(B; E \otimes \Omega_B). \tag{L5.14}$$

Namely, near a point $b \in B$ we can reduce ϕ to projection for the product $U \times Z$ to U, where U is a neighbourhood of $b \in B$. The density bundles behave well under products, so $\Omega_X = \Omega_U \otimes \Omega_Z$. Then (L5.14) is just locally in B the formula

$$\phi_*(u) = \left(\int_Z u(b,z)\nu(z) \right) \nu(b). \tag{L5.15}$$

In each fibre, i.e. for fixed b, $u(b,z)$ is a smooth map in z into the vector space E_b, the fibre of the bundle at b. Now, to get (L5.13) we just replace the integral in (L5.15) by the integral in (L5.8) after restricting to each fibre using (L5.10) and the result is smooth as claimed in (L5.13).

L5.5. Pseudodifferential operators. As already noted, we define the space of pseudodifferential operators, 'acting between' sections of two vector bundles E and F over X to be

$$\Psi^m(X; E, F) = I^m(X^2, \mathrm{Diag}; \mathrm{Hom}(E,F) \otimes \Omega_R). \tag{L5.16}$$

Here $\Omega_R = \pi_R^* \Omega_X$ is the pull-back of the density bundle from the right factor of X, $\pi_R(x,y) = y$, and $\mathrm{Hom}(E,F)$ is the 'big' homomorphism bundle. Thus $\mathrm{Hom}(E,F)$ is a vector bundle over X^2 with fibre at (x,y) the space $\hom(E_y, F_x)$ of linear maps from the fibre, E_y, of E at $y \in X$ to the fibre, F_x, of F at $x \in X$. Using standard identifications we can think of this bundle as

$$\mathrm{Hom}(E,F) = \pi_L^* F \otimes \pi_R^*(E'). \tag{L5.17}$$

Then the operator associated with (and indeed identified with) the kernel $A \in \Psi^m(X; E, F)$ is

$$\begin{gathered} (Au)(x) = \int_X A(x,y)u(y)dy, \ Au = (\pi_L)_* \left(A \cdot (\pi_R)^* u \right), \\ A : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F). \end{gathered} \tag{L5.18}$$

Here the left 'integral' is formal. The middle expression is supposed to be rigourous and yield the map as shown. Thus, for $u \in \mathcal{C}^\infty(X; E)$ the pull-back to X^2 under π_R is an element of $\mathcal{C}^\infty(X^2; \pi_R^* E)$. When we multiply it by the kernel $A \in I^m(X^2, \mathrm{Diag}; \mathrm{Hom}(E,F) \otimes \Omega_R)$ we get, using (L5.17), an element

$$A \otimes (\pi_R)^* u I^m(X^2, \mathrm{Diag}; \pi_L^* F \otimes (\pi_R)^*(E \otimes E')\Omega_R).$$

Now, we can pair E with E' to get the action of $\hom(E_y, F_x)$ on E_y and hence an element of $I^m(X^2, \mathrm{Diag}; \pi_L^* F \otimes \Omega_R)$. Finally we may apply (L5.13) to get the integral, mapping to $\mathcal{C}^\infty(X; F)$ as expected.

This means that the composite of two pseudodifferential operators acting on appropriate bundles is defined. It is of fundamental importance that the composite is again a pseudodifferential operator,

THEOREM 2. *On any compact manifold, X, and for any complex vector bundles, E, F and G*

$$\Psi^m(X; F, G) \circ \Psi^{m'}(X; E, F) \subset \Psi^{m+m'}(X; E, G). \tag{L5.19}$$

I will prove this after discussing the use of pseudodifferential operators to partially invert elliptic operators.

We also need to see what has happened to the symbol of our conormal distributions in this case. Namely the symbol map simplifies to give a short exact sequence

$$\Psi^{m-1}(X;E,F) \longrightarrow \Psi^m(X;E,F) \longrightarrow \mathcal{C}^\infty(S^*X;N_m \otimes \hom(E,F)). \tag{L5.20}$$

So, the density terms have disappeared, the manifold carrying the symbol has become the cosphere bundle of X, $S^*_xX = (T^*_xX\setminus 0)/\mathbb{R}^+$ and the bundle has become the usual homomorphism bundle, over X, lifted to S^*X.

L5.6. Action of differential operators. For the moment we can easily see that differential operators are special cases of pseudodifferential operators and more generally the restricted composition theorem

(L5.21)

$$\operatorname{Diff}^k(X;F,G) \circ \Psi^m(X;E,F) \subset \Psi^{k+m}(X;E,G),\ \sigma_{k+m}(PA) = \sigma_k(P) \circ \sigma_m(A)$$

is easy to deduce. This is enough for our application to Hodge theory.

[Needs proof]

5+. Addenda to Lecture 5

5+.1. The euler class.

CHAPTER 6

Ellipticity

6+.1. Bundles and sections.

Lecture 6: 4 October, 2005

First I want to talk about the basic properties of smoothing operators since to a large extent the study of more operators, particularly elliptic pseudodifferential operators, is ultimately reduced to the study of smoothing 'errors'.

Thus, if X is a compact manifold and E and F are complex vector bundles over X then the space of smoothing operators on X between sections of E and sections of F is

$$\Psi^{-\infty}(X;E,F) = \mathcal{C}^\infty(X^2;\operatorname{Hom}(E,F)\otimes\Omega_R). \tag{L6.1}$$

Here, $\operatorname{Hom}_{(x,x')}(E,F) = \hom(E_{x'},F_x)$ is the 'big' homomorphism bundle. Using the tensor product characterization of homomorphism it can also be identified with the 'exterior' tensor product $\pi_L^*F\otimes\pi_R^*E'$, the tensor product of the pull-back of F from the left fact with the pull-back of the dual of E from the right factor of X. The bundle Ω_R is the 'right density bundle' on X^2, just the pull-back from the right factor of the density bundle. It allows invariant integration.

As operators each $\Psi^{-\infty}(X;E,F)$ defines a linear map $A:\mathcal{C}^\infty(X;E)\longrightarrow\mathcal{C}^\infty(X;F)$ (with which we always identify it) given by

$$Af(x) = \int_X A(x,x')f(x'). \tag{L6.2}$$

Here, the product of $A(x,x')$ and $f(x')$ implicitly includes the action of A as a homomorphism from $E_{x'}$ to F_x. Thus, for fixed x, the integrand is a section of $F_x\otimes\Omega_R$ as a bundle over X in the variable x', i.e. F_x is a trivialized bundle and the integral makes invariant sense.

Basic properties of smoothing operators

- Smoothing operators are characterized (by standard distribution theory) as those continuous linear operators $A:\mathcal{C}^\infty(X;E)\longrightarrow\mathcal{C}^\infty(X;F)$ which extend by continuity to continuous linear operators $A:\mathcal{C}^{-\infty}(X;E)\longleftrightarrow\mathcal{C}^\infty(X;F)$ where $\mathcal{C}^{-\infty}(X;E)$ is the usual space of distributional sections of F over X. I will not use this characterization below, but it is sometimes handy.
- Smoothing operators extend by continuity to compact operators $A:L^2(X;E)\longrightarrow L^2(X;F)$. This is easy to prove using some form of the Ascoli-Arzela theorem which shows that the inclusion $\mathcal{C}^0(X;F)\longrightarrow L^2(X;F)$ is compact, or

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the usual form of Ascoli-Arzela which shows that $\mathcal{C}^1(X;F) \longrightarrow \mathcal{C}^0(X;F)$ is compact, and hence so is $\mathcal{C}^1(X;F) \longrightarrow L^2(X;F)$. From the integral formula (L6.2) it follows that smoothing operators define continuous maps $A : L^2(X;E) \longrightarrow \mathcal{C}^1(X;F)$ the compactness follows. Note that smoothing operators are *not* characterized as the continuous operators $A : L^2(X;E) \longrightarrow \mathcal{C}^\infty(X;F)$. However if an operator has this property and its adjoint, with respect to smooth inner products on the bundles and a smooth density, has the same property, $A^* : L^2(X;F) \longrightarrow \mathcal{C}^\infty(X;E)$ then A is smoothing.

- Now consider the special case $\Psi^{-\infty}(X;E) = \Psi^{-\infty}(X;E,E)$ of operators acting on sections of a fixed bundle. Then $\mathrm{Id} + A$ is *Fredholm* as an operator $A : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;E)$ or $A : L^2(X;E) \longrightarrow L^2(X;E)$. Namely
  (1) The null space is finite dimensional
  (2) The range is closed
  (3) The range has a finite dimensional complement.

Proof. The null space is

$$\mathrm{null}(\mathrm{Id} + A) = \{u \in L^2(X;E); u + Au = 0\} \tag{L6.3}$$

so for any element $u \in \mathrm{null}(\mathrm{Id} + A)$ it follows that $u = -Au \in \mathcal{C}^\infty(X;E)$. Thus the unit ball $\{u \in \mathrm{null}(\mathrm{Id} + A); \|u\| = 1\}$ is precompact in $L^2(X;E)$ and hence compact (since it is closed). It is a standard theorem that any Hilbert space with a compact unit ball is finite dimensional so proving (1) for L^2. The null space on $\mathcal{C}^\infty(X;E)$ is the same as the null space on $L^2(X;E)$ so this is also finite dimensional.

To see that the range is close, suppose $f_n \in L^2(X;E)$ and $f_n \to f$ in $L^2(X;E)$ and $f_n = (\mathrm{Id} + A)u_n$ for $u_n \in L^2(X;E)$. We can assume that $u_n \perp \mathrm{null}(\mathrm{Id} + A)$ and then we wish to show that $u_n \to u$ in $L^2(X;E)$ which implies that $f = (\mathrm{Id} + A)u$. So, suppose first that the sequence $\|u_n\|$ is unbounded. Passing to a subsequence, and relabelling, we may suppose that $\|u_n\| \to \infty$. Thus $v_n = u_n/\|u_n\|$ has unit norm and $(\mathrm{Id} + A)v_n = f_n/\|u_n\| \to 0$ in $L^2(X;E)$. Passing to a subsequence we may assume that $v_n \rightharpoonup v$ converges weakly (by the weak compactness of the unit ball in a Hilbert space). Then $v_n = Av_n + f_n$ must converge strongly, since A is a compact operator. Thus $v_n \to v$ with $\|v\| = 1$ and $v \in \mathrm{null}(\mathrm{Id} + A)$ which is a contradiction, since $u_n \perp \mathrm{null}(\mathrm{Id} + A)$ implies $v \perp \mathrm{null}(\mathrm{Id} + A)$. So in fact the assumption was false and $\|u_n\|$ is necessarily bounded. Then the same argument shows that on an subsequence $u_n \rightharpoonup u$ and hence $u_n = Au_n + f_n \to Au + f$ converges strongly and (2) follows.

Recall that the adjoint of a bounded operator is defined if one has a smooth (sesquilinear) inner product on the fibres of E and a smooth positive density ν on X – one needs these really to fix the inner product on $L^2(X;E)$,

$$\langle u, v \rangle = \int_X \langle u(x), \rangle_{E_x} d\nu(x) \tag{L6.4}$$

by

$$\langle Au, v \rangle = \langle u, A^* v \rangle \ \forall\ u, v \in L^2(X;E). \tag{L6.5}$$

In the case of a smoothing operator (and in fact in general) it follows that the kernel of A^* is $A^*(x', x)$ in terms of $*$ acting on $\mathrm{Hom}(E,E)$. Thus $A^* \in \Psi^{-\infty}(X;E)$ is also a smoothing operator.

Directly from the definition of the adjoint, the orthcomplement of the range of any bounded operator is always the null space of A^*

$$\langle Au, v\rangle = 0 \ \forall \ u \in L^2(X; E) \Longleftrightarrow A^* v = 0. \tag{L6.6}$$

Thus $\operatorname{null}(\operatorname{Id} + A^*)$ is a complement to the range of $\operatorname{Id} + A$ which is therefore finite dimensional, provign (3).

The range of $\operatorname{Id} + A$ is closed in $\mathcal{C}^\infty(X; E)$ by essentially the same argument. Namely if $(\operatorname{Id} + A)u_n = f_n \to f$ in $\mathcal{C}^\infty(X; E)$ then (since the null spaces on $L^2(X; E)$ and $\mathcal{C}^\infty(X; E)$ are the same) we may assume that $u_n \in \mathcal{C}^\infty(X; E)$ and $u_n \to u$ in $L^2(X; E)$ by the discussion above. Then $u_n = -Au_n + f_n \to u$ in $\mathcal{C}^\infty(X; E)$. It also follows that the range of $\operatorname{Id} + A$ has finite codimension in $\mathcal{C}^\infty(X; E)$, in fact $\operatorname{null}(\operatorname{Id} + A^*)$ is still a complement (in the algebraic sense that

$$(\operatorname{Id} + A)\mathcal{C}^\infty(X; E) + \operatorname{null}(\operatorname{Id} + A^*) = \mathcal{C}^\infty(X; E). \tag{L6.7}$$

In fact we know that the left side is a closed subspace of the right, so if they were not equal then there would be a non-trivial distributional section $v \in \mathcal{C}^{-\infty}(X; E)$ such that $\langle v, (\operatorname{Id} + A)u\rangle = 0$ for all $u \in \mathcal{C}^\infty(X; E)$ and $v(w) = 0$ for all $w \in \operatorname{null}(\operatorname{Id} + A^*)$. However the first condition is just $v + Av = 0$ as a distribution, but then $v = -Av$ and $A : \mathcal{C}^{-\infty}(X, E) \longrightarrow \mathcal{C}^\infty(X; E)$ so together these imply $v = 0$. □

Now consider differential opertors, $P \in \operatorname{Diff}^k(X; E, F)$. These are operators $P : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F)$ which are given everywhere locally, in terms of local coordinates and trivializations of the bundles, by a finite sum of derivatives composed with a matrix

$$P = \sum_{|\alpha| \le k} p_\alpha(x) D_x^\alpha. \tag{L6.8}$$

We say that such an operator is elliptic if the leading part of this sum

$$\sum_{|\alpha| = k} p_\alpha(x)\xi^\alpha \text{ is invertible for each } \xi \in \mathbb{R}^n \setminus \{0\} \tag{L6.9}$$

and for each x (i.e. is invertible as an $N \times N$ matrix).

The sum in (L6.9) makes invariant sense as a section over $T^*X \setminus \{0\}$ of the pull-back from the base of the bundle $\hom(E, F)$. To see this we simply have to give an invariant definition of its value at a point of T^*X! Choose such a point, $\Xi \in T^*_{\bar x}X$. Thus, near $\bar x \in X$ we may choose $f \in \mathcal{C}^\infty(X)$, real valued, such that $df(\bar x) = \Xi$. Now, given an element $\bar u \in E_{\bar x}$ choose $u \in \mathcal{C}^\infty(X; E)$ such that $u(\bar x) = \bar u$. Then, for $t \in \mathbb{R}$,

(L6.10)
$$P(ue^{itf}) = e^{itf}U(t, x),\ U(t, x) \in \mathcal{C}^\infty(\mathbb{R} \times X; F),\ U(t, \bar x) = t^k \sigma_k(P)(\bar x, df(\bar x) + O(t^{k-1}.$$

We can use (L6.8) to see this. Thus, $U(t)$ must be a polynomial of degree at most k in t and the leading term, of order k, at $\bar x$ is just

$$\sum_{|\alpha| = k} p_\alpha(\bar x)(df(\bar x)^\alpha \tag{L6.11}$$

which is just (L6.9). Thus in fact the principal symbol of a differential operator of order m, defined locally by (L6.9) is in fact a well defined section

$$\sigma_k(P) \in \mathcal{C}^\infty(T^*X; \hom(E, F)) \text{ is a fibre-polynomial of degree } k. \tag{L6.12}$$

Now recall that we defined pseudodifferential operators in terms of conormal distributions

$$\Psi^m(X;E,F) = I^{m'}(X^2,\mathrm{Diag};\mathrm{Hom}(E,F)\otimes\Omega_R) \tag{L6.13}$$

and showed that the acted on smooth sections

$$A : \mathcal{C}^\infty(X;E) \longleftrightarrow \mathcal{C}^\infty(X;F),\ Au(x) = \int_X A(x,x')u(x'). \tag{L6.14}$$

We also showed, locally, that differentiation of a conromal distribution gives a conormal distribution with the order increased by one. Since we also know that conormal distributions form a $\mathcal{C}^\infty$ module, it follows that

$$P(x,D_x) : I^{m'}(X^2,\mathrm{Diag};\mathrm{Hom}(E,F)\otimes\Omega_R) \longrightarrow I^{m'+k}(X^2,\mathrm{Diag};\mathrm{Hom}(E,F)\otimes\Omega_R). \tag{L6.15}$$

This in fact shows that

$$\mathrm{Diff}^k(X;E,F)\Psi^m(X;F,E) \subset \Psi^{m+k}(X;F). \tag{L6.16}$$

Now, consider what happens to the symbol of $A \in \Psi^m(X;F,E)$ under this action on the left by a differential operator. The symbol can be computed locally near a point of the diagonal and in terms of any normal fibration. In particular we can choose the normal fibration to be the 'right fibration with fibres given by the constancy of the second variable x'. That is a local fibre of the normal fibration (in local coordinates and with respect to a local trivialization of the bundles) is just $x' = \bar{x}$ is constant. Thus $P(x,D_x)$ just acts by differentiation on the fibre so the kernel of PA on this fibre is

$$P(x,D_x)A'(x-\bar{x}',\bar{x}') \tag{L6.17}$$

where the left variable has been shifted so that it vanishes at $\bar{x}'$, i.e. where the diagonal meets the fibre, and $A(x-\bar{x}',\bar{x}')$ is the kernel of A on this fibre. Now, it follows from (L6.8) that any lower order terms in P can only raise the order at most to $m+k-1$. Since we know that multiplication by $x_j - \xi'_j$ lowers the oder by 1 (since it vanishes at the singular point) we see that the symbol of PA, modulo lower order terms, is just

$$\sigma_k(P)(\bar{x},\xi)\sigma_m(A). \tag{L6.18}$$

Now, since we are assuming that P is elliptic everywhere, in particular $\sigma_k(\bar{x},\xi)$ is a homogeneous polynomial which does not vanish outside the origin. From the earlier discussion of this in the case of conormal distributions at a point, we know that we can solve the problem

$$PA = \mathrm{Id}_F + R,\ A \in \Psi^{-k}(X;F;E),\ B \in \Psi^{-\infty}(X;F) \tag{L6.19}$$

provided of course that $P \in \mathrm{Diff}^k(X;E,F)$ is elliptic.

Proposition 11. *If $P \in \mathrm{Diff}^k(X;E,F)$ is elliptic then there exists $A \in \Psi^{-k}(X;F,E)$ such that*

$$P\circ A = \mathrm{Id}_F + R_F,\ R_F \in \Psi^{-\infty}(X;F),\ A\circ P = \mathrm{Id}_E + R_E,\ R_E \in \Psi^{-\infty}(X;E) \tag{L6.20}$$

from which it follows that $P : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;F)$ is Fredholm.

PROOF. From the existence of a right parametrix, $A \in \Psi^{-k}(X; F, E)$, satisfying the first condition in (L6.20) we can conclude that the range is closed and of finite codimension. Indeed the range of P certainly contains the range of PA and this is equal to the range of $\mathrm{Id} + R_F$. This, as we know, is a closed subspace of $\mathcal{C}^\infty(X; F)$ of finite codimension, so any subspace of $\mathcal{C}^\infty(X; F)$ containing it must also be closed and of finite codimension.

To examine the null space we need the second condition in (L6.20). First we try to construct an element $A' \in \Psi^{-k}(X; F, E)$ satisfying this condition without worrying whether it is related to A. To do so, note that we may take adjoints and the condition becomes

$$P^* \circ (A')^* = \mathrm{Id} + R_E^*. \tag{L6.21}$$

From the local discussion above we see that for differential operators,

$$\sigma_k(P^*) = (\sigma_k(P))^* \tag{L6.22}$$

so P is elliptic if and only if P^* is elliptic. Thus we may apply the same construction as above to find $(A')^* \in \Psi^{-k}(X; E, F)$, satifying (L6.21) and then A' is a right parametrix. From this we conclude that the null space of P is finite dimensional, since it is contained in the null space of $\mathrm{Id} + R_E$.

So, it only remains to see that there is an element $A \in \Psi^{-k}(X; F, E)$ which is simultaneously a left- and a right-parametrix. Consider the left parametrix just constructed. From the identity for the right parametrix, and associativity of products, it satisfies

(L6.23)

$$A' = A'(PA - R_F) = (A'P)A - A'R_F = A + R_E A - A'R_F = A + S,\ S \in \Psi^{-\infty}(X; F, E).$$

Thus the left and right parametrices differ by a smoothing operator, either of them is a two-sided parametrix. □

In fact, and such elliptic operator has a 'generalized inverse'. If we choose inner products and densities so that the orthocomplement of the range of P may be identified with the null space of P^* and the orthocomplement of the null space of P may be identified with the range of P^* then there is a unique operator $A : \mathcal{C}^\infty(X; F) \longrightarrow \mathcal{C}^\infty(X; E)$ which vanishes on the null space of P^* has range exactly the range of P^* and which is a two-sided inverse of P as a map from the range of P^* to its own range. In fact, as we shall see next time, this is a pseudodifferential operator (i.e. differs from a parametrix A by a smoothing operator).

L6.2. Hodge theory. Next I want to remind you how the Fredholm properties of elliptic operators on $\mathcal{C}^\infty$ spaces lead to *Hodge theory*, either for the usual exterior differential complex or some other elliptic complex (such as the Dolbeault complex).

On a compact manifold, consider the exterior form bundle ΛX. Thus $\Lambda^k_x X$ is totally antisymmetric part of the k-fold tensor power of $T^*_x X$. Then, as is well-known (and this is really the reason for the definition)

$$d : \mathcal{C}^\infty(X; \Lambda^p X) \longrightarrow \mathcal{C}^\infty(X; \Lambda^{p+1} X),\ d^2 = 0 \tag{L6.24}$$

where we may think of $d : \mathcal{C}^\infty(X; \Lambda^* X) \longrightarrow \mathcal{C}^\infty(X; \Lambda^*)$ as the direct sum of these operators or write it out as a complex

$$\cdots \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^p X) \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^{p+1} X) \xrightarrow{d} \cdots. \tag{L6.25}$$

The definition of the symbol of a differential operator in (L6.10) can be applied since

$$d(e^{itf}u) = e^{itf}\,(itdf \wedge u + du) \Longrightarrow \sigma_1(d)(x,\xi) = i\xi \wedge . \tag{L6.26}$$

In particular of course $\sigma_1(d)^2 = 0$, but that follows directly from the property for d.

If we consider a general differential complex, so a sequence of differential operators $P_i \in \operatorname{Diff}^k(X; E_i, E_{i+1})$ (the orders can be taken to be different but it makes for heavier algebra) such that $P_{i+1} \circ P_i = 0$. Such a complex is said to be *elliptic* if

$$\cdots \xrightarrow{\sigma_k(P_{i-1})(x,\xi)} E_{i,x} \xrightarrow{\sigma_k(P_i)(x,\xi)} E_{i+1,x} \xrightarrow{\sigma_k(P_{i+1})(x,\xi)} \cdots \tag{L6.27}$$
$$\text{is exact } \forall\ (x,\xi) \in T^*X \setminus 0_X.$$

The deRham complex (L6.25) is elliptic in this sense, since for any $0 \neq \xi \in T^*_xX$ the elements $\alpha \in \Lambda^p_xX$ satisfying $\xi \wedge \alpha = 0$ are exactly those which are of the form $\xi \wedge \beta$ for some $\beta \in \Lambda^{k-1}X$ – to see this simply introduce coordinates in which $\xi = dx_1$ and decompose forms accordingly.

Such an elliptic complex is 'almost exact' in the sense that the cohomology (originally called the hypercohomology) of the complex is finite dimensional.

Proposition 12. *If*

$$\cdots \xrightarrow{P_{i-1}} \mathcal{C}^\infty(X; E_i) \xrightarrow{P_i} \mathcal{C}^\infty(X; E_{i+1}) \xrightarrow{P_{i+1}} \cdots \tag{L6.28}$$

is an elliptic complex of differential operators of order k then the range of each P_i is closed in $\mathcal{C}^\infty(X; E_{i+1})$ and

$$\operatorname{null}(P_i)/P_{i-1}\mathcal{C}^\infty(X; E_{i-1}) \text{ is finite dimensional.} \tag{L6.29}$$

Proof. Hodge's idea was to choose inner products and densities (well he actually did it in a very algebraic setting) and consider the adjoint complex. Since the adjoint of a product is the product of the adjoints in the opposite order, we get an elliptic complex going the other way

$$\cdots \xleftarrow{P^*_{i-1}} \mathcal{C}^\infty(X; E_i) \xleftarrow{P^*_i} \mathcal{C}^\infty(X; E_{i+1}) \xleftarrow{P^*_{i+1}} \cdots \tag{L6.30}$$

Now each of the operators

$$\Delta_i = P^*_iP_i + P_{i-1}P^*_{i-1} \in \operatorname{Diff}^{2k}(X; E_i) \tag{L6.31}$$

is elliptic. Indeed, its symobl at each point $(x,\xi) \in T^*_xX \setminus \{0\}$ is

$$\sigma_{2k}(\Delta_i) = \sigma_k(P_i)^*\sigma_k(P_i) + \sigma_k(P_{i-1})\sigma_k(P_{i-1})^*. \tag{L6.32}$$

This is a self-adjoint matrix and and element of its null space satisfies

$$\langle \sigma_{2k}(\Delta_i)u, u\rangle = |\sigma_k(P)_iu| + |\sigma_k(P_{i-1})u| = 0 \Longrightarrow \sigma_k(P_{i-1})^*u = 0 = \sigma_k(P_i)u. \tag{L6.33}$$

Since the null space of $\sigma_k(P_{i-1})^*$ is a complement to the range of $\sigma_k(P_{i-1})$, this implies u is zero.

Thus the null space of Δ_i is finite dimensional and its range is closed and has orthocomplement this same null space, by self-adjointness. Again by integration by parts on X, the null space of Δ_i is the intersection of the null spaces of P_i and P^*_{i-1}. It follows that for each i we may decompose

$$\mathcal{C}^\infty(X; E_i) \ni u = u_0 \oplus P_{i-1}v_{i-1} \oplus P^*_iv_{i+1},\ P_iu_0 = 0 = P^*_{i-1}u_0 \tag{L6.34}$$

where the decomposition is orthogonal and unique. The range of P_{i-1} must therefore be closed (since the closure in the $\mathcal{C}^\infty$ topology is contained in the closure in L^2). $\square$

Note that the 'Hodge decomposition' (L6.34) is a useful way to encapsulate the consequences of ellipticity for a complex. It shows in particular that (L6.29) can be seen in the stronger form that

$$\text{null}(\Delta_i) \longrightarrow \text{null}(P_i)/P_{i-1}\mathcal{C}^\infty(X; E_{i-1}) \text{ is an isomorphism} \tag{L6.35}$$

which is the *Hodge theorem.*

6+. Addenda to Lecture 6

CHAPTER 7

Localization and composition

Lecture 7: 6 October, 2005

L7.1. Localization. Finally I will connect the definition of pseudodifferential operators made here with the more standard approach, starting with a local definition on Euclidean space and proceeding by superposition. To break a pseudodifferential operator up into pieces it is convenient to use partitions of unity of the following type.

LEMMA 13. *If $\{U_i\}$ is an open cover of a compact manifold there is a partition of unity $\{\phi_{ij}\}$ subordinate to the cover, so*

$$0 \le \phi_{ij} \le 1, \ \forall\, i, j, \ \sum_{i,j} \phi_{ij} = 1, \ \operatorname{supp}(\phi_{ij}) \subset U_i,$$

which also satisfies

$$\operatorname{supp}(\phi_{ij}) \cap \operatorname{supp}(\phi_{i'j'}) \neq \emptyset \Longrightarrow \operatorname{supp}(\phi_{ij}) \cup \operatorname{supp}(\phi_{i'j'}) \subset U_i \cap U_{i'}. \tag{L7.1}$$

PROOF. First choose a partition of unity χ_i subordinate to the open cover $\{U_i\}$. Then each point $p \in X$ has an open neighbourhood V_p with the property

$$V_p \cap \operatorname{supp}(\chi_i) \neq \emptyset \Longrightarrow V_p \subset U_i. \tag{L7.2}$$

In fact we could take V_p to be the intersection of the U_i containing p. Pass from the V_p to a finite subcover, V_j, and choose a partition of unity ψ_j subordinate to this cover. Then set $\phi_{ij} = \chi_i \psi_j$. This is a partition of unity and the intersection condition in (L7.1) implies that the supports of ψ_j and $\chi_{i'}$ must meet, as well as those of $\psi_{j'}$ and χ_i. By (L7.2) this implies that $\operatorname{supp}(\psi_j) \subset U_{i'}$ and $\operatorname{supp}(\psi_{j'}) \subset U_i$ from which (L7.1) follows. □

We can use this to localize a pseudodifferential operator with respect to an open cover of X. Namely if $A \in \Psi^m(X; E, F)$ consider the decomposition obtained by mutliplying by the ϕ_{ij} on both the left and the right. That is, using the partition of unity $\phi_{ij}(x)\phi_{i'j'}(y)$ on X^2. This decomposes A (using the $\mathcal{C}^\infty$ module property) as a finite sum

$$A = \sum_{i,j,i',j'} \phi_{ij} A \phi_{i'j'} \tag{L7.3}$$

where we are thinking of the ϕ_{ij} as operators on $\mathcal{C}^\infty$ spaces, so (L7.3) is a composition of operators. The support of each term in (L7.3) is contained in $U_i \times U_{i'}$ but

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more importantly the support can only meet the diagonal if

(L7.4)
$$(\mathrm{supp}(\phi_{ij}) \times X) \cap (X \times \mathrm{supp}(\phi_{i'j'}) \cap \mathrm{Diag} \neq \emptyset \Longrightarrow \mathrm{supp}(\phi_{ij}) \cap \mathrm{supp}(\phi_{i'j'}) \neq \emptyset.$$

So, if we use the partition of unity from Lemma L7.2, then

$$\mathrm{supp}(\phi_{ij} A \phi_{i'j'}) \cap \mathrm{Diag} \neq \emptyset \Longrightarrow \mathrm{supp}(\phi_{ij} A \phi_{i'j'}) \subset U_i \times U_i. \tag{L7.5}$$

So, given an open cover $\{U_i\}$ of X we may decompose A into a sum of pseudodifferential operators of the same order

$$A = \sum_i A_i + A',\ \mathrm{supp}(A_i) \subset U_i \times U_i,\ A' \in \Psi^{-\infty}(X; E, F) \tag{L7.6}$$

where the last term comes from all the pieces which have support not meeting the diagonal.

L7.2. Local normal fibrations. In particular we can assume that the open cover $\{U_i\}$ with respect to which we get a decomposition (L7.6) consists of coordinate patches over each of which the bundles E and F are trivialized. Then the kernel of each A_i is a matrix of conormal distributions, with compact support and of order m, with respect to the diagonal in $U_i \times U_i$. The coordinate system identifies U_i with an open set U_i' in $\mathbb{R}^n$, $n = \dim X$. The density bundle on X is locally trivialized by the coordinate density $|dx|$ so it sufficies to consider 'scalar' pseudodifferential operators with kernels compactly supported on $\mathbb{R}^n \times \mathbb{R}^n$. This indeed is a typical starting point for the definition of pseudodifferential operators.

To specify the kernel as the inverse Fourier transform of a symbol we also need to choose a normal fibration of the diagonal

$$\mathrm{Diag}(\mathbb{R}^n) = \{x = y\} \subset \mathbb{R}^n_x \times \mathbb{R}^n_y. \tag{L7.7}$$

There are three standard choices for the normal fibration, which I will call the 'left' fibration, the 'right' fibration and the 'Weyl' fibration. These each give a global identification of the whole of $\mathbb{R}^{2n}$, as a neighbourhood of the diagonal, with $\mathbb{R}^n \times \mathbb{R}^n$, thought of as the normal bundle to the diagonal.

So first we have to identify the normal bundle to the diagonal. This is naturally the quotient of the tangent bundle to $\mathbb{R}^{2n}$, restricted to Diag, by the tangent bundle to Diag. The latter is easy to describe, namely

$$T\,\mathrm{Diag} = \{((x,x),(v,v)); (x,v) \in \mathbb{R}^{2n}\} \equiv \mathbb{R}^n \times \mathbb{R}^n \equiv \{(x,v) \in T\mathbb{R}^n\} \tag{L7.8}$$

where this identification is canonical. So the normal bundle can be identified with any subbundle of $T_{\mathrm{Diag}}\mathbb{R}^{2n}$ which is transversal to $T\,\mathrm{Diag}$. The standard choice is to take the 'left tangent bundle'

$$T\mathbb{R}^n \ni (x,w) \longmapsto ((x,x),(w,0)) \in T_{\mathrm{Diag}}\mathbb{R}^n \longrightarrow N\,\mathrm{Diag}\,. \tag{L7.9}$$

Notice that this is not really canonical. Namely we could 'just as well' take the right tangent vectors (but DO NOT DO THIS if you are easily confused)

$$T\mathbb{R}^n \ni (x,w) \longmapsto ((x,x),(0,w)) \in T_{\mathrm{Diag}}\mathbb{R}^{2n} \longrightarrow N\,\mathrm{Diag}\,.$$

The trouble is that modulo the tangent bundle to the diagonal $(0,w) - (w,w) = (-w,0)$ so this is almost the same identification but has the sign reversed. The identification (L7.9) is universally adopted, basically in the same sense that one writes compositions of operators on the left, i.e. AB means first apply B then A.

Once we have adopted (L7.9) as our identification of the normal bundle to the diagonal with the tangent bundle to the manifold (this works on a manifold as well) then there are still choices for the normal fibration. Now of course they correspond to maps from $\mathbb{R}^{2n}$ to $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with the right properties. The ones we consider each induce a linear isomorphism (linear in fact in all variables, not just the fibre variables). These are the left, the right and the Weyl fibrations:-

$$\begin{aligned} f_L &: \mathbb{R}^{2n} \ni (x,y) \longrightarrow (x, x-y) \in T\mathbb{R}^n \\ f_R &: \mathbb{R}^{2n} \ni (x,y) \longrightarrow (y, x-y) \in T\mathbb{R}^n \\ f_W &: \mathbb{R}^{2n} \ni (x,y) \longrightarrow (\frac{x+y}{2}, x-y) \in T\mathbb{R}^n. \end{aligned} \tag{L7.10}$$

Thus, for the left fibration we fix the variable x, so with the standard picture of x, y-space the fibres are the verticals, but we take the linear variable on each fibre which is $x - y$, the x being constant normalizes this to be zero at the point (x, x) on the diagonal, but the 'variable' is $-y$. This comes about because of the standard identification of the normal bundle to the diagonal with the tangent bundle. The right fibration is similar, except that y is held fixed, the fibres are 'horizontal' and the variable on them is still $x - y$. For the Weyl fibration, which I will not use for the moment, we hold $x + y$ fixed and the fibre variable is still $x - y$. There are plenty of other possibilities, but these are the usual ones.

So, what does our kernel $A \in \Psi^m(X)$, supported in a coordinate patch, look like with respect to these fibrations? It is always the inverse Fourier transform of a classical symbol, so the three representations (of the one kernel) are

$$\begin{aligned} A(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_L(x,\xi) d\xi |dy|, \\ A(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_R(y,\xi) d\xi |dy|, \\ A(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_W(\frac{x+y}{2},\xi) d\xi |dy|. \end{aligned} \tag{L7.11}$$

Here $|dy|$ is the coordinate trivialization of the right density bundle. In all three cases the ampllitude lies in $\rho^{-m}\mathcal{C}^\infty_c(\mathbb{R}^n \times \overline{\mathbb{R}^n})$.

For the moment, we are most interested in the two 'extreme' representations, the left and right representations. As noted above, in each case we are holding one of the variables x or y fixed. This means that there is a close relationship between the Fourier transform and the operator.

Lemma 14. *The left representation of a pseudodifferential operator with compactly supported kernel on $\mathbb{R}^n$ puts the operator in the form*

$$Af(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a_L(x,\xi) \hat{f}(\xi) d\xi, \ \forall \ f \in \mathcal{S}(\mathbb{R}^n), \tag{L7.12}$$

and similarly the right representation gives

$$\widehat{Af}(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} a_R(y,\xi) f(y) dy. \tag{L7.13}$$

Proof. □

L7.3. Composition. Almost as an immediate corollary of the representations (L7.12) and (L7.13) we deduce the basic composition property of pseudodifferential operators.

PROPOSITION 13. *If* $A \in \Psi^m(X; E, F)$ *and* $B \in \Psi^{m'}(X; F, G)$ *for complex vector bundles,* E, F *and* G *over a compact manifold* X *then as an operator*

$$BA : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; G),\ BA \in \Psi^{m+m'}(X; E, G). \tag{L7.14}$$

Furthermore

$$\sigma_{m+m'}(BA) = \sigma_{m'}(B)\sigma_m(A). \tag{L7.15}$$

PROOF. First we start with the 'easy case' where $m = -\infty$ or $m' = -\infty$ and one of the operators is smoothing. The composition is then very closely related to the action of pseudodifferential operators on smooth sections. In fact below I observe that it can be deduced directly from the continuity of this action after localizing.

However, one can also proceed directly and globally. I want to point out this argument, although I give a simpler alternative below, because it leads to an interesting geometric question which I will consider later.

Recall that we showed that $A \in \Psi^m(X; E, F)$ defines a map

$$A : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F)$$

by working on the kernel level. Namely we define the map (L7.3) by proceeding in steps. First lift an element $u \in \mathcal{C}^\infty(X; E)$ to the section $\pi_R^* u \in \mathcal{C}^\infty(X^2; \pi_R^* E)$ which is independent of the left, x, variable. Then use the $\mathcal{C}^\infty$-module property to 'multiply' the kernel by this smooth section (and compose in the bundle) to get

$$A\pi_R^* u \in I^m(X^2, \mathrm{Diag}; \pi_L^* F \otimes \Lambda_R). \tag{L7.16}$$

Then the 'action' of the operator is defined by integrating out the right, y, variables to get

$$Au = (\pi_L)_*(A \cdot \pi_R^* u). \tag{L7.17}$$

The push-forward theorem (using the freedom to choose the normal fibration) shows that this is an element of $\mathcal{C}^\infty(X; F)$.

Essentially the same argument works for composition of $B \in \Psi^{m'}(X; F, G)$ and $A \in \Psi^{-\infty}(X; E, F)$ except that we have three factors of X to worry about. However the right-most fact here can be viewed as a parameter space. The composition looks like

$$\int_X B(x, y)A(y, z)\text{`}dy' \tag{L7.18}$$

(where I have written 'dy' because the measure is already part of B) and we may interpret this as in (L7.17) by writing it

$$AB = (\pi_C)_*(\pi_S^* A \cdot \pi_F^* B). \tag{L7.19}$$

Here there are three projections from X^3 to X^2

(L7.20)

$$\pi_F : X^3 \ni (x, y, z) \longrightarrow (y, z) \in X^2,$$
$$\pi_C : X^3 \ni (x, y, z) \longrightarrow (x, z) \in X^2 \text{ and } \pi_S : X^3 \ni (x, y, z) \longrightarrow (x, y) \in X^2.$$

The first one drops the left variable, the second the middle variable and the last the right-most variable. The labels as supposed to correspond to the action of operators, as in $C = BA$, so A is the 'first' operator (in action) and corresponds to π_F, B is the 'second' operator and corresponds to π_S whereas C is the 'composite' operator and corresponds to π_C in (L7.18) and (L7.19); so you can think of this as the 'composite' projection or the 'central' projection.

Since these maps are smooth, $\pi_F^* A \in \mathcal{C}^\infty(X^3; \pi_M^* F \otimes \pi_R^* E')$ where

$$\begin{aligned} \pi_R &: X^3 \ni (x, y, z) \longrightarrow z \in X \\ \pi_M &: X^3 \ni (x, y, z) \longrightarrow y \in X \text{ and} \\ \pi_L &: X^3 \ni (x, y, z) \longrightarrow x \in X \end{aligned} \tag{L7.21}$$

are the three projections onto a single factor of X (corresponding to 'right', 'middle' and 'left'. We are using these projections mainly to pull bundles back. The pull-back theorem for conormal distributions proved above applies to show that

$$\pi_S^* A \in I^{m-\frac{1}{4}\dim X}(X^3, \pi_S^{-1} \operatorname{Diag}; \pi_L^* G \otimes \pi_M^* F \otimes \Omega_M). \tag{L7.22}$$

Thus the product in (L7.19) can be interpreted as an element

$$\pi_S^* A \cdot \pi_F^* B \in \mathcal{C}^\infty(X^2; \pi_L^* G \otimes \pi_R^* F \otimes \Omega_R) = \mathcal{C}^\infty(X^2; \operatorname{Hom}(E, G) \otimes \Omega_R). \tag{L7.23}$$

The global discussion of the composition when A is smoothing and B is pseudodifferential is similar. In fact it is not necessary to do it, since we know that the space of pseudodifferenetial operators is invariant under taking adjoints. Thus the discussion above then applies to $B^* A^*$ and this is $(AB)^*$.

Once we have taken care of the case where one of the factors is smoothing we can pass to the local setting. In fact, we can do that anyway. Thus if $\{U_i\}$ is an open cover of X we can decompose A and B into finite sums

$$A = \sum_{i,k} \psi_i A \Psi_k, \ B = \sum_{i',k'} \psi_{i'} B \Psi_{k'}. \tag{L7.24}$$

Then the composite decomposes into a big sum

$$(AB) = \sum_{i,k,i',k'} \psi_i A \Psi_k \psi_{i'} B \Psi_{k'}. \tag{L7.25}$$

Now, we have already discussed the case in which one of the factors is smoothing, which in particular covers the case where the support does not meet the diagonal. Let me prove this again by localization. Thus we can suppose that each element of the open cover $\{U_i\}$ is a coordinate neighbourhood over which the bundles E, F and G are trivial. The density bundle is trivialized by the coordinate density $|dx|$ so the kernels just become matrices of conormal distibutions with respect to the diagonal. The bundle composition is just matrix composition, so we are reduced to looking at each of the entries, just the composition of scalar kernels. In general there may have different coordinates in the various factors, but using Lemma 13 above we may assume that the middle patches, the left for B and the right for A, are the same. Now, if say the localized term on the right $A\Psi_k \psi_{i'} B \Psi_{k'}$ is smoothing, it can be regarded as a smooth map from $U_{k'}$ to smooth functions on $U_k = U_i$, using the fact that a smooth function on a product is the same as a smooth map from either factor into smooth functions on the other factor. Then applying $\psi_i A \psi_k$ on the left gives a smooth function on U_i, for each point in $U_{k'}$, where everything has compact support. The linearity and continuity of A means that it is a $\mathcal{C}^\infty$ map, so in fact this is a smooth map from $U_{k'}$ into $\mathcal{C}^\infty_c(U_i)$ and hence in fact an element of $\mathcal{C}^\infty_c(U_i \times U_{k'})$, i.e. the kernel of a smoothing operator. This gives the alternative proof of the composition formula where the right factor is smoothing, mentioned above. If the left factor is smoothing one can apply the discussion of adjoints as above.

Thus in the expansion of the product in (L7.25) we know that each term where one of the factors is smoothing is itself smoothing. Using a decomposition as in (L7.3) we arrive at (L7.5) and in fact by a similar argument we can see (changing the indexing) that if one of the terms in the product $A_l B_k$ is not smoothing then *both* factors have kernels supported in the product of a fixed element of the cover with itself, that is both have compact support in $U_i \times U_i$ for some i. This allows us to work in just one coordinate patch rather than two.

Thus, we are reduced to showing that the product AB in the case of compactly supported scalar pseudodifferential operators on $\mathbb{R}^n$. We choose to write B in right reduced form as in (L7.13) and A in left reduced form as in (L7.12)

$$\begin{aligned}\widehat{Bf}(\xi) &= \int_{\mathbb{R}^n} e^{-iy\cdot\xi} b_R(y,\xi) f(y)dy,\\ Af(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} a_L(x,\xi)\hat{f}(\xi)d\xi,\ \forall\ f\in\mathcal{S}(\mathbb{R}^n).\end{aligned} \tag{L7.26}$$

Inserting the formula for B into that of A we see that the kernel of the composite is

$$AB = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_L(x,\xi) b_R(y,\xi) d\xi. \tag{L7.27}$$

The product of the two symbols is a symbol itself, so this is almost of the form we expect, the inverse Fourier transform of a symbol. The problem is that it is not quite an inverse Fourier transform because both the variables x and y occur in the amplitude. However we have already effectively overcome this problem. Namely we can treat the dependence of the amplitude on, say, y as parameter and write (L7.27) in the form

$$AB(x,y) = \left((2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_L(x,\xi) b_R(z,\xi) d\xi\right)\big|_{z=y}. \tag{L7.28}$$

Now the inverse Fourier transform gives a conormal distribution on $\mathbb{R}^{3n}$, with variables x, y, z, with respect to the submanifold $x = y$. Then restriction to $z = y$ is transversal to the submanifold so we deduce that the kernel is conormal and of order $m + m'$. Putting all the terms back together we deduce (L7.14) and also (L7.15). □

L7.4. Ellipticity again. Now, we can prove the same result as I showed last time for elliptic differential operators but in the more general setting of elliptic pseudodifferential operators.

THEOREM 3. *If $P \in \Psi^m(X; E, F)$ is elliptic, in the sense that $\sigma_m(P)$ is invertible at each point of S^*X, then there exists $Q \in \Psi^{-m}(X; F, E)$ such that*

$$QP = \mathrm{Id}_E - R_E,\ R_E \in \Psi^{-\infty},\ PQ = \mathrm{Id}_F - R_F,\ R_F \in \Psi^{-\infty}(X; F). \tag{L7.29}$$

PROOF. The proof is the same as for differential operators above, except that we use the composition formula from Proposition 13. Still, let me take the time to go through the proof again.

□

L7.5. Index problem. As a direct result of Theorem 3 that proof that an ellipti element $P \in \Psi^m(X; E, F)$ is Fredholm on $\mathcal{C}^\infty$ sections is reduced to the same statement for operators of the form $\mathrm{Id} + A$ with A smoothing. Namely we want to show that

$$\begin{gathered}\mathrm{Nul}(P) = \{u \in \mathcal{C}^\infty(X; E); Pu = 0\} \text{ is finite dimensional} \\ \mathrm{Ran}(P) = \{f \in \mathcal{C}^\infty(X; F);\ \exists\ u \in \mathcal{C}^\infty(X; E),\ Pu = f\} \text{ is closed and} \\ \mathcal{C}^\infty(X; F) = \mathrm{Ran}(P) + V,\ V \subset \mathcal{C}^\infty(X; F) \text{ finite dimensional.}\end{gathered} \tag{L7.30}$$

From (L7.29) we see that

$$\mathrm{Nul}(P) \subset \mathrm{Nul}(\mathrm{Id}_E - R_E) \text{ and } \mathrm{Ran}(P) \supset \mathrm{Ran}(\mathrm{Id}_F - R_F). \tag{L7.31}$$

So if $\mathrm{Nul}(\mathrm{Id}_E - R_E)$ is finite dimensional, so is $\mathrm{Nul}(P)$ and if $\mathrm{Ran}(\mathrm{Id}_F - R_F)$ is closed with finite codimension then so is $\mathrm{Ran}(P)$ (check the algebra here for yourself); the point being that for smoothing perturbations of the identity, this is always true. As noted before, the fact that the range is closed follows from the last condition, the existence of a finite dimensional complement. I include it to avoid confusion with the weaker condition that the closure of the range has finite codimension. I will talk extensively about smoothing operators, next time.

Now the index of P is by definition the integer

$$\mathrm{ind}(P) = \dim \mathrm{Nul}(P) - \dim \left(\mathcal{C}^\infty(X; F) / \mathrm{Ran}(P)\right), \tag{L7.32}$$

(although it might have been better if it had been defined with the opposite sign). The problem solved by the index theorem of Atiyah and Singer (in its simplest form) is the computation of the index in terms of the symbol of P, via a topological formula.

The question arises as to why this integer is interesting. Of course the fundamental reason is that it is something that does not occur in finite dimensions. For a finite dimensional matrix, the corresponding integer is the difference between row rank and collum rank so it just the difference of dimension of source and target vector spaces.

Practically the index solves the problem of 'perturbative invertibility', as I will show next week. Namely we can ask whether there exists a smoothing operator $R \in \Psi^{-\infty}(X; E, F)$ such that $P + R$ is inverible, meaning for present purposes that it is injective and surjective.

Proposition 14. *For any elliptic pseudodifferential operator $P \in \Psi^m(X; E, F)$ there exists $R \in \Psi^{-\infty}(X; E, F)$ such that $P + \epsilon R$ is invertible for small $\epsilon \neq 0$ if and only if* $\mathrm{ind}(P) = 0$.

To analyse the index I will need to detour a little into K-theory. Suppose Y is any compact manifold and E is any vector bundle over Y. Then consider the operators of the form $\mathrm{Id} + A$, $A \in \Psi^{-\infty}(X; E)$ as we have been doing, but now look at those which are invertible (as an operator on $\mathcal{C}^\infty(X : E)$. The inverse is automatically of the same form, so this is a group which I will denote $G^{-\infty}(Y; E)$. In fact it is an open subset of $\mathcal{C}^\infty(X^2; \mathrm{Hom}(E) \otimes \Omega_R)$ so has a well-defined topology. I will define K-theory directly through the definition of odd K-theory. Thus for any compact manifold X set

$$K^{-1}(X) = [X; G^{-\infty}(Y; E)] \tag{L7.33}$$

the set of (smooth) homotopy classes of smooth maps into $G^{-\infty}(Y;E)$. Of course it is implicit in this definition that the result is independent of the choice of Y or E.

7+. Addenda to Lecture 7

CHAPTER 8

Smoothing operators

Lecture 8: 13 October, 2005

Now I am heading towards the Atiyah-Singer index theorem. Most of the results proved in the process untimately reduce to properties of smoothing operators, so let me review these today.

Recall that the space of smoothing operators on a compact manifold X acting between bundles E and F is identified with smooth sections of the 'big homomorphism bundle' over X^2 :

$$\Psi^{-\infty}(X;E,F)=\mathcal{C}^\infty(X^2;\pi_L^*F\otimes\pi_R^*E'\otimes\pi_R^*\Omega) \tag{L8.1}$$

where we identify $\mathrm{Hom}(E,F)=\pi_L^*F\otimes\pi_R^*E'$. These are bounded operators on L^2 sections as follows directly from the Cauchy-Schwarz inequality

$$\begin{gathered}\Psi^{-\infty}(X;E,F)\ni A:L^2(X;E)\longrightarrow L^2(X;F),\\ Au(x)=\int_X A(x,y)u(y),\ \|Au\|\le\|A\|_{L^2}\|u\|_{L^2}.\end{gathered} \tag{L8.2}$$

This just uses the square-integrability of the kernel.

Lemma 15. *If $A\in\Psi^{-\infty}(X;E)$ (so $E=F$) and its norm as a bounded operator on $L^2(X;E)$ is less than 1 then $(\mathrm{Id}+A)^{-1}=\mathrm{Id}+B$ for $B\in\Psi^{-\infty}(X;E)$.*

Proof. Since $\|A\|<1$ the Neumann series converges as a sequence of bounded operators so

$$B=\sum_{l=1}^{\infty}(-1)^lB^l \tag{L8.3}$$

is bounded on $L^2(X;E)$. As a 2-sided inverse $(\mathrm{Id}+A)(\mathrm{Id}+B)=\mathrm{Id}=(\mathrm{Id}+B)(\mathrm{Id}+A)$ which shows that

$$B=-A+A^2+ABA. \tag{L8.4}$$

From this it follows that $B\in\Psi^{-\infty}(X;E)$ since $ABA\in\Psi^{-\infty}(X;E)$. Indee the A on the right may be considered locally as a smooth map from X into $L^2(X;E)$ and hence remains so after applying B but then applying the second copy of A gives a smooth map into $\mathcal{C}^\infty(X;E)$ so the kernel of the composite is actually smooth on X^2. □

Corollary 2. *For any compact manifold and complex vector bundle E*

$$G^{-\infty}(X;E)=\big\{A\in\Psi^{-\infty}(X;E);(\mathrm{Id}+A)^{-1}=\mathrm{Id}+B,\ B\in\Psi^{-\infty}(X;E)\big\} \tag{L8.5}$$

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is an open subset of $\Psi^{-\infty}(X;E)$ *which is a topological group.*

Proof. For any point $A \in G^{-\infty}(X;E)$ the set $A+B$ such that $\|B\| < 1/\|(\mathrm{Id}+A)^{-1}\|$ is open in $\Psi^{-\infty}(X;E)$ and for such B it follows from the discussion above that $A+B \in G^{-\infty}(X;E)$ since $(\mathrm{Id}+A+B)^{-1} = (\mathrm{Id}+A)^{-1}(\mathrm{Id}+B(\mathrm{Id}+A)^{-1})^{-1}$. Similarly the maps $A \to (\mathrm{Id}+A)^{-1} - \mathrm{Id}$ and $(A,B) \longrightarrow (\mathrm{Id}+A)(\mathrm{Id}+B) - \mathrm{Id}$ are continuous. $\square$

Notice that I insist on $G^{-\infty}(X;E) \subset \Psi^{-\infty}(X;E)$ onto to make such statements easy to say. 'Really' of course you should think of $G^{-\infty}(X;E)$ as something like the invertible bounded operators on $L^2(X;E)$ which are of the form $\mathrm{Id}+A$ with $A \in \Psi^{-\infty}(X;E)$.

In fact, as we shall see later, $G^{-\infty}(X;E) \subset \Psi^{-\infty}(X;E)$ is actually an open dense subset, just like the invertible matrices in all matrices. As a topological algebra it is independant of X and E (provided $\dim X > 0$).

Definition 4. An operator has finite rank if its range is finite dimensional.

We are particularly interested in finite rank smoothing operators.

Lemma 16. *A smoothing operator* $A \in \mathcal{C}^\infty(X;E,F)$ *is of finite rank if and only if there are elements* $f_i \in \mathcal{C}^\infty(X;F)$, $e_i \in \mathcal{C}^\infty(X;E')$ $i = 1,\dots,N$ *and* $\nu \in \mathcal{C}^\infty(X;\Omega)$ *such that*

$$A = \sum_{i=1}^{N} f_i(x)e_i(y)\nu(y). \tag{L8.6}$$

Proof. By definition if $A \in \mathcal{C}^\infty(X;E,F)$ has finite rank, its range must be a finite dimensional subspace of $\mathcal{C}^\infty(X;F)$. Let the f_i be a basis of this space. Thus, we can write $Au = \sum\limits_{i=1}^{N}(A_iu)f_i$ where $A_i : \mathcal{C}^\infty(X;E) \longrightarrow \mathbb{C}$ is continuous. If the f_i are orthonormalized with respect to an hermitian inner product on F and a density on X then $A_iu = \langle Au, f_i\rangle$ so these functionals are given by pairing with the smooth density

$$A_i = \int_X \langle A(x,y), f_i\rangle_F \nu(x) \in \mathcal{C}^\infty(X;E'\otimes\Omega) \tag{L8.7}$$

Dividing by a fixed density $0 < \nu \in \mathcal{C}^\infty(X;\Omega)$ gives $e_i = A_i/\nu \in \mathcal{C}^\infty(X;E')$ and this shows that the kernel can be written in the form (L8.6). $\square$

If we insist that the e_i be independent, or even orthonormalized with respect to some choice of hermitian inner product on E (hence on E') and density on X then the kernel takes the form

$$A = \sum_{i=1}^{N} a_{ij}f_i(x)e_j(y)\nu(y). \tag{L8.8}$$

We may also use the antilinear isomorphism of E' and E in terms of the chosen inner product to think of the e_i as sections of $\mathcal{C}^\infty(X;E)$. Then (L8.8) can be written rather fancifully as

$$A = \sum_{i=1}^{N} a_{ij}f_i(x)\overline{e_j}(y)\nu(y),\ a_{ij} \in \mathbb{C}, \tag{L8.9}$$

where the operation $\overline{e_i}$ is the antilinear isomorphism. Then the action of A is through the inner product

$$Au(x) = \sum_{i,j} a_{ij} \int_X \langle e_i(y), u(y)(y)\rangle f_i(x). \tag{L8.10}$$

If $E = F$ then we can orthonormalize the collection of all the e_i and f_j together and denote the result as e_i. In this case we have embedded A inside the collection of $N \times N$ matrices via (L8.10) which now becomes

$$Au(x) = \sum_{i,j} a_{ij} \int_X \langle e_i(y), u(y)\rangle e_i(x). \tag{L8.11}$$

Notice in fact that these finite rank smoothing operators do form a subalgebra of $\Psi^{-\infty}(X; E)$ which is isomorphic *as an algebra* to $M(N, \mathbb{C})$.

Lemma 17. *The finite rank operators are dense in* $\Psi^{-\infty}(X; E, F)$.

I will give a rather uninspiring proof of this in which the approximation is done rather brutally. One can give much better approximation schemes, and I will, but first one needs to show that such approximation is possible (since this result is so basic it is actually used in the spectral theory which lies behind the better approximations. . .).

Proof. In the special case that $X\mathbb{T}^n$ is a torus and $E = \mathbb{C}^k$ and $F = \mathbb{C}^{k'}$ are trivial bundles we can use Fourier series. Let $\nu = |d\theta_1 \dots d\theta_n|$ be the standard density on the torus then and element $A \in \Psi^{-\infty}(\mathbb{T}^n; \mathbb{C}^k, \mathbb{C}^{k'})$ is a $k \times k'$ matrix with entries in $\Psi^{-\infty}(\mathbb{T}^n)$, so acting on functions. The kernel, using the trivialization of the density bundle, is just an element $a \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ which we can therefore expand in Fourier series. Let us write this expansion with the sign reversed in the second variable (in $\mathbb{T}^n$)

$$a(\theta, \theta') = \sum_{I,J} a_{IJ} e^{iJ\cdot\theta} e^{-iJ\cdot\theta'} \tag{L8.12}$$

where the sum is over all I, $J \in \mathbb{Z}^n$ and the coefficients are rapidly decreasing, because of the smoothness of a

$$a_{iJ} = (2\pi)^{-2n} \int_{\mathbb{T}^{2n}} e^{-iI\cdot\theta_i J\cdot\theta'} d\theta d\theta'. \tag{L8.13}$$

Since this double Fourier series converges rapidly the truncated kernels

$$a_N(\theta, \theta') = \sum_{|I|,|J|\le N} a_{IJ} e^{iJ\cdot\theta} e^{-iJ\cdot\theta'} \tag{L8.14}$$

converge to a in the $\mathcal{C}^\infty$ topology. Clearly a_N is a finite rank smoothing operator, so this proves the result in the case of the torus.

In the general case of a compact manifold X and bundles E, F, choose a covering of X by coordinate patches U_i over which both bundles are trivial and a partition of unity of the form ρ_p^2 subordinate to this cover. We may think of each of the U_p as embedded as an open subset of $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, by translating and scaling U_p until it is contained in $(0, 2\pi)^n$. Then we may apply the discussion above to the kernel $\rho_p A(x, y)\rho_q(y)$ which may be interpreted as acting between trivial bundles over the torus. Of course, from (L8.14) the resulting approximating finite rank kernels $a_{N,p,q}$ will not have support in $U_p \times U_q$ when regarded as subsets of $\mathbb{T}^{2n}$.

However $\rho_p(x)a_{N,p,q}\rho_q(y)$ does have such support and is of the form (L8.9) with the e_j being the $e^{iJ\cdot\theta'}\rho_q(y)$ and similarly for the f_i. Thus, summing these finitely many kernels we obtain a sequence of finite rank operators on X converging to A in the $\mathcal{C}^\infty$ topology. $\square$

We need to consider families of operators, so note that this proof of approximation works uniformly on compact sets with the e_i and f_j fixed, i.e. independent of the parameters so only the coefficients in the approximating kernels depend on the parameters.

Now, recall that I have defined the odd K-theory of a compact manifold as

$$K^{-1}(X) = [X, G^{-\infty}(Y;E)] = \pi_0(\mathcal{C}^\infty(X; G^{-\infty}(Y;E)) \tag{L8.15}$$

So this includes the claim that the result is independent of the choice of Y and the bundle E (provided that $\dim Y > 0$). Note that

$$\mathcal{C}^\infty(X; G^{-\infty}(Y,E)) = \{K \in \mathcal{C}^\infty(X \times Y^2; \operatorname{Hom}(E)\otimes\pi_L^*\Omega);\ \exists\,(\operatorname{Id} + K(x,\cdot))^{-1}\ \forall\, x \in X\}. \tag{L8.16}$$

So the equivalence relation defining $K^{-1}(X)$ is just that $K \equiv K'$ if there exists $\tilde{K} \in \mathcal{C}^\infty(X \times [0,1]; G^{-\infty}(Y;E))$ such that $\tilde{K}\big|_{t=0} = K$ and $\tilde{K}\big|_{t=1} = K'$.

The standard definition of odd K-theory is as the stable homotopy classes of (continuous) maps in $\operatorname{GL}(N,\mathbb{C})$. I will not work with this directly, but if you think a little about the proof below that $K^1(X)$ is independent of the choice of Y and E you will see how to show the equivalence of (L8.15) and the standard definition.

Proposition 15. *The groups $G^{-\infty}(Y;E)$ are connected and the set* (L8.15) *for any compact manifold X is independent of the choice of Y and E, so given two choices Y, E and Z, F there is a natural bijection between $[X; G^{-\infty}(Y;E)]$ and $[X, G^{-\infty}(Z,F)]$.*

Proof. That $G^{-\infty}(Y;E)$ is connected follows from the fact that that it is locally connected, so if $a_N \to a$ in $G^{-\infty}(Y;E)$ then for large N, a_N may be connected to a and the fact that $\operatorname{GL}(N;\mathbb{C})$ is connected. Or once can proceed more directly, as discussed below.

Let us choose a fixed 'model', namely $Y = \mathbb{S}$ and $E = \mathbb{C}$. Now, we may embed

$$G(N,\mathbb{C}) \subset G^{-\infty}(\mathbb{S}) \tag{L8.17}$$

by mapping the $N \times N$ matrices to the smoothing operators

$$M(N,\mathbb{C}) \ni a_{kl} \longmapsto A = \sum_{k,l=1,N} (a_{kl} - \delta_{kl})e^{ik\theta}e^{-l\theta'}|d\theta'|. \tag{L8.18}$$

The identity $N \times N$ matrix is subtracted here since we want $\operatorname{GL}(N,\mathbb{C})$ to be embedded as a subgroup of $G^{-\infty}(\mathbb{S})$, which it is for each N.

Given some compact manifold Y and bundle E any smooth map $A : X \ni x \longrightarrow A(x) \in G^{-\infty}(Y;E)$ may be approximated by finite rank operators $A_{(N)}$ as in Lemma 17. Choosing a basis as in (L8.11) we may identify the coefficients $\delta_{kl} + a_{kl}$ with an element of $\operatorname{GL}(N,\mathbb{C})$ and then use (L8.18) to map it to $\tilde{A} : X \longrightarrow G^{-\infty}(\mathbb{S})$. It is important to see that this procedure is well defined at the level of homotopy classes. That is, that the element $[\tilde{A}] \in \pi_0(X; G^{-\infty}(\mathbb{S})]$ is independnet of choices. With the approximations fixed, the procedure only depends on the choice of basis. Since (see the remarks following Lemma 17) the basis is independent of the parameters in X the choice only corresponds to a choice of basis (possibly including redundant elements). Add redundant elements to the basis does not change the family $\tilde{A}$ and changing the basis results in its conjugation by a fixed element of $G^{-\infty}(\mathbb{S})$, replacing $\tilde{A}$ by $B^{-1}\tilde{A}(x)B$, $B \in G^{-\infty}(\mathbb{S})$. Since we know that $G^{-\infty}(Y, E)$ is connected, B may be smoothly connected to the identity, so the conjugated element gives the same homotopy class. All families sufficiently close to a given family are in the same homotopy class so in fact for large enough N the homotopy class of $\tilde{A}$ only depends on the homotopy class of A. Applying the construction to $X \times [0, 1]$ shows that homotopic families lift to the same homotopy class, so the map

$$\pi_0(X; G^{-\infty}(Y; E)) \longrightarrow \pi_0(X; G^{-\infty}(\mathbb{S})) \tag{L8.19}$$

is well-defined. An inverse to it can be constructed in essentially the same way, so this is a bijection independent of choices. □

The trace of matrices may be defined as the sum of the diagonal elements

$$\operatorname{tr}(a_{ij}) = \sum_i a_{ii}. \tag{L8.20}$$

It is invariant under change of basis since if $a' = b^{-1}ab$ then

$$\operatorname{tr}(a') = \sum_i (b^{-1}ab)_{ii} = \sum_{i,j,k} b^{-1}_{ij} a_{jk} b_{ki} = \sum_{i,j,k} a_{jk} b_{ki} b^{-1}_{ij} = \operatorname{tr}(a). \tag{L8.21}$$

Thus, $\operatorname{tr} : \hom(V) \longrightarrow \mathbb{C}$ is a well-defined linear map for any vector space V.

If we apply this to the finite rank operators in (L8.11) we find, using the assume orthonormality of the basis, that

$$\sum_i a_{ii} = \sum_i a_{ii} \int_Y \langle e_i(y), e_i(y)\rangle = \int_Y \operatorname{tr}(A(y, y))\nu(y) \tag{L8.22}$$

in terms of the trace on $\hom(E)$ of which $A(y, y) = A\big|_{\mathrm{Diag}}$ is a section. Thus for general smoothing operators we may simply define

$$\operatorname{Tr}(A) = \int_Y \operatorname{tr}(A\big|_{\mathrm{Diag}}). \tag{L8.23}$$

Proposition 16. *The trace functional is a well-defined continuous linear map*

$$\operatorname{Tr} : \Psi^{-\infty}(Y; E) \longrightarrow \mathbb{C} \tag{L8.24}$$

which satisifies

$$\operatorname{Tr}([A, B]) = 0 \ \forall\ A,\ B \in \Psi^{-\infty}(Y; E). \tag{L8.25}$$

Proof. If A, $B\Psi^{-\infty}(Y; E)$ then

$$\operatorname{Tr}(AB) = \operatorname{Tr}(C),\ C(x, z) = \int_Y A(x, y) \cdot B(y, z)$$

where the $\cdot$ refers to composition in the '$Hom(E)$' bundles. Thus in fact

$$\operatorname{Tr}(AB) = \int_Y \operatorname{tr}\left(A(x, y) \cdot B(y, x)\right) = \int_Y \operatorname{tr}\left(B(y, x) \cdot A(x, y)\right) = \operatorname{Tr}(BA) \tag{L8.26}$$

using the same identity for $\hom(E)$. □

Note that it follows from (L8.22) that under approximation by smoothing operators,

$$\operatorname{Tr}(A) = \lim_{N\to\infty} \operatorname{Tr}(A_N). \tag{L8.27}$$

Using this one can show that the determinant extends to smoothing operators in the following sense.

THEOREM 4. *(Fredholm) There is a unique map*

$$\Psi^{-\infty}(Y;E) \ni A \longrightarrow \det(\mathrm{Id} + A) \in \mathbb{C} \tag{L8.28}$$

which is entire and satisfies

$$\begin{gathered} \det\left(\mathrm{Id} + A)(\mathrm{Id} + B)\right) = \det(\mathrm{Id} + A)\det(\mathrm{Id} + B) \\ \partial_s \det(\mathrm{Id} + sA)\big|_{s=0} = \operatorname{Tr}(A) \\ A \in G^{-\infty}(Y;E) \Longleftrightarrow A \in \Psi^{-\infty}(Y;E),\ \det(\mathrm{Id} + A) \neq 0. \end{gathered} \tag{L8.29}$$

From this it follows that $\Psi^{-\infty}(Y;E) \subset G^{-\infty}(Y;E)$ is an open *dense* subset. The determinant can be defined on $G^{-\infty}(Y;E)$ by using the connectedness to choose a smooth curve $\gamma_A : [0,1] \longrightarrow G^{-\infty}(Y;E)$ from Id to a given point A and then setting

$$\det(\mathrm{Id} + A) = \exp(\int_0^1 \operatorname{Tr}\left((\mathrm{Id} + \gamma_A(t))^{-1}\frac{d\gamma_A(t)}{dt}\right) dt. \tag{L8.30}$$

Of course it needs to be shown that this is independent of the choice of γ_A, that it extends smoothly to all of $\Psi^{-\infty}(Y;E)$ (as zero on the complement of $G^{-\infty}(Y;E)$ and that it satisfies (L8.29).

8+. Addenda to Lecture 8

There are many other results on smoothing operators which reinforce the sense in which they are 'infinite rank matrices.' Think for instance of the spectrum.

PROPOSITION 17. *If* $A \in \Psi^{-\infty}(X;E)$ *then*

(8+.31)

$$\operatorname{spec}(A) = \{z \in \mathbb{C} \setminus \{0\}; (z\,\mathrm{Id} - A) : L^2(X;E) \longrightarrow L^2(X;E) \textit{ is not invertible}\}$$
$$\textit{is discrete except (possibly) at } 0 \in \mathbb{C}$$

and for each $0 \neq z \in \operatorname{spec}(A)$ *the associated generalized eigenspace*

(8+.32)

$$E(z) = \{u \in \mathcal{C}^\infty(X;E); (z\,\mathrm{Id} - A)^N u = 0 \textit{ for some } N \in \mathbb{N}\} \textit{ is finite dimensional.}$$

PROOF. If we could use the Fredholm determinant – although at this stage I have not finished the proof of its properties – then the discreteness would be clear once since certainly

$$\operatorname{spec}(A) \subset \{z \in \mathbb{C}; \det(\mathrm{Id} - \frac{A}{z}) = 0\} \tag{8+.33}$$

and the latter is the set of zeros of a holomorphic function on $\mathbb{C} \setminus \{0\}$. So, we would only need to show that the determinant is not identically zero.

In any case we can proceed more directly, without using the determinant but instead using 'analytic Fredholm theory'. First of all, if we give E an inner product and choose a density on Y then we know that $\|A/z\| = \|A\|/|z|$ so for $|z| > \|A\|$

it follows that $(\mathrm{Id} - \frac{A}{z})^{-1}$ exists. Thus $\operatorname{spec}(A) \subset \{z; |z| \leq \|A\|\}$, meaning that $(A - z\,\mathrm{Id})^{-1}$ is a holomorphic family of bounded operators, and hence map in $G^{-\infty}$ for $\|z\| > \|A\|$. $\square$

CHAPTER 9

Homotopy invariance of the index

Lecture 9: 18 October, 2005

Let me first improve a little on the parametrix constructed in the case of an elliptic pseudodifferential operator.

Proposition 18. *If $A \in \Psi^m(X; E, F)$ is elliptic then there exists $B \in \Psi^{-m}(X; F, E)$ such that*

$$BA = \mathrm{Id}_E - \pi, \ AB = \mathrm{Id}_F - \pi' \tag{L9.1}$$

where $\pi \in \Psi^{-\infty}(X; E)$ is projection onto the null space of A and π' is projection onto the null space of B which is a complement to the range of A. Choosing inner products and smooth densities one can further arrange that $\pi^* = \pi$ and $(\pi')^* = \pi'$.

Proof. We know already, as a consequence of the assumption of ellipticity, that there exists a parametrix $B_0 \in \Psi^{-m}(X; F, E)$ such that $B_0 A = \mathrm{Id} - R$, $AB_0 = \mathrm{Id} - R'$ with R and R' smoothing operators on the appropriate bundles, E and F. Since the finite rank smoothing operators are dense in the smoothing operators, we can find a finite rank operator R_F such that $\tilde{R} = R - R_F$ has L^2 norm less than one. Thus $(\mathrm{Id} - \tilde{R})^{-1}$ exists as a bounded operator on $L^2(X; E)$ and is of the form $\mathrm{Id} - \tilde{S}$ with $\tilde{S} \in \Psi^{-\infty}(X; E)$. Composing on the right with this operator,

$$B'A = (\mathrm{Id} - \tilde{S})(\mathrm{Id} - R) = (\mathrm{Id} - \tilde{R})^{-1}(\mathrm{Id} - \tilde{R} - R_F) = \mathrm{Id} - (\mathrm{Id} - \tilde{S})R_F = \mathrm{Id} - S_F, \ B' = [(\mathrm{Id} - \tilde{S})B_0], \tag{L9.2}$$

where $S_F \in \Psi^{-\infty}(X; E)$ also has finite rank. On the null space of S_F, which has finite codimension, A is injective, since B' inverts it. It also follows from (L9.2) that the null space of A is contained in the null space of $\mathrm{Id} - S_F$, which is finite dimensional. Thus we may choose a finite dimensional subspace $U \subset \mathcal{C}^\infty(X : E)$ which complements $\mathrm{null}(S_F) + (A)$ in $\mathcal{C}^\infty(X; E)$. Setting $D = \mathrm{null}(S_F) + U$ it follows that

$$\mathcal{C}^\infty(X; E) = D + \mathrm{null}(A) \tag{L9.3}$$

and that $A : D \longrightarrow A(D) = A(\mathrm{null}(S_F) + A(U) \subset \mathcal{C}^\infty(X; F)$ is injective. Let $V \subset \mathcal{C}^\infty(X; F)$ be a complement to $A(D)$; thus V is finite-diemnsional and in terms of this, and the decomposition (L9.3),

$$A = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \tag{L9.4}$$

Then we may simply define B to be the inverse of A on $A(D)$ and to be zero on V. Note that B differs from B', which inverts A on $A(\mathrm{null}(S_F))$ by a finite rank smoothing operator and $BA = \mathrm{Id} - \pi$ where π is the projection onto null(A) which vanishes on D and that $AB = \mathrm{Id} - \pi'$ where π' is the identity on V and vanishes on $A(D)$. Thus we have arrived at (L9.1).

If we give E and F Hermitian inner products and choose a positive smooth density on X then we may consider the effect of replacing D in the discussion above by null$(A)^\perp$. Certainly null$(A)^\perp \cap D$ has finite codimension in null$(A)^\perp$ and the same codimension in D. We may replace D by null$(A)^\perp$ in the discussion above and choose V to be $A(\mathrm{null}(A)^\perp)^\perp$. This ensures that $\pi^* = \pi$ and $(\pi')^* = \pi'$. □

Now observe that for a finite rank, smoothing, projection $\mathrm{Tr}(\pi)$ is equal to its rank. Thus, with B the 'generalized inverse' of Proposition 18 we find that

$$\mathrm{ind}(A) = \mathrm{Tr}(\pi) - \mathrm{Tr}(\pi') = \mathrm{Tr}(\mathrm{Id}_E - BA) - \mathrm{Tr}(\mathrm{Id}_F - AB). \tag{L9.5}$$

Proposition 19. *For any parametrix, $B \in \Psi^{-m}(X; F, E)$ of an elliptic element $A \in \Psi^m(X; E, F)$*

$$\mathrm{ind}(A) = \mathrm{Tr}(\mathrm{Id}_E - BA) - \mathrm{Tr}(\mathrm{Id}_F - AB). \tag{L9.6}$$

Proof. Denote the generalized inverse of Proposition 18, for which (L9.5) holds, as $\tilde{B}$. Then for a parameterix as in the statement, $C = B - \tilde{B} \in \Psi^{-\infty}(X; F, E)$ and $B_t = \tilde{B} + tC$ is a smooth family of parametrices for $t \in [0, 1]$ with $B_0 = \tilde{B}$ and $B_1 = B$. Thus it suffices to show that the right side in (L9.6) is constant in t. Since

$$\frac{d}{dt}\left(\mathrm{Tr}(\mathrm{Id}_E - B_t A) - \mathrm{Tr}(\mathrm{Id}_F - AB_t)\right) = \mathrm{Tr}(AC) - \mathrm{Tr}(CA) = 0 \tag{L9.7}$$

since

Lemma 18. *For any $C \in \Psi^{-\infty}(X; F, E)$ and $A \in \Psi^m(X; E, F)$*

$$\mathrm{Tr}(AC) = \mathrm{Tr}(CA). \tag{L9.8}$$

Proof. If $C_i \longrightarrow C$ in $\Psi^{-\infty}(X; F, E)$ then $AC_i \longleftrightarrow AC$ and $C_i A \longrightarrow CA$ in $\Psi^{-\infty}(X; F)$ and $\Psi^{-\infty}(X; E)$ respectively. Since we may choose the C_i to be of finite rank, it suffices to prove (L9.8) for finite rank smoothing operators. Since the identity is linear in C it is enough to consider the case where C has rank 1, $Cf = v(f)w$ where $v(f) = \int_X v \cdot f$ for some $v \in \mathcal{C}^\infty(X; F' \otimes \Omega_X)$ and $w \in \mathcal{C}^\infty(X; E)$ is fixed. Then AC and CA are also of rank 1 (or 0)

$$AC(f) = v(f)Aw,\ CA(g) = v(Ag)w \tag{L9.9}$$

and

$$\mathrm{Tr}(AC) = \int_X v \cdot Aw = \mathrm{Tr}(CA). \tag{L9.10}$$

□

□

From this we deduce that

Proposition 20. *The index is a (smooth) homotopy invariant of elliptic operators.*

PROOF. Consider a smooth family of elliptic operators $A_t \in \mathcal{C}^\infty([0,1];\Psi^m(X;E,F))$, (the argument works equally well if we just assume continuity in t). Then, as shown above, we may construct a smooth family of parametrices $B_t \in \mathcal{C}^\infty([0,1];\Psi^{-m}(X;F,E)$. Thus $B_tA_t - \mathrm{Id}_E$ and $A_tB_t - \mathrm{Id}_F$ are both smooth families of smoothing operators. It follows from (L9.6) that the index itself depends smoothly on $t \in [0,1]$. However it takes integer values and so is constant. □

Since the index is homotopy invariant we can change the lower order terms freely and leave the index unchanged. Thus $\mathrm{ind}(A)$ actually only depends on $\sigma(A) \in \mathcal{C}^\infty(S^*X;\hom(E,F))$ since it two operators A, A' have the same symbol then $(1-t)A + tA'$ has constant symbol and hence remains elliptic, so the $\mathrm{ind}(A) = \mathrm{ind}(A')$.

In fact even some of the information in the symbol is irrelevant for the index and to state the index theorem we eliminate this extraneous data by passing to a topological object.

PROPOSITION 21. *For any compact manifold Y (of positive dimension) and any bundle G over Y*

$$K^{-1}(X) = [X;G^{-\infty}(Y;G)] \tag{L9.11}$$

the set of smooth homotopy classes of smooth maps, is an Abelian group naturally independent of the choice of Y and G.

PROOF. We know that we may deform a smooth map $F : X \longleftrightarrow G^{-\infty}(Y;G)$ to be of the form $\mathrm{Id} - \tilde{F}$ with $\tilde{F}$ of uniformly finite rank, i.e. acting on a fixed finite-dimensional subspace of $\mathcal{C}^\infty(Y;G)$. Choosing a basis of this space, this reduces the map to $\tilde{F} : X \longrightarrow M(N,\mathbb{C})$, $\mathrm{Id} - \tilde{F} \in \mathcal{C}^\infty(X;\mathrm{GL}(N,\mathbb{C})$.

Consider especially the case $Y = \mathbb{S}$, $G = \mathbb{C}$. Then we may identify $M(N,\mathbb{C})$, the algebra of $N \times N$ matrices, with the operators on finite Fourier series

$$M(N,\mathbb{C}) \ni \{a_{jk}\}_1^N \longmapsto a(\theta,\theta') = \frac{1}{2\pi}\sum_{j,k=1}^N a_{jk}e^{ij\theta}e^{-ik\theta'}, \tag{L9.12}$$

$$a\left(\sum_{p=1}^N u_p e^{ip\theta}\right) = \sum_k \left(\sum_l a_{kl}u_l\right)e^{ik\theta}.$$

Combined with the discussion above, this allows us to deform F to the finite rank perturbation $\tilde{F}$ and then embed into $G^{-\infty}(\mathbb{S})$:

$$[X;G^{-\infty}(Y,G)] \longmapsto [X;G^{-\infty}(\mathbb{S})]. \tag{L9.13}$$

Note that the homotopy class of the image is independent of the basis chosen, since $\mathrm{GL}(N,\mathbb{C})$ is connected. Similarly, it does not depend on N, increasing it results in a homotopic map.

This construction is reversible, so proving the first part of the proposition.

So, this is just a consequence of the possibility of finite rank approximation. In standard topological approaches $K^{-1}(X)$ is defined simiply by the stabilization of maps in $\mathrm{GL}(N,\mathbb{C})$, we 'avoid' this by passing to $G^{-\infty}$. Note that $G^{-\infty}$ is like $\mathrm{GL}(N,\mathbb{C})$, as non-commutative as can be. Nevertheless $K^{-1}(X)$ is an Abelian group with the product induced by the product in $G^{-\infty}$. Namely, after retracting both $F_i \in \mathcal{C}^\infty(X;G^{-\infty})$ to $\tilde{F}_i \in \mathcal{C}^\infty(X;\mathrm{GL}(N,\mathbb{C}))$ we may embed $\mathrm{GL}(N,\mathbb{C})$ as the upper left corner in $\mathrm{GL}(2N,\mathbb{C})$ as 2×2 matrices with entries in $M(N,\mathbb{C})$ (stabilized by the identity in the lower right corner) and then we may rotate using

$$\text{(L9.14)}\quad \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\tilde F & 0\\ 0 & \mathrm{Id}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}\tilde F & 0\\ 0 & \mathrm{Id}\end{pmatrix} \text{ at } \theta=0,\ \begin{pmatrix}\mathrm{Id} & 0\\ 0 & \tilde F\end{pmatrix} \text{ at } \theta=\pi/2.$$

This allows us to deform $\tilde F_2$ until it commutes with $\tilde F_1$. Thus the product is commutative. □

Lemma 19. *For any compact manifold with corners we may embed $K^{-1}(X)\longmapsto K^{-1}(X\times\mathbb{S})$ as the subgroup of homotopy classes of $\mathbb{S}$-constant maps and then $K^{-1}(X\times\mathbb{S})$ splits as a direct sum of groups*

$$\text{(L9.15)}\qquad K^{-1}(X\times\mathbb{S}) = K^{-1}(X)\oplus K^{-2}(X)$$

where

$$\text{(L9.16)}\qquad K^{-2}(X) = [X\times\mathbb{S}, X\times\{1\}, G^{-\infty}(Y;G),\mathrm{Id}]$$

may be identified as the homotopy classes of pointed maps.

Note that the identification (L9.15) can be seen at the level of maps as

$$\text{(L9.17)}\qquad [f]\longmapsto [f_1]+[f_2],\ f_1(x,\theta)=f(x,1),\ f_2(x,\theta)=f(x,1)^{-1}f(x,\theta)$$

which is clearly an isomorphism at the level of maps.

Proof. The map induced by (L9.17) gives an isomorphism (L9.15) since under homotopy of f both f_1 and f_2 undergo homotopies within their respective classes of maps, constant and pointed. □

There are other useful representations of $K^{-2}(X)$. One that will occur later corresponds to maps which are not only 'pointed' in the sense that $f(x,1)=\mathrm{Id}$ but are flat at this submanifold, that is they differ from the constant, identity, map by a map into $\Psi^{-\infty}(Y;G)$ which vanishes to infinite order at $X\times\{1\}$. Namely, if $F: X\times\mathbb{S}\longrightarrow\Psi^{-\infty}(Y;G)$ defines $\mathrm{Id}+F: X\times\mathbb{S}\longrightarrow G^{-\infty}(Y;G)$ and $F(x,1)=0$ then if $\phi\in\mathcal{C}^\infty(\mathbb{S})$ has $0\le\phi\le 1$ and $\phi(\theta)=1$ in $|\theta-1|\le\epsilon$, $\phi(\theta)=0$ if $|\theta-1|>2\epsilon$ for $\epsilon>0$ small enough,

$$\text{(L9.18)}\qquad \mathrm{Id}+(1-\rho)F: X\times\mathbb{S}\longrightarrow G^{-\infty}(Y;G)$$

is homotopic to $\mathrm{Id}+F$.

Definition 5. The (smooth, flat, pointed) loop group, $G^{-\infty}_{(1)}(Y;G)$, of $G^{-\infty}(Y;G)$ is the space of Schwartz maps

(L9.19)
$$G^{-\infty}_{(1)}(Y;G) = \left\{a\in{}^{\mathrm{b}}S(\mathbb{R};\Psi^{-\infty}(Y;G)) \text{ s.t. } (\mathrm{Id}+a(t))\in G^{-\infty}(Y;G)\ \forall\ t\in\mathbb{R}\right\}.$$

Lemma 20. *For any compact manifold Y and complex vector bundle G over Y, $G^{-\infty}_{(1)}(Y;G)$ is a topological group with the topology inherited from ${}^{\mathrm{b}}S(\mathbb{R};\mathcal{C}^\infty(Y^2;\mathrm{Hom}(G)\otimes\Omega_Y)$.*

PROOF. Since we already know that $G^{-\infty}(Y;G)$ is a topological group, this is straightforward. In fact $G^{-\infty}_{(1)}(Y;G)$ is an open subset of ${}^{\mathrm{b}}S(\mathbb{R};\mathcal{C}^\infty(Y^2;\operatorname{Hom}(G)\otimes\Omega_Y)$, since invertibility in $G^{-\infty}(Y;G)$ is the same as invertibility on $L^2(Y;G)$. Composition and inversion are continuous, since the are continuous on $G^{-\infty}(Y;G)$. $\square$

The smooth map $\mathbb{R}\ni t\longrightarrow \exp\left(i\frac{t}{1+t^2)^{\frac12}}\pi\right)$ identifies the complement of 1 in $\mathbb{S}$ with $\mathbb{R}$. Using this and the deformation above, we may identify

$$K^{-2}(X)=[X;G^{-\infty}_{(1)}(Y;G)], \tag{L9.20}$$

since this is just a restatement of flatness at the submanifold $X\times\{1\}$.

Thus, essentially by definition, $G^{-\infty}(Y;G)$ and $G^{-\infty}_{(1)}(Y;G)$ are classifying spaces for odd and even K-theory, respectively. Later I will reinterpret $G^{-\infty}_{(1)}(Y;G)$ as the 'symbol group' for elliptic Toeplitz operators on the circle (stabilized by having values in the smoothing operators on Y). This will lead to an exact *classifying sequence* for K-theory of the form

$$G^{-\infty}(\mathbb{S}\times Y;G)\longrightarrow *\longrightarrow G^{-\infty}_{(1),0}(Y;G) \tag{L9.21}$$

where $*$ is a contractible group (a group of invertible Toeplitz perturbations of the identity) and the extra '0' on the loop group means the component of the identity, on which the index vanishes. This is closely related to Bott periodicity. The sequence in (L9.21) is essentially the symbol sequence for Toeplitz operators, as a subalgebra of the pseudodifferential operators on the circle.

9+. Addenda to Lecture 9

CHAPTER 10

Chern forms and the Fredholm determinant

Lecture 10: 20 October, 2005

I showed in the lecture before last that the topological group $G^{-\infty} = G^{-\infty}(Y;E)$ for any compact manifold of positive dimension, Y, and and bundle E, is an open subset of the (infinite dimensional) vector space $\Psi^{-\infty}(Y;E)$. I also, by fiat, declared it to be a classifying space for odd K-theory. This would not be sensible except of course that it is such a classifying space. If you consult a standard book on topology you will see that my claim amounts to the assertion

$$\pi_k(G^{-\infty}) = \begin{cases} 0 & k \text{ even} \\ \mathbb{Z} & k \text{ odd.} \end{cases} \tag{L10.1}$$

This result, which I will prove later, justifies my declaring that for any smooth compact manifold

$$K^{-1}(X) = [X, G^{-\infty}] \tag{L10.2}$$

is the abelain group of (smooth) homotopy classes of (smooth) maps.

Back to the statement that $G^{-\infty}$ is open in $\Psi^{-\infty}$, where I drop the qualifying space Y and bundle E since they are irrelevant. This means that I can happily treat $G^{-\infty}$ as a manifold. In fact the tangent space to $G^{-\infty}$ at a point $A = \operatorname{Id} + a$ (I will try to stick to this notation of A as the whole operator and a as the smoothing part) defined as usual as the equivalence classes of smooth curves $\operatorname{Id} + a_t$, $a_0 = a$, under tangency, is just $\Psi^{-\infty}$,

$$dA : T_A G^{-\infty} \ni [\operatorname{Id} + a_t] \longmapsto \frac{da_t}{dt}\Big|_{t=0} \in \Psi^{-\infty}. \tag{L10.3}$$

The notation 'dA' really comes from Lie group theory. In fact we may think of this map as defined on the whole of the tangent bundle to $G^{-\infty}$ and hence also

$$A^{-1}dA : TG^{-\infty} \longrightarrow \Psi^{-\infty}. \tag{L10.4}$$

This is the universal left-invariant 1-form on $G^{-\infty}$. Under left multiplication by $B \in G^{-\infty}$

$$L_B : G^{-\infty} \ni A \longmapsto BA \in G^{-\infty},\ L_B^*(A^{-1}dA) = A^{-1}B^{-1}BdA = A^{-1}dA. \tag{L10.5}$$

From this form we may construct the (Unnormalized) odd Chern forms

$$u_{2k-1} = \operatorname{Tr}\left((A^{-1}dA)^{2k-1}\right),\ k = 1, 2, \dots. \tag{L10.6}$$

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Here the product is interpreted as the product in $\Psi^{-\infty}$ followed by antisymmetrization, i.e. the wedge product. Written out more formally this is

(L10.7)
$$u_{2k-1}(b_1,\dots,b_{2k-1}) = \sum_{\sigma\in\Sigma_{2k-1}} \operatorname{sgn}(\sigma)\operatorname{Tr}\left(A^{-1}b_{\sigma(1)}A^{-1}b_{\sigma(2)}\cdots A^{-1}b_{\sigma(2k-1)}\right)$$

where the sum is over the permutation group on $\{1,\dots,2k-1\}$. The smoothness of composition, inversion and the trace shows this to be a smooth form on $G^{-\infty}$. Of course this can also be done with an even number of factors but then the trace idenity shows that

(L10.8)
$$\operatorname{Tr}\left((A^{-1}dA)^{2k}\right) = \operatorname{Tr}\left((A^{-1}dA)\wedge(A^{-1}dA)^{2k-1})\right) = -\operatorname{Tr}\left((A^{-1}dA)^{2k-1})\wedge(A^{-1}dA)\right) = 0$$

since an odd number of transpositions occur.

The forms u_{2k-1} are left invariant, from the left invariance of $A^{-1}dA$ but also right-invariant, since under $R_B : G^{-\infty} \ni A \longmapsto AB \in G^{-\infty}$, $R_B^*(A^{-1}dA) = B^{-1}(A^{-1}dA)B$.

Now the standard formula $dA^{-1} = -A^{-1}(dA)A^{-1}$ is justified here as usual by differentiation the equality of smooth functions $A^{-1}A = \mathrm{Id}$. Rewriting the definition

(L10.9)
$$u_{2k-1} = \operatorname{Tr}\left((A^{-1}dA)\wedge(A^{-1}dAA^{-1}dA)^{k-1}\right) = (-1)^{k-1}\operatorname{Tr}\left((A^{-1}dA)\wedge(dA^{-1}\wedge dA)^{k-1}\right).$$

Thus,

(L10.10)
$$du_{2k-1} = (-1)^{k-1}\operatorname{Tr}\left((dA^{-1}\wedge dA)\wedge(dA^{-1}\wedge dA)^{k-1}\right) = -\operatorname{Tr}\left((A^{-1}dA)2k\right) = 0$$

and it follows that these forms are closed.

By definition in (L10.2), and odd K-class on a compact manifold X is represented by a smooth map $f : X \longrightarrow G^{-\infty}$. We may use f to pull back the forms u_{2k-1} to smooth forms on X. Since $df^*u_{2k-1} = f^*(du_{2k-1})$ these forms are necessarily closed.

PROPOSITION 22. *The deRham model of cohomology leads, for each $k \in \mathbb{N}$, to a well-defined and additive map*

$$U_{2k-1} : K^{-1}(X) \longrightarrow H^{2k-1}(X;\mathbb{C}). \tag{L10.11}$$

PROOF. The deRham cohomology class of the closed from f^*u_{2k-1} is constant under homotopy from $f : X \longrightarrow G^{-\infty}$ to $f : X \longrightarrow G^{-\infty}$. Indeed, such an homotopy is a smooth map $F : [0,1]\times X \longrightarrow G^{-\infty}$ with $F(0,\cdot) = f$ and $F(1,\cdot) = f'$. If $f_t = F(t,\cdot)$ then $dF^*u_{2k-1} =$ becomes the condition

$$\frac{df_t^*u_{2k-1}}{dt} = d_X v_t \Longrightarrow f_1^*u_{2k-1} - f_0^*u_{2k-1} = dv,\ v = \int_0^1 v_t dt. \tag{L10.12}$$

Thus the map (L10.11) is well-defined. Its additivity follows from the discussion last time which shows that two maps $f_i : X \longrightarrow G^{-\infty}$, $i = 1,2$ may be deformed homotopically to be each finite rank perturbations of the identity and in commuting $N\times N$ blocks in $G^{-\infty}$. For such maps the product $(f_1f_2)^*u_{2k-1} = f_1^*u_{2k-1} + f_2^*u_{2k-1}$ showing the additivity. □

Taking the correct constants in a formal sum

$$\sum_k c_k U_{2k-1} : K^{-1}(X) \longrightarrow H^{\mathrm{odd}}(X;\mathbb{C}) \tag{L10.13}$$

will give the 'odd Chern character' discussed later. Its range then spans $H^{\mathrm{odd}}(X;\mathbb{C})$ and its null space is the finite subgroup of torsion elements of $K^{-1}(X)$, those elements satisfying $p[f] = 0$ (represented by the constant maps) for some integer p depending on f.

Even Chern forms can be defined in the same way as forms on the group $G^{-\infty}_{(1)}$. Let me use the version of this group defined last time, were we consider (for some underlying manifold Y and bundle E) the space of smooth Schwartz maps

$$\begin{gathered} G^{-\infty}_{(1)}(Y,E) = \\ \{a \in \mathcal{S}(\mathbb{R}_t;\Psi^{-\infty}(Y;E)) = \mathcal{S}(\mathbb{R}\times Y^2;\mathrm{Hom}(E)\otimes\pi_R^*\Omega_Y); \mathrm{Id} + a_t \in G^{-\infty}(Y;E)\ \forall\, t\in\mathbb{R}\}. \end{gathered} \tag{L10.14}$$

Then again $G^{-\infty}_{(1)}$ is an open subspace of $\mathcal{S}(\mathbb{R}\times Y^2;\mathrm{Hom}(E)\otimes\pi_R^*\Omega_Y)$ and we set

$$u_{2k} = \int_{\mathbb{R}} \mathrm{Tr}\left((A^{-1}dA)^{2k}(A^{-1}\frac{dA}{dt}\right) dt. \tag{L10.15}$$

Since we may regard $G^{-\infty}_{(1)}$ as a subset of $\mathcal{C}^\infty(\mathbb{R};G^{-\infty})$ this may also be considered as the integral over $\mathbb{R}$ of the pullback of u_{2k+1}. In any case this is again a closed form, this can also be seen directly, and for the same reasons as in the odd case defines an additive map

$$K^{-2}(X) \longrightarrow H^{2k}(X;\mathbb{C}) \text{ for each } k\in\mathbb{N}_0. \tag{L10.16}$$

An appropriate combination of these forms gives the Chern character (now the 'usual' Chern character) which has image spanning over $\mathbb{C}$.

The simplest, and most fundamental, cases of these forms are the first odd Chern form

$$u_1 = \mathrm{Tr}(A^{-1}dA) \text{ on } G^{-\infty} \tag{L10.17}$$

and its integral in the even case

$$u_0 = \int_{\mathbb{R}} \mathrm{Tr}(A^{-1}\frac{dA}{dt})dt,\ A\in G^{-\infty}_{(1)}. \tag{L10.18}$$

Proposition 23. *The form $u_1/2\pi i$ is integral, i.e. for any smooth map $\gamma : \mathbb{S} \longrightarrow G^{-\infty}$,*

$$\int_\gamma u_1 \in 2\pi i\mathbb{Z}. \tag{L10.19}$$

Proof. We may prove this by finite rank approximation. Since the integral is a cohomological pairing, we know it is homotopy invariant. Thus it suffices to replace γ by an approximating loop which is a uniformly finite rank perturbation of the identity. Thus we can assume that $\gamma : \mathbb{S} \longrightarrow \mathrm{GL}(N,\mathbb{C})$ for some embedding of $\mathrm{GL}(N,\mathbb{C})$ in $G^{-\infty}$. Since the trace restricts in any such embedding we are reduced to the matrix case. Then (L10.19) follows from the standard formula for matrices that

$$d\log\det(A) = \mathrm{Tr}(A^{-1}dA) \tag{L10.20}$$

with the integer in (L10.19) being the variation of the argument of the determinant along the curve. $\square$

Conversely we may use (L10.19) to conclude the the definition of the determinant on $G^{-\infty}$ which I proposed earlier,

$$\det(A) = \exp\left(\int_0^1 \operatorname{Tr}(A_t^{-1}\frac{dA_t}{dt}dt\right), \tag{L10.21}$$

where $t \to A_t$ is a curve in $G^{-\infty}$ from $A_0 = \mathrm{Id}$ to $A_1 = A$, does indeed lead to a well-defined function

$$\det : G^{-\infty} \longrightarrow \mathbb{C}. \tag{L10.22}$$

Indeed, such a curve exists, by the connectedness of $G^{-\infty}$ and two such curves differ by a closed curve (admittedly only piecewise smooth but that is not a serious issue).

Furthermore it follows directly from the definition that det is multiplicative. Namely for AB we may use the product A_tB_t of the curves connection the factors to the identity. Then

$$\begin{gathered}(A_tB+t)^{-1}d(A_tB_t) = B_t^{-1}dB_t + B_t^{-1}(A_t^{-1}dA_t)B_t \Longrightarrow \\ \operatorname{Tr}(A_tB+t)^{-1}d(A_tB_t) = \operatorname{Tr}(B_t^{-1}dB_t) + \operatorname{Tr}(A_t^{-1}dA_t)\end{gathered} \tag{L10.23}$$

from which it follows that $\det(AB) = \det(A)\det(B)$ as in the finite dimensional case. Of course this also follows by approximation, given the continuity of det which follows from the same formula.

In fact the Fredholm determinant in (L10.22) extends to a smooth map

$$\Psi^{-\infty}(Y;E) \ni A \longmapsto \det(\mathrm{Id}+A)\mathbb{C} \tag{L10.24}$$

which is non-vanishing precisely on $G^{-\infty}$.

++++ Add definition near zeros (this is a good exercise!)

Of course it follows from Propositon 23 that

$$\frac{u_0}{2\pi i} : G^{-\infty}_{(1)} \longrightarrow \mathbb{Z}. \tag{L10.25}$$

We shall see below that this can be interpreted as the simplest case of the index formula and that this map faithfully labels the components of $G^{-\infty}_{(1)}$.

Next I turn to the Toeplitz algebra. This algebra is the basic object which leads to a short exact sequence of groups

$$G^{-\infty} \longrightarrow G^0 \longrightarrow G^{-\infty}_{(1),-}[[\rho]] \sim G^{-\infty}_{(1),0}. \tag{L10.26}$$

Here I will not explain the whole notation for the moment, but the normal subgroup on the left is one of our 'smoothing groups', the central group is supposed to be contractible and the group on the right is homotopic to the identity component (this is the extra 0 subscript, meaning the index is zero in (L10.25)) of the loop group $G^{-\infty}_{(1)}$.

Now, this sequence is supposed to come, after some work, from the short exact sequence arises from the symbol of a pseudodifferential operator

$$\Psi^{-1}(X;\mathbb{C}^N) \longrightarrow \Psi^0(Z;\mathbb{C}^N) \longrightarrow \mathcal{C}^\infty(S^*Z;M(N,\mathbb{C})). \tag{L10.27}$$

For the moment I will ignore the difference between Ψ^{-1} and $\Psi^{-\infty}$, when taken into account this will lead to the 'formal power series' parameter ρ on the right in (L10.26) – there are other more serious problems to be dealt with! To get from

(L10.27) to (L10.26) we first want to consider the set of elliptic and invertible elements of $\Psi^0(Z;\mathbb{C}^N)$. If we consider the normal subgroup of invertible perturbations of the identity we arrive at

$$G^{-1}(Z;\mathbb{C}^N) \longrightarrow G^0(Z;\mathbb{C}^N) \longrightarrow \mathcal{C}^\infty(S^*Z;\mathrm{GL}(N,\mathbb{C})). \tag{L10.28}$$

Here
(L10.29)

$$\begin{gathered} G^{-1}(Z;\mathbb{C}^N) = \{\mathrm{Id}+A; A\in\Psi^{-1}(Z;\mathbb{C}^N),\ (\mathrm{Id}+A)^{-1} = \mathrm{Id}+B,\ B\in\Psi^{-1}(Z;\mathbb{C}^N)\},\\ G^0(Z;\mathbb{C}^N) = \{A; A\in\Psi^0(Z;\mathbb{C}^N) \text{ elliptic and } A^{-1}\in\Psi^0(Z;\mathbb{C}^N)\}\end{gathered}$$

where we will finally replace the former by $G^{-\infty}$.

Now, in general the second map in (L10.28) is not surjective, since that would mean that every elliptic element can be perturbed to be invertible and we known that this means precisely that the index vanishes. Thus the index is the (only) obstruction to the exactness of (L10.28). Of course we want to discuss this in treating the index formula but for the moment I am after something else.

Namely, I would like to choose Z so that the central group in (L10.28) is contractible and the image group is essentially a $G^{-\infty}_{(1)}$. To arrange the latter we need to do two things. First we need to choose the manifold Z so that

$$S^*Z = \mathbb{S} \tag{L10.30}$$

and then to 'stabilize' things so that $\mathbb{C}^n$ is replaced by an infinite dimensional space in such a way that $\mathrm{GL}(N,\mathbb{C})$ becomes one of our $G^{-\infty}$ groups. This second step may seem the most daunting but it is not and I will discuss how to do this next time. So, let us think about how to arrange (L10.30). Of course the small problem here is that this is impossible, there is no such manifold. Indeed, it would have to be 1-dimensional and compact, hence just the circle if we demand it to be connected. However

$$S^*\mathbb{S} = \mathbb{S}\sqcup\mathbb{S} \tag{L10.31}$$

is the disjoint union of two copies of the circle.

There are two ways to overcome this problem (well I know a third which you can find in [**4**] if you look hard enough). Stated vaguely these are

(A) Replace the circle by the line $\mathbb{R}$ so that '$S^*\mathbb{R}$' is interpreted as the boundary of the radial compactification of $T^*\mathbb{R} = \mathbb{R}^2$ as a vector space (not a vector bundle over $\mathbb{R}$). In this sense we would arrive at (L10.30). I was going to do this in these lectures, and I may still do so. It requires going back to the beginning of the lectures and discussing a variant of the conormal distributions for subspaces of a vector space. This leads to the 'isotropic calculus' on $\mathbb{R}$ (or in fact on $\mathbb{R}^n$) which can be used to construct the sequence I am after.

(B) Kill off half of (L10.31) and work on the remaing half. This is what I will do, namely discuss the Toeplitz algebra and its variants. I find this approach less geometrically transparent but it has plenty of history behind it.

For the circle we can decomposte smooth functions as a direct sum

$$\mathcal{C}^\infty(\mathbb{S}) = \mathcal{C}^\infty_-(\mathbb{S}) + \mathcal{C}^\infty_+(\mathbb{S}) \tag{L10.32}$$

where these are limited by the Fourier coefficients

$$a \in \mathcal{C}^\infty_+(\mathbb{S}) \Longleftrightarrow a = \sum_{k\ge 0} a_k e^{ik\theta},\ \sum_{k\ge 0} |a_k| k^j < \infty\ \forall\ j. \tag{L10.33}$$

The Szegő projection is the linear map which excises the negative Fourier modes

$$S : \mathcal{C}^\infty(\mathbb{S}) \longrightarrow \mathcal{C}^\infty_+(\mathbb{S}),\ S(a) = \sum_{k\ge 0} a_k e^{ik\theta} \text{ if } a = \sum_{k\in\mathbb{Z}} a_k e^{ik\theta}. \tag{L10.34}$$

Clearly this is a projection, $S^2 = S$ with null space $\mathcal{C}^\infty_-(\mathbb{S})$ and range $\mathcal{C}^\infty_+(\mathbb{S})$.

Note that one can always recover a compact manifold, Z, from $\mathcal{C}^\infty(Z)$ with its multiplicative structure. Namely the points of Z can be identified with the valuations on the ring, the linear maps $p : \mathcal{C}^\infty(Z) \longrightarrow \mathbb{C}$ such that $p(fg) = p(f)p(g)$. The space $\mathcal{C}^\infty_+(\mathbb{S})$ is a ring, as follows easily from the definition, but it is not the space of smooth functions on a manifold since the set of valuations actually recovers $\mathbb{S}$. Still, the idea is that we can think of this 'Hardy space' $\mathcal{C}^\infty_+(\mathbb{S})$ as the space of functions on 'half of $\mathbb{S}$.' Note that the Fourier parameter k is closely related to the dual variable on the fibres of the cotangent space $T^*\mathbb{S} = \mathbb{S}\times\mathbb{R}$ which indicates that S restricts to the 'positive half of the cotangent bundle.' More concretely

LEMMA 21. *The Szegő projector $S \in \Psi^0(\mathbb{S})$.*

Consider the Toeplitz algebra

$$\mathcal{T} = \{A \in \Psi^0(\mathbb{S}); A = SAS\}. \tag{L10.35}$$

It is indeed a subalgebra of the algebra of pseudodifferential operators since

$$A_1,\ A_2 \in \mathcal{T} \Longrightarrow S(A_1A_2) = S(SA_1S)(SA_2S)S = (SA_1S)(SA_2S) = A_1A_2. \tag{L10.36}$$

To arrive at the algebra I will proceed in three steps.

(1) We need to replace $\Psi^0(\mathbb{S})$ by the corresponding algebra of operators 'valued in the smoothing operators' on some manifold Y. This can be identified with $\mathcal{C}^\infty(Y^2;\Psi^0(\mathbb{S}))$.
(2) The symbol space of this algebra consists of smooth functions on $S^*\mathbb{S} = \mathbb{S}\sqcup\mathbb{S}$ with values in $\mathcal{C}^\infty(Y^2)$. We will consider the subalgebra of functions which have (full) symbols vanishing to infinite order at one point $p \in \mathbb{S}_+$.
(3) We then consider the corresponding Toeplitz algebra SAS with A of this form and define G^0 to be the group of operators of the form $\mathrm{Id} + SAS$ which are elliptic on $\mathbb{S}_+$ and invertible.
(4) This group G^0 is actually contractible.

10+. Addenda to Lecture 10

CHAPTER 11

Toeplitz operators

Lecture 11: 25 October, 2005

Today I want to start working towards the contractibility of the group which I will call $G^0_{\mathcal{T}}$ and which I have not yet defined. As mentioned last time it is made up out of the Toeplitz algebra, hence the subscript $\mathcal{T}$. For the moment I will prove some preliminary results about the Toeplitz algebra and make a start on the contractibility.

The most basic result I will not prove in full detail – it is a good excerise!

LEMMA 22. *The Szegő projector $S : \mathcal{C}^\infty(\mathbb{S}) \longrightarrow \mathcal{C}^\infty_+(\mathbb{S})$ given explicitly in terms of the Fourier series expansion by*

$$Su(\theta) = \sum_{k\ge 0} c_k e^{ik\theta} \text{ if } u = \sum_{k\in\mathbb{Z}} c_k e^{ik\theta} \tag{L11.1}$$

is an element of $\Psi^0(\mathbb{S})$; it is a self-adjoint projection ($S^2 = S^* = S$) and its amplitude, the local Fourier transform of its kernel with respect to a normal fibration, vanishes rapidly at infinity in one (the negative) direction.

HINT ONLY, CARRIED OUT BELOW. Think of $\mathbb{S}$ as the boundary of the unit disk $\mathbb{D}$ in the complex plane. The elements of $\mathcal{C}^\infty_+(\mathbb{S})$ are actually those which have extension to $\mathcal{C}^\infty(\mathbb{D})$ (smooth up to the boundary that means) which are holomorphic in the interior. Then S can be obtained as the boundary value of the map

$$\tilde{S}(f)(z) = \frac{1}{2\pi i}\int_{\mathbb{S}} \frac{f(e^{i\theta})}{z - e^{i\theta}} d\theta, \ |z| < 1, \tag{L11.2}$$

interpreted as a contour integral. Applied to $e^{ik\theta} = \tau^k$ $k \ge 0$ it gives z^k in the interior and applied to $e^{-ik\theta} = z^{-k}$, $k > 1$, it gives zero as can be checked using Cauchy's formula. From this the kernel of S can be recovered in terms of the limit as $|z| \uparrow 1$ of $(z - \tau)^{-1}$. Certainly then the kernel is smooth away from the diagonal and one can compute the Fourier transform transversal to the diagonal of the kernel (cut off near the diagonal) and show that it is an element of $\mathcal{C}^\infty(\overline{T^*\mathbb{S}})$. A little contour shoving will show that it vanishes rapidly in the negative direction and approaches 1 in the positive direction. □

Now, the Toeplitz algebra

$$\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N) = \left\{A \in \Psi^0(\mathbb{S};\mathbb{C}^N); A = SAS\right\}, \tag{L11.3}$$

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which is regarded as an algebra of operators on $\mathcal{C}^\infty_+(\mathbb{S};\mathbb{C}^N)$, is topologically simpler than the whole of the algebra, as we shall see. I will proceed to prove some results for this and the whole algebra, leaving the 'stabilization' to next time.

First is the simplest basic result leading to the definition of the (analytic) families index of a family of elliptic pseudodifferential operators. I will do this for the circle but the proof will later be shown to extend almost unchanged to a general manifold. The circle is much simpler than the general case, at the moment, because we have a sequence of smoothing projections

$$\pi_r : \mathcal{C}^\infty(\mathbb{S}) \ni u = \sum_k c_k e^{ik\theta} \longmapsto \pi_r u = \sum_{|k|\le r} c_k e^{ik\theta} \in \mathcal{C}^\infty(\mathbb{S}). \tag{L11.4}$$

We extend these to act componentwise on vector-valued functions. The crucial property that these projections have is that

$$A \in \Psi^{-\infty}(\mathbb{S};\mathbb{C}^N) \Longrightarrow A\pi_r \to A \text{ in } \Psi^{-\infty}(\mathbb{S};\mathbb{C}^N). \tag{L11.5}$$

Proposition 24. *Suppose that $A : X \longrightarrow \Psi^0(\mathbb{S};\mathbb{C}^N)$ is a smooth family of elliptic pseudodifferential operators, parameterized by a compact manifold X, then there exists a smooth family $B : X \longrightarrow \Psi^{-\infty}(\mathbb{S};\mathbb{C}^N)$ such that $(A(x)+B(x))^{-1} \in \Psi^0(\mathbb{S};\mathbb{C}^N)$ exists for each $x \in X$, if and only if for large enough r the*

$$F_r(x) = \operatorname{null}((\operatorname{Id} - \pi_r)A^*(x)) \in \mathcal{C}^\infty(\mathbb{S};\mathbb{C}^N) \tag{L11.6}$$

form a smooth vector bundle over X which is bundle-isomorphic to a trivial bundle of dimension $(2r+1)N$.

Proof. First we show that for r large enough, the $F_r(x)$ do indeed form a smooth vector bundle over X. Since $A(x)$ is an elliptic family, there is a smooth family $Q : X \longrightarrow \Psi^0(\mathbb{S};\mathbb{C}^N)$ of parametrices for the $A(x)$, so

$$Q(x)A(x) = \operatorname{Id} - R(x),\ R \in \mathcal{C}^\infty(X;\Psi^{-\infty}(\mathbb{S};\mathbb{C}^N)). \tag{L11.7}$$

Composing on the right with $\operatorname{Id} - \pi_r$ we get

$$Q(x)A(x)(\operatorname{Id} - \pi_r) = (\operatorname{Id} - R'_r(x))(\operatorname{Id} - \pi_r),\ R'_r(x) = R(x)(\operatorname{Id} - \pi_r), \tag{L11.8}$$

where the fact that $(\operatorname{Id} - \pi_r)(\operatorname{Id} - \pi_r) = (\operatorname{Id} - \pi_r)$ has been used. Since $R(x)\pi_r \to R(x)$ uniformly as a family of smoothing operators (i.e. in the $\mathcal{C}^\infty$ toplogy) we know that for large enough r the inverse

$$(\operatorname{Id} - R'_r(x))^{-1} = \operatorname{Id} - S_r(x),\ S_r \in \mathcal{C}^\infty(X;\Psi^{-\infty}(\mathbb{S};\mathbb{C}^N) \tag{L11.9}$$

exists. Composing on the left with the inverse and then with the operator $\operatorname{Id} - \pi_r$ and setting $Q'(x) = (\operatorname{Id} - \pi_r)(\operatorname{Id} - S_r(x))Q(x)$ we find that

$$Q'(x)A'_r(x) = \operatorname{Id} - \pi_r,\ A'_r(x) = A(x) - A(x)\pi_r. \tag{L11.10}$$

From this it follows that the null space of A'_r is precisely

$$\operatorname{null}(A(x)(\operatorname{Id} - \pi_r)) = \operatorname{null}(\operatorname{Id} - \pi_r) \tag{L11.11}$$

the span of the Fourier terms with wavenumber $|k| \le r$. This is a trivial vector bundle over X of dimension $(2r+1)N$. Certainly, the left in (L11.11) contains the right and if $(\operatorname{Id} - \pi_r)u = u$ then (L11.10) shows that $A'_r(x)u \ne 0$.

It also follows from (L11.10) that the $F_r(x)$ form a smooth vector bundle over X. To see this, recall that we know the (numerical) index of $A(x)$ to be a homotopy invariant. In particular it is a fixed integer for all x (well, in each component of X) for $A'_r(x)$. Since

$$\operatorname{ind}(A'_r(x)) = (2r+1)N - \dim(F_r(x)) \tag{L11.12}$$

is locally constant, the $F_r(x)$ have locally constant dimensions and this is enough to guarantee that they vary smoothly with $x \in X$. In fact $Q'(x)$ has range the same as $\operatorname{Id} - \pi_r$ and hence has null space which is of constant dimension and $A'_r(x)Q'(x) = \operatorname{Id} - G(x)$ has null space which is a smooth bundle isomorphic to F_r.

Thus we have succeeded in 'stabilizing' the null spaces to a bundle and the complements to the range to a bundle by modifying $A(x)$ by a smoothing operator to $A'_r(x) = A(x) - A(x)\pi_r$. The 'families index of A' is the formal difference of the null bundle and complement to the range

$$[(A'_r(x) \ominus F_r] \in K^0(X) \tag{L11.13}$$

where for the moment I have not defined either the left or right sides of this inclusion.

Now, we can prove one direction of the Proposition. If for large r there is an isomorphism to a trivial bundle over dimension $(2r+1)N$ then we can interpret this as an isomorphism of F_r to $\operatorname{null}(A'_r(x))$ and in this sense it is given by a family of smoothing operators, which we can denote by $B'_r(x)$. Clearly $A'_r(x) + B'_r(x)$ is then a family of invertible operators, differing from the original family by smoothing operators as anticipated.

Conversely, suppose that such an invertible perturbation exists so $A(x) + B(x)$ is invertible for all $x \in X$. Since $B(x)\pi_r \to B(x)$ uniformly in the $\mathcal{C}^\infty$ topology, it follows that $A(x) + B(x)\pi_r$ is invertible for r large enough. Since this is equal to

$$A'_r(x) + B_r(x),\ B_r(x) = (A(x) + B(x))\pi_r \tag{L11.14}$$

where $A'_r(x) = A(x)(\operatorname{Id} - \pi_r)$ as before, it follows that $B_r(x)$ is an isomorphism from the null space, which is a trivial bundle of dimension $(2r+1)N$ to a complement to the range of $A'_r(x)$, and hence to F_r. $\square$

In fact this result is not restricted to the circle but extends to an arbitrary compact manifold (and more generally for a fibration by compact manifolds) once we can find appropriate replacements for the projections π_r.

The proof passes over to the Toeplitz case essentially unchanged, if we interpret π_r as the projection onto the span of the Fourier terms with $0 \le k \le r$.

Corollary 3. *Suppose that $A : X \longrightarrow \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ is a smooth family of elliptic Toeplitz pseudodifferential operators, parameterized by a compact manifold X, then there exists a smooth family $B : X \longrightarrow \Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ such that $(A(x) + B(x))^{-1} \in \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ exists for each $x \in X$, if and only if for large enough r the $F_r(x)$ defined by* (L11.6) *form a smooth vector bundle over X which is bundle-isomorphic to a trivial bundle of dimension $(r+1)N$.*

Corollary 4 (of proof). *If $A_t : X \longrightarrow \Psi^0_{\mathcal{T}}(X; \mathbb{C}^N)$ (or $\Psi^0(X; \mathbb{C}^N)$) is a curve of elliptic families, i.e. is a smooth map from $[0,1]_t \times X$ elliptic at each point, and is invertible for $t = 0$ then there exists a smooth family $B_t : X \longrightarrow \Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ with $B_0 = 0$ such that $A_t(x) + B_t(x)$ is invertible for all $t \in [0,1]$ and all X.*

Proof. As in the proof above, consider $A_t(x)(\operatorname{Id} - \pi_r)$. For r large enough this has null space equal to that of $\operatorname{Id} - \pi_r$ for all $t \in [0,1]$ and all $x \in X$ and there is then a smooth bundle over $[0,1]\times X$ complementary to the range. Applying Proposition 24 the restriction of this bundle to $t=0$ must be isomorphic to the null bundle, which is trivial and of dimension $(r+1)d$. Since $[0,1]\times X$ is contractible to $\{0\}\times X$ it follows that the bundle is trivial over the whole of $[0,1]\times X$ so applying the Proposition in the other direction there is a smoothing perturbation making the operator invertible. Following the last part of the proof, this perturbation can be chosen to vanish at $t=0$. □

We will use this result later to lift homotopies of smooth elliptic families to homotopies of invertible families.

Now, let me turn to the first substantial homotopy computation of the two needed to construct the classifying sequence for K-theory. In this I will use two 'shifts' in the Toeplitz algebra. Namely

$$\begin{gathered}U:\mathcal{C}^\infty_+(\mathbb{S})\longrightarrow\mathcal{C}^\infty_+(\mathbb{S}),\ Uu=\sum_{k\ge0}c_ke^{i(k+1)\theta}\text{ if }u=\sum_{k\ge0}c_ke^{ik\theta}\text{ and}\\ L:\mathcal{C}^\infty_+(\mathbb{S})\longrightarrow\mathcal{C}^\infty_+(\mathbb{S}),\ Lu=\sum_{k\ge1}c_ke^{i(k-1)\theta}\text{ if }u=\sum_{k\ge0}c_ke^{ik\theta}.\end{gathered}\tag{L11.15}$$

Both are elliptic elements of $\Psi^0_{\mathcal{T}}(\mathbb{S})$ since they can be written

$$U=Se^{i\theta}S,\ L=Se^{-i\theta}S\tag{L11.16}$$

and they are essential inverses of each other

$$LU=\mathrm{Id},\ UL=\mathrm{Id}-\pi_0.\tag{L11.17}$$

In particular L has null space exactly the constants and the constants form a complement to the range of U. Thus

$$\mathrm{ind}(U)=-1,\ \mathrm{ind}(L)=1.\tag{L11.18}$$

Set

$$\tilde G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)=\left\{A\in\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N);A\text{ is elliptic and }A^{-1}\in\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)\right\}.\tag{L11.19}$$

The tilde here is to distinguish it from a smaller group I will discuss later. We can inject $\mathrm{GL}(N,\mathbb{C})\longrightarrow\tilde G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ as the operators (of the form $\mathrm{Id}+a$, $a\in\Psi^{-\infty}_{\mathcal{T}}(\mathbb{S},\mathbb{C}^N)$)

$$\mathrm{GL}(N,\mathbb{C})\ni g\longrightarrow\mathrm{Id}-\pi_0+\pi_0g\pi_0.\tag{L11.20}$$

Let us also consider, in the standard way of 'stabilization' that

$$\mathrm{GL}(N,\mathbb{C})\subset\mathrm{GL}(2N,\mathbb{C})$$

as the upper left corner in a 2×2 block decomposition

$$\mathrm{GL}(N,\mathbb{C})\ni g\longmapsto\begin{pmatrix}g&0\\0&\mathrm{Id}_N\end{pmatrix}\in\mathrm{GL}(2N,\mathbb{C}).\tag{L11.21}$$

Proposition 25. *If* $\mathrm{GL}(N,\mathbb{C})\longrightarrow\tilde G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^{2N})$ *is embedded as a subgroup by combining* (L11.21) *and* (L11.20) *(for* $2N$ *in place of* N*) then the image subgroup is deformable to the identity.*

Proof. Dividing $\mathbb{C}^{2N}$ into $\mathbb{C}^N \oplus \mathbb{C}^N$ we can picture the operators as block 2×2 matrices with entries which are $N \times N$ matrices of Toeplitz operators. The subgroup $\mathrm{GL}(N,\mathbb{C})$ can then be identified with

$$M_0 = \begin{pmatrix} \operatorname{Id} - \pi_0 + \pi_0 g \pi_0 & 0 \\ 0 & \operatorname{Id} \end{pmatrix}. \tag{L11.22}$$

This is the intial value of a curve of operators in $\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^{2N})$

$$M_\theta = \begin{pmatrix} \cos\theta(\operatorname{Id} - \pi_0) + \pi_0 g \pi_0 & \sin\theta g U \\ -\sin\theta g^{-1} L & \cos\theta \operatorname{Id} \end{pmatrix},\ 0 \le \theta \le \frac{\pi}{2}. \tag{L11.23}$$

This is an elliptic family of Toeplitz operators (so the Id's can be read as S's) since it symbol is the invertible matrix

$$\begin{pmatrix} \cos\theta & \sin\theta g e^{i\theta} \\ -\sin\theta g^{-1} e^{-i\theta} & \cos\theta \end{pmatrix} \tag{L11.24}$$

(which has determinant 1). Now, M_θ has the property that for all $k > 0$,

$$M_\theta \begin{pmatrix} u_k e^{ik\theta} \\ v_k e^{i(k-1)\theta} \end{pmatrix} = \begin{pmatrix} f_k e^{ik\theta} \\ g_k e^{i(k-1)\theta} \end{pmatrix},\ \begin{pmatrix} f_k \\ g_k \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta g \\ -\sin\theta g^{-1} & \cos\theta \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} \tag{L11.25}$$

and maps $(u_0, 0)$ to $(gu_0, 0)$. From the invertibility of these matrices it follows that M_θ is a curve in $\tilde{G}^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^{2N})$.

At the end of this first deformation we have arrived at the initial point of the curve

$$M'_\theta = \begin{pmatrix} \cos\theta g(\operatorname{Id} - \pi_0) + \pi_0 g \pi_0 & \sin\theta U \\ -\sin\theta L & \cos\theta g^{-1} \operatorname{Id} \end{pmatrix}. \tag{L11.26}$$

(where now θ runs from $\pi/2$ back to 0). This has essentially the same properties as M_θ. Namely it is elliptic since the symbol matrix is

$$\begin{pmatrix} \cos\theta g & \sin\theta e^{i\theta} \\ -\sin\theta e^{-i\theta} & \cos\theta g^{-1} \end{pmatrix} \tag{L11.27}$$

which again has determinant 1 and satisfies the analogue of (L11.25) with the matrix replaced by

$$\begin{pmatrix} \cos\theta g & \sin\theta \\ -\sin\theta & \cos\theta g^{-1} \end{pmatrix} \tag{L11.28}$$

which is again invertible (and the same on the zero mode).

At the end of this second homotopy (all uniform on $\mathrm{GL}(N,\mathbb{C})$ of course) we have arrived at the 'Toeplitz operator' which is purely a matrix

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}. \tag{L11.29}$$

It is straightforward to see that this is homotopic to the identity in $\mathrm{GL}(2N,\mathbb{C})$ using a similar rotation but purely in matrices, namely

$$\begin{pmatrix} \cos\theta g & \sin\theta \\ -\sin\theta & \cos\theta g^{-1} \end{pmatrix},\ \theta \in [0, \pi/2] \tag{L11.30}$$

followed by

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},\ \theta \in [\pi/2, 0] \tag{L11.31}$$

finishing at the identity. $\square$

This result will allow us to show that the '$G^{-\infty}_{\mathcal{T}}$' part of the final group can be (weakly) deformed away. Next time I will start with Atiyah's proof of Bott periodicity modified to show how the invertible elliptic operators can be deformed into this smoothing subgroup. The combination of the two discussions will give the weak contractibility we are after.

11+. Addenda to Lecture 11

11+.1. Proof of (L11.5)**.** This is really just the convergence of Fourier series. Thus, for $f \in \mathcal{C}^\infty(\mathbb{S})$ the truncated Fourier series $\pi_r f \longrightarrow f$ in $\mathcal{C}^\infty(\mathbb{S})$ as $r \to \infty$. An element $A \in \Psi^{-\infty}(\mathbb{S})$ is represented by a smooth kernel, $A \in \mathcal{C}^\infty(\mathbb{S} \times \mathbb{S})$,

$$Af(\theta) = \int_{\mathbb{S}} A(\theta, \theta') f(\theta') d\theta'. \tag{11+.32}$$

Since π_r is self-adjoint and real,

$$A(\pi_r f)(\theta) = \int_{\mathbb{S}} A(\theta, \theta')(\pi_r f)(\theta') d\theta' = \int_{\mathbb{S}} A_r(\theta, \theta') f(\theta') d\theta' \tag{11+.33}$$

where A_r is obtained from A by the action of π_r in the second variable. For a smooth family of smooth functions, the Fourier series converges uniformly with all its derivatives. Thus

$$A\pi_r \longrightarrow A \in \mathcal{C}^\infty(\mathbb{S} \times \mathbb{S}) \tag{11+.34}$$

which is the topology on $\Psi^{-\infty}(\mathbb{S})$, as claimed in (L11.5).

11+.2. Proof of Lemma 22. Following the 'hint' of the lecture, we first observe that restriction to the boundary gives an isomorphism

$$\mathcal{C}^\infty_{\mathrm{hol}}(\mathbb{D}) = \{u \in \mathcal{C}^\infty(\mathbb{D}); (\partial_x + i\partial_y)u = 0\} \longrightarrow \mathcal{C}^\infty_+(\mathbb{S}). \tag{11+.35}$$

Surjectivity follows easily, as indicated in the lecture, since if $a \in \mathcal{C}^\infty_+(\mathbb{S})$ then its Fourier series converges uniformly with all derivatives on the circle and since $e^{ik\theta} = z^k$ restricted to the circle and $|z^k| \le 1$ in the disc

$$u_a(z) = \sum_{k \ge 0} a_k z^k$$

converges uniformly on $\mathbb{D}$, with all derivatives, to a holomomorphic function (since the terms are holomorphic) restricting to a on the boundary. Moreover, all elements of $\mathcal{C}^\infty_{\mathrm{hol}}(\mathbb{D})$ arise this way, since the Fourier coefficients of the boundary value of $u \in \mathcal{C}^\infty_{\mathrm{hol}}(\mathbb{D})$ can be written, for $k < 0$, as

$$a_k = \frac{1}{2\pi} \int_{\mathbb{S}} e^{-ik\theta} u(z) d\theta = \lim_{r \uparrow 1} \int_{|z|=r} z^{-k} u(z) \frac{dz}{z} = 0 \tag{11+.36}$$

by Cauchy's integral formula. Thus the boundary value is in $\mathcal{C}^\infty_+(\mathbb{S})$ and if it vanishes then extending u as 0 outside the disc gives a continuous function on $\mathbb{R}^2$ which satisfies $(\partial_x + i\partial_y)\tilde{u} = 0$, in the sense of distributions, everywhere. Thus (by elliptic regularity) it is in fact an entire function of compact support, which must vanish identically. Thus the map is also injective.

Consider the integral in (L11.2). For $|z| < 1$ this certainly converges for any $f \in \mathcal{C}^\infty(\mathbb{S})$ (since $z - e^{i\theta} \neq 0$ with all its derivatives and by differentiation under the integral sign it is holomorphic in $|z| < 1$. Using the rapid convergence of the Fourier series we may interchange series and integral and conclude that for any $f \in \mathcal{C}^\infty(\mathbb{S})$,

(11+.37)
$$\tilde{S}(f)(z) = \sum_{k \in \mathbb{Z}} a_k u_k(z),\ u_k(z) = \frac{1}{2\pi i} \int_{\mathbb{S}} \frac{e^{ik\theta}}{e^{i\theta} - z} d\theta = \frac{1}{2\pi i} \int_{|\tau|=1} \frac{\tau^k}{\tau - z} \frac{d\tau}{i\tau}.$$

For $k < 0$ there are no poles outside the unit disk, including at $\tau = \infty$, so by Cauchy's integral formula the $u_k(z) \equiv 0$, $k < 0$. For $z \geq 0$ there is a pole at ∞ and applying the residue formula, it evaluates to z^k. Thus in fact $\tilde{S}(f)(z)$ is the holomorphic extension, into $|z| < 1$, of $S(f)$. It is therefore smooth up to the boundary, by the discussion above, so indeed

$$S(f)(e^{i\theta}) = \lim_{r \uparrow 1} \tilde{S}(f)(z) \tag{11+.38}$$

as claimed.

CHAPTER 12

Linearization of symbols

Lecture 12: 27 October, 2005

Today I will go through the second homotopy that I will use next time to construct the classifying sequence for K-theory. This construction is due to Atiyah ([**1**]). The question is the extent to which one can simplify, or bring to normal form, a family of loops in $\mathrm{GL}(N,\mathbb{C})$. Thus, for a given smooth compact manifold X suppose we have a smooth map $a : X \longrightarrow \mathcal{C}^\infty(\mathbb{S};\mathrm{GL}(N,\mathbb{C}))$ which is the same thing as an element of $\mathcal{C}^\infty(X\times\mathbb{S};\mathrm{GL}(N,\mathbb{C}))$. I will assume that it satisfies the normalization condition

$$a\big|_{1\in\mathbb{S}} \equiv \mathrm{Id}\,. \tag{L12.1}$$

We are allowed to make deformations, i.e. homotopies, of the family and we are also permitted to *stabilize* the family by embedding $\mathrm{GL}(N,\mathbb{C}) \hookrightarrow \mathrm{GL}(M,\mathbb{C})$ for any $M \ge N$, as the subgroup

$$\mathrm{GL}(N,\mathbb{C}) \ni a \longrightarrow \begin{pmatrix} a & 0 \\ 0 & \mathrm{Id}_{M-N}\end{pmatrix} \in \mathrm{GL}(M,\mathbb{C}). \tag{L12.2}$$

The result shown by Atiyah is that by such stabilization and deformation (always through invertibles of course) we may arrive at a family

$$\tilde{a}(x) = \pi_-(x)e^{-i\theta} + \pi_0(x) + \pi_+(x)e^{i\theta} \tag{L12.3}$$

where π_-, π_0 π_+ are three smooth maps into the projections on $\mathbb{C}^M$ which mutually commute for each x and sum to the identity

$$\pi_-(x)+\pi_0(x)+\pi_+(x) = \mathrm{Id}\ \ \forall\ x\in X. \tag{L12.4}$$

Notice that this is just the normalization condition (L12.1) for a family of the form (L12.3).

To construct such a (stable) homotopy, we first consider the Fourier expansion of a

$$a(x,\theta) = \sum_{j=-\infty}^{\infty} a_j(x)e^{ij\theta}. \tag{L12.5}$$

The coefficients here are smooth functions valued in $N\times N$ matrices, namely

$$a_j(x) = \frac{1}{2\pi}\int_{\mathbb{S}} e^{-ij\theta}a(x,\theta)d\theta \tag{L12.6}$$

which vanish rapidly with j, so for any differential operator P on X and any $q \in \mathbb{N}$.

$$\sup_X |Pa_j(x)| \le C_q(1+|j|)^{-q}. \tag{L12.7}$$

Thus the series (L12.5) converges rapidly in any C^p norm and there exists q such that with
(L12.8)

$$a_{(q)}(x,\theta) = \sum_{|j|\le q} a_j(x)e^{ij\theta},\ a_t = (1-t)a + ta_{(q)} : [0,1]\times X\times \mathbb{S} \longrightarrow \mathrm{GL}(N,\mathbb{C}).$$

We can also maintain the normalization condition under the homotopy since $c_t(x) = a_t(x,1) : [0,1]\times X \longrightarrow \mathrm{GL}(N,\mathbb{C})$ is the identity at $t=0$ so $c_t^{-1}(x)a_t(x,\theta)$ is a new homotopy to a trigonometric polynomial satisfying the normalization condition. Thus a and $a_{(q)}$ are homotopic, so we can consider instead $a_{(q)}$ and just suppose that a is a trigonometric polynomial of some degree satisfying the normalization condition.

Thus $a(x,\theta) = b(x,z)\big|_{z=e^{-i\theta}}$ where

$$b(x,z) = z^{-q}b'(x,z),\ b' : X\times\mathbb{C} \longrightarrow M(N,\mathbb{C}) \text{ a polynomial of degree } 2q \tag{L12.9}$$

and of course b' is invertible on the circle and $b'(x,1) = \mathrm{Id}$. Now we will use a simple form of stabilization to separate off the z^{-q} factor. Add another $N\times N$ identity block and consider the 2×2 block rotation

$$R_\tau = \begin{pmatrix} \cos(\tau)\,\mathrm{Id}_N & \sin(\tau)\,\mathrm{Id}_N \\ -\sin(\tau)\,\mathrm{Id}_N & \cos(\tau)\,\mathrm{Id}_N \end{pmatrix}. \tag{L12.10}$$

Replacing a by

$$R_{-\tau}\begin{pmatrix} z^{-q}\,\mathrm{Id}_N & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix} R_\tau \begin{pmatrix} b'(x,z) & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix}$$

gives a homotopy for $\tau \in [0,\pi/2]$ which rotates the z^{-q} into the second block, finishing at

$$\begin{pmatrix} b'(x,z) & 0 \\ 0 & z^{-q}\,\mathrm{Id}_N \end{pmatrix}. \tag{L12.11}$$

We then proceed to discuss these two blocks separately, of course the first is a good deal more complicated than the second. We will stabilize the first block by another p blocks each $N\times N$, where $p = 2q$ is the degree of the polynomial b' (notice that a polynomial of degree p has $p+1$ terms.) Thus we replace the first block by

$$\begin{pmatrix} b'(x,z) & 0 & 0 & \dots & 0 \\ 0 & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{Id}_N \end{pmatrix} \tag{L12.12}$$

and what is crucial is that this is invertible on $X\times\mathbb{S}$ where the circle is $|z| = 1$ now. Since this matrix is block diagonal, we can keep invertibity while adding absolutely any terms above the diagonal. What I want to do is to choose polynomials valued in $N \times N$ matrices (no invertibility condition of course) and deform (L12.12) to

$$\begin{pmatrix} b'(x,z) & c_1(x,z) & c_2(x,z) & \dots & c_p(x,z) \\ 0 & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{Id}_N \end{pmatrix}. \tag{L12.13}$$

To do this, just put a $t \in [0,1]$ in front of the c_j's. We can imagine (L12.13) as postmuliplied by the identity, then deform the identity to

$$\begin{pmatrix} \mathrm{Id}_N & 0 & 0 & \dots & 0 \\ -z & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & -z & \mathrm{Id}_N & 0 \\ 0 & 0 & \dots & -z & \mathrm{Id}_N \end{pmatrix} \tag{L12.14}$$

which has $-z$ all along the 'subdiagonal'. This is a lower-triangular perturbation so is still invertible and homotopic to the identity. Thus, without having chosen the c_j, we have deformed the matrix to the product

(L12.15)

$$\begin{pmatrix} b'(x,z) & c_1(x,z) & c_2(x,z) & \dots & c_p(x,z) \\ 0 & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{Id}_N \end{pmatrix} \begin{pmatrix} \mathrm{Id}_N & 0 & 0 & \dots & 0 \\ -z & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & -z & \mathrm{Id}_N & 0 \\ 0 & 0 & \dots & -z & \mathrm{Id}_N \end{pmatrix}$$

$$= \begin{pmatrix} g_0(x) & g_1(x) & g_2(x) & \dots & g_p(x) \\ -z & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & -z & \mathrm{Id}_N \end{pmatrix} = p_+(x,z).$$

Here

$$g_0 = b' - zc_1,\ g_1 = c_1 - zc_2, \dots, g_{p-1} = c_{p-1} - zc_p,\ g_p = c_p. \tag{L12.16}$$

Observe that we can choose the c_j's successively to be polynomials of degree $p-j$ so that each of the g_j's is a constant matrix, i.e. does not depend on z at all. In fact the g_j's are just the coefficient matrices of b'.

At the end of this deformation the (enlarged) block corresponding to $b'(x,z)$ has been reduced to a linear, in z, matrix. We can proceed in the same way with the other, simpler block with entry $z^{-p}\,\mathrm{Id}_N$, but replacing z by $1/z$. This shows that there is a homotopy, after appropriate stabilization, to a matrix of the form

$$a(x,\theta) = a_-(x)e^{-i\theta} + a_0'(x) + a_0''(x) + a_+(x)e^{i\theta} \tag{L12.17}$$

through invertible matrices. Here the a_- and a_0' matrices form one block and the $a_0'' + a_+$ form another. As before we can enforce the normalization condition, that at the point $1 \in \mathbb{S}$ the matrix is the identity, simply by multiplying by the inverse of this matrix. Thus we can assume that *both* the blocks in the discussion above satify the normalization condition. Thus

$$a_-(x) + a_0'(x) = \mathrm{Id}_{p_-},\ a_0''(x) + a_+(x) = \mathrm{Id}_{p_+}. \tag{L12.18}$$

It follows that $a_-(x)$ and $a_0'(x)$ and $a_0''(x)$ and $a_+(x)$ commute for each x and these two block commute with each other. Thus, in the combined form (L12.17) it follows that $a_-(x)$, $a_0(x)$ and $a_+(x)$ are commuting matrices, for each x, summing to the identity.

Consider the matrices obtained by integration round the circle

$$\pi = \frac{1}{2\pi i}\int_{|z|=1} p_+(x,z)^{-1}\frac{dp_+(x,z)}{dz}\frac{dz}{z}. \tag{L12.19}$$

Since p_+ is invertible on the circle, this is a smooth matrix in x. Suppose for a moment that $a_+(x)$ is invertible. Then

$$p_+(x,z)^{-1} = (a_+(x))^{-1}(a_0''(x)+z)^{-1} \tag{L12.20}$$

and the contour integral (L12.19) may be evaluated by residues. In fact $\pi(x)$ is then the projection onto the span of those eigenvectors of $-a_+(x)^{-1}a_0''(x)$ with eigenvalues $|z|<1$ (and vanishing on the span of the eigenvectors with eigenvalues in $|z|>1$). We may always perturb $a_+(x)$ to $a_+(x)+s\operatorname{Id}$ for small s to make it invertible. So in the general case, without assuming that $a_+(x)$ is invertible, it follows that $\pi(x)$ is a projection (as the limit of a sequence of projections) and that it commutes with both $a_0''(x)$ and $a_+(x)$ (since these commute with the argument of the integral).

Decomposing $p_+(x,z)$ with respect to $\pi(x)$, with which it commutes, the term $p_+(x,z)\pi = a_1(x)+b_1(x)z$ has no zeros outside the unit circle so the matrix $(1-t)a_1(x)+b_1(x)z$ is invertible on the unit circle for all $t\in[0,1]$. Similarly $p_+(x,z)(\operatorname{Id}-\pi) = a_2(x)+b_2(x)z$ has no singular values inside the unit circle so $a_2(x)+(1-t)b_2(x)z$ remains invertible on the unit circle for all $t\in[0,1]$. Combining these two homotopies and premultiplying by the value at $z=1$ gives a homotopy of $p_+(x,z)$ to $\pi(x)+(\operatorname{Id}-\pi(x))z$ – indeed the end point is $a_2'(x)+b_1'(x)z$ where $a_2'(x)$ acts on the range of π and $b_1'(x)$ on the range of $\operatorname{Id}-\pi$ and the normalization condition holds.

Carrying out a similar analysis for $p_-(x,z)$ we obtain a homotopy, always keeping invertibility for $|z|=1$ from the initial map $a : X\times\mathbb{S}\longrightarrow \mathrm{GL}(N,\mathbb{C})$, after stabilization, to a family of the form (L12.3).

We will apply this homotopy to the symbols of a family of elliptic Toeplitz operators, $P : X\longrightarrow \Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$, allowing stabilization.

Proposition 26. *If P is a smooth family of invertible elliptic Toeplitz operators parameterized by the compact manifold X with symbols satisfying the normalization condition*

$$\sigma_0(P)\big|_{1\in\mathbb{S}} = \operatorname{Id} \tag{L12.21}$$

then, after stabilization, it may be smoothly deformed through invertible Toeplitz operators to $\tilde P : X\longrightarrow G^{-\infty}_{\mathcal{T}}(\mathbb{S};\mathbb{C}^M)$.

Proof. We can certainly apply the previous result to the symbol family, deforming it to the form (L12.3). We may then choose an elliptic family with these symbols which reduces to P at $t=0$. As shown above such a homotopy of families of elliptic operators which is invertible at $t=0$ may perturbed by a smoothing family, which vanishes at the intial point, to make the whole family invertible. Thus, we may suppose that we have an operator with symbol of the form (L12.4) and which is invertible. We can easily find an explicit family of operators with this symbol, namely

$$Q(x) = \pi_-(x)L + \pi_0(x) + \pi_+(x)U \tag{L12.22}$$

where L and U are the shift (down and up respectively) operators. Thus we can in fact suppose that $Q(x)$ is invertible after the addition of a smoothing family.

On the other hand we may easily compute the (stabilized) null bundles of $Q(x)$ and its adjoint. Namely (for any $k \ge 0$ it is not really necessary to stabilize here)

$$\begin{aligned} \operatorname{null}(Q(x)(\operatorname{Id} - \pi_k)) &= \operatorname{sp}\{e^{ij\theta}\mathbb{C}^M,\ 0 \le j \le k\} \\ \operatorname{null}((\operatorname{Id} - \pi_k)Q^*(x)) &= \operatorname{sp}\{e^{ij\theta}\mathbb{C}^M,\ 0 \le j \le k-1, \\ &\qquad e^{ik\theta}(\pi_0 + \pi_+(x))\mathbb{C}^M,\ e^{i(k+1)\theta}\pi_+(x)\mathbb{C}^M\}. \end{aligned} \tag{L12.23}$$

Now we know that the assumption that $Q(x)$ has an invertible perturbation means that these two bundles must be isomorphic for large k. The first of these is just the trivial bundle of rank $(k+1)M$ whilst the second is the trivial bundle of rank kM plus the range of $\pi_0 + \pi_+$ plus another copy of the range of π_+. Since π_- complements $\pi_0 + \pi_+$ to a trivial bundle of range M, adding the range of π_- to both sides (with the identity isomorphism) this means there must be a vector bundle isomorphism

$$\mathbb{C}^L + \operatorname{Ran}(\pi_-) \simeq \mathbb{C}^L + \operatorname{Ran}(\pi_+). \tag{L12.24}$$

Now, observe that the $2L \times 2L$ block matrix

$$\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tag{L12.25}$$

is homotopic to the identity using a simple rotation

$$\begin{pmatrix} \cos(\tau)e^{-i\theta} & \sin(\tau) \\ -\sin(\tau) & \cos(\tau)e^{i\theta} \end{pmatrix} \tag{L12.26}$$

to $\tau = \pi/2$, followed by the rotation back without the exponentials. Thus we can, by stabilizing, add such a matrix to the symbol of $Q(x)$. This replaces π_+ and π_- by trivially stabilized projections so that they have ranges which are bundle isomorphic. Finally then this allows us to perform a similar rotation to the identity. Namely, identifying the range of π_- with that of π_+ using a bundle isomorphism, F, we may consider the homotopy

$$\begin{pmatrix} \cos(\theta)e^{-i\theta}\pi_- & \sin(\tau)F & 0 \\ -\sin(\tau)F^{-1} & \cos(\tau)e^{i\theta} & 0 \\ 0 & 0 & \pi_0 \end{pmatrix} \tag{L12.27}$$

and then back again without the exponentials.

Thus the symbol can be deformed to the identity (after stabilization of course) which means that the operator can be deformed, through invertibles, to a family in $G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^L)$. $\square$

Next time I will show that this construction, together with the construction from last time, gives the weak contractibility of the stabilized, normalized Toeplitz group which I will now proceed to define.

Choose any compact manifold Y (as usual with positive dimension). Then we can consider the space

(L12.28)
$$\mathcal{C}^\infty(Y^2;\Psi^m(\mathbb{S})\otimes\Omega_R Y)=I^{m'}(\mathbb{S}^2\times Y^2,\mathrm{Diag}_{\mathbb{S}};\Omega_R(Y\times\mathbb{S}),\ m'=m-\frac{1}{2}\dim Y.$$

Apart from the (trivial) density factors this is just the space of smooth functions with values in the pseudodifferential operators on $\mathbb{S}$. However, we may also interpret it as the space of pseudodifferential operators on $\mathbb{S}$ 'with values in the smoothing operators on Y.' That is, there is a full operator composition on this space.

To see this, consider the Toeplitz action of $A\in\mathcal{C}^\infty(Y^2;\Psi^m(\mathbb{S})\otimes\Omega_R)$ on $u\in\mathcal{C}^\infty(\mathbb{S}\times Y)$

$$A(y,y')u(\theta,y'')\in\mathcal{C}^\infty(Y^3\times\mathbb{S};\Omega'_Y). \tag{L12.29}$$

Restricting to the diagonal and integrating gives

$$Au=\int_Y A(y,y')u(\theta,y')\in\mathcal{C}^\infty(\mathbb{S}\times Y) \tag{L12.30}$$

and this is a continuous linear operator. Operator composition therefore works in the obvious way, if $A\in\mathcal{C}^\infty(Y^2;\Psi^{m_1}(\mathbb{S})\otimes\Omega_R Y)$, $B\in\mathcal{C}^\infty(Y^2;\Psi^{m_2}(\mathbb{S})\otimes\Omega_R Y)$ then

$$AB(y,y')=\int_Y A(y,z)\circ_{\mathbb{S}} B(z,y')\in\mathcal{C}^\infty(Y^2;\Psi^{m_1+m_2}(\mathbb{S})\otimes\Omega_R Y) \tag{L12.31}$$

and with this product we will denote the space as

$$\mathcal{C}^\infty(Y^2;\Psi^m(\mathbb{S})\otimes\Omega_R Y)=\Psi^{m,-\infty}(\mathbb{S},Y). \tag{L12.32}$$

It is straightforward to do the same thing for operators between sections of any two vector bundles over Y or $Y\times\mathbb{S}$. We can also look at the elements which are valued in the Toeplitz operators and consider the algebra

$$\Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y)=S\Psi^{0,-\infty}(\mathbb{S},Y)S. \tag{L12.33}$$

We 'really' view this algebra as a stabilization of all the $\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ each of which can be embedded in it as a subalgebra by taking a corresponding finite dimensional subspace of $\mathcal{C}^\infty(Y)$ and considering only operators acting on it. These can also be thought of in the form (L12.33) in that if $\pi_N\in\Psi^{-\infty}(Y)$ is a projection onto a subspace of dimension N then

$$\pi_N\Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y)\pi_N\simeq\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N). \tag{L12.34}$$

The symbol maps in all these cases are surjective maps onto the corresponding spaces of smooth functions

$$\begin{gathered}\Psi^0(\mathbb{S})\xrightarrow{\sigma_0}\mathcal{C}^\infty(\mathbb{S}_+\sqcup\mathbb{S}_-)\\ \Psi^0_{\mathcal{T}}(\mathbb{S})\xrightarrow{\sigma_0}\mathcal{C}^\infty(\mathbb{S}),\ \mathbb{S}=\mathbb{S}_+\\ \Psi^0(\mathbb{S};\mathbb{C}^N)\xrightarrow{\sigma_0}\mathcal{C}^\infty(\mathbb{S};M(N,\mathbb{C}))\\ \Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S})\xrightarrow{\sigma_0}\mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Y))\end{gathered} \tag{L12.35}$$

and these are homomorphism of the corresponding algebras.

Now, we want to consider the group of invertible perturbations of the identity of this type. Notice that the fact that these operators are valued in smoothing operators means that they cannot be invertible, say acting on $\mathcal{C}^\infty(\mathbb{S}\times Y)$, on their own. We add a normalization condition for the same reason as it was added to the homotopy result above and consider

(L12.36)
$$G^{0,-\infty}_{\mathcal{I}}(\mathbb{S},Y)=\big\{A\in\Psi^{0,-\infty}_{\mathcal{I}}(\mathbb{S});\operatorname{Id}+A\text{ is elliptic (i.e. }\sigma_0(A)\in\mathcal{C}^\infty(\mathbb{S};G^{-\infty}(Y))\\ (\operatorname{Id}+A)^{-1}-\operatorname{Id}\in\Psi^{0,-\infty}_{\mathcal{I}}(\mathbb{S}),\ \sigma(A)\big|_{1\in\mathbb{S}}=0\big\}.$$

Sometimes I will get carried away and just denote this as $G^0_{\mathcal{I}}(\mathbb{S})$ even though it does depend on Y.

So, next time I will prove

Proposition 27. *The topological group $G^{0,-\infty}_{\mathcal{I}}(\mathbb{S},Y)$ is weakly contractible.*

The proof, as I said before, is obtained by combining the two homotopies that I have talked about today and last time. We will get some other things out of these as well. Why should we care about this? For one thing it means that the short exact sequence of groups

$$G^{-\infty}_{\mathcal{I}}(\mathbb{S}\times Y)\longrightarrow G^{0,-\infty}_{\mathcal{I}}(\mathbb{S},Y)\longrightarrow Q \tag{L12.37}$$

where Q is the quotient, is a classifying sequence for K-theory.

12+. Addenda to Lecture 12

CHAPTER 13

Classifying sequence for K-theory

Lecture 13: 1 November, 2005

Today I will discuss some of the consequences of the two homotopies I described last week.

Recall the second of these results. Let X be a compact manifold and consider

$$A : X \longrightarrow \mathcal{C}^\infty(\mathbb{S}; \mathrm{GL}(N, \mathbb{C})),$$

a family of smooth maps, so $A \in \mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$, satisfying the normalization condition that $A(x, 1) = \mathrm{Id}$ for all $x \in X$. We are permitted to stabilize the family by embedding $\mathrm{GL}(N, \mathbb{C})$ in $\mathrm{GL}(M, \mathbb{C})$ for $M \ge N$. Then for M sufficiently large we can find a homotopy, which is to say a family $A_t \in \mathcal{C}^\infty(X \times [0, 1]_t \times \mathbb{S}; \mathrm{GL}(M, \mathbb{C}))$, such that $A_0 = A$ and

$$A_1(x) = \pi_-(x)e^{-i\theta} + \pi_0(x) + \pi_+(x)e^{i\theta} \tag{L13.1}$$

where π_-, π_0 and π_+ are three smooth families of projections which are mutually commuting and sum to the identity.

L13.1. Numerical index for the circle.

Corollary 5. *If $P \in \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ is an elliptic Toeplitz operator, so $\sigma_0(P) \in \mathcal{C}^\infty(\mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$ then*

$$\mathrm{ind}(P) = \dim(\mathrm{null}(P)) - \dim(\mathrm{null}(P^*)) = \frac{i}{2\pi} \int_{\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P)^{-1}\frac{d\sigma_0(P)}{d\theta}\right) d\theta. \tag{L13.2}$$

Proof. For a single symbol (i.e. $X = \{\mathrm{pt}\}$) of the form (L13.1) we can prove (L13.2) directly. Namely

$$P = \pi_- L + \pi_0 + \pi_+ U \tag{L13.3}$$

is a Toeplitz operator with this symbol, since L has symobl $e^{-i\theta}$ and U has symbol $e^{i\theta}$. The null space of P is

$$\mathrm{null}(P) = \pi_-(\mathbb{C}^N) \tag{L13.4}$$

and since the adjoint is $P^* = \pi_- U + \pi_0 + \pi_+ L$

$$\mathrm{null}(P^*) = \pi_+(\mathbb{C}^N) \Longrightarrow \mathrm{ind}(P) = \mathrm{rank}(\pi_-) - \mathrm{rank}(\pi_+). \tag{L13.5}$$

On the other hand, with $\sigma_0(P)$ given by A_1,

$$A_1^{-1}\frac{dA_1}{d\theta} = -i\pi_- + i\pi_+ \Longrightarrow \int_{\mathbb{S}} \mathrm{Tr}\left(A_1^{-1}\frac{dA_1}{d\theta}\right) d\theta = -2\pi i\, \mathrm{ind}(P) \tag{L13.6}$$

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which is (L13.2) in this special case.

The homotopy argument shows that every elliptic symbol $p \in \mathcal{C}^\infty(\mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$ normalized by $p(1) = \mathrm{Id}$ is stably homotopic to one of the form (L13.1). Setting $Q = p(1) \in \mathrm{GL}(N, \mathbb{C})$ it follows that any elliptic operator may be written as a product $P = QP'$ where $\sigma(P')$ satisfies the normalization condition and $Q \in \mathrm{GL}(N, \mathbb{C})$. Since Q is an isomorphism, the index of P is equal to the index of P'. Moreover, since Q is independent of θ,

$$\begin{aligned}\text{(L13.7)}\quad \int_{\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P)^{-1}\frac{d\sigma_0(P)}{d\theta}\right) d\theta &= \int_{\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P')^{-1}Q^{-1}\frac{dQ\sigma_0(P')}{d\theta}\right) d\theta \\ &= \int_{\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P')^{-1}\frac{d\sigma_0(P')}{d\theta}\right) d\theta.\end{aligned}$$

Thus, it suffices to prove the index formula for P', i.e. to assume the normalization condition for P. Now, the index of a curve of elliptic operators is constant and we also know, from Proposition 22, that the right side of (L13.2) is homotopy invariant, i.e. is constant along a curve of elliptic symbols and holds at the end point. Thus (L13.2) must hold in general. $\square$

A similar argument works for elliptic pseudodifferential operators on the circle, with the resulting formula being 'the same' except there are now two circles forming the boundary of $\overline{T^*\mathbb{S}}$.

Proposition 28. *If $P \in \Psi^0(\mathbb{S}; \mathbb{C}^N)$ is elliptic*

(L13.8)
$$\mathrm{ind}(P) = \dim(\mathrm{null}(P)) - \dim(\mathrm{null}(P^*)) = \frac{i}{2\pi}\int_{S^*\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P)^{-1}\frac{d\sigma_0(P)}{d\theta}\right) d\theta.$$

As a consequence of this one can see that the index of any differential operator on the circle vanishes. Namely, the principal symbol of a differential operator is a homogeneous polynomial $p(\theta)\tau^k$ so the restrictions to $\pm\infty$ are $(-1)^k p(\theta)$ as sections of the trivial homogeneity bundle. The signs cancel in (L13.8) and the orientations are opposite, so the terms cancel each other.

L13.2. Contractibility of the Toeplitz group. The central consequence of the two homotopies discussed last week is the weak contractibility of the normalized and stabilized group of invertible Toeplitz operators. Let me recall the definition. We start with the Szegő projector, $S \in \Psi^0(\mathbb{S})$ which projects a smooth function on the circle to its non-negative-frequency part, $S : \mathcal{C}^\infty(\mathbb{S}) \longrightarrow \mathcal{C}^\infty_+(\mathbb{S})$. Then the Toeplitz algebra is the compression of the pseudodifferential algebra to the range of S :

$$\text{(L13.9)}\qquad \Psi^0_{\mathcal{T}}(\mathbb{S}) = S\Psi^0(\mathbb{S})S$$

which we think of as operators on $\mathcal{C}^\infty_+(\mathbb{S})$. There is no problem considering matrices of such operators, forming the algebras $\Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ but we want to consider the 'fully stabilized' algebra which is the Toeplitz algebra 'with values in the smoothing operators' on another compact manifold Y (and maybe acting on a bundle E.)

So, consider

$$\text{(L13.10)}\qquad \mathcal{C}^\infty(Y^2; \Psi^0(\mathbb{S}) \otimes E) = I^{-\dim Y/2}(Y^2 \times \mathbb{S}^2, \mathrm{Diag}_{\mathbb{S}} \otimes \mathrm{Hom}(E))$$

where for simplicity of notation I am leaving out the density bundles, since they are trivial anyway. From the results we have proved for conormal distributions, this is an algebra where the product can be interpreted in several equivalent ways. Perhaps the clearest is to do the composition in $\mathbb{S}$ first. Thus, if $A, B \in \mathcal{C}^\infty(Y^2; \Psi^0(\mathbb{S}) \otimes E)$ then

$$A(y, y') \circ_{\mathbb{S}} B(z, z') \in \mathcal{C}^\infty(Y^4; \Psi^0(\mathbb{S}) \otimes \operatorname{Hom}(E)_L \otimes \operatorname{Hom}(E)_R) \tag{L13.11}$$

where the two copies of $\operatorname{Hom}(E)$ are on the left two and the right two copies of Y. We can then restrict to $y' = z$, compose in the two copies of $\operatorname{Hom}(E)$ and integrate out the z variable giving the composite

$$(A \circ B)(y, y') = \int_Y A(y, z) \circ_{\mathbb{S}} B(z, y') \tag{L13.12}$$

where we really do need to be carrying the densities along to do the integration invariantly.

Now, we can compress the operators onto the range of S as before, or equivalently consider directly the smooth maps into the Toeplitz algebra $\mathcal{C}^\infty(Y^2; \Psi^0_{\mathcal{T}}(\mathbb{S}) \otimes E)$. I will denote this space with the product (L13.12) as $\Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E)$. The symbol map on $\Psi^0_{\mathcal{T}}(\mathbb{S})$ extends to give a symbol map which is multiplicative and takes values in the loops in smoothing operators

$$\sigma_0 : \Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E) \longrightarrow \mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}(Y; E)),\ \sigma_0(AB) = \sigma_0(A)\sigma_0(B). \tag{L13.13}$$

This algebra does effectively stabilize the matrix-valued Toeplitz operators since we can embed the $N \times N$ matrices as a subalgebra of $\Psi^{-\infty}(Y; E)$, just by choosing an N-dimensional subspace of $\mathcal{C}^\infty(Y; E)$, and then

$$\Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N) \hookrightarrow \Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E) \tag{L13.14}$$

as a subalgebra acting on the subspace. Of course such an inclusion is not natural, but any two choices are homotopic through such embeddings, simply by rotating one subspace of $\mathcal{C}^\infty(Y; E)$ into the other.

Finally then we come to the group which is

$$\begin{aligned} G^0_{\mathcal{T}}(\mathbb{S}, Y; E) = \big\{ & A \in \Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E);\ \operatorname{Id} + \sigma(A) \in G^{-\infty}_{(1)}(Y; E), \\ & (\operatorname{Id} + A)^{-1} - \operatorname{Id} \in \Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E) \text{ and } \sigma_0(A)\big|_{1 \in \mathbb{S}} = 0 \big\}. \end{aligned} \tag{L13.15}$$

The first condition is ellipticity (recall that $G^{-\infty}_{(1)}(Y; E)$ is the loop group for $G^{-\infty}(Y; E)$, corresponding to maps from the circle). The last condition is the normalization condition. Since the symbol, fixed at a point on the circle, takes values in $G^{-\infty}(Y; E)$ this effectively kills off a whole classifying space for odd K-theory. We need this to get the result we are after, namely

Theorem 5. *The topological group $G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ is weakly contractible, i.e. if $f : X \longrightarrow G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ is any smooth map from a compact manifold then there is a smooth homotopy $f_\cdot : X \times [0, 1] \longrightarrow G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ with $f_0 = f$ and $f_1 \equiv \operatorname{Id}$.*

It is easy to see that continuous maps are approximable by smooth maps – or indeed the proof below carries through in the continuous case with only a little extra care.

Conjecture 1. *The group $G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ is dominated by a CW complex and as a result is actually contractible.*

Proof. Basically this amounts to putting the two homotopies, discussed earlier, together. First however we need to discuss the topology, to check that we do indeed have a topological group – in the infinite dimensional case such as this one needs to be careful. The topology on the space of conormal distributions of any fixed order, for a fixed submanifold, is very like the $\mathcal{C}^\infty$ topology. Namely we know that a conormal distribution is the sum of a smooth term and the inverse Fourier transform of a symbol and we can write this as

$$I^m(X,Z;E) \ni u \Longrightarrow \phi u = F^*\mathcal{F}^{-1}(a),\ a \in \mathcal{C}^\infty(\overline{N^*Z}; E\big|_Z \otimes N_{-m'} \otimes \Omega_{\text{fib}}) \tag{L13.16}$$

where $\phi \in \mathcal{C}^\infty(X)$ cuts off close to Z in the collar neighbourhood fixed by F. With such choices (including the identification of E on the collar neighbourhood with $E\big|_Z$) made, a and $\phi u \in \mathcal{C}^\infty(X;E)$ are determined and we can impose the usual $\mathcal{C}^\infty$ topology on them. That is, the seminorms on $I^m(X,Z;E)$ are those giving uniform convergence of all derivatives for a and ϕu. This gives a metric topology on $I^m(X,Z;E)$ with respect to which it is complete. Of course it is necessary to check that different choices of cutoff, normal fibration and bundle identification lead to the same topology but this follows directly from the earlier proofs (and I should have mentioned it ...).

The spaces of pseudodifferential operators are just special cases of conormal distributions so they also have such topologies. Moreover the proof of the composition theorem shows the continuity of composition with respect to this topolgy, so we have the first condition needed for a topological group, that

$$G^0_{\mathcal{T}}(\mathbb{S},Y;E) \times G^0_{\mathcal{T}}(\mathbb{S},Y;E) \ni (A,B) \longmapsto AB \in G^0_{\mathcal{T}}(\mathbb{S},Y;E) \tag{L13.17}$$

is continuous with respect to the topology inherited from $\Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y;E)$. We also need to check that the same is true for

$$G^0_{\mathcal{T}}(\mathbb{S},Y;E) \ni A \longmapsto A^{-1} \in G^0_{\mathcal{T}}(\mathbb{S},Y;E). \tag{L13.18}$$

This is the usual stumbling block. In fact, the way we constructed the inverse was to first use the ellipticity to construct a parametrix and then the parameterix was 'corrected ' to the inverse by adding a smoothing operator. The construction of the parametrix is locally uniform on compact sets – it involves summation of the Taylor series for the symbol. The construction of the compensating smoothing term is also locally uniform. The uniqueness of the inverse (given that it exists) gives continuity on compact sets. This is enough to give the continuity in (L13.18) since the topology is metrizable, so it is enough to prove sequential continuity. In fact the set of invertible elliptic elements is open (within the subspace fixed by the normalization condition).

Now we proceed in 5 steps.

1) Given such a smooth map $f : X \longleftarrow G^0_{\mathcal{T}}(\mathbb{S},Y;E)$ we first approximate closely, and uniformly on X, by elements of $G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ using (L13.14) and hence deform into this smaller algebra. This follows exactly as in the approximation of smoothing operators by finite rank operators discussed earlier, the only difference is that in (L13.10) our smoothing operators are valued in the Toeplitz operators. So, simply decompose Y^2 into small product sets $U_i \times U_j$ over which the bundle E is trivial and which are embedded in the torus. Using the product of a partition of unity from Y and Fourier expansion on the torus allows us to approximate f arbitrarily closely. Note that the fact that the smooth functions are valued in the linear space $\Psi^0_{\mathcal{T}}(\mathbb{S})$ makes very little differece, since this is essentially the same as $\mathcal{C}^\infty(Z)$ for a compact manifold Z (in fact we can reduce to that case for the symbol and the smoothing error). It follows that the approximation is uniform on X and when enough terms in the Fourier series are taken the resulting finite rank family

(on Y) lies in $G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ and is homotopic to f in $G^0_{\mathcal{T}}(\mathbb{S},Y;E)$. Notice that we can maintain the normalization condition by first ignoring it and then afterwards composing with the inverse of $\sigma_0(f.)\big|_{1\in\mathbb{S}}$ thought of as a map from $X\times[0,1]$ into $G^{-\infty}(Y;E)$.

2) Now we are reduced to a smooth map from X into $G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$. This was the setting in which the homotopy given by Atiyah was discussed above. By first approximating the symbol by its truncated Fourier expansion and then stabilizing (depending on the order of the truncated symbol as a trigonomentric polynomial) we get a homotopy for the symbol, stabilized to an element of $\mathcal{C}^\infty(\mathbb{S};\mathrm{GL}(M,\mathbb{C}))$, for M large, to a symbol of the form

$$\pi_-(x)e^{-i\theta}+\pi_0(x)+\pi_+(x)e^{i\theta}\in\mathcal{C}^\infty(\mathbb{S};\mathrm{GL}(M,\mathbb{C})). \tag{L13.19}$$

Here the smooth families of projections π_-, π_0 and π_+ are mututally commuting and sum to the identity.

A smooth family of operators with the symbol (L13.19) is

$$A(x)=\pi_-(x)L+\pi_0(x)+\pi_+(x)U\in\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^M). \tag{L13.20}$$

This is certainly elliptic and we know that we may stabilize the null spaces to a bundle by considering $A(x)(\operatorname{Id}-\pi_k)$ for large enough k, where π_k is projection onto the span of the $e^{ij\theta}$ for $0\le j\le k$. The null space is then equal to that of $\operatorname{Id}-\pi_k$ and we are interested in the null bundle of the adjoint

$$\begin{aligned}\text{(L13.21)}\quad \operatorname{null}((\operatorname{Id}-\pi_k)A(x)^*)&=\operatorname{null}\left((\operatorname{Id}-\pi_k)(\pi_-(x)U+\pi_0(x)+\pi_+(x)L)\right)\\ &=\operatorname{sp}\{e^{ij\theta}\mathbb{C}^M,\ 0\le j\le(k-1),\ (\pi_0(x)+\pi_+(x))\mathbb{C}^Me^{ik\theta},\ \pi_+(x)\mathbb{C}^Me^{i(k+1)\theta}\}.\end{aligned}$$

3) Now, the original family was invertible and we know that along a curve of elliptics, which is initially invertible, we may perturb by a smoothing family (initially zero) to maintain invertibility. Thus the family we arrive at, of the form (L13.20) can be perturbed to be invertible by a smoothing operator. As shown earlier this means that the null bundle and null bundle of the adjoint are bundle isomorphic once they are sufficiently stabilized. In this case this just means that the bundle (L13.21) is trivial, i.e. isomorphic to a trivial bundle of the same rank, for large enough k. Writing out (L13.21) this means

$$\operatorname{null}((\operatorname{Id}-\pi_k)A^*(\cdot))=\mathbb{C}^{kM}\oplus\left(\mathbb{C}^M\setminus\operatorname{Ran}(\pi-(\cdot)\right)\oplus\operatorname{Ran}(\pi_+(\cdot))\simeq\mathbb{C}^{(k+1)M}. \tag{L13.22}$$

This in turn means that there exists

$$F:\operatorname{Ran}(\pi_-(\cdot))\oplus\mathbb{C}^{(k+1)M}\longleftrightarrow\operatorname{Ran}(\pi_+(\cdot))\oplus\mathbb{C}^{(k+1)M}, \tag{L13.23}$$

i.e. that the ranges of these two projections are stably isomorphic.

Now, for any bundle, with projector π it is straightforward to see that the symbol

$$\begin{pmatrix}\pi(x)e^{-i\theta}+(\operatorname{Id}-\pi(x)) & 0\\ 0 & (\operatorname{Id}-\pi(x))+\pi(x)e^{i\theta}\end{pmatrix} \tag{L13.24}$$

is homotopic to the identity through invertible symbols. Indeed one such homotopy is

$$\begin{pmatrix}\cos(\theta)\pi(x)e^{-i\theta}+(\operatorname{Id}-\pi(x)) & \sin(\theta)\pi(x)\\ -\sin(\theta)\pi(x) & (\operatorname{Id}-\pi(x))+\cos(\theta)\pi(x)e^{i\theta}\end{pmatrix} \tag{L13.25}$$

rotating to $\pi/2$ and then back again without the exponentials. It follows that by using such a homotopy from the identity (in some other matrix block) the symbol in (L13.20) can be connected to one in which π_- and π_+ are increased by the same trivial projection corresponding to $\mathbb{C}^{kM}$. Then the isomorphism in (L13.23) can be used to deform this symbol to the identity. Namely, simplifying the notation by identifying $\pi_\pm$ with the stabilized projections, we may identify F as an isomorphism from the range of π_- to the range of π_+. Splitting the space into three, the ranges of π_-, π_0 and π_+ we may consider the homotopy (where the $\pi_\pm$'s are now redundant) from $\tau = 0$ to $\pi/2$

$$\begin{pmatrix} \cos(\tau)\pi_- e^{-i\theta} & 0 & \sin(\tau)F^{-1} \\ 0 & \pi_0 & 0 \\ -\sin(\tau)F & 0 & \cos(\tau)\pi_+ e^{i\theta} \end{pmatrix} \tag{L13.26}$$

and then back again without the exponentials, finishing at the identity.

This we have deformed the family of symbols to the identity after sufficient stabilization. As already noted this can be lifted to a deformation of invertibles, i.e. in $G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ which finishes at an element of $G^{-\infty}_{\mathcal{T}}(\mathbb{S} \times Y; E)$ (which is of finite rank in Y.)

4) At this stage in the deformation the symbol has been trivialized and we are reduced to a family $A \in \mathcal{C}^\infty(X; G^{-\infty}_{\mathcal{T}}(\mathbb{S} \times Y; E))$ which can be taken to be of finite rank in Y, i.e. to have image in a subgroup $G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ for large N. Even if it were not the case we can achieve this result directly by finite rank approximation in Y as before. Now, we further make a finite rank approximation in $\mathbb{S}$ by replacing the family by $(\mathrm{Id} - \pi_k)A(x)(\mathrm{Id} - \pi_k)$ which converges uniformly to $A(x)$ as $k \to \infty$. Taking k sufficiently large, the family may now be assumed to act on the finite dimensional subspace of $\mathcal{C}^\infty_+(\mathbb{S} \times Y; E)$ spanned by

$$e^{ij\theta} e_l,\ 0 \le j \le k,\ 0 \le l \le N. \tag{L13.27}$$

Now, again stabilize the group by expanding N to $(k+1)N$ by choosing k other independent subspaces of $\mathcal{C}^\infty(Y; E)$ of the same dimension. Then the basis in (L13.27) is expanded to

$$e^{ij\theta} e_{l,p},\ 0 \le j \le k,\ 0 \le l \le N,\ 0 \le p \le k \tag{L13.28}$$

where $e_{l,0} = e_l$ and of course the operator is the identity on the terms with $p > 0$. Then consider the rotation of basis elements in 2 dimensional spaces for $1 \le j \le k$, $1 \le l \le N$

$$\cos(\theta) e^{ij\theta} e_l + \sin(\theta) e_{l,j},\ -\sin(\theta) e^{ij\theta} e_l + \cos(\theta) e_{l,j},\ \theta \in [0, \pi/2] \tag{L13.29}$$

with all other elements held fixed. This has the effect of rotating all the non-trivial parts of the matrix into the 0 Fourier term with everything outside the constants on the circle being the identity.

5) The final step is then to follow the first homotopy of last week which allows such a matrix in $\mathrm{GL}(N, \mathbb{C}) \subset G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$, identified as the zero Fourier terms, to be deformed to the identity in $G^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^{2N})$. This completes the deformation to the identity. □

L13.3. Classifying sequence for K-theory. One reason this weakly contractible group is of interest here is that it gives a smooth classifying sequence for K-theory.

Theorem 6. *There is a short exact sequence of topological groups*

$$G^{-\infty}_{\mathcal{T}}(\mathbb{S}\times Y;E)\longrightarrow G^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y;E)\longrightarrow G^{-\infty}_{(1),0}(Y;E)[[\rho]] \tag{L13.30}$$

in which the first group is classifying for odd K-theory, the second is weakly contractible and the third is (therefore) a reduced classifying group for even K-theory (i.e. the identity component of such a classifying group). The quotient group is a formal countable sum (i.e. the elements are sequences, written as power series in the indeterminant ρ) with leading term an element of $G^{-\infty}_{(1),0}(Y;E)$, the subgroup of the loop group $\mathcal{C}^\infty(\mathbb{S};G^{-\infty}(Y;E))$ consisting of the pointed loops (taking 1 to the identity) of index zero and with lower order terms which are arbitrary elements of $\mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Y;E))$.

Since (L13.30) is a short exact sequence of groups, there is a product induced on the quotient. This will show up a bit later.

Proof. There is actually not too much to prove here since we have shown the weak contractibility. The leading term of the projection map is just the principal symbol σ. Thus, if $A\in G^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y;E)$ then we know that $\sigma_0(A)\in\mathcal{C}^\infty(\mathbb{S};G^{-\infty}(Y;E))$ has index zero (this follows from our first result today) and $\sigma_0(A)(1)=0$ is the normalization condition on the symbol. This is precisely the definition of $G^{-\infty}_{(1),0}(Y;E)$ and the map is surjective since any such symbol of index zero is the symbol of an invertible operator.

To get the second map in (L13.30) we just consider a normal fibration around the diagonal in $\mathbb{S}$. Then the corresponding 'full symbol map' takes a conormal distribution in (L13.10) and maps it to the Taylor series at the circle at infinity of the transverse Fourier transform of the kernel (cut off near the diagonal of $\mathbb{S}$). This gives a short exact sequence of linear maps

$$\Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}\times Y;E)\longrightarrow\mathcal{C}^\infty(Y^2;\Psi^0_{\mathcal{T}}(\mathbb{S})\otimes E)\longrightarrow\sum_{j=0}^{\infty}\rho^j\mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Y;E)). \tag{L13.31}$$

The only constraint on an elliptic operator to be perturbable is the already-noted requirement that the index vanish. Thus, for invertible perturbations of the identity we arrive at (L13.30). $\square$

There are two other closely related theorems that I will prove next time, again as consequences of the homotopies discussed earlier. To state them I need to define the 'usual' K-group $K^0(X)$ for a compact manifold X. Traditionally this is the starting point for topological K-theory, but I have instead approach the subject through

$$\begin{aligned} &K^{-1}(X)=[X;G^{-\infty}(Y;E)]\text{ and}\\ &\qquad K^{-2}(X)=[X;G^{-\infty}_{(1)}(Y;E)]=[X\times\mathbb{S},X\times\{1\};G^{-\infty}(Y;E),\mathrm{Id}]. \end{aligned} \tag{L13.32}$$

We define $K^0(X)$ as the Grothendieck group associated to stable vector bundles (under direct sum). Thus if $E\longrightarrow X$ and $F\longrightarrow X$ are two vector bundles over X they are isomorphic if there is a diffeomorphism between the total spaces $E\longleftrightarrow F$ which maps the fibre E_x linearly to the fibre F_x; denote this relationship $E\equiv F$. To define $K^0(X)$ consider pairs of vector bundles (E_+,E_-) (also thought of as $\mathbb{Z}_2$-graded vector bundles) and the equivalence relation of stable isomorphism. That is

$$(E_+, E_-) \sim (F_+, F_-) \Longleftrightarrow \exists\, H \text{ s.t. } E_+ \oplus F_- \oplus H \equiv E_- \oplus F_+ \oplus H. \tag{L13.33}$$

It is straightforward to check that this is an equivalence relation and the set it defines, $K^0(X)$, is an abelian group under direct sum

$$[(E_+, E_-)] + [(F_+, F_-)] = [(E_+ \oplus F_+, E_- \oplus F_-)]. \tag{L13.34}$$

THEOREM 7. *[Families index for the Toeplitz algebra] Given* $[a] \in K^{-2}(X)$, *represented by* $a \in \mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$ *with* $a(1) = \mathrm{Id}$, *choosing any smooth family of operators* $A \in \mathcal{C}^\infty(X; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N))$ *with* $\sigma_0(A) = a$, *the stabilized 'families index'*

$$[(\mathrm{null}(A(x)(\mathrm{Id} - \pi_k), \mathrm{null}((\mathrm{Id} - \pi_k)A^*(x))] \in K^0(X) \tag{L13.35}$$

is well-defined for large k, *independent of the choice of* A, *and defines an isomorphism of abelain groups*

$$K^{-2}(X) \longrightarrow K^0(X). \tag{L13.36}$$

THEOREM 8. *[Bott periodicity] For any representative* $[(E_+, E_-)] \in K^0(X)$ *one can choose smooth families of commuting projections* $\pi_-(x)$, $\pi_0(x)$, $\pi_+(x)$ *on* $\mathbb{C}^N$ *for large* N *such that* $E_\pm$ *are isomorphic to the ranges of* $\pi_\pm$ *and* $\pi_-(x) + \pi_0(x) + \pi_+(x) = \mathrm{Id}$ *and then the element*

$$\pi_-(x)e^{-i\theta} + \pi_0(x) + \pi_+(x)e^{i\theta} \in \mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N, \mathbb{C})) \tag{L13.37}$$

projects to a well-defined map

$$K^0(X) \longrightarrow K^{-2}(X) \tag{L13.38}$$

which is an isomorphism.

The maps in these two theorems are just inverses of each other (assuming that I have not messed up the signs).

13+. Addenda to Lecture 13

13+.1. Proof of Proposition 28. First choose an element $P_+ \in \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ with $\sigma(P_+) = \sigma(P)\big|_{S^*_+\mathbb{S}}$. Then the operator $P_+ + (\mathrm{Id} - S) \in \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ has the same index as P_+ (the latter acting on $\mathcal{C}^\infty_+(\mathbb{S}; \mathbb{C}^N)$) so the formula (L13.2) applies. We can also choose a 'negative' Toeplitz operator, $P_- \in \Psi^0_{-\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$, the Toepltiz algebra for the opposite orientation, with $\sigma(P_-) = \sigma(P)\big|_{S^*_-\mathbb{S}}$. Extending it as the identity on the positive side, P_+P_- is an elliptic operator with the same index as P and this index is $\mathrm{ind}(P_+) + \mathrm{ind}(P_-)$. This proves (L13.8).

CHAPTER 14

Bott periodicity

Lecture 14: 3 November, 2005

Recall that I defined the standard K-theory of a compact manifold as the set of equivalence classes of pairs of complex vector bundles

$$K^0(X) = \{(E_+, E_-)\}/\sim \tag{L14.1}$$

where equivalence is the existence of a stable isomorphism. In particular $(E_+, E_-) \sim (E_+ \oplus H, E_- \oplus H)$ so these really are formal differences in the sense that we can 'cancel' an H from both terms.

Although the equivalence relation here is stable bundle isomorphism, it is important to realize that it implies the equivalence of homotopic bundles.

Lemma 23. *If E is a complex vector bundle over $[0,1] \times X$ then as bundles over X, $E_0 = E\big|_{\{0\}\times X}$ and $E_1 = E\big|_{\{1\}\times X}$ are isomorphic.*

L14.1. Proof of Theorem 7. We have also defined

$$K^{-2}(X) = [X \times \mathbb{S}, X \times \{1\}; G^{-\infty}(E;E), \mathrm{Id}] \tag{L14.2}$$

as the homotopy classes of pointed maps from $X \times \mathbb{S}$ into the 'smoothing group'. Theorem 7 asserts that these two abelian groups are isomorphic where the map between them is constructed by regularizing the null bundle of an elliptic family of Toeplitz operators as follows

(L14.3)

$$\begin{tikzcd}
{[\sigma_0(A)] \in K^{-2}(X)} \arrow[ddrr, dotted] & & \\
\mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N,\mathbb{C})) \arrow[u] & & \\
A \in \mathcal{C}^\infty(X; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)) \arrow[u, "\sigma_0"] \arrow[rr] & & {[\mathrm{null}(A(\mathrm{Id} - \pi_k)), \mathrm{null}((\mathrm{Id} - \pi_k)A^*] \in K^0(X)}
\end{tikzcd}$$

So we have to show first that this is really does define a map

$$\mathrm{ind} : K^{-2}(X) \longrightarrow K^0(X). \tag{L14.4}$$

We first check that the element of $K^0(X)$ does not depend on k and it does not depend on the choice of A with fixed symbol $\sigma_0(A) = a \in \mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N,\mathbb{C}))$. This will give us a map

$$\mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N,\mathbb{C})) \longrightarrow K^0(X). \tag{L14.5}$$

Two operators A_0 and A_1 with the same symbol are homotopic through the linear homotopy $A_t = (1-t)A_0 + tA_0$. Choosing k large it follows from our earlier arguments that $((\operatorname{Id} - \pi_k)A_t)$ is a smooth bundle over $X \times [0,1]$ and hence that the pairs of bundles

$$[\operatorname{null}(A_0(\operatorname{Id} - \pi_k)), \operatorname{null}((\operatorname{Id} - \pi_k)A_1^*)] \text{ and } [\operatorname{null}(A_1(\operatorname{Id} - \pi_k)), \operatorname{null}((\operatorname{Id} - \pi_k)A_1^*)]$$

(in which the null bundles are constant and trivial) define the same element in $K^0(X)$.

Thus it remains to consider the effect of taking different values of k. By assumption k is chosen large enough that $A(\operatorname{Id} - \pi_k)$ has null bundle equal to that of $\operatorname{Id} - \pi_k$. So it is enough to consider the effect of increasing k to $k+1$. The null bundle of $A(\operatorname{Id} - \pi_{k+1})$ is just increased by the trivial bundle $e^{i(k+1)\theta}\mathbb{C}^N$. Since none of these elements are annihilated by $A(x)$, the range of $A(\operatorname{Id} - \pi_k)$ is just the range of $A(x)(\operatorname{Id} - \pi_{k+1})$ plus $A(x)(e^{i(k+1)\theta}\mathbb{C}^N)$. Since A is a smooth isomorphism onto this space, it is a trivial bundle of rank N, with the trivialization given by $A(x)$ itself. Thus the null space of $(\operatorname{Id} - \pi_{k+1})A^*$, being the annihilator of the range with respect to the chosen innner product, must be equal to the null space of $(\operatorname{Id} - \pi_{k+1})A^*$ plus a trivial bundle of rank N. Thus increasing k by 1 does not change the element in $K^0(X)$.

Thus we do have a map (L14.5). A homotopy of symbols can be lifted to a homotopy of operators and as we have already seen, this results in the same element in $K^0(X)$, so (L14.5) descends to the desired map (L14.4). So it remains to show that this is an isomorphism.

So, suppose $A \in \mathcal{C}^\infty(X; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N))$ has symbol $a \in \mathcal{C}^\infty(X \times \mathbb{S}; \operatorname{GL}(N, \mathbb{C}))$ with $\operatorname{ind}(a) = 0 \in K^0(X)$. We can assume that $a(x,1) = \operatorname{Id}$, since $b(x) = a(x,1) \in \mathcal{C}^\infty(X; \operatorname{GL}(N, \mathbb{C})$ is a smooth family of matrices, hence trivially an element of $\mathcal{C}^\infty(X; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N))$, which is invertible. Thus $A(x)$ and $b^{-1}(x)A(x)$ have the same index. Now, if $\operatorname{ind}(a) = 0$ then we know that there is a family $A \in \mathcal{C}^\infty(X; G^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N))$ with symbol a. Thus from the argument of last time we know that there is then an homotopy from a suitably stabilized a to the identity. Stabilizing a corresponds to stabilizing the operator by the identity on a bundle and so does not change the index. This if $\operatorname{ind}(a) = 0$ then a can be deformed to the identity and hence $[a] = 0 \in K^{-2}(X)$, so the map (L14.4) is injective.

Surjectivity of the index map also follows easily. First recall that any smooth complex bundle E over X can be complemented to a trivial bundle, i.e. can be embedded as a subbundle of a trivial bundle $\mathbb{C}^N$ (and hence for any larger N). Taking a pair of vector bundles, (E_+, E_-), let π_+ be the projection onto E_+ as a subbundle of $\mathbb{C}^N$ and similarly let π_- be projection onto E_- as a subbundle of $\mathbb{C}^M$. Then the symbol

$$a(x,\theta) = \pi_+(x)e^{-i\theta} + (\operatorname{Id}_N - \pi_+) + (\operatorname{Id}_M - \pi_-) + \pi_-(x)e^{i\theta} \tag{L14.6}$$

has index $[(E_+, E_-)]$. Indeed, it is the symbol of the elliptic family

$$\pi_+(x)L + (\operatorname{Id}_N - \pi_+) + (\operatorname{Id}_M - \pi_-) + \pi_-(x)U \in \mathcal{C}^\infty(X; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^{N+M})) \tag{L14.7}$$

which has null space $\pi_+(x)\mathbb{C}^N$ (constant on the circle) and the null space of its adjoint

$$\pi_+(x)U + (\operatorname{Id}_N - \pi_+) + (\operatorname{Id}_M - \pi_-) + \pi_-(x)L \tag{L14.8}$$

is similalry $\pi_-(x)\mathbb{C}^M$ so indeed $\operatorname{ind}(a) = [(E_+, E_-)]$ shows the surjectivity.

An elliptic element, such as L, with index $1 = [(\mathbb{C}, 0)] \in L(\{\mathrm{pt}\})$, in the Toeplitz algebra is sometimes called a Bott element and the inverse $K^0(X) \longrightarrow K^{-2}(X)$ just constructed is the Bott map.

More generally, if X is a possibly non-compact manifold we still want definitions of $K_{\mathrm{c}}^{-1}(X)$, $K_{\mathrm{c}}^{-2}(X)$ and $K_{\mathrm{c}}^{0}(X)$ reducing to $K^{-1}(X)$, $K^{-2}(X)$ and $K^0(X)$ in the compact case. The natural choice for the first two is to take maps into the same spaces as before but which now reduce to the identity outside a compact set (depending on the map). Homotopies are also required to be constant (and hence equal to the identity) outside some compact subset of X in

(L14.9)
$$K_{\mathrm{c}}^{-1}(X) = \Big\{ f \in \mathcal{C}^\infty(X; G^{-\infty}(Y; E)); f\big|_{X \setminus K} = \mathrm{Id},\ K \Subset X \Big\} \,/\mathrm{homotopy}$$
$$K_{\mathrm{c}}^{-2}(X) =$$
$$\Big\{ f \in \mathcal{C}^\infty(X \times \mathbb{S}; G^{-\infty}(Y; E)); f\big|_{(X\setminus K)\times \mathbb{S}} = \mathrm{Id} = f\big|_{X \times \{1\}},\ K \Subset X \Big\} \,/\mathrm{homotopy}.$$

Of course, this is consistent with out defintion for compact spaces.

For $K^0(X)$ we need to take a similar definition in which the two bundles (E_+, E_-) are isomorphic outside a compact set, where the isomorphism needs to be included in the data defining the element. Thus we consider triples (E_+, E_-, a) where $a \in \mathcal{C}^\infty(X \setminus K; \hom(E_+, E_-))$ is invertible for some compact set $K \Subset X$. Thus

$$K_{\mathrm{c}}^{0}(X) = \{(E_+, E_-, a)\} \,/ \sim \tag{L14.10}$$

for such data, where the equivalence relation is that

(L14.11)
$$(E_+, E_-, a_0) \sim (E_+, E_-, a_1)$$
$$\text{if } \exists \text{ a homotopy of isomorphisms } a_t : E_+ \longrightarrow E_- \text{ over } [0,1]_t \times (X \setminus K)$$
$$\text{and } (E_+, E_-, a) \sim (F_+, F_-, b)$$
$$\text{if } \exists\ H \text{ and } F : E_+ \oplus F_- \oplus H \longleftrightarrow E_+ \oplus F_- \oplus H$$
$$\text{s.t. } F = a \oplus b \oplus \mathrm{Id}_H \text{ on } X \setminus K,\ K \Subset X.$$

Note that a triple (E_+, E_-, a) defines the zero element in $K_{\mathrm{c}}(X)$ if and only if there is a bundle H and an isomorphism $b : E_+ \oplus H \longrightarrow E_- \oplus H$ over X restricting to $a \oplus \mathrm{Id}_H$ outside some compact subset.

Exercise 18. Show that the index isomorphism (L14.4) carries over to the case of non-compact manifolds.

An important consequence of the existence of this index isomorphism is

Proposition 29. *[Bott periodicity, usual form] For any manifold X there is a natural isomorphism*

$$K_c^0(X) \longrightarrow K_c^0(\mathbb{R}^2 \times X). \tag{L14.12}$$

Proof. In our original definition of $K^{-2}(X)$ we can perturb any representative slightly so that the normalization condition $f\big|_{X \times \{1\}} = \mathrm{Id}$ can be arranged to hold on $X \times I$ for some neighbourhood I of $1 \in \mathbb{S}$ and similarly for homotopies. Identifying $\mathbb{S} \setminus \{1\}$ with $\mathbb{R}$ this shows that in terms of the non-compact notion

$$K_{\mathrm{c}}^{-2}(X) = K_{\mathrm{c}}^{-1}(\mathbb{R} \times X). \tag{L14.13}$$

Next consider $K^0_{\mathrm{c}}(\mathbb{R}\times X)$, say for X compact; we shall show that

$$K^0_{\mathrm{c}}(\mathbb{R}\times X) = K^{-1}(X). \tag{L14.14}$$

A bundle over $\mathbb{R}\times X$ is necessarily isomorphic to the lift of a bundle from X so any element is represented by two bundles E_+, E_- over X and isomorphisms between then over $(-\infty,-N)\times X$ and $(N,\infty)\times X$ for some N. By homotopy invariance, these isomorphisms can also be taken to be constant in the real variable. Then the isomorphism at $-N$ may be used to identify the bundles and the isomorphism at N becomes an isomorphism of a fixed bundle to itself. Stabilizing such an isomorphism by the identity on a complementary bundle gives an element of $K^{-1}_{\mathrm{c}}(X)$ and it is straightforward to check that this element is well defined and leads to the isomorphism (L14.14).

Combining these two identifications we see that

$$K^0_{\mathrm{c}}(\mathbb{R}^2\times X) = K^{-1}_{\mathrm{c}}(\mathbb{R}\times X) = K^{-2}_{\mathrm{c}}(X) = K^0_{\mathrm{c}}(X) \tag{L14.15}$$

where the last identification is using the index map. $\square$

From this we can deduce that (for $k\ge 1$)

$$K^{-1}(\mathbb{S}^k) = \begin{cases}\mathbb{Z} & k \text{ odd}\\ \{0\} & k \text{ even.}\end{cases} \tag{L14.16}$$

In fact we shall show that $K^{-1}(\mathbb{S}^k) = K_{\mathrm{c}}(\mathbb{R}^k)$ then from (L14.14) $K_{\mathrm{c}}(\mathbb{R}^k) = \mathrm{K}^{-1}_{\mathrm{c}}(\mathbb{R}^{k+1})$ and (L14.16) follows. There is a map

$$K^{-1}_{\mathrm{c}}(\mathbb{R}^k) \longrightarrow K^{-1}(\mathbb{S}^k) \tag{L14.17}$$

defined by identifying a point on the sphere as the point at infinity on $\mathbb{R}^k$. Then a map from $\mathbb{R}^k$ to $G^{-\infty}(Y;E)$ required to be the identity near infinity defines an element of $K^{-1}(\mathbb{S}^k)$. Homotopy with the value fixed near infinity as the identity implies homotopy on the sphere so this gives (L14.17). Moreover, using the connectedness of $G^{-\infty}(Y;E)$ every element of $K^{-1}(\mathbb{S}^k)$ must arise this way, since the value at the chosen point can be deformed to Id. Thus (L14.17) is surjective. An element can only go to zero if it is homotopic to the identiy through families which are constant near infinity. But then multiplying everywhere by the inverse of the value at infinity gives a homotopy which is the identity near infinity, so (L14.17) is an isomorphism and (L14.16) follows.

Corollary 6. *The homotopy groups of $G^{-\infty}$ are*

$$\pi_k(G^{-\infty}(Y;E)) = \begin{cases}\mathbb{Z} & k \textit{ odd}\\ \{0\} & k \textit{ even.}\end{cases} \tag{L14.18}$$

This is one justification for the statement that $G^{-\infty}$ is a classifying group for K-theory.

Next I want to give at least a preliminary statement of the Atiyah-Singer index theorem. I will discuss both the 'numerical' index and the families index. The formula for the former and the formula for the Chern character for the latter are of particular interest.

Given a compact manifold, Z, and two complex vector bundles E_+, and E_- over Z any elliptic operator $P\in\Psi^0(Z;E_+,E_-)$ (if one exists) has finite dimensional null space and its range has finite dimensional complement. The difference between these two integers is the (numerical) index of P

$$\text{(L14.19)} \qquad \operatorname{ind}(P) = \dim(\operatorname{null}(P)) - \dim(\mathcal{C}^\infty(Z; E_-)/P\mathcal{C}^\infty(Z; E_+)).$$

We already know that this function is homotopy invariant, so it can only depend on the geometric data (Z, E_+, E_-) and the symbol $\sigma_0(P) \in \mathcal{C}^\infty(S^*Z; \hom(E_+, E_-))$.

PROPOSITION 30. *The index defines a map*

$$\text{(L14.20)} \qquad \operatorname{ind}_{\mathrm{a}} : K_c(T^*Z) \longrightarrow \mathbb{Z},\ \operatorname{ind}(P) = \operatorname{ind}_{\mathrm{a}}([(\pi^*E_+, \pi^*E_-, \sigma_0(P))]).$$

PROOF. Since $K_{\mathrm{c}}(T^*Z)$ is defined as the set of equivalence classes of triples (E_+, E_-, a), with a an isomorphism outside a compact set, we need to show first that, for T^*Z, every such classes arises from the symbol of an elliptic operator. Notice that the fibres of the cotangent bundle are contractible, being vector spaces. So it is a standard fact (and easy enough to check) that every vector bundle over T^*Z is bundle isomorphic to π^*E for some bundle over Z. Using the invariance under bundle isomorphisms in the definition of $K_{\mathrm{c}}(T^*Z)$ it follows that every element is represented by a triple corresponding to an elliptic operator – note that by the homotopy invariance in the definition of the equivalence relation we may assume that a is homogeneous of degree 0 (or any other degree you might choose). So it only remains to show that the index is constant on equivalence classes. As for the bundles themselves, bundle isomorphism over T^*Z are homotopic to their values at the zero section, i.e. to bundle isomorphisms over Z. Such a bundle isomorphism is invertible and hence has zero index as a (rather trivial) pseudodifferential operator. This, with the homotopy invariance, shows that the index map does project to a well-defined map (L14.20). $\square$

Not only is this map, the 'analytic index map' well defined but it is clearly a homomorphism, since we know that $\operatorname{ind}(AB) = \operatorname{ind}(A) + \operatorname{ind}(B)$.

Gelfand around 1960 asked what amounts to the question of identifying this map in topological terms and in particular to find a formula for it.

An answer to this, given by Atiyah and Singer, is to define another map, the topological index, and show that the two are equal. This second map is defined by 'trivializing the topology' of the space. Namely by embedding Z as a submanifold of a simple manifold, either Euclidean space or a sphere according to your taste. Then, and this is where most of the work is, an operator on the larger space is constructed which has the same index and with symbol which is 'derived' from that of the original operator. For appropriate operators on the sphere (trival near the point at infinity) the index can again be seen to be an isomophism and this allows the topological index to be defined, or the analytic index to be computed depending on how you look at it.

14+. Addenda to Lecture 14

CHAPTER 15

Fibrations and families

Lecture 15: 8 November, 2005

The usual geometric setting for the families version of the index theorem of Atiyah and Singer is in terms of operators on the fibres of a fibration. Thus, rather than simply consider a parameterized family $A_b \in \Psi^m(Z; E, F)$ depending on $b \in B$ we shall allow the family to be 'twisted' by diffeomorphisms of Z depending on B.

L15.1. Fibrations. Such twisting is to be interpreted in terms of a fibration of compact manifolds

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B. \end{tikzcd} \tag{L15.1}$$

A fibration of compact manifolds is just a submersion. For simplicity of notation we will assume that the 'base' B is connected. Then a smooth map

$$\phi : M \longrightarrow B \tag{L15.2}$$

is a *submersion* if

$$\phi_* : T_m M \longrightarrow T_{\phi(m)} B \tag{L15.3}$$

is surjective for each $m \in M$.

THEOREM 9. *If $\phi : M \longrightarrow B$ is a smooth submersion between compact manifolds with B connected then*

1. *For each $b \in B$, $\phi^{-1}(b) = Z_b \subset M$ is an embedded compact submanifold diffeomorphic to a fixed manifold Z.*
2. *Each $b \in B$ has an open neighbourhood $b \in U_b \subset B$ such that there exists a diffeomorphism f_b giving a commutative diagramme*

$$\begin{tikzcd} \phi^{-1}(U_b) \arrow[rr, "f_b"] \arrow[dr, "\phi"'] & & Z \times U_b \arrow[dl, "\pi_L"] \\ & U_b. & \end{tikzcd} \tag{L15.4}$$

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(3) *For each intersectiong pair of such open sets it follows that there is a commutative diagramme*

(L15.5)
$$\begin{tikzcd} Z\times(U_b\cap U_{b'}) \arrow[dr, "\pi_L"'] & \phi^{-1}(U_b\cap U_{b'}) \arrow[l, "f_{b'}"'] \arrow[r, "f_b"] \arrow[d, "\phi"] & Z\times(U_b\cap U_{b'}) \arrow[dl, "\pi_L"] \\ & U_b\cap U_{b'} & \end{tikzcd}$$

which shows that $f_{b'b}=f_{b'}f_b^{-1}\in\mathcal{C}^\infty(U_b\cap U_{b'};\mathrm{Difm}(Z))$ *is a smooth map into the diffeomorphisms of* Z *and also that the cocycle condition*

(L15.6)
$$f_{b''b'}f_{b'b}=f_{b''b} \text{ holds on } U_{b''}\cap U_{b'}\cap U_b.$$

PROOF. (Brief) The implicit function theorem shows that $Z_b=\phi^{-1}(b)$ is an embedded compact submanifold of M. Indeed, if t_i are local coordinates near b on B then the $\phi^*(t_i)$ are defining functions for Z_b in M. One can choose commuting vector fields in $\phi^{-1}(U_b)$ for a sufficiently small neighbourhood U_b of b, T_i on $\phi^{-1}(U_b)$ such that $\phi_*(T_i)=\partial_{t_i}$ and then by integration along the T_i one can define f_b with $Z=Z_b$. Having done this on an open covering of B it follows that all the Z_b are diffeomorphic, so Z_b can be replaced by a fixed Z in (L15.4). This proves (1) and (2) and (3) follow directly from (2). □

One can recover the fibration, thought of here as a fibre bundle with fibre Z and structure group $\mathrm{Difm}(Z)$ (the diffeomorphism group of Z), from (2) and (3). If the maps $f_{b'b}$ can be chosen, for some covering of B, to lie in a subgroup $G\subset\mathrm{Difm}(Z)$ of the diffeomorphism group, then the structure group is 'reduced to G.'

Fibrations have various functoriality properties. The most important for us is that we may restrict to a submanifold of the base or more generally we may 'pull-back' a fibration.

PROPOSITION 31. *If* $F:\tilde B\longrightarrow B$ *is any smooth map, with* $\tilde B$ *compact, and* $\phi:M\longrightarrow B$ *is a fibration, then*

(L15.7)
$$\tilde M=\{(m,\tilde b)\in M\times\tilde B;\phi(m)=F(\tilde b)\} \text{ is an embedded submanifold of } M\times B \text{ and}$$
$$\tilde\phi:\tilde M\longrightarrow\tilde B,\ \tilde\phi(m,\tilde b)=\tilde b \text{ is a fibration.}$$

Equally important is that the composite of two fibrations is a fibration.

PROPOSITION 32. *If* $\phi':M'\longrightarrow M$ *is a fibration with typical fibre* Z' *and* $\phi:M\longrightarrow B$ *is a fibration with typical fibre* Z *then* $\phi\phi':M'\longrightarrow B$ *is a fibration with typical fibre* $Z\times Z'$.

It is also easy to see that the direct product of two fibrations, $\phi_i:M_i\longrightarrow B_i$, $i=1,2$ is a fibration

(L15.8)
$$\phi_1\times\phi_2:M_1\times M_2\longrightarrow B_1\times B_2.$$

In the proof of the Atiyah-Singer index theorem discussed below, a given fibration is trivialized by embedding, so it is important to see that this is always possible.

EXERCISE 19. Given a fibration (of compact manifolds always) $\phi:M\longrightarrow B$ show that there is an embedding $e:M\longrightarrow\mathbb{S}^N\times B$ with range in $(\mathbb{S}^N\setminus\{p\})\times B$

for fixed point $p \in \mathbb{S}^N$ such that

$$\begin{tikzcd} M \arrow[rr, "e"] \arrow[dr, "\phi"'] & & \mathbb{S}^N \times B \arrow[dl, "\pi_L"] \\ & B & \end{tikzcd} \tag{L15.9}$$

commutes.

Hint: This is not hard, just use Whitney's theorem to embed M in a big sphere, staying away from one point, and then define e as the product of that embedding and ϕ. Check that this is an embedding and that (L15.9) holds.

L15.2. Pseudodifferential operators on the fibres. Next we turn to the definition of pseudodifferential operators 'on the fibres' of a fibration. We could proceed locally from (L15.4) and (L15.5), using the definition of pseudodifferential operators on Z depending on a parameter and then the invariance under diffeomorphisms to piece these together between patches. However we are in a position to proceed more directly than this.

The standard notation for pseudodifferential operators on the fibres of a fibration $\phi : M \longrightarrow B$ is $\Psi^m(M/B; E, F)$, where E and F are bundles over M. Note that the fibration ϕ does not appear explicitly in the notation, which is designed (I suppose) to suggest the the operators are on 'M/B' which does not mean anything but could only be interpreted as the fibre.

Given a fibration $\phi : M \longrightarrow B$ we first define the fibre-product of this fibration with itself, $M^2_\phi \longrightarrow B$. Namely, M^2_ϕ is the restriction of $M \times M$, as a fibration over $B \times B$, to the diagonal $B \equiv \text{Diag} \subset B \times B$. The total space in then

$$M^2_\phi = M \times_\phi M = \{(m, m') \in M \times M; \phi(m) = \phi(m')\}. \tag{L15.10}$$

Thus the fibres of $M^2_\phi \longrightarrow B$, (where I use the same letter for the new fibration) are modelled on $Z \times Z$. Clearly the diagonal in M^2 is contained in M^2_ϕ where we may think of it as the 'fibre diagonal' Diag_ϕ so we have the embedded submanifold

$$M \equiv \text{Diag}_\phi \hookrightarrow M^2_\phi. \tag{L15.11}$$

Definition 6. The space of pseudodifferential operators on the fibres of a fibration $\phi : M \longrightarrow B$ is identified as

$$\Psi^m(M/B; E, F) = I^{m'}(M^2_\phi, \text{Diag}_\phi; \text{Hom}(E, F) \otimes \Omega_R),\ m' = m - \frac{1}{4}\dim B \tag{L15.12}$$

for any complex vector bundles E and F over M; here Ω_R is the bundle of fibrewise densities on the right, discussed more below.

Note that if $M = Z \times B$ is 'trivial' and E, F are the lifts of bundles over Z then

$$I^{m'}(M^2_\phi, \text{Diag}_\phi; \text{Hom}(E, F) \otimes \Omega_R) = \mathcal{C}^\infty(B; \Psi^m(Z; E, F)).$$

Thus, locally over $U \subset B$ over which the fibration is trivial and so small that E and F are the pull-backs of their restrictions E_b and F_b to Z_b, $\Psi^m(M/B; E, F)$ reduces to $\mathcal{C}^\infty(U; \Psi^m(Z; E_b, F_b))$. From this, and the definition, we can deduce all the basic properties.

In general, elements of $\Psi^m(M/B; E, F)$ cannot be elements of $\Psi^m(M; E, F)$ since the latter have kernels singular only on the diagonal of M whereas the kernels of the fibrewise operators are supported on M^2_ϕ. In fact the only elements in both are fibrewise differential operators (assuming the dimension of the base is positive).

(1) (Action) The elements of $\Psi^m(M/B;E,F)$ are continuous linear operators

$$\Psi^m(M/B;E,F) \ni A : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;F). \tag{L15.13}$$

Indeed, locally over any small open set, $U \subset B$, $u \in \mathcal{C}^\infty(M,E)$ becomes $u_b \in \mathcal{C}^\infty(U \times Z; E_b)$ and A maps this to $\mathcal{C}^\infty(U \times Z; F_b)$. Alternatively we can go back to the proof in the case of a single manifold and use the push-forward theorem.

These operators clearly act on the fibres. That is, if $u \in \mathcal{C}^\infty(M;E)$ and $u\big|_{Z_b} = 0$ then $Au\big|_{Z_b} = 0$. Hence A_b is well-defined by

$$A_b v = (Au)\big|_{Z_b},\ u \in \mathcal{C}^\infty(M;E) \text{ s.t. } u\big|_{Z_b} = v \text{ and } A_b \in \Psi^m(Z_b;E_b,F_b). \tag{L15.14}$$

(2) (Smoothing operators) The smoothing families are

$$\Psi^{-\infty}(M/B;E,F) = \mathcal{C}^\infty(M^2_\phi; \operatorname{Hom}(E,F) \otimes \Omega_R). \tag{L15.15}$$

(3) (Symbol map) For each point $b \in B$ the symbol of A_b, where $A \in \Psi^m(M/B;E,F)$, is an element of $\mathcal{C}^\infty(S^*Z_b; \hom(E_b,F_b) \otimes N_m)$ and in terms of a local trivialization of the fibration and bundles (i.e. local reduction to a product) depends smoothly on b. Let $T(M/B)$ be the subbundle of TM consisting of the vectors tangent to the fibre at each point. Thus, restricted to Z_b, $T(M/B)$ reduces to TZ_b. Let $T^*(M/B)$ be the dual bundle and $S^*(M/B)$ be the corresponding sphere bundle. Then, from the local properties, the symbol map becomes

$$\sigma_m : \Psi^m(M/B;E,F) \longrightarrow \mathcal{C}^\infty(S^*(M/B); \hom(E,F) \otimes N_m) \tag{L15.16}$$

where as usual, N_m is the bundle of functions homogeneous of degree m on $T^*(M/B)$ (as a bundle over $S^*(M/B)$).

(4) (Symbol sequence) The symbol map leads immediately to the short exact sequence

$$\Psi^{m-1}(M/B;E,F) \longrightarrow \Psi^m(M/B;E,F) \xrightarrow{\sigma_m} \mathcal{C}^\infty(S^*(M/B); \hom(E,F) \otimes N_m). \tag{L15.17}$$

(5) (Composition) Of course one of the most important properties of pseudodifferential operators is that they compose to give such operators. Agian it follows directly from the local picture, or using the same proofs as there but in the more global setting, that

$$\begin{gathered} A \in \Psi^m(M/B;F,G),\ B \in \Psi^{m'}(M/B;E,F) \Longrightarrow AB \in \Psi^{m+m'}(M/B;E,G) \text{ and} \\ \sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B). \end{gathered} \tag{L15.18}$$

(6) (Ellipticity) $A \in \Psi^m(M/B;E,F)$ is said to be elliptic (as a family) if each A_b is elliptic, which is the same as saying that the symbol has an inverse

$$(\sigma_m(A))^{-1} \in \mathcal{C}^\infty(S^*(M/B); \hom(F,E) \otimes N_m) \tag{L15.19}$$

Then, as in the case of a single operator, the ellipticity of A is equivalent to the existence of a two-sided parametrix $Q \in \Psi^{-m}(M/B;F,E)$ such that

$$QA - \operatorname{Id}_F \in \Psi^{-1}(M/B;F),\ AQ - \operatorname{Id}_E \in \Psi^{-1}(M/B;E). \tag{L15.20}$$

(7) (Asymptotic completeness) Using just the corresponding fact for conormal distributions we know that given a sequence $A_j \in \Psi^{m-j}(M/B; E, F)$ for $j \in \mathbb{N}_0$,

(L15.21)
$$\exists\ A \in \Psi^m(M/B; E, F) \text{ s.t. } A \sim \sum_j A_j \Longleftrightarrow A - \sum_{j=0}^{N} A_j \in \Psi^m(M/B; E, F)$$

and A is unique modulo $\Psi^{-\infty}(M/B; E, F)$.

Using these properties we can improve the parametrix for an elliptic operator from (L15.20). Namely let Q_0 be that operator, so

$$Q_0 A = \operatorname{Id} - R,\ R \in \Psi^{-1}(M/B; E). \tag{L15.22}$$

Then the formal Neumann series for $(\operatorname{Id} - R)^{-1}$, $\sum_j R^j$ is asymptotically summable, as is the product on the left with Q_0. Thus we can find

$$Q \sim \sum_j R^j Q_0 \in \Psi^{-m}(M/B; F, E) \Longrightarrow QA - \operatorname{Id}_E \in \Psi^{-\infty}(M/B; E). \tag{L15.23}$$

Similarly a right parametrix modulo smoothing operators can be constructed and shown to be equal to Q modulo smoothing, so Q also satisfies

$$AQ - \operatorname{Id}_F \in \Psi^{-\infty}(M/B; F).$$

L15.3. The analytic index. Now, we can proceed very much as in the case of the Toeplitz operators to discuss the families index theorem. Of course the geometry of the fibration M will cause complications. In fact we need another basic fact to proceed (the way I want, there are other approaches), namely a replacement for the projections on the first k terms in the Fourier series on the circle.

PROPOSITION 33. *For any fibration of compact manifolds, $\phi : M \longrightarrow B$ there is a sequence of projections $\pi_N \in \Psi^{-\infty}(M/B; E)$, for any vector bundle E, satisfying*

$$\operatorname{rank}(\pi_N) \le N,\ A\pi_N \longrightarrow A \text{ for any } A \in \Psi^{-\infty}(M/B; E, F) \tag{L15.24}$$

in terms of the usual topology on $\mathcal{C}^\infty(M^2_\phi; \operatorname{Hom}(E, F) \otimes \Omega_R)$.

Note that I am not assuming here that the projections are increasing, so it may be that $\pi_N \pi_{N+1} \neq \pi_N$ (and this product may not even be a projection).

PROOF. Missing – I do not yet have a reasonably elementary proof of this. There is one using Kuiper's theorem which I will resort to if necessary but I am still hoping to find something a bit better than that! It is pretty easy to do this in case of a product $Z \times B$ but the twisting of the bundle causes some trouble. □

Assuming the existence of such a family of projections we can proceed as in the Toeplitz case to construct the analytic index. Thus, given an elliptic family $A \in \Psi^m(M/B; E, F)$ choose a parametrix $Q \in \Psi^{-m}(M/B; F, E)$ as above, i.e. satisfying (L15.23), so $QA = \operatorname{Id} - R$, with $R \in \Psi^{-\infty}(M/B; E)$. Then $R\pi_N \longrightarrow R$ for a family of projections as in Proposition 33 and hence for N sufficiently large, $(\operatorname{Id} - R(\operatorname{Id} - \pi_N))^{-1}$ exists and is of the form $\operatorname{Id} - S$ with $S \in \Psi^{-\infty}(M/B; E)$. Thus, for N sufficiently large,

$$(\operatorname{Id} - S)QA(\operatorname{Id} - \pi_N) = \operatorname{Id} - \Pi_N \tag{L15.25}$$

from which it follows that

$$\operatorname{null}(A(\operatorname{Id}-\pi_N)) = \operatorname{Ran}(\pi_N) \text{ is a bundle over } B. \tag{L15.26}$$

It follows from this (just work locally as usual) that

$$\operatorname{null}((\operatorname{Id}-\pi_N)^* A^*) \text{ is a bundle over } B \tag{L15.27}$$

again of finite rank – for any choice of inner products and smooth densities used to define the adjoints. This latter bundle is a complement to the range of $A(\operatorname{Id}-\pi_N)$ as a subbundle of $\mathcal{C}^\infty(M/B; E)$ thought of as a bundle over B.

Proposition 34. *For any elliptic family in* $\Psi^m(M/B; E, F)$ *the symbol determines an element of* $K_c^*(T^*(M/B))$ *and the regularized null bundles in* (L15.26) *and* (L15.27) *determine an element of* $K^0(B)$ *and this correspondence projects to a well-defined map*

$$\operatorname{ind}_{\mathrm{a}} : K_c^0(T^*(M/B)) \longrightarrow K^0(B). \tag{L15.28}$$

Proof. We need to show independence of the choice of π_N, independence of the choice of A, given the symbol, homotopy invariance under deformation of the symbol (which amounts to homotopy invariance for A) constancy of the index class under stablization and under composition of the symbol with bundle isomorphisms; of course we also need to check that every compactly supported K-class on $T^*(M/B)$ arises from a symbol. All of this is pretty straightforward and pretty much as in the Toeplitz case. □

Next time I will introduct the algebra of product-type pseudodifferential operators on a fibration which I will use to identify this analytic index map with the topological index map defined by embedding of the fibration. This is the index theorem of Atiyah and Singer.

15+. Addenda to Lecture 15

15+.1. Some more details.

15+.2. The analytic index map (L15.28).

CHAPTER 16

Product-type symbols

Lecture 16: 10 November, 2005

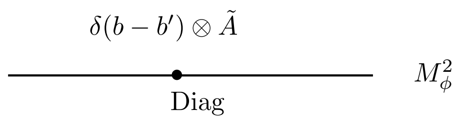
Last time I described the space of $\Psi^m(M/B; E, F)$ of pseudodifferential operators acting on the fibres of a fibration $\phi : M \longrightarrow B$. This is defined directly in terms of conormal distributions, as $I^{m'}(M^2_\phi, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R)$ where M^2_ϕ is the fibre diagonal, the set of pairs $(m, m') \in M^2$ such that $\phi(m) = \phi(m')$ and Diag is the diagonal of M^2, so the set of pairs $\{(m, m); m \in M\}$. Such an operator defines a map

$$\Psi^m(M/B; E, F) \ni A : \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; F), \tag{L16.1}$$

just as pseudodifferential operators on M do. It therefore has a Schwartz kernel on $M \times M$. This is easily seen to be, in terms of a local trivialization of ϕ

$$K_A(m, m') = \tilde{A}(m, m')\delta(b - b') \tag{L16.2}$$

where $\tilde{A}$ is the conormal distribution defining (and usually identified with) A. Thus there are two submanifolds of M^2 in the picture here, namely M^2_ϕ and Diag . These are nested as in the simple picture



Thus, in this simplified picture the kernels of elements of $\Psi^m(M/B; E, F)$ are singular all along the bigger submanifold, with a delta-singularity normal to it whereas the elements of $\Psi^m(M; E, F)$ have conormal singularities just at the smaller submanifold, and so are smooth outside it. It is then rather easy to see the following

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EXERCISE 20. Show that the elements of $\Psi^m(M;E,F)\cap\Psi^m(M/B;E,F)$ for a fibration with base and fibre of positive dimension are the fibrewise differential operators of order m (so for instance this intersection is empty if $m\in\mathbb{R}\setminus\mathbb{N}_0$).

For arguments in the proof of the index theorem, and for other reasons too, I want to define a larger class of 'pseudodifferential operators of product type' with respect to any fibration which is to include both the fibrewise pseudodifferential operators and the usual pseudodifferential operators on the total space of the fibration. To do this we return to the beginning and use the same pattern of definition as before. Namely, the operators will be defined, through their Schwartz kernels, in terms of a corresponding class of product-type conormal distributions

$$\Psi^{m,m'}_{\mathrm{pt}-\phi}(M;E,F)=I^{m-N,m'-N'}(M^2,M^2_\phi,\mathrm{Diag};\mathrm{Hom}(E,F)\otimes\Omega_R). \tag{L16.3}$$

Here m is the 'main order', m' is the 'fibre order' and on the right I am using as yet undefined notation for the conormal distibutions with respect to a nested pair of submanifolds; N and N' are dimension shifts as before.

So, to define (L16.3) we wish to define

$$I^{m,m'}(X,Y,Z;E)\subset\mathcal{C}^{-\infty}(X;E) \tag{L16.4}$$

the space of (product-type) conormal distributions (distributional sections of the bundle E) with respect to two embedded submanifolds

$$Z\subset Y\subset X. \tag{L16.5}$$

Here, somewhat confusingly, m is the 'order at Z' whereas m' is the 'order at Y.'

Following backwards through the previous argument, to define (L16.4) we will want to carefully discuss a model case which we take to be a vector space $\mathbb{R}^n$ with two subspaces. The variables along the smaller manifold Z in (L16.4) are intended to be 'smooth parameters' so we can take the smaller subspace to be $\{0\}$ and so consider as the model for a nested pair of submanifolds

$$\{0\}\subset\mathbb{R}^k_y\subset\mathbb{R}^n_{y,z}. \tag{L16.6}$$

Here (y,z) are linear coordinates, with $z=0$ being the larger of the subspaces so the $y=(y_1,\dots,y_k)$ are coordinates in it.

So now, we want to define

$$I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\}) \tag{L16.7}$$

where the subscript $\mathcal{S}$ is supposed to mean that the elements will have some sort of 'rapid decay' at infinity to compensate for the fact that $\mathbb{R}^n$ is not compact. Let me try to motivate the definition a little more. We want these spaces (for appropriate orders) to include both

$$I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})\text{ and }\mathcal{S}(\mathbb{R}^k_y;I^{m'}_{\mathcal{S}}(\mathbb{R}^{n-k},\{0\})) \tag{L16.8}$$

which we defined before, and the latter space being a natural candidate for the space of conormal distributions associated to $\mathbb{R}^k$ with 'rapid decrease at infinity'.

Now, the first space in (L16.8) is by definition $\mathcal{F}^{-1}\rho^{-M}(\overline{\mathbb{R}^n})$, in terms of the radial compactification of $\mathbb{R}^n$, which is a ball (which is where we started). The second space is defined by Fourier (inverse) transform on $\mathbb{R}^k$ so is

$$\mathcal{F}^{-1}_{\zeta\to z}\left(\mathcal{S}(\mathbb{R}^k;\rho^{-M'}\mathcal{C}^\infty(\overline{\mathbb{R}^{n-k}}_\zeta))\right),$$

To compare these two it is natural to take the Fourier transform in the y variables in the second space as well; since it is just Schwartz in these variables this gives the same space again.

So, assuming that we want to define our new space, (L16.7), as the inverse Fourier transform of some class of 'symbols' and we want it to 'include' (for appropriate orders) the two older spaces then the symbol space should include

$$\rho^{-M}(\overline{\mathbb{R}^n}_{\eta,\zeta}) \text{ and } \mathcal{S}(\mathbb{R}^k_\eta;\rho^{-M'}\mathcal{C}^\infty(\overline{\mathbb{R}^{n-k}}_\zeta)). \tag{L16.9}$$

One of the points I am trying to make in this course is that in such circumstances one should look for an appropriate compactification, of $\mathbb{R}^n$ in this case. I already briefly describe the 'correct' compactification in the addenda to Lecture 1, when I talked mostly about the radial compactification of vector space. The one I have in mind is the 'relative' compactification of a vector space with respect to a subspace. In this case

$$\mathbb{R}^n \hookrightarrow {}^V\overline{W},\ W=\mathbb{R}^n,\ V=\mathbb{R}^{n-k}=\{\eta=0\}\subset\mathbb{R}^n. \tag{L16.10}$$

Note that we have taken the Fourier transforms, so the symbols are defined on the dual of the original space. So the well-defined subspace is the annihilator of $\mathbb{R}^k_y\subset\mathbb{R}^n$, i.e. $\mathbb{R}^{n-k}_\zeta=\{\eta=0\}\subset\mathbb{R}^n$.

Let me recall the definition of this compactification from (1+.30), changing (and reversing the order of) the variables to fit (L16.10)

$$\begin{aligned} &R_V : W\ni w=(\eta,\zeta)\longmapsto(t,s,\eta',\zeta')=\\ &\left(\frac{1}{(1+|\eta|^2)^{\frac12}},\frac{(1+|\eta|^2)^{\frac12}}{(1+|\eta|^2+|\zeta|^2)^{\frac12}},\frac{\eta}{(1+|\eta|^2)^{\frac12}},\frac{\zeta}{(1+|\eta|^2+|\zeta|^2)^{\frac12}}\right)\in\mathbb{R}^2\times W. \end{aligned} \tag{L16.11}$$

As noted there, the image lies in the compact manifold with corners (product of two half-spheres)

$${}^V\overline{W}=\{(t,s,\eta',\zeta')\in\mathbb{R}^{2+n}; t^2+|\eta'|^2=1=s^2+|\zeta'|^2,\ t\ge0,\ s\ge0\}. \tag{L16.12}$$

In fact the image is precisely the interior ($s,t>0$) since the inverse there can be written

$$\eta=\eta'/t,\ \zeta=\zeta'/st, \tag{L16.13}$$

that is, (L16.11) is a diffeomorphism onto the interior of (L16.12) which is therefore a compactification.

Our symbol spaces will be the same type as before, just 'Laurent' functions, meaning smooth functions except for a (possibly non-integral) overall power behaviour at each boundary face. So, we arrive at the basic definition of (model) product-type conormal distributions

$$\begin{aligned} I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k_y\times\{0\},\{0\}) &= \mathcal{F}^{-1}(t^{-M}s^{-M'}\mathcal{C}^\infty({}^V\overline{W}),\\ & W=\mathbb{R}^n_{\eta,\zeta},\ V=\{\eta=0\},\ M=m,\ M'=m'. \end{aligned} \tag{L16.14}$$

To check that this is consistent with what I claim above, we want to know that

$$u\in I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k_y\times\{0\},\{0\})\Longrightarrow u\big|_{(y,z)\neq0}\in I^m(\mathbb{R}^n\setminus\{0\},\mathbb{R}^k\setminus\{0\}), \tag{L16.15}$$

$$I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})\subset I^{m,m+}(\mathbb{R}^n,\mathbb{R}^k,\{0\}), \tag{L16.16}$$

$$\mathcal{S}(\mathbb{R}^k_y;I^{m'}(\mathbb{R}^{n-k},\{0\})\subset I^{-\infty,m'}(\mathbb{R}^n,\mathbb{R}^l,\{0\}), \tag{L16.17}$$

Before trying to check these results and more, we need to look at the properties of the relative compactification. In particular we want to show some linear invariance as a prelude to eventual coordinate invariance, as in Lemma 2. The definition itself corresponds to choosing a transversal subspace to V and writing W as a product

$$W = V \times U,\ V = \{\zeta = 0\},\ U = \{\eta = 0\}. \tag{L16.18}$$

We want to show that, up to a diffeomorphism, ${}^V\overline{W}$ does not depend on the choice of U, otherwise the notation is defective (to say the least)!

Points of ${}^V\overline{W}$ fall into four classes, those in the interior, those at which $t = 0$ but $s > 0$, those at which $t > 0$ but $s = 0$ and those at which $t = 0, s = 0$. We can introduce local coordinates near each such point, although it is simpler just to introduct a local generating system (i.e. a set of functions which are smooth and which contain a coordinate system). We can safely ignore the interior, since this is just $\mathbb{R}^n$ with global coordinates η, ζ. As in (1+.35), (1+.36) observe that

$$\text{Near } t = 0, s > 0,\ \frac{1}{|\eta|},\ \frac{\zeta}{|\eta|},\ \frac{\eta}{|\eta|} \text{ generate,} \tag{L16.19}$$

$$\text{near } t > 0, s = 0,\ \frac{1}{|\zeta|},\ \frac{\zeta}{|\zeta|},\ \eta \text{ generate} \tag{L16.20}$$

$$\text{and near } t = 0, s = 0,\ \frac{1}{|\eta|},\ \frac{|\eta|}{|\zeta|}, \frac{\eta}{|\eta|},\ \frac{\zeta}{|\zeta|} \text{ generate} \tag{L16.21}$$

(where 'generate' can be read as 'are smooth and generate').

In fact, to see the first of these, observe that $t = 0$ implies $|\eta| = \infty$, meaning that in a sufficiently small neighbourhood (in ${}^V\overline{W}$) of such a point $|\eta| > R$ for any preassigned R ($R > 10$ say below.) Since $s > 0$ at the point, $s \geq s_0 > 0$ in a neighbourhood for some $s_0 > 0$, so

$$s = \left(1 + \frac{|\zeta|^2}{1 + |\eta|^2}\right)^{-\frac{1}{2}} > s_0 \Longrightarrow |\zeta| < C|\eta| \tag{L16.22}$$

where $C > 0$ depends on R and s_0, especially the latter. Thus as we approach the first type of boundary point, $|\eta| \to \infty$, maybe $|\zeta| \to \infty$ but no faster than a multiple of $|\eta|$ and (L16.19) follows since we can replace t by $1/|\eta|$, s by $|\zeta|/|\eta|$ etc. Similarly at the second type of boundary point $t \geq t_0 > 0$ in some neighbourhood so $|\eta|$ is bounded above, i.e. η is finite. Hence $|\zeta| \to \infty$ as we approach the point, since $s \to 0$. Thus we can replace (t, η') by η itself, then s by $1/|\zeta|$ and ζ' by $\zeta/|\zeta|$, giving (L16.20). In the third case, of a point on the corner, along any sequence approaching such a point, $|\eta| \to \infty$, since $t \to 0$ and $|\zeta| \to \infty$ since $s \to 0$ (using (L16.22)). In fact $|\zeta|/|\eta| \to \infty$ for the same reason. From this (L16.21) follows.

Note that near any particular point on the boundary we just need to drop one, or in the last case two, of the 'spherical' variables to get a coordinate system.

As noted back in (1+.34) this allows us to see which of the constant and linear vector fields lift. Namely

$$\partial_{\eta_i},\ \partial_{\zeta_k},\ \eta_i\partial_{\zeta_k},\ \zeta_l\partial_{\zeta_k},\ \eta_i\partial_{\eta_j} \tag{L16.23}$$

all lift to be smooth on ${}^V\overline{W}$ (for the obvious range of the indices) but $\zeta_l\partial_{\eta_i}$ does not. To see this, we can use homogeneity in terms of the coordinates derived from the generating functions. For instance, look at the corner and denote the first two functions in (L16.21) as r and R and an appropriate choice of coordinates from the spherical variables as ω. Then each of the vector fields lifts to be of the form

$$a(r,R,\omega)\partial_r + b(r,R,\omega)\partial_R + V(r,R,\omega) \tag{L16.24}$$

where V is a vector field in the ω's. The vector field is certainly smooth in $r > 0$, $R > 0$ since that is the interior. Moreover the scaling $r \to \lambda r$, $\lambda > 0$ corresponds precisely to $\eta \to \eta/\lambda$ and $\zeta \to \zeta/\lambda$. Under this combined scaling, all of the vector fields in (L16.23) are homogeneous, or degrees 0 or 1 (as is $\zeta_l\partial_{\eta_i}$). On the other hand the scaling $R \to \lambda R$ (with other variables fixed) corresponds precisely to $\zeta \to \zeta/\lambda$. Under this scaling all the vector fields in (L16.23) are homogeneous of degrees 0 or 1 still (whereas $\zeta_l\partial_{\eta_i}$ is homogeneous of degree -1). This homogeneity translates to homogeneity of the individual terms in (L16.24) and shows that the coefficients are all homogeneous of positive degrees, hence the vector fields lift to be smooth (and if you look a little more carefully, $\zeta_l\partial_{\eta_i}$ definitely does not.) They are all tangent to both boundary hypersurfaces. I have just been talking about a neighbourhood of the corner but the other regions of the boundary are similar with the discussion simpler (basically one of these homogeneities persists at each).

This proves Lemma 2. From (L16.11) we see immediately that the definition only depends on V, U, and the choice of Euclidean metrics on these spaces. That is, the group $\mathrm{O}(n-k) \times \mathrm{O}(k)$, which acts on W once the decomposition (L16.18) and choice of Euclidean metrics is fixed, lifts to act smoothly on ${}^V\overline{W}$, namely (O_η, O_ζ) acts through $(t, s, \eta', \zeta') \longmapsto (t, s, O_\eta\eta', O_\zeta\zeta')$. To show that the whole group

$$\{A \in \mathrm{GL}(W); AV = V\} \text{ lifts to act smoothly on } {}^V\overline{W} \tag{L16.25}$$

lifts to act smoothly on ${}^V\overline{W}$ observe that in terms of a splitting $W = V \oplus U$, this group consists of the lower triangular block matrices

$$\begin{pmatrix} A' & 0 \\ S & A'' \end{pmatrix},\ A' \in \mathrm{GL}(U),\ A'' \in \mathrm{GL}(V),\ S \in \hom(U,V). \tag{L16.26}$$

We have already seen the invariance under the block diagonal, orthogonal, matrices and modulo those (needed just to make sure that A' and A'' are both positively oriented) such a matrix can be connected to the identity in the group. Thus, it can be written as a product of exponentials of elements of the Lie algebra. However, the Lie algebra is spanned by the linear vector fields in (L16.23) so these exponentials are given by the integration of smooth vector fields on ${}^V\overline{W}$ and so all lift to diffeomorphisms.

Thus in fact the definition of ${}^V\overline{W}$ does not depend on the choices made in the explicit map (L16.11). This justifies the notation ${}^V\overline{W}$ for the compactification of a vector space W with respect to a subspace V. Note that there really is asymmetry in the definition, as there has to be if it is independent of the choice of U, the transversal, but not of V. One can also see this in terms of the important map back to the radial compactification.

Lemma 24. *The identification of the interiors of ${}^V\overline{W}$ and $\overline{W}$ with W extends to a smooth surjective map*

$$\beta : {}^V\overline{W} \longrightarrow \overline{W}. \tag{L16.27}$$

Proof. We only need to compare the compactification map (L16.11) with that corresponding to the radial compactification expressed in terms of these variables

$$R : W \ni w = (\eta, \zeta) \longmapsto (\tau, \eta'', \zeta'') = \left(\frac{1}{(1+|\eta|^2+|\zeta|^2)^{\frac12}}, \frac{\eta}{(1+|\eta|^2+|\zeta|^2)^{\frac12}}, \frac{\zeta}{(1+|\eta|^2+|\zeta|^2)^{\frac12}}\right) \in \mathbb{R} \times W. \tag{L16.28}$$

Clearly

$$\tau = st,\ \eta'' = s\eta',\ \zeta'' = \zeta' \tag{L16.29}$$

which shows that the map (L16.27) exists and is smooth. □

Notice from (L16.29) that β maps the boundary hypersurface $\{t = 0, s > 0\}$ in ${}^V\overline{W}$ onto the boundary of $\overline{W}$ except for the part $\partial\overline{V}$ where we regard $\overline{V} \subset \overline{W}$. This is actually the alternative construction of ${}^V\overline{W}$ which I will record here even though I have not defined the notion of blow up. It means 'introduce polar coordinates around the submanifold.'

Lemma 25. *The relative compactification ${}^V\overline{W}$ is canonically identified with the manifold obtained by blowing up the boundary $\partial\overline{V}$ in $\overline{W}$ (denoted by me $[\overline{W}, \partial\overline{V}]$).*

Now we know that the space of (model) product-type conormal distributions defined by (L16.14) is also invariant under linear transformations which preserve $\mathbb{R}^k$ (as a subspace of $\mathbb{R}^n$) because the Fourier transform converts this to the action of the transpose, which preserves the annihilator in the dual and we may use Lemma 2 which implies in particular that on the 'symbolic side'

$$A^*\left(t^{-M}s^{-M'}\mathcal{C}^\infty({}^V\overline{W})\right) = t^{-M}s^{-M'}\mathcal{C}^\infty({}^V\overline{W})\ \forall\ M, M' \in \mathbb{R} \text{ (or indeed } \mathbb{C}). \tag{L16.30}$$

Recall that the whole thrust of this definition is towards (L16.15) – (L16.17). So, consider (L16.16) first. This is a consequence of (L16.27) and (L16.29). Namely we are to show that

$$u \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \Longrightarrow u \in I^{m,m+}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k) \tag{L16.31}$$

where of course this must be true for any choice of k. By definition

$$\hat{u} = R^*a,\ a \in \rho^{-M}\mathcal{C}^\infty(\overline{W}),\ W = \mathbb{R}^n \tag{L16.32}$$

and then from (L16.27) and (L16.29) (which shows that $\beta^*\rho = st$)

$$\beta^*a \in t^{-M}s^{-M}\mathcal{C}^\infty({}^V\overline{W}),\ V = \mathbb{R}^{n-k},\ W = \mathbb{R}^n. \tag{L16.33}$$

However, β is just the canonical extension of the identification of the interiors so of course, $\beta R_V = R$ since they are equal on the interiors. Thus(L16.33) and (L16.32) mean that

$$\hat{u} = R_V^*b,\ b \in t^{-M}s^{-M}\mathcal{C}^\infty({}^V\overline{W}) \tag{L16.34}$$

which is (L16.31) and hence (L16.16) (for the moment all the order-normalizations are messed up or omitted here).

Next consider (L16.17). We want to do much the same as (L16.33) but we do not quite have the right map (with a little more blow-up technology we could get it). So, let me proceed more by hand. We already observed in the run up to (L16.9) that

$$u \in \mathcal{S}(\mathbb{R}^k_y; I^m_{\mathcal{S}}(\mathbb{R}^{n-k}, \{0\})) \Longrightarrow \hat{u} \in \mathcal{S}(\mathbb{R}^k_\eta; \rho_\zeta^{-M}\mathcal{C}^\infty(\overline{\mathbb{R}^{n-k}_\zeta})). \tag{L16.35}$$

Ignoring the factor of ρ_ζ^{-M} for the moment we want to show that (L16.35) implies that $\hat{u}$ extends from the interior (i.e. $\mathbb{R}^n$) to be smooth on ${}^V\overline{W}$. To do this we can consider the three regions of the boundary in (L16.19). Near the boundary $s = 0$, away from the corner in (L16.20), $\hat{u}$ is smooth, since it is a smooth function of η and of the generating functions $1/|\zeta|$ and $\zeta/|\zeta|$ of $\overline{\mathbb{R}_\zeta^{n-k}}$. Near the remainder of the boundary, covered by (L16.19) and (L16.21), $|\eta| \to \infty$. from (L16.35) we know that $\hat{u}$ is uniformly rapidly decreasing as $|\eta| \to \infty$, i.e.

$$|\hat{u}(\eta,\zeta)| \le C_N |\eta|^{-N} \text{ in } |\eta| > 1. \tag{L16.36}$$

Since the entries of the Jacobian of the singular changes of variables from $\zeta/|\eta|$ to $\zeta/(1+|\zeta|^2)^{\frac12}$ and $1/(1+|\zeta|^2)^{\frac12}$ are bounded by powers of $|\eta|$ it follows that $\hat{u}(\eta,\zeta)$ is in fact smooth down to the boundary $t = 0$ at which it vanishes to infinite order. That is,

$$\hat{u} \in t^\infty s^{-M} \mathcal{C}^\infty({}^V\overline{W}) \longrightarrow u \in I^{-\infty,m}(\mathbb{R}^n, \mathbb{R}^k, \{0\}). \tag{L16.37}$$

Additonal factors of ρ_ζ present no extra problems.

In fact we will later make use of the fact that

Lemma 26. *Under the identification as functions on the interior*

$$\mathcal{S}(\mathbb{R}^k_\eta; \rho_\zeta^{-M} \mathcal{C}^\infty(\overline{\mathbb{R}^{n-k}_\zeta}) \equiv t^\infty s^{-M} \mathcal{C}^\infty({}^V\overline{W}),\ W = \mathbb{R}^n,\ V = \mathbb{R}^{n-k}. \tag{L16.38}$$

I will prove the partial converse of this, (L16.15) next time and go through the extension to vector bundles and submanifolds, much as before, leading to the definition of the pseudodifferential operators through (L16.3).

16+. Addenda to Lecture 16

16+.1. More on the relative compactification. The relative compactification ${}^V\overline{W}$ is given by the map and image in (L16.11) for the vector spaces V and W in (L16.10). Observe that as well as the map (L16.27) there is a natural map

$${}^V\overline{W} \ni (t, s, \eta', \zeta') \longmapsto (t, \eta') \in \overline{W/V}. \tag{16+.39}$$

Certainly this map is smooth and surjective in the model setting. Furthermore it follows from the form of the general element of $\mathrm{GL}(W, V)$, i.e. an element of $A \in \mathrm{GL}(W)$ such that $A(V) \subset V$,

$$\mathrm{GL}(W,V) \ni A = \begin{pmatrix} A' & 0 \\ Q & A'' \end{pmatrix},\ A(\eta,\zeta) = (A'\eta, Q\eta + A''\zeta) \tag{16+.40}$$

that the map (16+.39) and the actions of $\mathrm{GL}(W,V)$ and $\mathrm{GL}(V)$ give a commutative diagram

$$\begin{tikzcd} \mathrm{GL}(W,V) \arrow[r] \arrow[d, dash] & \mathrm{GL}(V) \arrow[d, dash] \\ {}^V\overline{W} \arrow[r] & \overline{W/V}. \end{tikzcd} \tag{16+.41}$$

Thus the map (16+.39) is natural.

LEMMA 27. *The map (16+.39) is a fibration with fibre diffeomorphic to* $\overline{V}$. *Although not naturally a product the fibration is trivial and induces a fibration of the boundary face* $H_W = \{t = 0\}$ *of* ${}^V\overline{W}$,

$$H_W \longrightarrow S(W/V) = \partial\overline{W/V} \text{ with fibre diffeomorphic to } \overline{V}. \tag{16+.42}$$

The other boundary hypersurface, $H_V = \{s = 0\}$ *naturally decomposes as a product*

$$H_V = SV \times \overline{W/V},\ (t, 0, \eta', \zeta') \longmapsto ((0, \zeta'), (t, \eta')). \tag{16+.43}$$

PROOF. The map (16+.43) corresponds to the quotient of the group $\mathrm{GL}(W, V)$ by the normal subgroup $\begin{pmatrix} \mathrm{Id} & 0 \\ * & \mathrm{Id} \end{pmatrix}$ in (16+.40). Namely it gives a commutative diagram

$$\begin{tikzcd} \mathrm{GL}(W,V) \arrow[r] \arrow[d, no head] & \mathrm{GL}(V) \times \mathrm{GL}(W/V) \arrow[d, no head] \\ B_V \arrow[r, leftrightarrow] & SV \times \overline{W/V} \end{tikzcd} \tag{16+.44}$$

which shows that the product decomposition is natural.

On the other hand in (16+.42), on restriction to H_W the off-diagonal part of $\mathrm{GL}(W, V)$ still acts non-trivially, so the map is only naturally a trivial fibration (that it is a fibration follows from the explicit form of (16+.42) which presents it as a product). $\square$

The invariance of these maps shows that they extend directly to the corresponding bundle settings. In the geometric case discussed below, where

$$Z \subset Y \subset X \tag{16+.45}$$

are submanifolds, the vector space W and subspace V are replaced by the bundle and subbundle

$$N^*Y_Z \subset N^*Z \tag{16+.46}$$

with the conormal bundles being relative to X. Note that the quotient N^*Z/N^*_ZY may be naturally identified with the conormal bundle of Z as a submanifold of Y. Perhaps I will finally admit a relative notation for normal/conormal bundles and write

$$ {}^XN^*Z/\,{}^XN^*_ZY = {}^YN^*Z. \tag{16+.47}$$

Then the relative compactification becomes the manifold with corners

$$ {}^{N^*Y_Z}\overline{N^*Z} \tag{16+.48}$$

which has the two boundary hyersurfaces we can now associate to Z (corresponding to W above) and Y (corresponding to V) which are respectively fibred and have a product-bundle structure:-

$$\begin{tikzcd} \overline{{}^XN^*_zY} \arrow[r, no head] & H_Z \arrow[d] \\ & {}^YSN^*Z, \end{tikzcd} \tag{16+.49}$$

$$H_Y \equiv \overline{{}^YN^*Z} \times_Z {}^XSN^*_ZY.$$

We shall see later that there is a natural idenfication with the blow up

$$H_Z \equiv [\,{}^X SN^* Z,\, {}^X S^*_Z Y], \tag{16+.50}$$

of ${}^X S^*_Z Y$ as a submanifold of ${}^X SN^* Z$.

CHAPTER 17

Product-type conormal distributions

Lecture 17: 15 November, 2005

The aim today is to complete the definition of the spaces of product-type conormal distributions, $I^{m,m'}(X,Y,Z;E)$ where $Z \subset Y \subset X$ are embedded compact submanifolds (of positive codimension!) As for the case where Z is absent, we shall find symbol maps which capture the leading singularity. Here there are two symbol maps, one corresponding to singularities on Z associated, perhaps unfortunately, with the first order m and the second one associated with the second order m' and the singularity on Y (which does include Z but the singularities captured by this second symbol are only those 'conormal to Y).'

The idea, as for $I^m(X,Y;E)$, is to reduce to the case of the normal bundle to Y. Here, however it is useful to discuss first the normal bundle to Z in X and how it is related to Y.

The most significant difference between the old spaces $I^m(X,Y;E)$ and the new $I^{m,m'}(X,Y,Z;E)$ is that the symbol map for Y itself takes values in conormal distributions. Recall that the old symbol map was

(L17.1) $$\sigma_m : I^{m'}(X,Y;E) \longrightarrow \mathcal{C}^\infty(SN^*Y;E_Y \otimes N_{-m'})$$

where SN^*Y is the sphere bundle (thought of as the compactifying surface at infinity for $\overline{N^*Y}$) of the conormal bundle to Y in X. In the present case this has a submanifold corresponding to Z, namely

(L17.2) $$SN^*_ZY = \bigcup_{z\in Z} SN^*_zY \subset SN^*Z$$

just the union of the fibres over Z, i.e. the restriction of the bundle to Z; as indicated in (L17.2) this is a subbundle of the conormal bundle to Z itself. Then our modified 'Y-symbol' is to be part of a short exact sequence

(L17.3)
$$I^{m,m'-1}(X,Y,Z;E) \hookrightarrow I^{m,m'}(X,Y,Z;E) \xrightarrow{\sigma_Y} I^m(SN^*Y,SN^*_ZY;E_Y\otimes N_{-m'}),$$
$$\sigma_Y = \sigma_{Y,m,m'}.$$

There should be a picture here.

The other symbol, the Z-symbol, is more like the previous one

(L17.4) $$\sigma_m : I^m(X,Z;E) \longrightarrow \mathcal{C}^\infty(SN^*Z;E_Z \otimes N_{-m}).$$

The extra singularity still shows up in the replacement for this map, in that SN^*Z is to be replaced by the part of the boundary of the relative compactification ${}^{N^*_ZY}\overline{N^*Z}$

which corresponds to it under the 'blow-down map'

$$\beta : {}^{N^*_Z Y}\overline{N^* Z} \longrightarrow \overline{N^* Z} \tag{L17.5}$$

discussed last time. I denoted the 'lift' or 'proper transform' of the boundary, SN^*Z, of the radial compactification under β as $[SN^*Z, SN^*_Z Y]$. Note that this is *not* the preimage under β. Rather it is the closure of the preimage of $SN^*Z \backslash SN^*_Z Y$. The notation $[X, Y]$ makes sense for any embedded submanifold of any manifold, but I am using it here without full explanantion – I will add something to the addenda about this. So the modified form of the symbol map for Z becomes the short exact sequence

$$\begin{aligned} I^{m-1,m'}(X,Y,Z;E) &\hookrightarrow I^{m,m'}(X,Y,Z;E) \\ &\xrightarrow{\sigma} \mathcal{C}^\infty([SN^*Z, SN^*_Z Y]; E_Z \otimes N_{-m,-m'}),\ \sigma = \sigma_{Y,m,m'}. \end{aligned} \tag{L17.6}$$

So, in this new setting, the Z-symbol is a smooth function (or section of a trivial line bundle) over a compact manifold with boundary.

The total symbol is the combination of these two. Even though each of these symbols is surjective there is a compatibility condition between them. Namely the symbol for Z in (L17.6) restricts to the boundary of the blown-up manifold to define the 'corner symbol'

$$\sigma_{Y,m,m'}\big|_{S^*N(SN^*_Z Y)},\ \partial[SN^*Z, SN^*_Z Y] \equiv S^*N(SN^*_Z Y). \tag{L17.7}$$

Here $S^*N(SN^*_Z Y)$ is the sphere bundle of the conormal bundle to $SN^*_Z Y$ as a submanifold of SN^*Y. On the other hand, this is exactly where the symbol of an element of the image of $\sigma_{Y,m,m'}$ lives. The compatibility condition between the two symbols is then precisely

$$\gamma_{m,m'}(u) = \sigma_{Y,m,m'}(u)\big|_{S^*N(SN^*_Z Y)} = \sigma_m(\sigma_{Y,m,m'}(u)). \tag{L17.8}$$

That is, together these two maps give one joint symbol map giving a short exact sequence

$$\begin{gathered} I^{m-1,m'-1}(X,Y,Z;E) \hookrightarrow I^{m,m'}(X,Y,Z;E) \stackrel{\sigma_{m,m'}}{\longrightarrow} J^{m,m'}(Y,Z;E) \\ J^{m,m'}(Y,Z;E) = \big\{(a,v); a \in \mathcal{C}^\infty([SN^*Z, SN^*_Z Y]; E_Z \otimes N_{-m,-m'}), \\ v \in I^m(SN^*Y, SN^*_Z Y; E_Y \otimes N_{-m'}) \text{ s.t. } a\big|_{S^*N(SN^*_Z Y)} = \sigma_m(v)\big\}. \end{gathered} \tag{L17.9}$$

This does capture the 'full leading singularity' because

$$\bigcap_k I^{m-k,m'-k}(X,Y,Z;E) = \mathcal{C}^\infty(X;E) \tag{L17.10}$$

so in an iterative argument one would expect to finish up with smooth errors if all went well.

Note that we can also think in terms of the corner symbol in (L17.8) as being another symbol map. It corresponds to the short exact sequence

$$\begin{aligned} I^{m,m'-1}(X,Y,Z;E) + I^{m-1,m'}(X,Y,Z;E) &\hookrightarrow I^{m,m'}(X,Y,Z;E) \\ &\stackrel{\gamma_{m,m'}}{\longrightarrow} \mathcal{C}^\infty(SN^*(SN^*_Z Y); E_Z \otimes N_{-m,-m'}). \end{aligned} \tag{L17.11}$$

So, it remains to define the spaces $I^{m,m'}(X,Y,Z;E)$ and prove all these things. For the most part this goes through following the earlier model for conormal distributions with respect to a single submanifold. I will therefore concentrate on the new twists which arise and relegate many of the proofs to the addenda.

We start with the model case where Z is a point and Y is a linear subspace of Euclidean space, $\{0\} \subset \mathbb{R}^k \subset \mathbb{R}^n$. Now, last time I recalled the definition of the relative compactification ${}^V\overline{W}$ where $V \subset W$ is a linear subspace of a vector space. We want to consider it here for the *dual* spaces. Thus $V = (\mathbb{R}^k)^\circ \subset \mathbb{R}^n = W$ is the inclusion $\mathbb{R}^{n-k} \subset \mathbb{R}^n$. Recall that there is a smooth map

$$ {}^V\overline{W} \longrightarrow \overline{W} \tag{L17.12}$$

so $\mathcal{C}^\infty({}^V\overline{W})$ is 'bigger than' $\mathcal{C}^\infty(\overline{W})$ in the sense that the latter is naturally included in the former. With these identifications we defined

$$ I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\}) = \mathcal{F}^{-1}\left(\rho_W^{-M}\rho_V^{-M'}\mathcal{C}^\infty({}^V\overline{W})\right),\ M=m,\ M'=m'. \tag{L17.13}$$

Here the defining function for the boundary of $\overline{W}$ pulls back under the map (L17.12) to $\rho_V\rho_W$ where both are elements of $\mathcal{C}^\infty({}^V\overline{W})$, the one, ρ_W, defining the 'main face' at infinity (the one whose image is the whole of the boundary of $\overline{W}$) and the other defining the 'product-type' face which corresponds to V, hence the notation ρ_V.

For simiplicity of notation, set $m = m' = 0$ so that the powers in (L17.13) are removed. In this case the two symbols of $u \in I^{0,0}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\})$ are just the restrictions of $\mathcal{F}(u)$ to the two faces $\rho_W = 0$ and $\rho_V = 0$. Referring back to the defining map for ${}^V\overline{W}$ in (L16.11) to see what these two boundary faces are with the compactification given by the closure of the image in (L16.12), corresponding to a choice of splitting $W = V \times U$. Here

$$ \rho_W = t,\ \rho_V = s. \tag{L17.14}$$

From (L16.12) we see that there is an identification

$$ {}^V\overline{W} \supset \{s=0\} \longrightarrow \overline{U}\times SV,\ SV = \partial\overline{V}. \tag{L17.15}$$

At least in coordinates, the 'Y-symbol' can therefore first be identified with an (arbitrary) element of $\mathcal{C}^\infty(\overline{U}\times SV)$. If we now take the inverse Fourier transform on U we will get an element of $I^m(U',\{0\})$ (ignoring as usual niceties about the shifts in the order of conormal distributions). Since V is the dual of $\mathbb{R}^k$, we may identify the dual, U' of U with V and hence identify $SV \times U'$ with $SN^*Y = \mathbb{R}^k \times \mathbb{S}^{n-k-1}$, the sphere bundle of the conormal bundle of $Y = \mathbb{R}^k$ in $\mathbb{R}^n$. With this identification, which we have to check behaves properly under linear transformations,

$$ \begin{gathered} \sigma_Y : I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\}) \ni u \longmapsto a = \mathcal{F}(u)\big|_{s=0} \longrightarrow \\ \mathcal{F}_u^{-1}(a) \in \mathcal{C}^\infty(\mathbb{S}^{n-k-1}; I^{m'}_{\mathcal{S}}(\mathbb{R}^k,\{0\}) = I^{m'}_{\mathcal{S}}(\mathbb{R}^k\times\mathbb{S}^{n-k-1},\{0\}\times\mathbb{S}^{n-k-1}) \end{gathered} \tag{L17.16}$$

is exactly what we have anticipated for the 'Y-symbol'. It is a conormal distribution on the spherical conormal bundle to Y with respect to the submanifold given by the fibre over $Z = \{0\}$.

We have already shown that any linear transformation of a real vector space W which preserves a subspace $V \subset W$ lifts to a diffeomorphism of $\bar{[}V]W$. If $L \in \mathrm{GL}(n,\mathbb{R})$ preserves the subspace $\mathbb{R}^k = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ then for any $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$ \mathcal{F}(L^*u) = |\det(L)|^{-1}(L^t)^{-1}\mathcal{F}(u). \tag{L17.17}$$

Since the transpose preserves the annihlator $(\mathbb{R}^k)^\circ \subset \mathbb{R}^n$, we see directly from the definition that

(L17.18)
$$L^* : I_{\mathcal{S}}^{m,m'}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \longrightarrow I_{\mathcal{S}}^{m,m'}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \text{ if } L(\mathbb{R}^k) = \mathbb{R}^k,\ L \in \mathrm{GL}(n, \mathbb{R}).$$

Thus we do have linear invariance in the sense that under a general linear transformation

$$\text{(L17.19)} \qquad L^* : I_{\mathcal{S}}^{m,m'}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \longrightarrow I_{\mathcal{S}}^{m,m'}(\mathbb{R}^n, L^{-1}\mathbb{R}^k, \{0\}),\ L \in \mathrm{GL}(n, \mathbb{R}).$$

The other symbol map is surjective, essentially by definition, onto the space $\mathcal{C}^\infty([SW, SV])$ where this manifold with boundary, which looks like $SU \times \overline{V}$, is identified (by definition) with $\{t = 0\}$ in ${}^V\overline{W}$. Then the properties of the symbol maps corresponding to (L17.3) and (L17.6) are the exactness of

(L17.20)
$$I_{\mathcal{S}}^{0,-1}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \hookrightarrow I_{\mathcal{S}}^{0,0}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \xrightarrow{\sigma_Y} I_{\mathcal{S}}^{m'}(\mathbb{R}^k \times \mathbb{S}^{n-k-1}, \{0\} \times \mathbb{S}^{n-k-1})$$

and

$$\text{(L17.21)} \qquad I_{\mathcal{S}}^{-1,0}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \hookrightarrow I_{\mathcal{S}}^{0,0}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \xrightarrow{\sigma_Z} \mathcal{C}^\infty([SW, SV]).$$

Namely, in each case, a function in $\mathcal{C}^\infty({}^V\overline{W})$ is in $\rho_Y \mathcal{C}^\infty({}^V\overline{W})$ or $\rho_Z \mathcal{C}^\infty({}^V\overline{W})$ if and only if it vanish on $s = \rho_Y = 0$ or $t = \rho_V = 0$ respectively, and this means exactly that the restriction of the function to that boundary face vanishes.

The joint symbol $\gamma(u)$ is obtained simply by combining of these two symbols. The only compatibility condition between them is that the have the same restriction to the corner of ${}^V\overline{W}$, which is $SU \times SV$ and which defines the 'corner symbol' $\gamma(u) \in \mathcal{C}^\infty(SU \times SV)$. The vanishing of the joint symbol means that $\mathcal{F}(u)$ can be written as *stb* with b smooth giving the analogue of (L17.9) in this vector space setting:-

$$\text{(L17.22)} \quad \begin{gathered} I_{\mathcal{S}}^{m-1,m'-1}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \hookrightarrow I^{m,m'}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \xrightarrow{\sigma_{0,0}} J_{\mathcal{S}}^{0,0}(\mathbb{R}^n, \mathbb{R}^k) \\ J_{\mathcal{S}}^{0,0}(\mathbb{R}^n, \mathbb{R}^k) = \big\{(a, v); a \in \mathcal{C}^\infty([SW, SV]), \\ v \in I^m(\mathbb{R}^k \times \mathbb{S}^{n-k-1}, \{0\} \times \mathbb{S}^{n-k-1}) \text{ s.t. } a\big|_{SU \times SV} = \sigma(v)\big\}. \end{gathered}$$

The vanishing of the corner symbol γ implies that $\mathcal{F}(u) \in \mathcal{C}^\infty({}^V\overline{W})$ can be written as the sum of a smooth function vanishing at $t = 0$ and one vanishing at $s = 0$ (check this yourself!) giving the analogue of (L17.11)

(L17.23)
$$I_{\mathcal{S}}^{0,-1}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) + I_{\mathcal{S}}^{-1,0}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \hookrightarrow I_{\mathcal{S}}^{0,0}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \xrightarrow{\gamma_{0,0}} \mathcal{C}^\infty(SU \times SV).$$

Note that by iterative use of the 'Z-symbol' one would expect errors in

$$\text{(L17.24)} \qquad \bigcap_k I_{\mathcal{S}}^{m-k,m'}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) = I_{\mathcal{S}}^{m'}(\mathbb{R}^n, \mathbb{R}^k) = \mathcal{S}(\mathbb{R}^k; I_{\mathcal{S}}^{m'}(\mathbb{R}^{n-k}, \{0\}).$$

To see this equality note first that the middle space needs some comment even as regards its definition – simply because I did not define a 'tempered' conormal space in the general case of a subspace of a vector space (because I did not need it). However, the last equality serves as a reasonable definition of the middle space. If we take variables in $\mathbb{R}^n = \mathbb{R}_y^k \times \mathbb{R}_z^{n-k}$ then taking the Fourier transform to realize the conormal distributions at the origin in $\mathbb{R}^{n-k}$ as symbols we get

(L17.25)
$$\mathcal{F}_{z\to\zeta}\mathcal{S}(\mathbb{R}^k_y; I^{m'}_{\mathcal{S}}(\mathbb{R}^{n-k}_z,\{0\})) = \mathcal{S}(\mathbb{R}^k_y; \rho^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^{n-k}_\zeta})) = \rho_y^\infty\rho_\zeta^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^k_y}\times\overline{\mathbb{R}^{n-k}_\zeta}).$$

Now, taking the Fourier transform in y gives again Schwartz functions in the dual variable η. Thus the right two spaces in (L17.24) can be identified under Fourier transform in all variables with

$$\rho_\eta^\infty\rho_\zeta^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^k_\eta}\times\overline{\mathbb{R}^{n-k}_\zeta}). \tag{L17.26}$$

Thus, it remains to see that this is the same as the space on the left in (L17.24). In fact, by definition, Fourier transform gives a symbol in

$$\rho_V^{-m'}\rho_W^\infty\mathcal{C}^\infty({}^V\overline{W}) \tag{L17.27}$$

so it remains to see that these two spaces are the same (as spaces of functions on $\mathbb{R}^n_{\eta,\zeta}$). This is Lemma 26 from last time (unproven then, with proof in the addenda).

As an intermediate case (which I did not have time to include in the lecture) suppose that $N\longrightarrow Z$ is a vector bundle over a compact manifold Z. I am thinking here of the normal bundle to Z as a submanifold of some compact manifold X. Then we replace Y by its linearization in N, so suppose that $M\subset N$ is a subbundle over Z. To fit a little with the earlier notation, let W be the dual bundle of N and V the annihilator of M in W. Then, for any bundle E over Z we wish to define, and explore the properties of, $I^{m,m'}_{\mathcal{S}}(N,M,Z;E)$. This is rather easy (which is why I skipped it), since we may always take local trivializations of the bundle N in which it becomes $\mathbb{R}^n\times O$ over an open subset $O\subset Z$ with the identification such that $M=\mathbb{R}^k\times O$. If we assume that E is trivial over O as well, then we are reduced to smooth functions on O with values in (the direct sum of rank E copies of) $I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\})$. From the linear invariance of $I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\})$ the result space is independent of choice of trivialization and patches to give $I^{m,m'}_{\mathcal{S}}(N,M,Z;E)$.

We can alternatively proceed more directly, by taking the fibre Fourier transform on N and defining more directly

(L17.28)
$$I^{m,m'}_{\mathcal{S}}(N,M,Z;E) = \mathcal{F}^{-1}_{\text{fib}}\left(\rho_V^{-m'}\rho_W^{-m}\mathcal{C}^\infty({}^V\overline{W};E\otimes\Omega_{\text{fib}})\right),\ W=N^*,\ V=M^\circ.$$

Here we just observe that the fibre-by-fibre relative compactification gives a well-defined compact manifold with corners, fibred over Z so this makes good sense. Clearly this gives the same space as before. The second definition has the advantage that the symbol maps discussed above carry over directly and we get the analogous short exact sequences, except that everything is now fibred over Z.

Having briefly discussed the case of a bundle over Z we now consider the case of a bundle N over Y, a compact manifold with a given submanifold $Z\subset Y$. Of course, N will be the normal bundle for Y in a manifold X. Now the set up is completely global in Y and we will define the space by reduction to the previous case.

Lemma 28. *For a real vector bundle N over Y any submanifold $Z\subset Y$, identifed as a subset of the zero section of N, has a normal fibration F in N in which the image of the zero section of N lies in a subbundle $M_Y\subset NZ$.*

Given such a normal fibration, we can set, for any vector bundle over Y,

$$\begin{aligned} I_{\mathcal{S}}^{m,m'}(N,Y,Z;E) &= I_{\mathcal{S}}^{m'}(N,Y;E) \\ &+ F^*\{u \in I_{\mathcal{S}}^{m,m'}(NZ,M_Y,Z;E_F); \operatorname{supp}(u) \subset D(F) \subset NZ\}. \end{aligned} \tag{L17.29}$$

Here $D(F)$ is the open neighbourhood of Z in NZ which is the image of F and E_F is a bundle over Z with an identification to E over $D'(F)$, the domain of F. Of course to use this as a definition we need to check that the right side is independent of the normal fibration. This follows the usual pattern and will be included in the addenda (when I get around to it).

Naturally we also wish to show that the symbol maps extend to these spaces and have the properties which will lead to those displayed above. In fact we are now in the general case, except for more coordinate invariance. That is, we need to show that we can set

$$\begin{aligned} I^{m,m'}(X,Y,Z;E) &= \mathcal{C}^\infty(X;E) \\ &+ G^*\{u \in I_{\mathcal{S}}^{m,m'}(NY,Y,Z;E_Y); \operatorname{supp}(u) \subset D(F) \subset NY\} \end{aligned} \tag{L17.30}$$

where G is a normal fibration of Y in X with the usual identifications of bundles. Then the properties will reduce to the case in (L17.29). The main issues are to show that the symbol maps are well-defined, that they are surjective and that they have the null spaces as required to give the short exact sequences. For the symbol associated to Z this is rather clear. We already know it is unaffected by what happens away from Z so, apart from coordinate invariance, it drops back to the case (L17.28) of a bundle over Z where we already understand it.

So, it is more productive to talk about the Y symbol. This is global so needs to be discussed carefully. To see that it is well-defined we can proceed to make the decomposition in (L17.29) a little more definitive. Thus, we can choose a function $\psi \in \mathcal{C}_c^\infty(N)$ which is equal to one in a neighbourhood of Z and which has support in the domain of the normal fibration of Z in N. Then we the decomposition $u = \psi u + (1-\psi)u$ gives and element in $I^{m'}(N,Y;E)$ supported away from Z and an element in $v \in I_{\mathcal{S}}^{m,m'}(NZ,M_Y,Z;E_F)$ with compact support such that $\psi u = F^* v$. We may then define the symbol as the sum

$$\sigma_Y(u) = \sigma((1-\psi)u) + (F_*)^*\sigma_Y(v). \tag{L17.31}$$

These may both be directly interpreted as elements of the expected space

$$I^m(SN^*Y, SN_Z^*Y; E_Y \otimes N_{-m'}). \tag{L17.32}$$

Indeed the first term in (L17.31) is a smooth section of this bundle supported away from Z and the second is in this space from the discussion above. To prove that the result is well-defined we only need check that change of ψ does not affect the result. This just means showing that if u is supported away from Z but in the domain of the normal fibration then the two symbols are the same. This however follows from the definitions, which are the same away from Z.

This argument also shows surjectivity of σ_Y. Namely the second term in (L17.31) is of the form $\psi(F_*)^*\sigma_Y(v')$ and hence every conormal distribution arises this way. Conversely, if $\sigma_Y(u) = 0$ then u is certianly of order $m' - 1$ away from Z. Hence subtracting a term in $I^{m'-1}(X,Y;E)$ with support away from Z replaces u by a distribution supported in the domain of the normal fibration. Since its symbol can be computed from the bundle model and it vanishes by hypothesis, it must lie in $I^{m,m'-1}(X,Y,Z;E)$ and this proves the exactness of (L17.3).

The other claims may be established in much the same way.

The two inclusion which can be seen directly from the definition of product-type symbols are

$$I^m(X,Z;E) \subset I^{m,m}(X,Y,Z;E) \text{ for any } Y \supset Z,$$
$$I^{m'}(Y;E) \subset I^{-\infty,m'} \text{ for any } Z \subset Y. \tag{L17.33}$$

There is a third inclusion which I will use below. Namely (I will put something about this back in the addenda) for any submanifold $Y \subset X$ of any manifold (compact for simplicity of notation) there is an inclusion

$$\mathcal{C}^{-\infty}(Y;E_Y \otimes \Omega(N^*Y)) \ni v \longrightarrow v \otimes \delta_Y \in \mathcal{C}^{-\infty}(X;E) \tag{L17.34}$$

corresponding to 'tensoring with a delta function in the normal direction' and the density factor is there because the delta 'function' wants to be a density in the normal direction (and Ω stands for the absolute value of the maximal exterior power of the dual of any real bundle). In local coordinates in which Y is given by $z_1 = \cdots = z_{n-k} = 0$, the map (L17.34) is just

$$v(y) \longmapsto v(y)\delta(z_1)\ldots\delta(z_{n-k}). \tag{L17.35}$$

I leave you to check that it is independent of coordinates.

Then there is an inclusion

$$I^{m'}(Y,Z;E_Y) \longrightarrow I^{0,m'}(X,Y,Z;E) \tag{L17.36}$$

which extends to the more obvious inclusion

$$\mathcal{C}^{\infty}(Y;E_Y \otimes \Omega(N^*Y)) \longrightarrow I^0(X,Y;E) \tag{L17.37}$$

and in which the '0' arises as the order of the delta function as a conormal distribution.

17+. Addenda to Lecture 17

17+.1. Linear invariance.

CHAPTER 18

Product-type pseudodifferential operators

Lecture 18: 17 November, 2005

The main application I will make of the product-type conormal distribution, that I discussed last time, is to product-type pseudodifferential operators. Since these operators are associated to a fibration, let me start with a short discussion of the geometry of fibrations.

L18.1. Product-type operators defined. Thus consider a fibration,

(L18.1)
$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B. \end{tikzcd}$$

If we take the product fibration $M^2 \longrightarrow B^2$ and map the diagonal of B into the product, $B = \mathrm{Diag}_B \longrightarrow B^2$ then pulling back the product gives us the fibre product

(L18.2)
$$\begin{tikzcd} Z^2 \arrow[r, dash] & M^2_\phi \arrow[r, hook] \arrow[d, "\phi^2"] & M^2 \arrow[d, "\phi^2"] \\ & B \arrow[r, hook, "\mathrm{Diag}"] & B^2. \end{tikzcd}$$

Since the points of $M^2_\phi \subset M^2$ are exactly those mapped to the diagonal in B^2 under the fibration, we have a pair of embedded submanifolds

(L18.3)
$$\mathrm{Diag}_M \subset M^2_\phi \hookrightarrow M^2.$$

Note that M^2_ϕ is often called the fibre diagonal. In local coordinates z, y and z', y near different points in M but above the same point in B with respect to which the fibration is projection onto the second factor,

(L18.4)
$$\mathrm{Diag}_M = \{z = z',\ y = y'\} \subset M^2_\phi = \{y = y'\} \subset M^2.$$

Definition 7. The pseudodifferential operators on M of product type with respect to the fibration ϕ and acting between sections of bundle, E and F over M are identified as a space of kernels with the product-type conormal distributions

(L18.5)
$$\Psi^{m,m'}_{\phi-\mathrm{pt}}(M; E, F) = I^{\tilde m, \tilde m'}(M^2, M^2_\phi, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R).$$

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LEMMA 29. *The product-type pseudodifferential operators act continuously on smooth sections:*

$$\Psi^{m,m'}_{\phi-\mathrm{pt}}(M;E,F) \ni A : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;F). \tag{L18.6}$$

PROOF. This is a direct application of the simple push-forward theorem for product-type conormal distributions, as in the standard case before. Namely the product-type conormal distributions on any manifold X are a module over $\mathcal{C}^\infty(X)$:

$$\mathcal{C}^\infty(X) \cdot I^{m,m'}(X,Y,Z;E) \longrightarrow I^{m,m'}(X,Y,Z;E).$$

If $\pi : X \longrightarrow X'$ is a fibration which is transversal to *both* submanifolds and E is the lift of a bundle from the base then

$$\pi_* : I^{m,m'}(X,Y,Z;\pi^*E \otimes \Omega) \longrightarrow \mathcal{C}^\infty(X';\Omega)$$

is continuous.

In this case we consider the projection $\pi_L : M^2 \longrightarrow M$ as a fibration. For any bundles E and F over M the module product (L18.1) followed by composition in the fibres gives

$$\begin{aligned}\mathcal{C}^\infty(M^2;E) \cdot I^{m,m'}(M^2, M^2_\phi, \mathrm{Diag}; \mathrm{Hom}(E,F) \otimes \Omega_R) &\longrightarrow \\ &I^{m,m'}(M^2, M^2_\phi, \mathrm{Diag}; \pi_L^* F \otimes \Omega_R).\end{aligned} \tag{L18.7}$$

The cancellation of left densities, as in the standard case, allows us to interpret (L18.6) as the composition of the maps (L18.1) and (L18.1) (for π_L) with pull-back:

$$\begin{aligned}A : \mathcal{C}^\infty(M;E) \ni u \longmapsto \pi_R^* u \in \mathcal{C}^\infty(M^2;E) &\xrightarrow{\cdot A} \\ I^{m,m'}(M^2, M^2_\phi, \mathrm{Diag}; \pi_L^* F \otimes \Omega_R) &\xrightarrow{(\pi_L)_*} \mathcal{C}^\infty(M;F).\end{aligned} \tag{L18.8}$$

□

L18.2. Symbol maps. The symbol, acting on the space of conormal distributions, associated to the smaller submanifold M^2_ϕ takes values in (conormal) sections of a bundle over the the sphere bundle of the conormal bundle, $SN^*M^2_\phi$.

LEMMA 30. *For any fibration,* (L18.1)*, the sphere bundle of the conormal bundle to the fibre diagonal, may be naturally identified as the pull-back*

$$SN^*(M^2_\phi) = \pi^*(M^2_\phi) \text{ where } \pi : S^*B \longrightarrow B, \tag{L18.9}$$

as a fibration over S^*B with fibre Z^2 giving a commutative diagramme

$$\begin{tikzcd} Z^2 \arrow[r, dash] & SN^*(M^2_\phi) \arrow[r, "\pi"] \arrow[d] & M^2_\phi \arrow[r, hook] \arrow[d, "\phi^2"] & M^2 \arrow[d, "\phi^2"] \\ & S^*B \arrow[r, "\pi"] & B \arrow[r, hook, "\mathrm{Diag}"] & B^2. \end{tikzcd} \tag{L18.10}$$

PROOF. At any point $p \in M^2_\phi$ the conormal fibre in M^2, $N^*_p M^2_\phi$ is the space of differentials of functions vanishing on M^2_ϕ. Since M^2_ϕ is the preimage of Diag_B under the product fibration, this is just $(\phi^2)^* N^*_{\phi(p)} \mathrm{Diag}_B = T^*_{\phi(p)} B$. The same is therefore true of the spherical quotient, $SN^*_p M^2_\phi$ which is therefore identified with the pull-back of the fibration, $SN^*_p M^2_\phi = \pi^*_p S^* B$. In local coordinates this is just saying that $N^*_p M^2_\phi$ is spanned by the dy_j in terms of product coordinates. □

For any fibration manifold $SN^*(\mathrm{Diag}_M) = S^*M$ as usual where the identification comes from pull-back from the left factor of M.

LEMMA 31. *For a fibration, the boundary of the relative compactification of $N^*\,\mathrm{Diag}_M$ with respect to the subbundle $N^*_{\mathrm{Diag}_M} M^2_\phi$ may be identified with the blow-up $[S^*M, \phi^*S^*B]$ and this fibres*

$$\gamma : [S^*M, \phi^*S^*B] \longrightarrow S^*(M/B) \tag{L18.11}$$

with fibre, over $b \in B$, modelled on $\overline{T^*_bB}$; the boundary of $[S^*M, \phi^*S^*B]$ is naturally identified as

$$\partial[S^*M, \phi^*S^*B] = \pi^*S^*(M/B), \tag{L18.12}$$

the pull-back to S^*B of the bundle $S^*(M/B)$ over B.

PROOF. This follows from the earlier discussion of the relative compactificaiton of a vector bundle U with respect to a subbundle V. Namely, the 'main' boundary component of the relative compactification of U with respect to V may be identified with blow-up $[SU, SV]$, that this fibres over SU/V with fibres modelled on $\overline{V_p}$ (at any point p) and has boundary naturally diffeomorphic $\partial[SU, SV] \equiv \pi^*_V S(U/V)$ to the pull-back of $S(U/V)$ to SV. In the present case the base is M, $U = T^*M$ and $V = \phi^*T^*B$, the pull-back to M of the cotangent bundle to the base. Thus, $SU = S^*M$ and $SB = \phi^*S^*B$ and the 'main' boundary face, $[S^*M, \phi^*S^*B]$, fibres over the 'vertical sphere bundle' $S^*(M/B)$ with fibre modelled on the fibres of $\overline{T^*B}$ with the boundary, which is to say the corner of ${}^V\overline{U}$, being the pull-back to S^*B of $S^*(M/B)$. $\square$

With these reinterpretations of the manifolds on which the symbols of product-type conormal distributions are defined we may reinterpret the general symbol maps in the case of pseudodifferential operators to give

$$\begin{aligned} \sigma_{0,0} : \Psi^{0,0}_{\phi-\mathrm{pt}}(M; E, F) &\longrightarrow \mathcal{C}^\infty([S^*M, \phi^*S^*B]; \hom(E, F)), \\ \beta_{0,0} : \Psi^{0,0}_{\phi-\mathrm{pt}}(M; E, F) &\longrightarrow \Psi^0(\pi^*M/S^*B; E, F). \end{aligned} \tag{L18.13}$$

For operators of double order other than $0, 0$ we need to add appropriate 'homogeneity bundles' to the symbol maps

$$\begin{aligned} \sigma_{m,m'} : \Psi^{m,m'}_{\phi-\mathrm{pt}}(M; E, F) &\longrightarrow \mathcal{C}^\infty([S^*M, \phi^*S^*B]; \hom(E, F)) \otimes N_{m.m'}, \\ \beta_{m',m} : \Psi^{m,m'}_{\phi-\mathrm{pt}}(M; E, F) &\longrightarrow \Psi^{m'}(\pi^*M/S^*B; E, F \otimes N_{-m}). \end{aligned} \tag{L18.14}$$

These two maps are therefore separately surjective and have joint range the compatible subset

$$\sigma_{m,m'}(\beta_{m',m}(A)) = \sigma_{m,m'}(A)\big|_{\partial[S^*M, \phi^*S^*B]}. \tag{L18.15}$$

I will generally call $\sigma_{m,m'}(A)$ the 'usual symbol' since it is a fairly obvious extension of the standard symbol map. On the other hand I will call $\beta_{m',m}$ the 'base symbol'. This may be a rather contrarian name, since the base symbol is actually a family of fibre-wise pseudodifferential operators. However, these depend on the cotangent varriables in the base and this is why I think of it as the 'base' sybmol – it looks like the symbol of an operator on the base except that it takes values in pseudodifferential operators on the fibres instead of simply bundle homomorphisms.

L18.3. Composition. Also very much as in the standard case, the composite of two pseudodifferential operators, as maps (L18.6) is again a pseudodifferential operator of product type.

PROPOSITION 35. *For any fibration of compact manifolds* (L18.1) *and any three bundles E, F and G over M,*

$$\Psi^{m_1,m_1'}_{\phi-\mathrm{pt}}(M;F,G)\circ\Psi^{m_2,m_2'}_{\phi-\mathrm{pt}}(M;E,F)\subset\Psi^{m_1+m_2,m_1'+m_2'}_{\phi-\mathrm{pt}}(M;E,G) \tag{L18.16}$$

and the symbol maps are both homomorphisms, i.e. map products to products

$$\begin{aligned}\sigma_{m_1+m_2,m_1'+m_2'}(AB)=&\sigma_{m_1,m_1'}(A)\sigma_{m_2,m_2'}(B),\\ &A\in\Psi^{m_1,m_1'}_{\phi-\mathrm{pt}}(M;F,G),\ B\in\Psi^{m_2,m_2'}_{\phi-\mathrm{pt}}(M;E,F)\end{aligned} \tag{L18.17}$$

$$\beta_{m_1'+m_2'}:(AB)=\beta_{m_1'}(A)\beta_{m_2'}(B)\in\Psi^{m_1'+m_2'}(\pi^*M/S^*B;E,G\otimes N_{m_1+m_2}).$$

PROOF. This is basically the same as in the standard case – I did not go through it carefully in the lecture, but it is written out below in the addenda. ☐

L18.4. Ellipticity. If both symbols are invertible then A is said to be fully elliptic and then (in fact iff) it has a parametrix.

PROPOSITION 36. *If $A\in\Psi^{m,m'}_{\phi-\mathrm{pt}}(M;E,F)$ is fully elliptic in the sense that $\sigma_{m,m'}(A)$ has an inverse in $\mathcal{C}^\infty([S^*M,\phi^*S^*B];\hom(F,E))\otimes N_{-m,-m'}$ and $\beta_{m',m}(A)$ has an inverse in $\Psi^{-m'}(M/B;F,E\otimes N_{-m})$ then there exists $B\in\Psi^{-m,-m'}_{\phi-\mathrm{pt}}(M;F,E)$ such that*

$$A\circ B=\mathrm{Id}_F-R',\ B\circ A=\mathrm{Id}_E-R,\ R\in\Psi^{-\infty}(M;E),\ R'\in\Psi^{-\infty}(M;F). \tag{L18.18}$$

PROOF. This is a good opportunity to review the construction of a parametrix for an elliptic operator in the standard case, since the argument is almost precisely the same. ☐

Homotopy invariance of the index follows as before. Namely, if A_t is a smooth (in $t\in[0,1]$) family of elliptic operators then we can find a smooth family of parametrices B_t up to smoothing errors. The arguments leading to the formula

$$\mathrm{ind}(A_t)=\mathrm{Tr}(\mathrm{Id}_E-BA)-\mathrm{Tr}(\mathrm{Id}_F-AB) \tag{L18.19}$$

carry over directly to this more general setting and show that the index is smooth and integer-valued, hence constant.

REMARK 1. This suggests a harder index problem, which I hope to come back to before the end of the semester, namely what is the (families) index of A of product-type; it depends only on the (invertible) joint symbol $\sigma_{m,m'}(A)$, $\beta_{m',m}(A)$. Of course it is also the case that full ellipticity is quite a strong condition, since it requires the invertibility of a family of operators. On the other hand the index theorem in the standard case gives us a good hold on invertibility, after smoothing purturbation.

L18.5. Subalgebras. For the application to the index of ordinary pseudodifferential operators we need three important inclusions (see (L18.11)). The first is of the fibrewise operators.

Proposition 37. *For any fibration of compact manifolds*

(L18.20)
$$\Psi^{m'}(M/B;E,F)\subset\Psi^{0,m'}_{\phi-\mathrm{pt}}(M;E,F),\ \sigma_{m,m'}(A)=\gamma^*\sigma_{m',0}(A),\ \beta_{0,m'}(A)=A.$$

Proof. This is just the corresponding inclusion of conormal distributions discussed last time

$$I^{m'}(Y,Z;E\otimes\Omega_Y)\ni u\hookrightarrow u\cdot\delta_Y\in I^{0,m'}(X,Y,Z;E\otimes\Omega_X) \tag{L18.21}$$

in which a (conormal) distribution on Y, with respect to Z, is extended to X as a 'Dirac delta' in the normal variables. Locally (for the fibration case) this is rather obvious, since in product coordinates z,y and z',y (the same in the base, but possibly near different points in the fibre)

$$\begin{aligned}\Psi^{m'}(M/B;E)\ni A=A(y,z,z')\in I^m(Z^2;E\otimes\Omega_Z)\longrightarrow&\\ \delta(y-y')A(y,z,z')\in I^{\tilde m,\tilde 0}(M^2;M^2_\phi,\mathrm{Diag};E).&\end{aligned} \tag{L18.22}$$

As usual the densities take care of themselves (which one needs to check of course) and the symbol behaves as indicated in (L18.21). Namely, the base symbol comes from the (local) Fourier transform in y so recovers the operator and the usual symbol comes from the full Fourier transform on y,z which is constant in the dual to y. □

Similarly the inclusion of the standard pseudodifferential operators corresponds to the inclusion
$$I^m(X,Z;E)\longleftrightarrow I^{m,m}(X,Y,Z;E)$$
for any embedded submanifold $Z\subset Y$.

Proposition 38. *For any fibration of compact manifolds*

(L18.23)
$$\Psi^m(M;E,F)\subset\Psi^{m,m}_{\phi-\mathrm{pt}}(M;E,F),\ \sigma_{m,m}(A)=\sigma_m(A),\ \beta_{m,m}(A)=\sigma_m(A)\big|_{\phi^*S^*B}.$$

Thus in the second case the 'base symbol' is just the ordinary symbol – so acts as a bundle isomorphism on the fibres.

Perhaps the most important inclusion for us is that of pseudodifferential operators on the base. For any bundle E over M we may view $\mathcal{C}^\infty(M;E)$ as an infinite-dimensional bundle over B, it could be denoted $\mathcal{C}^\infty(M/B;E)$, with fibre isomorphic to $\mathcal{C}^\infty(Z;E\big|_Z)$. Suppose we have a family of smoothing projections, hence of finite rank,

$$\pi\in\Psi^{-\infty}(M/B;E),\ \pi^2=\pi(\text{ and }\pi^*=\pi\text{ if you want.}) \tag{L18.24}$$

Then the range of π is a finite dimensional bundle which sits inside $\mathcal{C}^\infty(M/B;E)$.

Proposition 39. *If $\pi_1\in\Psi^{-\infty}(M/B;E)$ has range isomorphic to a bundle $\tilde E$ over B and $\pi_2\in\Psi^{-\infty}(M/B;F)$ has range isomorphic to $\tilde F$ over B then*

$$\begin{aligned}\Psi^m(B;\tilde E,\tilde F)\ni A\longrightarrow\pi_2A\pi_1\in\Psi^{-\infty,m}_{\phi-\mathrm{pt}}(M;E,F),&\\ \sigma_{-\infty,m}(\pi_FA\pi_1)=0\ \textit{(by definition)},\ \beta_{m,-\infty}(\pi_2A\pi_1)=\pi_2\sigma(A)\pi_1.&\end{aligned} \tag{L18.25}$$

Proof. This corresponds to the general inclusion for product-type conormal distributions

$$I^m(X,Y;E)\subset I^{-\infty,m}(X,Y,Z;E). \tag{L18.26}$$

□

I have inserted smoothing operators in (L18.25) 'compressing' the pseudodifferential operator on the base so that it acts on a finite subbundle on the fibres because I felt this was clearer in the application below. One can instead consider an operator on the base as acting on the lifted bundles and then one arrives at

PROPOSITION 40. *For any fibration of compact manifolds there is a natural inclusion*

$$\Psi^m(B; E, F) \subset \Psi^{m,0}(M; \phi^*E, \phi^*F),\ \sigma_{m,0}(A) = \sigma_m(A),\ \beta_{0,m} = \sigma_m(A). \tag{L18.27}$$

L18.6. Connection.

DEFINITION 8. A connection on a fibration is a choice of complementary bundle to $T(M/B) \subset TM$ where

$$T_p(M/B) = \left\{v \in T_pM; v \text{ is tangent to } Z_{\phi(p)} = \phi^{-1}(\phi(p))\right\}. \tag{L18.28}$$

The complement corresponding to a connection is necessarily isomorphic to the lift of the tangent bundle to the base, $\phi^*(TB)$, corresponding to the short exact sequence

$$T(M/B) \longrightarrow TM \longrightarrow \phi^*TB. \tag{L18.29}$$

Thus a connection is a splitting of (L1.2) as a sequence of bundles over M.

L18.7. Tensor product construction. Finally, with this ammuntion (unverified as a lot of it is) we come to the main construction of Atiyah and Singer, at least from this point of view.

PROPOSITION 41. *If $B \in \Psi^0(M/B; E_+, E_-)$ is an elliptic family with trivial index bundle of rank 1 – more specifically which is surjective and has null bundle trivial of rank 1 – then for any elliptic operator $A \in \Psi^0(B; F_+, F_-)$ (having chosen inner products and densities) the operator*

$$P_A = A \otimes B = \begin{pmatrix} B & 0 \\ \pi_{\text{null}(B)} A \pi_{\text{null}(B)} & B^* \end{pmatrix} \in \Psi^{0,0}_{\phi-\text{pt}}(M; H_+, H_-), \tag{L18.30}$$
$$H_+ = E_+ \otimes F_+ \oplus E_- \otimes F_-,\ H_- = E_+ \otimes F_- \oplus E_- \otimes F_+$$

is elliptic with

$$\operatorname{ind}(A \otimes B) = \operatorname{ind}(A) \tag{L18.31}$$

and P_A is deformable, through fully elliptic elements of $\Psi^{0,0}_{\phi-\text{pt}}(M; H_+, H_-)$ to an element

$$\tilde{A} \in \Psi^0(M; H_+, H_-),\ \sigma(\tilde{A}) = \begin{pmatrix} \chi_1\sigma_0(B) & -\chi_2\sigma_0(A)^* \\ \chi_2\sigma_0(A) & \chi_1\sigma_0(B^*) \end{pmatrix} \tag{L18.32}$$

where $\chi_i \in \mathcal{C}^\infty(S^*M)$ form a partition of unity subordinate to the cover.

The operator P_A can be thought of as the 'Clifford tensor product' of A and B.

How are we going to use this? Given $A \in \Psi^m(B; \mathbb{E})$ (where I will start using 'superbundle' notation, with $\mathbb{E} = (E_+, E_-)$ and B acting between them) then given an embedding $B \hookrightarrow \mathbb{S}^N$ we may take a normal fibration to B. The normal bundle NB is itself is a bundle over B and if we take its 1-point compactification ${}^1\overline{NB}$ we get a fibration over B. The result above is applied to lift A to a pseudodifferential operator on ${}^1\overline{NB}$ with the same index (and the 'same' symbol in the sense of (L18.32). We can actually arrange that the lifted operator is completely trivial near the section 'at infinity' of the 1-point compactification and so extend it to $\mathbb{S}^N$, to be trivial outside the collar neighbourhood of B. This effectively reduces the index problem to $\mathbb{S}^N$, we we can solve it using Bott periodicity.

18+. Addenda to Lecture 18

18+.1. Fredholm condition and ellipticity. In the general mixed order case Sobolev spaces are needed to characterize ellipticity.

PROPOSITION 42. *If* $A \in \Psi^{0,0}(M;E,F)$ *then* A *is Fredholm as a map* $A : L^2(M;E) \longrightarrow L^2(M;F)$ *if and only if it is fully elliptic.*

18+.2. Proof of Proposition 40.

CHAPTER 19

Multiplicativity and excision

Lecture 19: 22 November, 2005

L19.1. Multiplicativity. Last time I set up the following single operator version of multiplicativity but did not complete the proof. The general case is no harder, the notational overhead is just heavier.

PROPOSITION 43. *[Multiplicativity] Consider a 'tower' of compact fibrations*

(L19.1)
$$\begin{tikzcd} S \arrow[r, dash] & \tilde{M} \arrow[d, "\tilde{\phi}"] \\ Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B \end{tikzcd}$$

and suppose that $P \in \Psi^0(\tilde{M}/M; \mathbb{E})$ is an elliptic family with trivial one-dimensional index bundle then for any ellptic family $A \in \Psi^0(M/B; \mathbb{F})$

(L19.2)
$$\operatorname{ind}(A) = \operatorname{ind}(P \otimes A) = \operatorname{ind}(A_P) \in K^0(B)$$

where $P \otimes A$ is the product-type family

(L19.3)
$$P \otimes A = \begin{pmatrix} P & 0 \\ A_{\operatorname{null}(P)} & P^* \end{pmatrix}$$

and $A_P \in \Psi^0(\tilde{M}/B; \mathbb{E} \otimes \mathbb{F})$ is any family with symbol

(L19.4)
$$\begin{pmatrix} \chi_1\sigma_0(P) & -\chi_2\sigma_0(A)^* \\ \chi_2\sigma(A)) & \chi_1\sigma_0(P)^* \end{pmatrix}$$

where χ_1, χ_2 is a partition of unity on $S^*(\tilde{M}/B)$ subordinate to the cover by the complements of $\tilde{\phi}^*(S^*(M/B))$ and $S^*(\tilde{M}/M)$ for some choice of connection on $\tilde{\phi}$.

PROOF. I set this up last time in the single operator case, where the bottom fibration just has one fibre and B is a point. Formally the general case is not very different. Thus, by assumption, the null space of the family P is a trivial line bundle over M. We can make A act between sections of $E_+ \otimes \tilde{\phi}^* F_+$ and $E_+ \otimes \tilde{\phi}^* F_-$ by considering it as the composite

(L19.5)
$$A_{\operatorname{null}(P)} = \pi_{\operatorname{null}(P)} A \pi_{\operatorname{null}(P)} : \Psi^{0,-\infty}_{\tilde{\phi}\text{-pt}}(\tilde{M}/B; E_+ \otimes \mathbb{F})$$

where I am using $\mathbb{F}$ to stand for the 'superbundle' $(F_+, F-)$ and $E_+ \otimes \mathbb{F}$ stands for $(E_+\otimes F_+, E_+\otimes F_-)$. Then (L19.3) is a well-defined family of product-type operators (it is a family over B, and of product-type with respect to $\tilde{\phi}$.)

We know by direct computation that its index is $\operatorname{ind}(A)$, at least if A has been stabilized to have a smooth null bundle. Namely, the null space of $P \otimes A$ consists of pairs $(u, v) \in \mathcal{C}^\infty(\tilde{M}; E_+ \otimes F_+) \oplus \mathcal{C}^\infty(\tilde{M}; E_+ \otimes F_-)$ satisfying

$$\begin{aligned} Pu = 0 &\Longrightarrow u \in \mathcal{C}^\infty(M; E_+ \otimes \operatorname{null}(P)), \\ A_{\operatorname{null}(P)}u + P^*v = 0 &\Longrightarrow Au = 0,\ v = 0. \end{aligned} \tag{L19.6}$$

In the second line we use the fact that the range of P^* is orthogonal to the null space of P so the two terms must vanish separately. Then $Au = 0$ just recovers the null space of A. For the adjoint

$$(P \otimes A)^* = \begin{pmatrix} P^* & A^*_{\operatorname{null}(P)} \\ 0 & P \end{pmatrix} \tag{L19.7}$$

we similarly conclude that (u', v') in the null space implies that $u = 0$, $v \in \mathcal{C}^\infty(M; E_- \otimes \operatorname{null}(P))$ and then $u \in \operatorname{null}(A^*)$.

We also 'know' (I only did it in the single operator case in fact) that a homotopy of totally elliptic product-type pseudodifferential operators has constant index in K-theory of the base; I will add this to the addenda. So we proceed to deform the family (L19.3) but keeping total ellipticity. Recall that the family $P \otimes A$ is totally elliptic because it symbol and base symbol are respectively

$$\begin{aligned} \sigma_{0,0}(P \otimes A) &= \begin{pmatrix} \sigma_0(P) & 0 \\ 0 & \sigma_0(P)^* \end{pmatrix} \\ \beta_0(P \otimes A) &= \begin{pmatrix} P & 0 \\ \sigma_0(A)\pi_{\operatorname{null}(P)} & P^* \end{pmatrix}. \end{aligned} \tag{L19.8}$$

Now, we have the partition of unity χ_1, χ_2 on $S^*(\tilde{M}/B)$ in which χ_2 is supported near the lift of $\overline{T^*(M/B)}$ under $\tilde{\phi}$. This means that $\chi_2\sigma_0(A)$ is a well-defined symbol in a neighbourhood of the 'non-commutative' front face – on the fibres of which it is constant. Take an element $\tilde{A} \in \Psi^0(\tilde{M}/B; E_+ \otimes \mathbb{F})$ which has symbol $\chi_2\sigma_0(A)$ and consider the curve of operators

$$\begin{pmatrix} P & -t\tilde{A}^* \\ (1-t)A_{\operatorname{null}(P)} + t\tilde{A} & P^* \end{pmatrix} \in \Psi^{0,0}_{\tilde{\phi}\text{-pt}}(\tilde{M}/B; \mathbb{E} \otimes \mathbb{F}). \tag{L19.9}$$

The claim is that this remains elliptic. Its usual symbol is just

$$\begin{pmatrix} \sigma_0(P) & -t\chi_2\sigma_0(A)^* \\ t\chi_2\sigma(A) & \sigma(P)^* \end{pmatrix}. \tag{L19.10}$$

The crucial property of this (Clifford) tensor product matrix is that it is invertible because the 'diagonal' part is invertible. Consider an element (α, α') of the null space. Note that $\sigma(P)$ and $\sigma(A)$ commute, because the act on different factors of the tensor product, so

$$\begin{aligned} &\sigma_0(P)\alpha - t\chi_2\sigma_0(A)^*\alpha' = 0,\ t\chi_2\sigma(A)\alpha + \sigma(P)^*\beta = 0 \Longrightarrow \\ &\qquad \alpha = t\chi_2\sigma_0(A)^*\sigma_0(P)^{-1}\alpha' = -t^2\chi_2^2\sigma_0(A)^*\sigma_0(A)(\sigma(P)^*)^{-1}\sigma_0(P)^{-1}\alpha. \end{aligned} \tag{L19.11}$$

Thus the null space is trivial, because of the invertibility of $\sigma_0(P)$ and hence the operator is 'symbolically' elliptic. The non-commutative, or base symbol is

$$\begin{pmatrix} P & -t\chi_2\sigma_0(A) \\ (1-t)\sigma_0(A)_{\mathrm{null}(P)} + t\sigma_0(A) & P^* \end{pmatrix} \tag{L19.12}$$

since (as a fibre family) P is its own base symbol. We know this to be invertible for $t = 0$ and for $t > 0$ a similar argument applies. The symbol preserves the decomposition coming from the null space of P. On it, it is invertible because it is actually constant in t. Off the null space of P it is invertible because of the invertibility of P and an argument just like (L19.11) but now using P instead of its symbol. Note that P and $\sigma_0(A)$ commute because the latter is fibre constant for $\tilde{\phi}$ and acts on a different bundle in the tensor product. Thus we arrive at the operator with $t = 1$. Now choose an element

$$\tilde{P} \in \Psi^0(\tilde{M}/B; \mathbb{F} \otimes \mathbb{E}_+) \text{ with } \sigma_0(\tilde{P}) = \chi_1\sigma_0(P) \tag{L19.13}$$

and perform the homotopy

$$\begin{pmatrix} (1-t)P + t\tilde{P} & -\tilde{A}^* \\ \tilde{A} & (1-t)P^* + t\tilde{P}^* \end{pmatrix} \in \Psi^{0,0}_{\tilde{\phi}\text{-pt}}(\tilde{M}/B; \mathbb{E} \otimes \mathbb{F}). \tag{L19.14}$$

The 'commutative' symbol of this is

$$\begin{pmatrix} (1-t)\sigma_0(P) + t\chi_1\sigma_0(P) & -\chi_2\sigma_0(A)^* \\ \chi_2\sigma(A) & (1-t)\sigma(P)^* + t\chi_2\sigma_0(P)^* \end{pmatrix}$$

which remains invertible everywhere. Similarly the 'non-commutative' symbol is

$$\begin{pmatrix} (1-t)P & -\sigma_0(A) \\ \sigma_0(A)_{\mathrm{null}(P)} & (1-t)P^* \end{pmatrix}$$

which is invertible because of the invertibilty of $\sigma_0(A)$. Thus the family remains elliptic throughout the deformation and we finally arrive at

$$\begin{pmatrix} \tilde{P} & -\tilde{A}^* \\ \tilde{A} & \tilde{P}^* \end{pmatrix} \in \Psi^0(\tilde{M}/B; \mathbb{E} \otimes \mathbb{F}) \tag{L19.15}$$

which is an elliptic family in the usual sense with symbol (L19.4). Thus (L19.2) follows. $\square$

Corollary 7. *For an iterated fibration* (L19.1), *if* $b \in K^0_c(T^*(\tilde{M}/M))$ *has* $\mathrm{ind}(b) = 1 \in K^0(M)$ *then tensor product gives a commutative diagramme*

$$\begin{tikzcd} K^0_c(T^*(M/B)) \arrow[rr, "\otimes b"] \arrow[dr, "\mathrm{ind}"'] & & K^0_c(T^*(\tilde{M}/B)) \arrow[dl, "\mathrm{ind}"] \\ & B & \end{tikzcd} \tag{L19.16}$$

where for $[(\mathbb{E}, a)] \in K^0_c(T^*(M/B))$, $b \otimes [(\mathbb{E}, a)]$ *is represented by*

$$\begin{pmatrix} \chi_1 b & -\chi_2 a \\ \chi_2 a & \chi_1 b^* \end{pmatrix} \tag{L19.17}$$

where $b = [(\mathbb{F}, b)]$.

Proof. The main thing to check is that the top map in (L19.16) is well-defined, using (L19.17). This is straightforward. $\square$

L19.2. Excision.

PROPOSITION 44. *[Excision.] Let $M_i \longrightarrow B$ be fibrations of compact manifolds and suppose $i_j : E \hookrightarrow M_j$, $j = 1, 2$, are inclusions of a non-compact manifold as an open subset giving a commutative diagramme,*

(L19.18)
$$\begin{tikzcd} M_1 \arrow[dr, "\phi_1"'] & U \arrow[l, hook', "i_1"'] \arrow[r, hook, "i_2"] & M_2 \arrow[dl, "\phi_2"] \\ & B & \end{tikzcd}$$

then the diagramme

(L19.19)
$$\begin{tikzcd} & K^0_c(T^*(M_1/B)) \arrow[dr, "\mathrm{ind}"] & \\ K^0_c(T^*(U/B)) \arrow[ur, "(i_1)_*"] \arrow[dr, "(i_2)_*"'] & & K^0(B) \\ & K^0_c(T^*(M_2/B)) \arrow[ur, "\mathrm{ind}"'] & \end{tikzcd}$$

is commutative.

PROOF. The main issue is to understand the maps $(i_j)_*$ induced by the inclusions. A representative of an element of $K^0_c(T^*(U/B))$ is a triple (E_+, E_-, a) where $E_\pm$ are bundles over U and a is an isomophism between the lifts of them outside a compact subset $K \subset T^*(U/B)$. The fact that the two fibrations are the same on U means that $T^*(U/B)$ is a well-defined bundle over U, identified by the i_j with $T^*)i_j(U)(M_j/B)$. The image, K', of K under projection to U is compact and a is therefore defined over the whole of the bundle $T^*_{U\setminus K'}(U/B)$. We can use the restriction of a to the zero section to identify the two bundles E_+ and E_- over $U\setminus K''$, where K'' is the image of a slightly larger compact subset of $T^*(U/B)$ which contains K' in its interior and having done this use the fibre homogeneity of the bundle to give a homotopy between a and a' which is now the identity isomorphism between E_+ and E_- in $U \setminus K''$. Now, recall from the definition that E_+ and E_- are in any case supposed to be trivial outside a compact set, so we may replace (E_+, E_-, a) by a representative in which $E_\pm$ are trivial outside a compact subset of U and $a = \mathrm{Id}$ outside such a set. Of course a need not be the identity outside a compact subset of $T^*(U/B)$. Then the maps are given by extending $E_\pm$ and a trivially outside U to give well-defined maps

(L19.20)
$$(i_j)_* : K^0_c(T^*(U/B)) \longrightarrow K^0_c(T^*(M_j/B).$$

Now, the index is defined by quantizing the 'symbol' a – deformed to be homogeneous of degree 0 outside the zero section of $T^*(M_j/B)$ to a family of pseudodifferential operators. We know that the result is independent of the family chosen with the given symbol, so we may choose the families to be of the form $P_j \in \Psi^0(M_j/B; E_+, E_-)$ and to be equal to the identity outside $M_j \setminus K_j$ for $K = i_j(K)$ the image of a compact subset of U. Thus, $P_j - \mathrm{Id}$ is to have its Schwartz kernel supported in $K_j \times K_j$. Now, in this sense the two families of operators are 'exactly the same'. We only have to make sure that nothing goes wrong in the stabilization process to define the index as the difference of the null and conull bundles. Of course we may start with the 'same' parameterices for the P_j – each being $\mathrm{Id} - Pj'$ where P_j' has kernel support in $K_j \times K_j$ where they are same. The remaining problem here is that I did not do the stabilization procedure fully in the families case. Here I will refer to an alternative stabilization procedure – the relationship between this and the other one (which I did not complete!) will be added to the addenda.

To define the families index we need to stabilize the null space, or the range, to a bundle. One way to do this is to add an auxilliary finite dimensional map. Namely

Lemma 32. *If $P \in \Psi^m(M/B;\mathbb{E})$, $\mathbb{E} = (E_+, E_-)$, is elliptic then there is a smooth map $S \in \mathcal{C}^\infty(B \times M; \hom(\mathbb{C}^N; E_-))$ such that*

$$\big(P \oplus S\big) : \begin{matrix} \mathcal{C}^\infty(M;E_+) \\ \oplus \\ \mathcal{C}^\infty(B;\mathbb{C}^N) \end{matrix} \longrightarrow \mathcal{C}^\infty(M;E_-) \text{ is surjective.} \tag{L19.21}$$

Proof. For each point $b \in B$ we know that the range of P has finite codimension. We can therefore find a finite number $v_i \in \mathcal{C}^\infty(Z_b; E_-)$, of smooth sections which span a complement. Extending them to smooth sections of E_- over M (say supported close to b) will mean, by continuity, that the v_j span the range of $P_{b'}$ for b' in a neighbourhood of b. Now, by compactness we may cover B by a finite number of such neighourhoods with corresponding $v_{j,k} \in \mathcal{C}^\infty(M; E_-)$ as k ranges over some finite set. Now, let N be the total number of such sections and let S be the linear map from $\mathbb{C}^N \ni a_{j,k}$ to $\sum\limits_j a_{j,k} v_{j,k}$. The sum $P + S$ is surjective at each point of the base, since it is constructed to be surjective when to the subspace with $a_{j,k} = 0$ for all but one value of k. Now, the fact that $P + S$ is surjective leads, by the same argument as before, to the conclusion that the null spaces form a smooth finite dimensional subbundle of the bundle $\mathcal{C}^\infty(M/B; E_+) \oplus \mathbb{C}^N$ as a bundle over B. The claim (or definition depending on how you look at it) is that

$$\operatorname{ind}(P) = [\operatorname{null}(P + S), \mathbb{C}^N] \in K^0(B). \tag{L19.22}$$

In fact it is easy to see that two choices or S are stabily homotopic – just put all the choices together, maybe refine the covering to a common one such that one each set one of each stabilizations works, and then do an appropriate homotopy]. □

With this 'definition' of the families index, we may complete the proof of excision. Namely the stabilizing sections can always be chosen to have support in K_j and we may take the same stabilization for the two operators P_1 and P_2. □

L19.3. Atiyah-Singer index theorem. Now we can state the first form of the Atiyah-Singer families index theorem – in K-theory.

Theorem 10. *If $\phi : M \longrightarrow B$ is a fibration of compact manifold then the analytic index map*

$$\operatorname{ind} : K^0(T^*(M/B)) \longrightarrow K^0(B) \tag{L19.23}$$

defined by quantization of symbols, is equal to the topological index map, i.e. can be factored through any embedding of the fibration

$$\begin{tikzcd} M \arrow[rr, "e"] \arrow[dr, "\phi"'] & & \mathbb{R}^n \times B \arrow[dl, "\pi_B"] \\ & B, & \end{tikzcd} \quad \textit{so}$$

(L19.24)
$$\begin{tikzcd} K^0(T^*(M/B)) \arrow[r, "\otimes b"] \arrow[dr, "\mathrm{ind}"'] & K^0_c(T^*(NM/B)) \arrow[r, "i_*"] & K_c(\mathbb{R}^{2N} \times B) \arrow[dl, "\mathrm{ind}"] \\ & K^0(B) & \end{tikzcd} \quad \textit{commutes}$$

where $\otimes b$ is the product with the Bott element for the normal fibration.

19+. Addenda to Lecture 19

CHAPTER 20

Chern character

Lecture 20: 29 November, 2005

We have the families index theorem in K-theory and now I want to discuss the image in cohomology.

Recall that in terms of K-theory we have shown that for any fibration of compact manifolds $Z \longrightarrow M \xrightarrow{\phi} B$ an elliptic element $A \in \Psi^m(M/B; E_+, E_-)$ can be stabilized by the addition of $A' \in \Psi^{-\infty}(M/B; E_+, E_-)$ so that the null spaces form a bundle and then

$$\text{(L20.1)} \qquad \operatorname{ind}(A) = [(\operatorname{null}(A + A'), \operatorname{null}((A + A')^*))] \in K^0(B)$$

is the analytic index. As an element of the K-group it only depends on the image of the symbol of A in $K^0_{\mathrm{c}}(T^*(M/B))$.

Then for an embedding of the fibration

$$\begin{tikzcd} M \arrow[rr, "e"] \arrow[dr, "\phi"'] & & B \times \mathbb{S}^N \arrow[dl, "\pi_1"] \\ & B & \end{tikzcd}$$

we can replace A with a family $P_A \in \Psi^0(B \times \mathbb{S}^N/B; G_+, G_-)$ having symbol given in terms of the Bott element and cut-offs

$$\text{(L20.2)} \qquad \begin{pmatrix} \chi_1 b & -\chi_2 a^* \\ \chi_2 a & \chi_2 b^* \end{pmatrix}$$

in a collar neighbourhood of M and extended outside as the identity, with the property that $\operatorname{ind}(A) = \operatorname{ind}(P_A)$ in $K^0(B)$. This constructs a commutative diagram

$$\text{(L20.3)} \qquad \begin{tikzcd} K^0_{\mathrm{c}}(T^*(M/B)) \arrow[rr, "e_!"] \arrow[dr, "\operatorname{ind}"'] & & K^0_{\mathrm{c}}(\mathbb{R}^{2N} \times B) \arrow[dl, "\operatorname{ind}"] \\ & K^0(B) & \end{tikzcd}$$

where the index map on the right we 'understand completely' in the sense that it is given by repeated application of Bott periodicity, the index isomorphism for the Toeplitz calculus.

The traditional interpretation of (L20.3) is that the embedding construction defines the topological index, so the commutativity of (L20.3) is the equality of analytic and topological indexes. We can also think of it as an effective tool for

0.7E; Revised: 29-11-2006; Run: November 29, 2006

computing the index. This will be more apparent in the generalization to product-type operators below.

L20.1. Review of Chern-Weil theory. Let $E \longrightarrow X$ be a complex vector bundle over a compact manifold. Then E always admits an affine connection which is to say a first order differential operator $\nabla \in \mathrm{Diff}^1(X; E, \Lambda^1 \otimes E)$:

$$\mathcal{C}^\infty(X; E) \xrightarrow{\nabla} \mathcal{C}^\infty(X; \Lambda^1 \otimes E) \tag{L20.4}$$

which has the property

$$\nabla(fu) = df \otimes u + f\nabla u \ \forall \ f \in \mathcal{C}^\infty(X),\ u \in \mathcal{C}^\infty(X; E). \tag{L20.5}$$

If ∇ is a connection on E and $a : E \longrightarrow \tilde{E}$ is a bundle isomorphism then $\tilde{\nabla}\tilde{u} = a\nabla(a^{-1}\tilde{u})$ is a connection on $\tilde{E}$. If $E = \mathbb{C}^N$ is trivial then d itself, acting on the coefficients, is a connection. If $\rho_i \in \mathcal{C}^\infty(X)$ is a partition of unity and ∇_i is a connection on E over an open set containing the support of ρ_i then

$$\nabla = \sum_i \rho_i \nabla_i \tag{L20.6}$$

is a connection on E. Combining these observations we see that any complex bundle does indeed admit a connection.

Any connection has a natural extension to a superconnection, which is to say an operator $\nabla \in \mathrm{Diff}^1(X; \Lambda^* X \otimes E)$ which satisfies

$$\nabla(\alpha \otimes u) = d\alpha \otimes u + (-1)^k \alpha \otimes \nabla u \ \forall \ \alpha \in \mathcal{C}^\infty(X; \Lambda^k),\ u \in \mathcal{C}^\infty(X; E),\ \forall\ k. \tag{L20.7}$$

The superconnection corresponding to an ordinary connection clearly satisfies the grading condition

$$\nabla \in \mathrm{Diff}^1(X; \Lambda^k \otimes E, \Lambda^{k+1} \otimes E) \ \forall \ k. \tag{L20.8}$$

The sign change corresponds to anticommuting ∇ past k wedge factors. Namely we can just insist on (L20.7) to get the superconnection with the connection on the right side; of course one still needs to check that the result is consistent. I will not distinguish between the connection ∇ and the superconnection it defines.

This allows us to define the curvature as the square of the connection which is always a bundle map

$$\mathcal{C}^\infty(X; \Lambda^2 \otimes \hom(E)) \ni \omega_\nabla = \nabla^2 \in \mathrm{Diff}^2(X; E, \Lambda^2 \otimes E). \tag{L20.9}$$

To see this, just observe that ∇^2 commutes with multiplication by any smooth function

$$\nabla^2(fu) = \nabla(df \otimes u + f\nabla u) = d^2 f \otimes u + (df \otimes \nabla u - df \otimes \nabla u) + f\nabla^2 u = f\nabla^2 u.$$

If $a : E \longmapsto \tilde{E}$ is a bundle isomorphism and $\tilde{\nabla} = a\nabla a^{-1}$ is the transformed connection then the curvature of $\tilde{\omega}$ of $\tilde{\nabla}$ is $a\omega_\nabla a^{-1}$. A connection on E induces a connection on the dual bundle E^* by demanding that

$$du^*(u) = \nabla^* u^*(u) + u^*(\nabla u) \ \forall \ u^* \in \mathcal{C}^\infty(X; E^*),\ u \in \mathcal{C}^\infty(X; E). \tag{L20.10}$$

The curvature of ∇^* is the transpose of the curvature of ∇. Similarly if E and F are bundles with connections ∇_E and ∇_F then the direct sum has the obvious connection $\nabla_E + \nabla_F$ with curvature $\omega_E + \omega_F$. Connections on E and F also induce a connection on $E \otimes F$ where for any sections

$$\nabla_{E\otimes F} u \otimes v = \nabla_E u \otimes v + u \otimes \nabla_F v. \tag{L20.11}$$

The curvature of this connection is easily computed

$$\omega_{E\otimes F} = \omega_E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \omega_F. \tag{L20.12}$$

Combining these two constructions we see that $\hom(E, F)$ also acquires a connection from connections on E and on F. Namely if we identify $\hom(E, F) = F \otimes E^*$ the connection is the tensor product of ∇_F and ∇_E^*. Alternatively one can see that the commutation formular

$$(\nabla a)u = \nabla_F(au) - a(\nabla_E)u,\ \forall\ u \in \mathcal{C}^\infty(X; E) \tag{L20.13}$$

defines the action of the connection on $a \in \mathcal{C}^\infty(X; \hom(E, F)$. Bianchi's identity, which comes from computing ∇_E^3 in two ways, then becomes the identity

$$\nabla_E \omega_E = 0 \tag{L20.14}$$

where ∇_E is also written for the (super) connection action on $\hom(E)$.

So, having defined the curvature of a connection we may define the Chern character form, or just the Chern character, of the bundle with connection as

$$\lambda_E = \operatorname{tr} \exp(\frac{i}{2\pi}\omega_E). \tag{L20.15}$$

The normalizing constant, $i/2\pi$, is put in for reasons of rationality (and is sometimes left out). To understand (L20.15) note first that the tensor product $\Lambda^* \otimes \hom(E)$ is a bundle of algebras over X. The product is just the tensor product of wedge and matrix products

$$(\alpha_p \otimes a_p) \cdot (\beta_p \otimes b_p) = \alpha_p \wedge \beta_p \otimes (a_p \circ b_p),\ \alpha_p,\ \beta_p \in \Lambda_p^*,\ a_p,\ b_p \in \hom(E_p). \tag{L20.16}$$

Then the exponential in (L20.15) is computed with respect to this product

$$\exp(\frac{i}{2\pi}\omega) = \mathrm{Id} + \sum_{k=1}^{\infty} \frac{i^k}{(2\pi)^k k!}\omega^k. \tag{L20.17}$$

Since ω takes values in 2-forms the sum is finite, since the power vanishes identically for $2k > \dim X$. Thus each term in the sum in (L20.17) is a smooth section of the bundle $\Lambda^{2k} \otimes \hom E$ over X. The trace functional, defined on $\hom(E)$ extends naturally to the tensor product

$$\operatorname{tr} : \mathcal{C}^\infty(X; \Lambda^j \otimes \hom E) \longrightarrow \mathcal{C}^\infty(X; \Lambda^j) \tag{L20.18}$$

and this is the meaning of (L20.15)

$$\lambda_E = r + \sum_{k=1}^{\infty} \frac{i^k}{(2\pi)^k k!} \operatorname{tr}(\omega^k) \in \mathcal{C}^\infty(X; \Lambda^{\mathrm{evn}}) \tag{L20.19}$$

where r is the rank of E (and the trace of the identity acting on it). Note that under a bundle isomorphism $a : E \longrightarrow \tilde{E}$ the form λ_E for a connection ∇ on E is the same as the form for the connection $a\nabla a^{-1}$ on $\tilde{E}$.

Lemma 33. *For any $a \in \mathcal{C}^\infty(X; \Lambda^k \otimes \hom E)$ and any connection*

$$d \operatorname{tr}(a) = \operatorname{tr}(\nabla a). \tag{L20.20}$$

Proof. We can cover X by open sets U_i over each of which E is trivial. Over these sets tr is given as the sum of the diagonal entries of the (form-valued) matrix a_i representing a. The connection on E over U_i can be compared to the trivial connection d and written $\nabla = d + \gamma_i$ where γ_i is a matrix valued in 1-forms (this follows directly from the definition of a connection); the action of the connection on a homomorphism, represented as a matrix, is then just

$$\nabla a = da + [\gamma_i, a]. \tag{L20.21}$$

Using a partition of unity ρ_i subordinate to the cover

$$\begin{aligned} d\,\mathrm{tr}(a) &= \sum_i d\,\mathrm{tr}(\rho_i a_i) = \sum_i \mathrm{tr}(d(\rho_i a_i)) \\ &= \sum_i \mathrm{tr}(d\rho_i a_{jk} + [\gamma_i, \rho_i a_i]) = \sum_i \mathrm{tr}(\nabla_E \rho_i a) = \mathrm{tr}(\nabla a). \end{aligned} \tag{L20.22}$$

□

From this lemma if follows immediately that

$$d\lambda_E = \mathrm{tr}\left(\nabla_E \exp(\frac{i}{2\pi}\omega_E)\right) = 0 \tag{L20.23}$$

since $\nabla\,\mathrm{Id} = 0$ and ∇ (acting on homomorphism) distributes over products, so $\nabla_E \omega_E^k = 0$ for every k.

PROPOSITION 45. *The cohomology class of λ_E in $H^{evn}(X;\mathbb{C})$ is independent of the connection on E used to define it and this defines a group homomorphism*

$$\mathrm{Ch} : K^0(X) \longrightarrow H^{evn}(X;\mathbb{C}), \tag{L20.24}$$

the Chern character.

PROOF. To show the independence of the choice of connection we use a standard 'transgression' analysis. Suppose ∇ and ∇' are two connections on E. Then

$$\tilde{\nabla} = (1-t)\nabla + t\nabla' + \partial_t dt \tag{L20.25}$$

is a connection on the bundle $\pi^* E$ over $[0,1] \times X$ where π is the projection onto X. Let $\tilde{\lambda}$ be the Chern form of this connection. From the discussion above, $\tilde{\lambda}$ is a (sum of) closed form(s) on $[0,1] \times X$ so, decomposing in terms of t-dependent forms on X

$$\tilde{\lambda} = \lambda' + dt \wedge \mu,\ d\tilde{\lambda} = 0 \Longrightarrow \partial_t \lambda = d\mu. \tag{L20.26}$$

Now, the Chern forms of ∇ and ∇' are respectively $\lambda'\big| + t = 0$ and $\lambda'\big|_{t=1}$ which are cohomologous since

$$\lambda'\big|_{t=1} - \lambda'\big|_{t=0} = \int_0^1 \partial_t \lambda' dt = d\int_0^1 \mu dt. \tag{L20.27}$$

For the direct sum of two bundle $E \oplus F$ we can choose a direct sum connection. Then, as noted above, the curvature is the sum of the curvatures, the one acting on E the other on F. As such an product of the two curvatures vanishes, so

$$\exp(\frac{i}{2\pi}(\omega_E + \omega_F)) = \exp(\frac{i}{2\pi}\omega_E) + \exp(\frac{i}{2\pi}\omega_F) \Longrightarrow \lambda_{E\oplus F} = \lambda_E + \lambda_F. \tag{L20.28}$$

This shows that the map

$$K^0(X) \ni [(E_+, E_-)] \longrightarrow [\lambda_{E_+} - \lambda_{E_-}] \in H^{\mathrm{evn}}(X;\mathbb{C}) \tag{L20.29}$$

is well-defined, since it is invariant under the addition of the same bundle to both E_+ and E_- and under bundle isomorphisms. □

As well as being an Abelian group, $K^0(X)$ is a ring with the product being induced by the tensor product of bundles. In fact we have already used this in the construction of P_A above. Suppose that $\mathbb{E}$ and $\mathbb{F}$ are superbundles, just $\mathbb{Z}_2$-graded bundles, $\mathbb{E} = (E_+, E_-)$ and $\mathbb{F} = (F_+, F_-)$. Then the graded tensor product is the bundle $\mathbb{G} = (G_+, G_-)$ where

$$G_+ = (E_+ \otimes F_+) \oplus (E_- \otimes F_-),\ G_- = (E_+ \otimes F_-) \oplus (E_- \otimes F_+).$$

It is straightforward to check that the equivalence class of $\mathbb{E} \otimes \mathbb{F}$ is determined by the classes of $\mathbb{E}$ and $\mathbb{F}$ and that this product on $K^0(X)$ is Abelian.

Since we know that for the tensor product of connections on E and F the curvature of $E \otimes F$ is $\omega_E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \omega_F$ it follows directly that

$$\lambda_{E\otimes F} = \lambda_E \wedge \lambda_F. \tag{L20.30}$$

Using the formula for direct sums as well and setting $\lambda_{\mathbb{E}} = \lambda_{E_+} - \lambda_{E_-}$ it follows that

$$\lambda_{\mathbb{E}\otimes\mathbb{F}} = \lambda_{\mathbb{E}} \wedge \lambda_{\mathbb{F}} \tag{L20.31}$$

as well. Thus in fact the Chern character is a multiplicative map

$$\mathrm{Ch} : K^0(X) \longrightarrow H^{\mathrm{evn}}(X;\mathbb{C}),\ \mathrm{Ch}(a \cdot b) = \mathrm{Ch}(a) \wedge \mathrm{Ch}(b)\ \forall\ a, b \in K^0(X) \tag{L20.32}$$

where the wedge product in deRham theory is the usual cup product. With a little more care it can be seen that Ch is well defined mapping into rational cohomology. It is important to know

Theorem 11. *(Atiyah-Hirzebruch) After tensoring with $\mathbb{C}$ the Chern character becomes and isomorphism*

$$K^0(X) \otimes \mathbb{C} \xrightarrow{\simeq} H^{evn}(X;\mathbb{C}). \tag{L20.33}$$

I will not discuss the proof of this (nor use it), although I hope that there is a treatment in the present spirit – at the moment I do not know one.

L20.2. Toeplitz families index. Recall that for elliptic families of Toeplitz operators, $A : B \longrightarrow \Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ the families index theorem gives us Bott periodicity

$$\mathrm{ind} : K^{-2}(B) \longrightarrow K^0(B). \tag{L20.34}$$

Namely we can stabilize the symbol of the Toeplitz family

$$a = \sigma(A) \in \mathcal{C}^\infty(B \times \mathbb{S}; \mathrm{GL}(N,\mathbb{C})) \hookrightarrow \mathcal{C}^\infty(B \times \mathbb{S}; G^{-\infty})$$

and we can compose with the inverse of $\sigma(A)(b,1)$, as a bundle isomorphism over B, to normalize the symbol so that $A(b,1) = \mathrm{Id}$. This normalization does not change the index and a defines an element $[a] \in K^{-2}(B)$, as the homotopy class of $a : B \longrightarrow G^{-\infty}_{(1)}$, the pointed loop group. This gives the map (L20.34) which we know to be an isomorphism.

Thus the Chern character as discussed above on $K^0(B)$ induces a similar map from K^{-2} :

$$\begin{tikzcd} K^{-2}(B) \arrow[r, "\mathrm{ind}_{\mathcal{T}}"] \arrow[dr, dotted, "\mathrm{Ch}"'] & K^0(B) \arrow[d, "\mathrm{Ch}"] \\ & H^{\mathrm{evn}}(B;\mathbb{C}). \end{tikzcd} \tag{L20.35}$$

I hope the notation is not be too confusing.

What we want is an explicit representative of this map in terms of $a \in \mathcal{C}^\infty(B \times S; \mathrm{GL}(N,\mathbb{C}))$, the Toeplitz symbol.

PROPOSITION 46. *For any $a \in \mathcal{C}^\infty(B \times \mathbb{S}; \mathrm{GL}(N,\mathbb{C}))$ or $a \in \mathcal{C}^\infty(B \times \mathbb{S}; G^{-\infty})$ the Chern character is*

$$\mathrm{Ch}([a]) = \sum_{k=0}^{\infty} \frac{i^{k+1}k!}{(2\pi)^{k+1}(2k+1)!} \int_{\mathbb{S}} \mathrm{Tr}((a^{-1}da)^{2k+1}. \tag{L20.36}$$

The integrand in (L20.36) is a form on $B \times \mathbb{S}$ and the integral means push-forward. That is the form is $\alpha \wedge d\theta + \beta$ where α and β are θ-dependent forms on B and (L20.36) means the integral, with respect to θ, of α.

PROOF. Recall that by stabilizing and extensively deforming a we reduced it to the form

$$a = \pi_-(b)e^{-i\theta} + \pi'_-(b) + \pi'_+(b) + \pi_+(b)e^{i\theta} \tag{L20.37}$$

where $\pi_\pm(b)$ are smooth families of projections on two trivial bundles $\mathbb{C}^{M_\pm}$ and $\pi'_\pm(b) = \mathrm{Id} - \pi_\pm(b)$ are their complementary projections. Thus, a is an elliptic symbol acting on $\mathbb{C}^M$, $M = M_- + M_+$. We know that if we quantize a to the family of Toeplitz operators

$$A = \pi_-(b)L + \pi'_-(b) + \pi'_+(b) + \pi_+(b)U \in \mathcal{C}^\infty(B; \Psi^0_T(\mathbb{S}; \mathbb{C}^M)) \tag{L20.38}$$

then its null spaces for the bundle $E_- = \pi_-(b)\mathbb{C}^M_-$ realized in the constant functions on the circle and similarly for the adjoint, so the index is

$$\mathrm{ind}(a) = [\mathbb{E}] = [(\pi_-\mathbb{C}^{M_-}, \pi_+(b)\mathbb{C}^{M_+})] \in K^0(B). \tag{L20.39}$$

So, we need to compute the Chern forms for these two bundles, presented as the ranges of smooth families of projections on trivial bundles. For simiplicity of notation I will drop the signs for the moment and consider a subbundle $E = \pi(x)\mathbb{C}^N$ of a trivial bundle over a manifold X. Notice that this bundle is in no way special. So we need a connection on E and the obvious one is

$$\mathcal{C}^\infty(X; E) \ni u \longmapsto \pi(x)du \in \mathcal{C}^\infty(X; \Lambda^1 \otimes E). \tag{L20.40}$$

Here d acts on the coefficients. Now, we can write this operator as

$$\pi(x)d = d + (\mathrm{Id} - \pi(x))d = d + \pi'(x)d\pi(x) : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; \Lambda^1 \otimes E)$$

where d acts on the coefficients of $\pi(x)$ as a matrix. The superconnection takes the same form so the curvature is

(L20.41)

$$\omega_E u = (d+\pi'(x)d\pi(x))^2 u = d^2u + d(\pi'(d\pi)u) + \pi'(d\pi)du + \pi'(d\pi)\pi'(d\pi)u = (d\pi' \wedge d\pi)u$$

where I have used the identities that come from differentiating $\pi^2 = \pi$, namely $\pi'(d\pi)\pi' = \pi(d\pi)\pi = 0$. Here the wedge product is to be understood in terms of antisymmetrizing the value on the tangent space, not commutation of homomoprhisms. Since the curvature is acting on E we can write it out more fully as

$$\omega_E = -\pi(d\pi)(\pi')(d\pi)\pi \tag{L20.42}$$

and as already noted the product is in Λ^* hom. Thus the Chern character form for these connections on the index bundle is

(L20.43) $\quad \operatorname{Ch}(\operatorname{ind}(a)) = \operatorname{tr}(\pi_-) - \operatorname{tr}(\pi_+)$

$$+\sum_{k=1}^{\infty} \frac{i^k(-1)^k}{(2\pi)^k k!} \operatorname{tr}\left((\pi_-(d\pi_-)(\pi_-')(d\pi_-)\pi_-)^k\right) - \sum_{k=1}^{\infty} \frac{i^k(-1)^k}{(2\pi)^k k!} \operatorname{tr}\left((\pi_+(d\pi_+)(\pi_+')(d\pi_+)\pi_+)^k\right).$$

where the trace is on $\mathbb{C}^{M_\pm}$.

Now, we proceed to compute the correspondint terms in (L20.36). From (L20.37) we can compute the total differential, on $B \times \mathbb{S}$, which is what appears in (L20.36) but I will write here as $d' = d + d\theta\partial_\theta$ where d is the differential on B :

$$d'a = (-i\pi_- e^{-i\theta} + i\pi_+ e^{i\theta})d\theta + d\pi_- e^{-i\theta} - d\pi_- - d\pi_+ + d\pi_+ e^{i\theta}.$$

The inverse of a is simply $a(-\theta)$ and the composite is seen to be

(L20.44) $\quad a^{-1}d'a = (-i\pi_- + i\pi_+)d\theta$

$$+ ((e^{-i\theta} - 1)\pi_-' + (1 - e^{i\theta})\pi_-)d\pi_- + ((e^{i\theta} - 1)\pi_+' + (1 - e^{-i\theta})\pi_+)d\pi_+.$$

There is no interaction between the two terms so

(L20.45)

$$\operatorname{tr}\left((a^{-1}d'a)^{2k+1}\right) = \lambda_k' \wedge d\theta + \mu_k',$$

$$\lambda_k' = -i(-1)^k(2k+1)(2 - e^{-i\theta} - e^{i\theta})^k \left(\pi_-(d\pi_-)\pi_-'(d\pi_-)\pi_-)^k - \pi_+(d\pi_+)\pi_+'(d\pi_+)\pi_+)^k\right).$$

Here the constant term in θ, with factor $d\theta$, which is what the integral will pick out, is computed by noting that the first term in (L20.44) must arise from exactly one factor. There are $2k + 1$ choices for this and commuting the chosen factor to the front results in no overall change of sign. Since $\pi_- d\pi_- \pi_- = 0$ the next factor can be replaced by the π_-' part, and so on alternatively through the remaining factors. So we arrive at (L20.36) in the special case that a is given by (L20.37). So, to compute the constant we need to evaluate

(L20.46)
$$\int_0^{2\pi} (2 - e^{-i\theta} - e^{i\theta})^k d\theta = (-1)^k \int_0^{2\pi} (e^{-i\theta/2} - e^{i\theta/2})^{2k} d\theta = 2\pi \frac{(2k)!}{(k!)^2}.$$

However, from the earlier discussion of the forms in (L20.36), we know the cohomology classes to be stable under homotopy, and the forms are unchanged under stabilization by the identity. So in fact (L20.36) must always hold. □

Of course what we have computed is the Chern character of the index bundle for Toeplitz families.

Corollary 8. *If* $A \in \mathcal{C}^\infty(B; \Psi_T^0(\mathbb{S}; \mathbb{C}^N))$ *is an elliptic family of Toeplitz operators then the Chern character of its index bundle (in* $K^0(B)$*) is given by* (L20.36) *with* $a = \sigma(A)$.

Next time I will consider the relative Chern character, as a map from compactly supported K-theory. In particular we need to understand the map

(L20.47) $$\operatorname{Ch} : K_c(T^*(M/B)) \longrightarrow H_c^{\text{evn}}(T^*(M/B); \mathbb{C})$$

since this is what appears in the cohomological version of the families index theorem

(L20.48) $$\operatorname{Ch}(\operatorname{ind}(A)) = \int \operatorname{Td} \operatorname{Ch}(\sigma(A)) \in H^{\text{evn}}(B)$$

for an elliptic family $A \in \Psi^m(M/B; \mathbb{E})$. In fact we can already see that the map to cohomology of the base must be of this form, for some class Td which is independent of the operator. Next time I will identify Td.

20+. Addenda to Lecture 20

CHAPTER 21

Families Atiyah-Singer index theorem

Lecture 21: 1 December, 2005

L21.1. Relative Chern character. For the Atiyah-Singer formula, we wish to associate with the symbol $[\sigma(A)] \in K^0_{\mathrm{c}}(T^*(M/B))$ of a family of elliptic operators a cohomology class $\mathrm{Ch}(\sigma(A)) \in H^{\mathrm{evn}}_{\mathrm{c}}(T^*(M/B))$. This enters crucially into the formula for the Chern character of the index bundle,

$$\mathrm{Ch}(\mathrm{ind}(A)) = \int \mathrm{Td} \cdot \mathrm{Ch}(\sigma(A)) \in H^{\mathrm{evn}}(B) \tag{L21.1}$$

where the integral is the pushforward operation for the overall fibration $T^*(M/B) \longrightarrow M \longrightarrow B$.

I will define this relative Chern character in the context of the interior of a compact manifold with boundary; the model case being $\overline{T^*(M/B)} \supset T^*(M/B)$. From a topological point of view there is not difficulty in defining this relative Chern character quite generally. This if the Chern character is defined for a general class of compact topological spaces then for non-compact spaces U with 1-point compactification ${}^1\overline{U}$ in this class one can (and indeed this is the standard way to do it) define the K-theory of U in terms of the K-theory of ${}^1\overline{U}$

$$K(U) = \mathrm{null}\left(K({}^1\overline{U}) \longrightarrow K(\{\mathrm{pt}\})\right) \tag{L21.2}$$

where the map is restriction to the point at infinity. Then if one has a topological Chern character the Chern character on $K(U)$ is defined as the composite.

However, I want a smooth version of this with explicit forms, since later I need to generalize the set up substantially. For the interior of a compact manifold with boundary, the definition (L21.2) reduces to the one I have been using. Namely elements of $K_{\mathrm{c}}(\mathrm{int}(X))$ are equivalence classes of pairs of bundles $[(E_+, E_-, a)]$ with a bundle isomorphism between then outside a compact set, i.e. in a neighbourhood of the boundary. In fact in this case we are free to assume that the bundles are smooth up to the boundary and a is just an identification of them over the boundary. For the moment however I will assume that a is defined near the boundary. The equivalence relation imposed identifies triples related by bundle isomorphisms and homotopies as previously discussed. So, we will associated a deRham form on $\mathrm{int}(X)$ with such a triple (and choice of connections) which vanishes near the boundary, and so defines a relative cohomology class, and show that this gives a map

$$\mathrm{Ch} : K_{\mathrm{c}}(\mathrm{int}(X)) \longrightarrow H^{\mathrm{evn}}_{\mathrm{c}}(\mathrm{int}(X)) = H^{\mathrm{evn}}(X, \partial X) \tag{L21.3}$$

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where the compact supported cohomology of the interior may be identified with the cohomology of X relative to its boundary.

Let $\nabla_\pm$ be connections on $E_\pm$. We can use the isomorphism a to relate a connection on E_+ to that on E_-. Thus, if $\rho \in \mathcal{C}^\infty_c(\mathrm{int}(X))$ is such that $1-\rho \in \mathcal{C}^\infty(X)$ has support in the neighbourhood of the boundary over which a is defined (and is an isomorphism) then

$$\nabla = \rho\nabla_+ + (1-\rho)a^{-1}\nabla_- a \tag{L21.4}$$

is a connection on E_+. The Chern form we consider is

$$\lambda = \mathrm{Tr}\left(\exp(\frac{i}{2\pi}\omega)\right) - \mathrm{Tr}\left(\exp(\frac{i}{2\pi}\omega_-)\right),\ \omega = \nabla^2,\ \omega_- = (\nabla_-)^2. \tag{L21.5}$$

That this is closed follows immediately from the discussion of last lecture. In this case $\lambda = 0$ as a form near the boundary and its class in $H^{\mathrm{evn}}(X, \partial X)$ is independent of choices. In fact I want to get a reasonably explicit formula for a representative of the class of λ which does not have the cut-off function in it.

First we need to compute the curvature of ∇. First recall that the connections $\nabla_\pm$ on $E_\pm$ determine a natural connection on $\hom(E_+, E_-)$ as a bundle over X. Namely, if a is such a homorphism then

$$(\nabla a)u = \nabla_-(au) - a(\nabla_+ u)\ \forall\ u \in \mathcal{C}^\infty(X; E_+) \tag{L21.6}$$

defines the connection which perhaps should be denoted ∇_{-+} since has nothing much to do with the ρ-dependent connection in (L21.4). In fact, we can express that connection in terms of it since

$$\nabla = \nabla_+ + (1-\rho)a^{-1}\nabla a \text{ on } \mathcal{C}^\infty(X; E_+). \tag{L21.7}$$

Thus the curvature of ∇, which is what appears in (L21.5) is

$$\begin{aligned} &\omega u = \nabla^2 u = (\omega_+ + (1-\rho)a^{-1}\nabla a)^2 u \\ &= \omega_+ + \nabla_+((1-\rho)a^{-1}(\nabla a)u) + (1-\rho)a^{-1}(\nabla a)\nabla_+ u + (1-\rho)^2(a^{-1}\nabla a)^2 u \Longrightarrow \\ &\quad \omega = -d\rho a^{-1}(\nabla a) + (1-\rho)\left(a^{-1}\omega_- a\right) + \rho\omega_+ - \rho(1-\rho)a^{-1}(\nabla a)a^{-1}(\nabla a). \end{aligned} \tag{L21.8}$$

Here I have used the identities

$$(\nabla^2 a)u = \omega_+ a - a\omega_- \text{ and } \nabla a^{-1} = -a^{-1}(\nabla a)a^{-1} \tag{L21.9}$$

which follow from the definitions.

Consider the form

$$\mathrm{Tr}\exp(\frac{i}{2\pi}\omega) = \sum_k \frac{i^k}{(2\pi)^k}\mathrm{Tr}(w^k). \tag{L21.10}$$

To remove ρ we will let it approach the characteristic function of the manifold. Choose a boundary defining function $x \in \mathcal{C}^\infty(X)$, $\partial X = \{x = 0\}$, $x \geq 0$, $dx \neq 0$ on ∂X and for $\chi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(t) = 0$ in $t < \frac{1}{2}$, $\chi(t) = 1$ in $t \geq 1$, set

$$\rho = \chi(x/\epsilon). \tag{L21.11}$$

For $\epsilon > 0$ small enough ρ satisfies the conditions required above. The curvature form in (L21.8) can be written as the sum

$$\omega = \alpha + \frac{\chi'(x)}{\epsilon}dx\beta \tag{L21.12}$$

where α and β are localy integrable uniformly as $\epsilon \downarrow 0$. Inserting this into (L21.10) gives a similar decomposition

(L21.13) $$\operatorname{Tr}\exp(\frac{i}{2\pi}\omega) = A + \frac{\chi'(x/\epsilon)}{\epsilon}dx \wedge B$$

where A and B have uniformly locally integrable coefficients.

LEMMA 34. *As $\epsilon \downarrow 0$ the form* (L21.13) *converges as a (supported) distibutional form on X to*

(L21.14) $$\operatorname{Tr}\exp(\frac{i}{2\pi}\omega_+) - \delta(x)dx \wedge \frac{i}{2\pi}\iota^*_{\partial X}$$
$$\operatorname{Tr}\left(a^{-1}(\nabla a)\int_0^1 \exp\left(\frac{i}{2\pi}(s\left(a^{-1}\omega_- a\right) + (1-s)\omega_+ - s(1-s)a^{-1}(\nabla a)a^{-1}(\nabla a)\right)\right)ds.$$

PROOF. From (L21.8) the first term in (L21.13) converges in the sense of supported distributions to the first term in (L21.14) – that is after integrating agains a smooth section (up to the boundary) of the dual bundle tensored with the density bundle. Thus, it is only necessary to prove the convergence of the second term to the second term in (L21.14).

Expanding out the second term, using the trace identity to bring each possible factor $d\rho$ to the front, shows that B in (L21.13) is

$$B = \operatorname{Tr}\left(\sum_k \frac{i^k}{(2\pi)^k(k-1)!}((1-\rho)\left(a^{-1}\omega_- a\right) + \rho\omega_+ - \rho(1-\rho)a^{-1}(\nabla a)a^{-1}(\nabla a))^{k-1}\right)$$

The coefficient of B tends to $\delta(x)dx$, supported on the boundary and apart from the explicit dependence on ρ the form is uniformly smooth up to the boundary. Replacing the smooth coefficients in B by their restrictions to the boundary leaves an error of the form $x/\epsilon\chi'(x/\epsilon)dxB'$, with B' smooth, and this vanishes, as a distribution, in the limit as $\epsilon \to 0$. Thus we may assume that B coefficients in B are replaced by their restrictions to the boundary, extended to be independent of x in a product decomposition near the boundary. As a result the distribution limit is the same as the integral against a smooth x-independent factor. The x integral becomes

(L21.15)
$$\frac{i}{2\pi}\operatorname{Tr}(a^{-1}(\nabla a)\int_0^{2\epsilon}\exp\left(\frac{i}{2\pi}(1-\rho)\left(a^{-1}\omega_- a\right) + \rho\omega_+ - \rho(1-\rho)a^{-1}(\nabla a)a^{-1}(\nabla a)\right)$$
$$\chi'(\frac{x}{\epsilon})\frac{dx}{\epsilon}$$

which reduces to (L21.14) after the change of variable $s = \chi(x/\epsilon)$. □

This gives the form on the relative Chern character (due, I believe, to Fedosov), as a distribution deRham class

(L21.16) $$\lambda = \operatorname{Tr}\exp(\frac{i}{2\pi}\omega_+) - \operatorname{Tr}\exp(\frac{i}{2\pi}\omega_-) - \delta(x)dx\wedge$$
$$\frac{i}{2\pi}\iota^*_{\partial X}\operatorname{Tr}(a^{-1}(\nabla a)\int_0^1\exp\left(\frac{i}{2\pi}(s\left(a^{-1}\omega_- a\right) + (1-s)\omega_+ - s(1-s)a^{-1}(\nabla a)a^{-1}(\nabla a))\right)ds.$$

Note that this sort of 'conormal representation' gives a cohomology class with an explicit transgression. That is, a distribution form

$$\alpha + \delta(x)dx \wedge \beta, \tag{L21.17}$$

where α and β are smooth forms, respectively up to and on the boundary, is closed *as a supported differential form* (dual to smooth sections) if and only if α is closed (and so defines an absolute cohomology class on X) and also

$$\iota^*_{\partial X}\alpha = d\beta \text{ on } \partial X. \tag{L21.18}$$

Note that this formula can be compared to the formula for last time for the Chern character in the Toeplitz case.

L21.2. Bott element.

21+. Addenda to Lecture 21

CHAPTER 22

Eta forms

Lecture 22: 6 December, 2005

The index formula for product-type will involve a 'regularized Chern character' which we interpret as an 'eta' form. To construct these forms, by regularization, we use holomorphic families of pseudodifferential operators. This leads us to a discussion of the residue trace, the regularized trace and the trace-defect formula and then finally to η-forms.

L22.1. Trace functional. For smoothing operators I have already discussed the trace. Namely

$$\mathrm{Tr} : \Psi^{-\infty}(Z;E) \longrightarrow \mathbb{C},\ \mathrm{Tr}(A) = \int_Z \mathrm{tr}_E(A(z,z)) \tag{L22.1}$$

where tr_E is the trace functional on the fibres of $\hom(E) = \mathrm{Hom}(E)\big|_{\mathrm{Diag}}$. It is straightforward to extend the trace to low order operators, for which the kernel is continuous (and a little more) across the diagonal.

THEOREM 12. *The trace functional extends canonically to*

$$\mathrm{Tr} : \Psi^s(Z;E) \longrightarrow \mathbb{C},\ s \in \mathbb{C},\ \mathrm{Re}(s) < -\dim Z. \tag{L22.2}$$

PROOF. To see this, and derive a formula for the extended functional, observe that the trace vanishes on any smoothing operator with kernel having support not meeting the diagonal. Since we can decompose and pseudodifferential operators as

$$A = A_1 + A_2,\ A_2 \in \Psi^{-\infty}(X;E),\ \mathrm{supp}(A_2) \cap \mathrm{Diag} = \emptyset \tag{L22.3}$$

we only need to consider the part, A_1, of A with support near the diagonal. Directly from our original definition of pseudodifferential operators, this is given as the inverse Fourier transform of a symbol on the cotangent bundle and then transferred to Z^2 using a bundle isomorphism (from Hom to hom) covering a normal fibration of the diagonal:

$$A_1 = F^*\mathcal{F}^{-1}(a),\ a \in \rho^{-s}\mathcal{C}^\infty(\overline{T^*Z}; \hom(E)). \tag{L22.4}$$

This is the case even for a smoothing operator, when $a \in \dot{\mathcal{C}}^\infty(\overline{T^*Z}; \hom(E))$ is Schwartz on the fibres of T^*Z.

By definition of a normal fibration, the diagonal is carried to the zero section of TZ under F. Thus, for a smoothing operator in (L22.4), the trace may be written

$$\mathrm{Tr}(A_1) = \int_{O \subset TM} \mathcal{F}^{-1}(a) = (2\pi)^{-d} \int_{T^*Z} a\omega^d,\ d = \dim Z. \tag{L22.5}$$

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Here I am just using the fact that, fibre by fibre, the value of a function at 0 is the integral of its Fourier transform. We also really need to check that the measures behave correctly in (L22.5) – but that is something I have sloughed over anyway. Here ω^d is the symplectic measure on T^*Z, just the maximal exterior power of the symplectic form. In local coordinates one can check (L22.5) directly. Notice that the (full) symbol a is by no means invariantly defined, but we see from (L22.5) that its integral is, and that it continues to make sense provided $\mathrm{tr}_E(a)$ is integrable, which is just the condition $\mathrm{Re}\, s < -n$ in (L22.2). □

As I shall show in the addenda, under this condition A_1 is indeed trace class so (L22.5) does represent the trace of an operator in the usual sense.

L22.2. Holomorphic families (of holomorphic order). For topological vector spaces such as $\Psi^m(Z;E)$, the topoolgy here being very similar to that on $\mathcal{C}^\infty(M)$, there is no difficulty in defining (strongly) holomorphic families, i.e. holomorphic maps from some open set

$$\mathbb{C} \supset \Omega \longrightarrow \Psi^m(Z;E). \tag{L22.6}$$

Namely, this is just a smooth function of the parameter, i.e. and element of $\mathcal{C}^\infty(\Omega;\Psi^m(Z;E))$ which satifies the Cauch-Riemann equtions

$$\overline{\partial} A = (\partial_x + i\partial_y)A = 0 \text{ in } \Omega. \tag{L22.7}$$

We do want to consider such maps, but we need something more. Namely holomorphic families where the order is changing holomorphically as well. These are *not* holomorphic maps into a fixed topological vector space, so we need to be a little careful about their properties. In fact it is probably better to think of them as 'yet-another-variant' of the spaces of pseudodifferential operators. Note that we have defined the space of pseudodifferential operators of complex order, it is the holomorphy that needs to be analyzed.

Definition 9. A map $A : \Omega \longrightarrow \Psi^s(Z;E,F)$ is said to be a holomorphic family of order $\mu : \Omega \longrightarrow \mathbb{C}$, a given holomorphic function on an open set $\Omega \subset \mathbb{C}$, if for any function $\chi \in \mathcal{C}^\infty(Z^2)$ with $\mathrm{Diag} \cap \mathrm{supp}(\chi) = \emptyset$,

$$\chi A : \Omega \longrightarrow \Psi^{-\infty}(Z;E,F) \text{ is holomorphic} \tag{L22.8}$$

(in the usual sense) and for some (any) normal fibration and bundle trivialization and an appropriate cutoff

$$\mathcal{F}(G^*(1-\chi)A) = \rho^{-\mu(s)}a,\ \Omega \ni s \longrightarrow a(s) \in \mathcal{C}^\infty(\overline{T^*Z};\mathrm{hom}(E,F)). \tag{L22.9}$$

Thus coefficinet a in (L22.9) is itself is holomorphic in the usual sense, as a smooth function on $\Omega \times \overline{T^*Z}$ and only the factor $\rho^{-\mu(s)}$ is 'extraordinary'. Note that changing to another boundary defining function ρ' merely multiplies a by $(\rho/\rho')^{-\mu(s)}$ which is holomorphic in the usual sense, since it is $b^{\mu(s)}$ for a positive smooth function b. We are mostly interested in the case $\mu(s) = \pm s$; the case $\mu(s) = m$, constant, is the usual notion of holomorphy.

Of course, it needs to be checked that this definition is independent of the normal fibration and the bundle isomorphism. This however proceed exactly as before so I pass over it without too much comment. The crucial point being that the space of functions $\rho^{-s}\,\mathrm{Hom}(\Omega \times X)$ for any compact manifold with boundary and any open set $\Omega \subset \mathbb{C}$ is invariant under the action of smooth vector fields on X which are tangent to the boundary. It is also necessary to do the asymptotic summation lemma, not only uniformly in Ω but holomorphically as well – this is quite straightforward.

Since the proof of the product formula follows from the freedom to change normal fibrations and bundle isomorphisms, it is also straightforward to check that composition makes sense.

LEMMA 35. *If $A \in \Psi^{\mu(s)}(Z; E, F)$ and $B \in \Psi^{\nu(s)}(F, G)$ are holomorphic families in an open set $\Omega \subset \mathbb{C}$ then $BA \in \Psi^{\mu(s)+\nu(s)}(Z; E, G)$ is holomorphic.*

Ellipticity of such a family is just pointwise ellipticity and a useful result is a version of 'holomorphic Fredholm theory'.

LEMMA 36. *If $A(s) \in \Psi^{m(s)}(X; E, F)$ is an elliptic holomorphic family on a connected open set Ω such that $A(s_0)^{-1} \in \Psi^{-\mu(s_0)}(Z; F, E)$ exists for one $s_0 \in \Omega$ then $A(s)^{-1} \in \Psi^{-\mu(s)}(Z; F, E)$ exists for $s \in \Omega \setminus D$, with D discrete in Ω, and there is a holomorphic family $B(s) \in \Psi^{-\mu(s)}(Z; F, E)$ and a meromorphic map on $E : \Omega \longrightarrow \Psi^{-\infty}(Z; F, E)$ with poles only at D and of finite rank, such that*

$$A^{-1}(s) = B(s) + E(s), \ \forall \ s \in \Omega \setminus D. \tag{L22.10}$$

The standard examples of such holomorphic families are the complex powers of a positive, self-adjoint, elliptic operator. For instance if Δ is the Laplacian on some compact manifold then $(\Delta + 1)^s$ is a holomorphic family of order $2s$. In fact Δ^s, defined correctly, is itself a holomorphic family of order $2s$. Althouth the residue trace was defined using such complex poweres this is by no means necessary (as was shown originally by Victor [**3**]). Instead the following is enough for our purposes:-

PROPOSITION 47. *For any bundle E on any compact manifold Z there is an entire family (i.e. holomorphic on $\mathbb{C}$) $E(s) \in \Psi^s(Z; E)$ which is everywhere elliptic and satisfies*

$$E(0) = \mathrm{Id}\,. \tag{L22.11}$$

Using complex powers (or otherwise) one can show that there is such a family which is everywhere invertible as well.

PROOF. For any normal fibration and bundle isomorphism, the identity is always represented by the full symbol Id_E . Thus if we simply choose a boundary defining function $\rho \in \mathcal{C}^\infty(\overline{T^*Z})$ and take the quantization of the symbol $a = \rho^{-s}\,\mathrm{Id}_E$,

$$E(s) = (1 - \chi) F^* \mathcal{F}^{-1}(\rho^{-s}\,\mathrm{Id}_E) \tag{L22.12}$$

we get such a family. □

L22.3. Seeley's theorem on the trace. The important relationship of holomorphic families and the trace functional is given by a theorem of Seeley, originally in the context of zeta functions.

THEOREM 13 (Seeley). *For any holomorphic family of order s on a connected open set $\Omega \subset \mathbb{C}$ such that $\Omega' = \Omega \cap \{\mathrm{Re}(s) < -\dim Z\}$ is non-empty and connected,*

$$\mathrm{Tr}(E(s)) : \Omega' \longrightarrow \mathbb{C} \tag{L22.13}$$

extends to a meromorphic function with at most simple poles at the divisor $-\dim Z + \mathbb{N}$

$$\mathrm{Tr}(E(s)) : \Omega \setminus \{-d + \mathbb{N}\} \longrightarrow \mathbb{C}, \ d = \dim Z. \tag{L22.14}$$

Proof. From the discussion above, if we take the full symbol of $E(s)$ localized near the diagonal

$$\rho^{-s}a(s) = \mathcal{F}(G^*((1-\chi)E(s)) \tag{L22.15}$$

then $a(s) \in \mathcal{C}^\infty(\overline{T^*Z}; \hom(E))$ is holomorphic in Ω and

$$\operatorname{Tr}(E(s)) = \int_{T^*Z} \rho^{-s} \operatorname{tr}_E(a(s))\Omega^{2d} \text{ in } \Omega'. \tag{L22.16}$$

So we need only show that this integral extends meromorphically to Ω with the stated poles, since the uniqueness follows from the uniqueness of holomorphic extensions.

The integral (L22.16) can be decomposed using a partition of unity on Z and the invariance of the trace under conjutation means that we may replace E by a trivial bundle. Since the symbol $a(s)$ is itself holomorphic, the integral over any fixed compact region is holomorphic. Thus we may take $\rho = r = 1/R$ the inverse of a polar coordinate in $T^*Z \equiv U \times \mathbb{R}^d$ locally and reduce $\operatorname{Tr}(E(s))$ to a finite sum of integrals of the form

$$T_j(s) = \int_{S^*Z}\int_0^1 r^{-s}a(s,r,z,\omega)r^{-d-1}drdzd\omega. \tag{L22.17}$$

Here a is a smooth function of all variales, down to $r = 0$ and holomorphic in s and the singular factor comes for the usual formula for Lebesgue measure in polar coordinates, $R^{d-1}dR = -r^{-d-1}dr$. Here the local cutoff makes a compactly supported in z so the z and $\omega \in \mathbb{S}^{d-1}$ integrals may be carried out, leaving the single integral

$$T_j(s) = \int_0^1 r^{-s}a'(s,r)r^{-d-1}drdzd\omega. \tag{L22.18}$$

The integral converges uniformly for $\operatorname{Re}(s) < -d$, which is the initial domain of it existence (inside Ω). If $a' = r^k a''(s,r)$ where $a''(s,r)$ is also smooth and holomorphic in s then the integral (L22.18) converges uniformly for $\operatorname{Re} s < -d+k$. Thus, if we replace a' by its Taylor series at $r = 0$ to high order

$$a'(s,r) = \sum_{j=0}^{k-1} a'(s)_j r^j + r^k a''(s,r) \tag{L22.19}$$

we get just such a remainder term, so

$$T_j(s) - T'_j(s) = \sum_{j=0}^{k-1} a'(s)_j \int_0^1 r^{-s+j-d-1}dr = \sum_{j=0}^{k-1} \frac{a'(s)_j}{-s+j-d} \tag{L22.20}$$

with $T'_j(s)$ holomorphic in $\Omega \cap \{\operatorname{Re} s < -d+k\}$. This proves the stated meromorphy and shows that the extension only has simple poles and only at the points $s = -d+j$, $j \in \mathbb{N}_0$. $\square$

L22.4. Residue trace. If we take an element $A \in \Psi^m(Z; E)$ for some $m \in \mathbb{Z}$ and a holomorphic family $E(s) \in \Psi^s(Z; E)$ satisfying (L22.11) then $A(s) = AE(s) \in \Psi^{s+m}(Z; E)$ and $\operatorname{Tr}(AE(s))$ can only have poles at the points $-d+m+\mathbb{N}_0$. Since $A(0) = A$ the pole at $s = 0$ is of particular interest. Wodzick observed that the residue is actually well-defined.

PROPOSITION 48. *For any holomorphic family $A(s) \in \Psi^{m+s}(Z;E)$ the residue of the holomorphic extension of the trace from* $\operatorname{Re} s < -d$,

$$\operatorname{Tr}_{\mathrm{R}}(A(0)) = \lim_{s\to 0} s\operatorname{Tr}(A(s)) \tag{L22.21}$$

is independent of the choice of $A(s)$, with $A(0) = A$, and so defines a continuous functional

$$\operatorname{Tr}_{\mathrm{R}} : \Psi^m(Z;E) \longrightarrow \mathbb{C},\ m \in \mathbb{Z}, \tag{L22.22}$$

which vanishes identically if $m < -d$ and satisfies

$$[A,B] = 0\ \forall\ A \in Psi^m(Z;E),\ B \in \Psi^{m'}(Z;E),\ m,\ m' \in \mathbb{Z}. \tag{L22.23}$$

The functional (L22.21) is called the residue trace.

PROOF. By Seeley's computation above, the residue in (L22.21) certainly exists.

To see that it does not depend on the holomorphic family of order s chosen so that $A(0) = A$, suppose that $A'(s)$ is another such family. Thus $B(s) = A'(s) - A(s)$ is a holomorphic family of order s such that $B(0) = 0$. Consider what this means. For the part away from the diagonal, the kernel as a family of smoothing operators must vanish at $s = 0$. By Taylors formula the kernel
$chiB(s) = sB'(s)$ where $B'(s)$ is also holomorphic. For the part near the diagonal, passing to the symbol $\rho^{-s}b(s)$ with b holomorphic, it follows that $b(0) = 0$ and hence, from the same reasoning, that $b(s) = sb'(s)$. So in fact $B(s) = sB'(s)$ where $B'(s)$ is again a holomorphic family of order s. Now applying Seeley's computation again,

$$\operatorname{Tr}(B(s)) = s\operatorname{Tr}(B'(s)) \text{ is regular at } s = 0 \tag{L22.24}$$

since $\operatorname{Tr}(B'(s))$ can have at most a simple pole at the origin. Thus $\operatorname{Tr}_{\mathrm{R}}(A(0))$ defined by (L22.21) is indeed independent of the holomorphic family (of order s) used to define it.

In particular we may choose or basically family $E(s)$ satisfying (L22.11) and then

$$\operatorname{Tr}_{\mathrm{R}}(A) = \lim_{s\to 0} s\operatorname{Tr}(AE(s))\ \forall\ A \in \Psi^m(Z;E),\ m \in \mathbb{Z}. \tag{L22.25}$$

For a commutator,

$$\begin{aligned}\operatorname{Tr}_{\mathrm{R}}([A,B]) &= \lim_{s\to 0} s\operatorname{Tr}([A,B]E(s)) = \lim_{s\to 0}\operatorname{Tr}(ABE(s) - BAE(s))\\ &= \lim_{s\to 0} s\operatorname{Tr}(A[B,E(s)]) - \lim_{s\to 0} s\operatorname{Tr}(B[A,E(s)]) = 0.\end{aligned} \tag{L22.26}$$

Here, $A[B,E(s)]$ and $B[A,E(s)]$ are both holomorphic families of order s which vanish at $s = 0$ (since $E(0) = \mathrm{Id}$) so the residues must vanish. □

The discussion of Seeley's theorme above allows us to derive a formula for the residue trace. Namely, there can be no singularity in $\operatorname{Tr}(A(s))$ arising from the smoothing terms. I leave it as an exercise (probably disussed more in the addenda) to show that

$$\operatorname{Tr}_{\mathrm{R}}(A) = \int_{S^*M} \operatorname{tr}_E(a_{-d}) \tag{L22.27}$$

where a_{-d} is the term or degree $-d$ in the expansion of the symbol, made into a density by multiplying by the term of homogeneity d in the corresponding expansion of ω^d. Note that this is true for the full symbol computed with respect to *any* normal fibration. Not that a_{-d} is well-defined, but the integral of its bundle trace is.

L22.5. Regularized trace. As well as the residue trace we are interested in the regularization of the trace functional itself. Having chosen a holomorphic family $E(s)$ we set

$$\overline{\mathrm{Tr}}_E(A) = \lim_{s\to 0}\left(\mathrm{Tr}(AE(s) - \frac{1}{s}\,\mathrm{Tr}_{\mathrm{R}}(A)\right) \tag{L22.28}$$

where the limit exists exactly because we have removed the singular term. As the notation indicates this functional *does* depend on the family $E(s)$ chosen to define it; further more it is *not* a trace. Rather it is precisely the trace-defect which we want to compute.

As shown above, if $E_i(s) \in \Psi^s(Z;E)$, $i = 1, 2$, are two holomorphic families satisfying (L22.11) then

$$E_1(s) = E_2(s) = sB(s),\ \ B(s) \in \Psi^s(Z;E) \text{ holomorphic.} \tag{L22.29}$$

Thus, we can set $D(E_1, E_2) = B(0) \in \Psi^0(Z;E)$. Then
(L22.30)
$$\mathrm{Tr}(AE_2(s)) = \mathrm{Tr}(AE_1(s)) + s\,\mathrm{Tr}(AB(s)) \Longrightarrow \overline{\mathrm{Tr}}_{E_1}(A) = \overline{\mathrm{Tr}}_{E_2}(A) + \mathrm{Tr}_{\mathrm{R}}(AD(E_1, E_2)$$

since $AB(s)$ is a holomorphic family with value $AD(E_1, E_2)$ at $s = 0$. This shows (see the addenda):

Lemma 37. *The regularized traces, defined by* (L22.28) *on* $\Psi^{\mathbb{Z}}(Z, E)$*, by holomorphic families satisfying* (L22.11)*, form an affine space modelled on* $\Psi^0(Z, E)/\Psi^{-\infty}(Z;E)$.

L22.6. Trace-defect formula. There is another important operation [1] which arises from the properties of the holomorphic family satisfying (L22.11). Namely, as we have already remarked, $[A, E(s)] = sB(s)$ is a holomorphic family vanishing at the origin. Thus

$$D_E : \Psi^{\mathbb{Z}}(Z;E) \ni A \longmapsto [A, E(s)]/s\big|_{s=0} \in \Psi^{\mathbb{Z}}(Z;E) \tag{L22.31}$$

is a well-defined linear map.

Proposition 49. *The map* (L22.31) *is an exterior derivation mapping* $\Psi^m(Z;E)$ *to* $\Psi^{m-1}(Z;E)$ *for any* $m \in \mathbb{Z}$ *(actually for any* $m \in \mathbb{C}$*) which is well-defined up to interior derivations,*

$$D_{E_1}A = D_{E_2}A + [D(E_1, D_2), A] \tag{L22.32}$$

and which is in fact the unique continuous exterior derivation (up to constant multiplies and addition of interior derivations).

Proof. That D_E is a derivation follow immediately from the identity

$$[AB, E(s)] = A[B, E(s)] + [A, E(s)]B. \tag{L22.33}$$

The difference formula (L22.32) follows from the definition of $D(E_1, E_2)$.

That D_E is not itself an interior derivtion follows easily from the fact that The uniqueness is not so simple, maybe it will be/is discussed in the addenda. □

[1] which I did not quite emphasize enough during the lecture

Formally, $D_E A$ is the commutator $[\log Q, A]$ for some positive operator Q of order 1 in the algebra. There is no such element, in the pseudodifferential algebra as it is defined above, so this is an exterior derivation (this is only supposed to be a plausibility argument). In fact it is easy enough to construct an operator which does represent the derivation as a commutator, it is just not in the algebra but rather is in an extension of the algebra.

One relationship that is easy to see is that the residue trace vanishes on the range of D_E – of course it vanishes on the range of interior derivations by (L22.26)

$$\mathrm{Tr_R}(D_E A) = 0 \ \forall \ A \in \Psi^{\mathbb{Z}}(Z; E). \tag{L22.34}$$

Indeed this just follows from the definition of D_E in (L22.31) since

$$\mathrm{Tr_R}(D_E A) = \lim_{s\to 0} s(\frac{[A, E(s)]}{s}) = 0. \tag{L22.35}$$

More importantly for computations in the sequel

Lemma 38. *For all A, $B \in \Psi^{\mathbb{Z}}(Z; E)$,*

$$\overline{\mathrm{Tr}}_E([A, B]) = \mathrm{Tr_R}(BD_E A). \tag{L22.36}$$

Proof. By definition the regularized trace is the value at $s = 0$ of

(L22.37)

$$\overline{\mathrm{Tr}}_E([A, B]) = \lim_{s\to 0} \mathrm{Tr}([A, B]E(s)) = \lim_{s\to 0} s\,\mathrm{Tr}(B\frac{[E(s), A]}{s}) = \mathrm{Tr_R}(BD_E A),$$

where there is no pole at the origin, since $\mathrm{Tr_R}([A, B]) = 0$ and the identity

$$\mathrm{Tr}(ABE(s)) = \mathrm{Tr}(BE(s)A)$$

holds because it holds in the trace class region. □

L22.7. The circle. For pseudodifferential operators on the circle it is easy to make some of these operations explicit (this can in fact be done in general, although it is not necessarily enlightening). First we can take as our holomorphic family

$$E(s)e^{ik\theta} = (k^2 + 1)^{s/2}e^{ik\theta} \in \Psi^s(\mathbb{S}). \tag{L22.38}$$

That this can be checked following the arguments for the Szegő projector. Then the exterior derivation is seen to satisfy

$$\sigma_{m-1}(D_E A) = \pm r\partial_\theta\sigma_m(A), \ \forall \ A \in \Psi^m(\mathbb{S}) \tag{L22.39}$$

where the sign refers to the component of the cosphere bundle $S^*\mathbb{S} = \mathbb{S}_+ \sqcup \mathbb{S}_-$. The residue trace we already know to be

$$\mathrm{Tr_R}(A) = \int_{\mathbb{S}_+} \sigma_{-1}(A)d\theta - \int_{\mathbb{S}_-} \sigma_{-1}(A)d\theta \ \forall \ A \in \Psi^{-1}(\mathbb{S}). \tag{L22.40}$$

L22.8. Toeplitz η forms. Recall that on the Toeplitz smoothing group, stabilized by the smoothing operators on some other compact manifold,

(L22.41)

$$G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(Z; E)) = \{a \in \Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(Z; E)); (\mathrm{Id} + a)^{-1} = \mathrm{Id} + b, \ b \in \Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(Z; E))\}$$

is a classifying group for odd K-theory (in this approach by definition) and that the forms

$$\mathrm{Ch_{odd}}(a) = \sum_{k=0}^{\infty} c_k\beta_{2k+1}, \ \beta_{2k+1}(a) = \mathrm{Tr}\left(((\mathrm{Id} + a)^{-1}da)^{2k+1}\right). \tag{L22.42}$$

Consider the inclusion of $G^{-\infty}$ as a normal subgroup of the group of (stabilized) invertible Toeplitz operators of order 0 :

(L22.43)

$$G^0_{\mathcal{T}}(\mathbb{S};\Psi^{-\infty}(Z;E)) = \left\{A \in \Psi^0_{\mathcal{T}}(\mathbb{S};\Psi^{-\infty}(Z;E)); (\mathrm{Id}+A)^{-1} = \mathrm{Id}+B,\ B \in \Psi^0_{\mathcal{T}}(\mathbb{S};\Psi^{-\infty}(Z;E))\right\}.$$

The subgroup satifying the normalization condition $\sigma_0(A)(1) = 0$ is contractible, but for the moment we will ignore this.

Using the regularized trace introduced above, we can extend the forms in (L22.42) from the normal subgroup to the whole group. So we set

$$\eta = \sum_{k=0}^{\infty} c_k \eta_{2k+1},\ \eta_{2k+1}(A) = \overline{\mathrm{Tr}}((\mathrm{Id}+A)^{-1}dA)^{2k+1}) \tag{L22.44}$$

where we drop the suffix indicating the regularizing family, since we will just use (L22.38) for definiteness sake.

Proposition 50. *The forms in* (L22.44) *are well-defined on* $G^0_{\mathcal{T}}(\mathbb{S})$, *restrict to* $G^{-\infty}_{\mathcal{T}}(\mathbb{S})$ *to the forms in* (L22.42) *and are such that*

$$d\eta_{2k+1}(a) = \sigma_0^*(\beta_{2k} + d\gamma_{2k+1}),\ \textit{where} \tag{L22.45}$$
$$\beta_{2k}(b) = -\frac{1}{2}\int_{\mathbb{S}} \mathrm{Tr}\left((b^{-1}db)^{2k} b^{-1}\frac{\partial}{\partial\theta}b\right),$$
$$\gamma_{2k+1}(b) = \frac{1}{2}\int_{\mathbb{S}} \mathrm{Tr}\left((b^{-1}db)^{2k+1} b^{-1}\frac{\partial}{\partial\theta}b\right)$$

are defined on the loop group

$$\{b \in \mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Z;E); (\mathrm{Id}+b)^{-1} = \mathrm{Id}+b',\ b' \in \mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Z;E)\}. \tag{L22.46}$$

Proof. The functional $\overline{\mathrm{Tr}}$ is linear and continuous, so we can commute d through it to compute $d\eta_{2k+1}$. The argument in (L22.44) can be written

$$((\mathrm{Id}+A)^{-1}dA)^{2k+1} = (-1)^k A^{-1}dA(d(A^{-1})dA)^k \tag{L22.47}$$

where we use the identity $dA^{-1} = -A^{-1}(dA)A^{-1}$. Thus, only the first factor in (L22.47) is not exact, so

$$d\eta_{2k+1}(A) = -\overline{\mathrm{Tr}}((A^{-1}dA)^{2k+2}). \tag{L22.48}$$

The argument can now be written as a 'supercommutator' – really it is a commutator when we take the antisymmetry of the exterior product into account. Namely

$$d\eta_{2k+1}(A) = -\frac{1}{2}\overline{\mathrm{Tr}}\left([A^{-1}dA, (A^{-1}dA)^{2k+1}]\right). \tag{L22.49}$$

Then, using a 'super' version of the trace defect formula we conclude that

$$d\eta_{2k+1}(A) = \frac{1}{2}\mathrm{Tr}_{\mathrm{R}}\left((A^{-1}dA)^{2k+1}D_E(A^{-1}dA)\right). \tag{L22.50}$$

All the products $A^{-1}dA$ are of order zero and D_E lower the order by one, so we know from (L22.40) that the residue trace here is just the integral of the principal symbol,

$$d\eta_{2k+1}(A) = \frac{1}{4\pi}\int_{\mathbb{S}} \sigma_{-1}\left((A^{-1}dA)^{2k+1}D_E(A^{-1}dA)\right)d\theta. \tag{L22.51}$$

Now, D_E expands to

$$D_E(A^{-1}dA) = -A^{-1}(D_EA)A^{-1}dA + A^{-1}dD_EA. \tag{L22.52}$$

Now, using (L22.39) to evaluate the leading term in D_E, we arrive at (L22.45). Indeed, in the first term arising from inserting (L22.52) into (L22.51), the last factor $A^{-1}dA$ can be commuted to the front, giving β_{2k}. Similarly, in the term arising from the second part of (L22.52) the factor premultiplying dD_EA,

$$(A^{-1}dA)^{2k+1}A^{-1} = (-1)^{k+1}(dA^{-1}\cdot dA)^k dA^{-1}, \tag{L22.53}$$

is exact, so this reduces to $d\gamma_{2k+1}$ with γ_{2k+1} as in (L22.45). $\square$

The pointed loop group, which is denote above

(L22.54)

$$G^{-\infty}_{(1)} = \{b \in \mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}(Z;E); b(1) = \mathrm{Id},\ (\mathrm{Id} + b)^{-1} = \mathrm{Id} + b',\ b' \in \mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}(Z;E)\}$$

is a normal subgroup of the full loop group in (L22.46) and it in turn has the normal subgroup of index 0 loops

$$G^{-\infty}_{(1),0}(Z;E) = \{b \in G^{-\infty}_{(1),0}(Z;E); \frac{1}{2\pi}\int_{\mathbb{S}} \mathrm{Tr}((\mathrm{Id} + b)^{-1}\frac{\partial}{\partial\theta}b)d\theta = 0\} \tag{L22.55}$$

which is the leading part of our classifying sequence. What has been shown above can be pictured like this

(L22.56)

$$\begin{tikzcd}
\mathrm{Ch}_{\mathrm{odd}} \arrow[d, dotted, no head] & \eta_{\mathrm{odd}} \arrow[l, "\big|_{G^{-\infty}}"'] \arrow[r, "d"] \arrow[d, dotted, no head] & \mathrm{Ch}_{\mathrm{evn}} + d\Gamma \arrow[dd, dotted, no head] \\
G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(Z;E)) \arrow[r] & G^0_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(\mathbb{S}; \Psi^{-\infty}(Z;E)) \arrow[d] & \\
& G^{-\infty}_{(1),0}(Z;E) + \rho\mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}(Z;E))[[\rho]] \arrow[r] & G^{-\infty}_{(1),0}(Z;E).
\end{tikzcd}$$

The fact that $d\eta_{2k+1}$ descends to the quotient group and represents there the (even) Chern character is what is fundamental. The fact that this differential actually lifts to the leading part of the quotient, and does not depend on the lower order symbols at all, is (a higher order extension of) the 'miracle of the loop group' of Pressley and Segal [**5**]. This latter behaviour does not carry over to higher dimensions or the 'geometric case'.

22+. Addenda to Lecture 22

22+.1. Proof of Lemma 37. We have already seen in (L22.30) that the difference between two regularized traces if given by $\mathrm{Tr}_{\mathrm{R}}(AD)$ where $D \in \Psi^0(Z;E)$ is the difference element discussed above. Since the residue trace vanishes on smoothing operators, this certainly vanishes if $D \in \Psi^{-\infty}(Z;E)$ and so only depends on the 'full symbol' element $D \in \Psi^0(Z;E)/\Psi^{-\infty}(Z;E)$. Every element D can appear as $D(E_1,E_2)$ since if $E_1(s)$ is a given holomorphic family satisfying (L22.11) then $E_2(s) = E_1(s) + sDE_1(s)$ is another family of this type with $D(E_1,E_2) = D$. Finally the image of D in $\Psi^0(Z;E)/\Psi^{-\infty}(Z;E)$ can be recoverd from the difference of the functionals, namely

(22+.57)

$$\Psi^{\mathbb{Z}}(Z;E) \ni A \longrightarrow \mathrm{Tr}_{\mathrm{R}}(AD) \text{ determines } [D] \in \Psi^0(Z;E)/\Psi^{-\infty}(Z;E) \text{ uniquely.}$$

To see this (and we need to use all the integral orders, or at least arbitrarily large ones) start with A of order $-d$. Then the term in the symbol of AD of order $-d$ is just the product of the principal symbols, so in this case

$$\operatorname{Tr}_{\mathrm{R}}(AD) = \int_{S^*M} \operatorname{tr}_E(\sigma_{-d}(A)\sigma_0(D)). \tag{22+.58}$$

If we think of $\sigma_0(D)$ as being a distribution on S^*M this determines it, since $\operatorname{tr}(ab)$ is non-degenerate as a bilinear form on the fibre of $\hom(E)$ at each point. Thus $\sigma_0(D)$ can be recovered from the difference functional. This determines D modulo $\Psi^{-1}(Z;E)$ so can subtract from $\operatorname{Tr}_{\mathrm{R}}(AD)$ the functional $\operatorname{Tr}_{\mathrm{R}}(A\tilde{D})$ where $\tilde{D} \in \Psi^0(Z;E)$ is some operator with the same principal symbol. Thus we can suppose that $D \in \Psi^{-1}(Z;E)$, or proceeding inductively that $D \in \Psi^{-k}(Z;E)$ and then repeat the argument, now with $A \in \Psi^{-d+k}(Z;E)$ so that the residue trace still comes out in terms of the principal symbol of AD. Thus, D is indeed determined modulo $\Psi^{-\infty}(Z;E)$.

CHAPTER 23

Index for product-type families

Lecture 23: 8 December, 2005

It is probably a good thing that no one gave me notes of this lecture. This allows me to write something closer to what I should have said and try to undo some of the confusion I must have sown!

NB. At this stage, I have paid no serious attention to the coefficients in the expansion of the odd Chern character and correspondingly the eta forms. As a result you will find some discrepancies with the constants below — at some point I will track down these constants (and quite a few ealier ones!)

L23.1. Product-type Toeplitz algebra. I wanted to finish this course with an example of an index formula showing how the eta forms discussed last time enter as 'regularized Chern forms'.[1] To make thing reasonably simple[2] I will consider a product $M \times \mathbb{S}$ where M is an arbitrary compact manifold.[3] To further simplify things I will consider operators on a fixed bundle and indeed a trivial one (this is not much of a restriction since one can always complement a bundle to be trivial, with the identity operator on the complement). Thus, the fibration is actually a product

$$\begin{tikzcd} M \times \mathbb{S} \arrow[d, "\pi"] \\ M. \end{tikzcd} \tag{L23.1}$$

Rather than consider a general elliptic element $A \in \Psi^{0,0}_{\pi-\mathrm{pt}}(M; \mathbb{C}^N)$ initially I will further restrict the problem by considering the Toeplitz algebra in this sense. Recall that the fibrewise operators

$$\mathcal{C}^\infty(M; \Psi^0(\mathbb{S}; \mathbb{C}^N)) \subset \Psi^{0,0}_{\pi-\mathrm{pt}}(M; \mathbb{C}^N). \tag{L23.2}$$

In particular the Szegő projector, S, on $\mathbb{S}$ lifts to an element of the product-type algebra. Thus we can consider the Toeplitz subalgebra of the product-type algebra

$$\Psi^{0,0}_{\pi-\mathrm{pt},\mathcal{T}}(M; \mathbb{C}^N) = S\Psi^{0,0}_{\pi-\mathrm{pt}}(M; \mathbb{C}^N)S. \tag{L23.3}$$

0.7E; Revised: 29-11-2006; Run: November 29, 2006

[1] In fact I had hoped to do a reasonably general case, but there is still some work required for this.

[2] And let's face it I had enough trouble with this as it is.

[3] If you are feeling energetic a natural generalization to understand would be that of a circle bundle, more precisely the circle bundle of a complex line bundle (hence oriented), over a compact manifold.

The symbol algebra for $\Psi^{0,0}_{\pi-\mathrm{pt}}(M;\mathbb{C}^N)$ takes values in smooth maps in matrices on the blown-up cosphere bundle $[S^*(M\times\mathbb{S}),S^*M]$. In this case, because $S^*\mathbb{S}$ is just two copies of $\mathbb{S}$,

$$[S^*(M\times\mathbb{S}),S^*M]=S^*_+(M\times\mathbb{S})\sqcup S^*_-(M\times\mathbb{S}) \tag{L23.4}$$

is just the disjoint of the two compact manifolds with boundary, consisting of the upper half-sphere bundle and the lower half-sphere bundle, separated by the blow-up rather than meeting at the equatorial sphere bundle. For a fibrewise family the symbol is the lift of the symbol of the family, so for the Szegő projector it is the identity on the upper half-sphere bundle and 0 on the lower half. Thus, symbol map gives a short exact sequence

$$\Psi^{-1,0}_{\pi-\mathrm{pt},\mathcal{T}}(M;\mathbb{C}^N)\longrightarrow\Psi^{0,0}_{\pi-\mathrm{pt},\mathcal{T}}(M;\mathbb{C}^N)\longrightarrow\mathcal{C}^\infty(S^*_+(M\times\mathbb{S});M(N,\mathbb{C})). \tag{L23.5}$$

The base family for a fibrewise family is just the family itself (lifted to the cosphere bundle of the base). So for the Toeplitz algebra in this product case we get a short exact sequence

$$\Psi^{0,-1}_{\pi-\mathrm{pt},\mathcal{T}}(M;\mathbb{C}^N)\longrightarrow\Psi^{0,0}_{\pi-\mathrm{pt},\mathcal{T}}(M;\mathbb{C}^N)\longrightarrow\mathcal{C}^\infty(S^*M;\Psi^0_{\mathcal{T}}(\mathbb{S};M(N,\mathbb{C}))). \tag{L23.6}$$

So, consider an elliptic element $A\in\Psi^{-1,0}_{\pi-\mathrm{pt},\mathcal{T}}(M;\mathbb{C}^N)$, meaning that the symbol $a=\sigma_0(A)\in\mathcal{C}^\infty(S^*_+(M\times\mathbb{S});\mathrm{GL}(N,\mathbb{C}))$ and the base family $\beta=\beta(A)\in\mathcal{C}^\infty(S^*M;\Psi^0_{\mathcal{T}}(\mathbb{S};M(N,\mathbb{C})))$ are invertible, so $\beta^{-1}\in\mathcal{C}^\infty(S^*M;\Psi^0_{\mathcal{T}}(\mathbb{S};M(N,\mathbb{C})))$. Under these conditions we may easily check that A defines a Fredholm operator[4]

$$A:\mathcal{C}^\infty(M\times\mathbb{S};\mathbb{C}^N)\longrightarrow\mathcal{C}^\infty(M\times\mathbb{S};\mathbb{C}^N) \tag{L23.7}$$

and we wish to compute the index.

For an elliptic element of $\Psi^0(M\times\mathbb{S};\mathbb{C}^N)$ the index is given by the formula of Atiyah and Singer. The assumptions we have made above mean that the formula simplifies. First, the relative Chern character reduces to the odd Chern character, since we are working on a trivial bundle and may take the trivial connection. Thus the topological image of the symbol is

$$\mathrm{Ch}(a)=\sum_k c_k\operatorname{Tr}\left((a^{-1}da)^{2k+1}\right) \tag{L23.8}$$

then the index formula becomes

$$\mathrm{ind}(A)=\int_{S^*(M\times\mathbb{S})}\mathrm{Td}(M)\wedge\mathrm{Ch}(a). \tag{L23.9}$$

In principle the Todd class of $M\times\mathbb{S}$ enters here, but this can easily be seen to reduce to the Todd class of M. Although I have not discussed Td in detail here, we may take it as the lift of a cohomology class on M, represented by some explicit deRham class, $\mathrm{Td}(M)\in\mathcal{C}^\infty(M;\Lambda^{\mathrm{evn}})$ pulled back to $S^*(M\times\mathbb{S})$.

So, what I want to show is how the eta forms enter in the corresponding formula for the elliptic elements of the Toeplitz algebra.

[4] Construct a parameterix!

THEOREM 14. *For an elliptic element* $A \in \Psi^{-1,0}_{\pi-\mathrm{pt},\mathcal{T}}(M;\mathbb{C}^N)$ *the index is given by*

$$\operatorname{ind}(A) = -\frac{1}{2}\int_{S^*M} \operatorname{Td}(M)\wedge\eta(\beta) + \int_{S^*_+(M\times\mathbb{S})} \operatorname{Td}(M)\wedge\operatorname{Ch}(a), \tag{L23.10}$$
$$\text{where } \eta(\beta) = \sum_k c_k\eta_{2k+1}(\beta) = \sum_k c_k\overline{\operatorname{Tr}}\left((\beta^{-1}d\beta)^{2k+1}\right)$$

and $\operatorname{Ch}(a)$ *is given by* (L23.8).

L23.2. Variation formulæ. Let me deduce variation formulæ for the (odd) eta forms and the odd Chern character which follow from the computation, done eariler, of the exterior derivative in each case.

For the eta forms on the group $G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)\hookrightarrow G^0_{\mathcal{T}}(\mathbb{S};\Psi^{-\infty}(Z;E))$ of invertible Toeplitz operators with values in $N\times N$ matrices

$$\eta_{2k+1}(B) = \overline{\operatorname{Tr}}((B^{-1}dB)^{2k+1}) \tag{L23.11}$$

we know, from Proposition 50, that

$$d\eta_{2k+1}(B) = \sigma_0^*(\beta_{2k+2} + d\gamma_{2k+1}) \tag{L23.12}$$

where the forms on the right are both defined on the loop group $G^{-\infty}_{(1)}$ but in this case we have descended to $\mathrm{GL}(N,\mathbb{C})$:

$$\beta_{2k+2}(b) = -\frac{1}{2}\int_{\mathbb{S}}\operatorname{tr}((b^{-1}db)^{2k+2}b^{-1}\partial_\theta b)d\theta, \tag{L23.13}$$
$$\gamma_{2k+1}(b) = -\frac{1}{2}\int_{\mathbb{S}}\operatorname{tr}((b^{-1}db)^{2k+1}b^{-1}\partial_\theta b)d\theta,\ b\in\mathcal{C}^\infty(\mathbb{S};\mathrm{GL}(N,\mathbb{C})).$$

So, if we suppose that X is an oriented compact manifold without boundary and that

$$F\in\mathcal{C}^\infty(X\times(0,1)_t;G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)) \tag{L23.14}$$

then we can consider the function

$$h_1(t) = \int_X T\wedge F_t^*\eta_{2k+1} \tag{L23.15}$$

where $T\in\mathcal{C}^\infty(X;\Lambda^{d-2k-1})$ is a fixed closed form. Certainly h_1 is a smooth function and

$$h_1'(t) = \int_X T\wedge\frac{d}{dt}F_t^*\eta_{2k+1}. \tag{L23.16}$$

Consider the pull back $F^*\eta_{2k+1}$ to $X\times(0,1)$. It can be decomposed with respect to dt as

$$F^*\eta_{2k+1} = \eta_{2k+1}(F,t) + dt\wedge\eta'_{2k}(F,t),\ \eta_{2k+1}(F,t) = F_t^*\eta_{2k+1}, \tag{L23.17}$$
$$\eta'_{2k}(F,t) = \sum_{j=0}^{2k}(-1)^jF_t^*\overline{\operatorname{Tr}}((F^{-1}d_XF)^jF^{-1}\frac{dF}{dt}(F^{-1}d_XF)^{2k-j})$$

where the coefficients are t-dependent forms on X. If we write out the formula for $dF^*\eta_{2k+1}$ obtained by pulling back (L23.12) we find

$$dF^*\eta_{2k+1} = d_X\eta_{2k+1}(F,t) + dt \wedge \left(\frac{d}{dt}\eta_{2k+1}(F,t) - d_X\eta'_{2k}(F,t)\right) = (F\circ\sigma)^*_0\left(\beta_{2k+2} + d\gamma_{2k+1}\right). \tag{L23.18}$$

So, expanding the last pull-back with respect to dt gives

$$\frac{d}{dt}\eta_{2k+1}(F,t) = d_X\eta'_{2k}(F,t) + \beta'_{2k+1}(f,t) + d_X\gamma'_{2k+1}(f,t) \tag{L23.19}$$

where $f = \sigma\circ F$ is the symbol map for the family F and

(L23.20)

$$\beta'_{2k+1}(f,t) = -\frac{1}{2}\sum_{j=0}^{2k+1}(-1)^j\int_{\mathbb{S}}\mathrm{Tr}((f^{-1}d_Xf)^j(f^{-1}\frac{df}{dt})(f^{-1}d_Xf)^{2k+1-j}f^{-1}\partial_\theta f)d\theta,$$

$$\gamma'_{2k+1}(f,t) = -\frac{1}{2}\sum_{j=0}^{2k}\int_{\mathbb{S}}\mathrm{Tr}((f^{-1}d_Xf)^jf^{-1}\frac{df}{dt}(f^{-1}d_Xf)^{2k-j}f^{-1}\partial_\theta f)d\theta,\ f=\sigma_0\circ F.$$

So now, if we insert this formula into (L23.16) we find that the exact terms integrate to zero, so only β'_{2k+1} survives and

$$h'_1(t) = \int_X T\wedge\beta'_{2k+1}(f,t) \tag{L23.21}$$

with β'_{2k+1} given by (L23.20).

Next we make a similar computation for the Chern character. Suppose that Y is a compact manifold with boundary of dimension q, that $T'\in\mathcal{C}^\infty(Y;\Lambda^{q-2k-1})$ is closed and that $G : Y\times(0,1)\longrightarrow G^{-\infty}$ is smooth. Then consider the function

$$h_2(t) = \int_Y T'\wedge G_t^*\beta_{2k+1} \tag{L23.22}$$

where β_{2k+1} are the component forms for the odd Chern character,

$$\beta_{2k+1} = \mathrm{Tr}\big((a^{-1}da)^{2k+1}\big)\,. \tag{L23.23}$$

We know that $d\beta_{2k+1}=0$ and from this we find

$$\frac{d}{dt}G_t^*\beta_{2k+1} = d_Y\beta'_{2k},\ \beta'_{2k} = \mathrm{Tr}\left((G^{-1}d_YG)^{2k}G^{-1}\frac{dG}{dt}\right). \tag{L23.24}$$

Using this and Stokes' theorem

$$h'_2(t) = \int_Y T'\wedge d_Y\beta'_{2k} = \int_{\partial Y} i^*_{\partial Y}T'\wedge\beta'_{2k}. \tag{L23.25}$$

In the application of these formulæ below, $\partial Y = X\times\mathbb{S}$. Thus the form β'_{2k} is pulled back to the product $X\times\mathbb{S}$. In this case, decomposing the total differential on ∂Y and carrying out the integral over the circle, first, shows that

(L23.26)

$$h'_2(t) = \sum_{j=0}^{2k-1}(-1)^j\int_{X\times\mathbb{S}} i^*_{\partial Y}T'\wedge\mathrm{Tr}\left((g^{-1}d_Xg)^jg^{-1}d_\theta g(g^{-1}d_Xg)^{2k-1-j}g^{-1}\frac{dg}{dt}\right)d\theta,$$

$$g = G\big|_{\partial Y = X\times\mathbb{S}}.$$

Compare this to (L23.20), first increasing k to $k+1$.

L23.3. Proof of Theorem 14. We first show that the right side of the formula (L23.10) is homotopy invariant. Thus, for an elliptic 1-parameter family $A_t \in \mathcal{C}^\infty((0,1);\Psi^{-1,0}_{\pi-\mathrm{pt},\mathcal{T}}(M;\mathbb{C}^N))$ set

$$I_1(t) = -\frac{1}{2}\int_{S^*M} \mathrm{Td}(M)\wedge\eta(\beta),\ I_2(t) = \int_{S^*_+(M\times\mathbb{S})} \mathrm{Td}(M)\wedge\mathrm{Ch}(a). \tag{L23.27}$$

The discussion above allows us to compute the derivatives of these functions. Namely if we set

$$b_t = \sigma_0(\beta(A_t)) = a_t\big|_{S^*M\times\mathbb{S}}, \tag{L23.28}$$

with equality being the consistency between the symbol and the base family, then from (L23.21), (L23.24) and (L23.25)[5]

$$I_1'(t) + I_2'(t) = 0. \tag{L23.29}$$

Thus the sum

$$I = I_1(t) + I_2(t) \text{ is constant.} \tag{L23.30}$$

So, to prove the index formula we may make a homotopy to some operator for which we can compute the index, in this case using the Atiyah-Singer theorem.[6]

Consider the symbol $a = \sigma_0(A) \in \mathcal{C}^\infty(S^*_+(M\times\mathbb{S};\mathrm{GL}(N,\mathbb{C}))$. This is a ball bundle, where the origin of each fibre can be taken as the fibre vertical, the point corresponding to

$$M\times\mathbb{S}\ni(m,\theta)\longmapsto(m,\theta,0,d\theta) \text{ that is } (0,1)\in T^*_mM\times T^*_\theta\mathbb{S}. \tag{L23.31}$$

Thus the fibres are contractible. This allows us to construct a smooth family of symbols

$$\tilde a : [0,1]_t\times S^*_+(M\times\mathbb{S})\longrightarrow\mathrm{GL}(N,\mathbb{C}),\ \tilde a\big|_{t=0}=a,\ \tilde a\big|_{t=1}=\pi^*(a\big|_{M\times\mathbb{S}}). \tag{L23.32}$$

For instance, one can first smoothly deform the symbol so that it is fibre-constant near the 'centre' in (L23.31) and then radially translate the symbol to expand the constant region.

In particular the boundary symbols

$$\tilde b = \tilde a\big|_{S^*M\times\mathbb{S}=\partial S^*_+(M\times\mathbb{S})} : [0,1]\times S^*M\times\mathbb{S}\longrightarrow\mathrm{GL}(N,\mathbb{C}) \tag{L23.33}$$

form a 1-parameter family of symbols of Toeplitz operators, in $\mathcal{C}^\infty(\mathbb{S};\mathrm{GL}(N,\mathbb{C}))$ parameterized by S^*M. The compatibility condition between symbols and base families shows that at $t=0$, for the given operator, these are the symbols of an invertible family of Toeplitz operators. It follows from our earlier results about such families that we can lift to a family of invertible Toeplitz operators

$$\tilde\beta : [0,1]\times\mathbb{S}^*M\longrightarrow\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N),\ \tilde\beta^{-1}\in\mathcal{C}^\infty([0,1]\times S^*N;\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)),\quad \tilde\beta\big|_{t=0}=\beta(A). \tag{L23.34}$$

[5]For the moment I am completely ignoring the constants – there is good reason to believe they work out correctly!

[6]In the lecture I was trying to run the homotopy in the wrong direction, which accounts for some of my discomforture.

Finally then we may select a smooth family $\tilde{A}_t \Psi^0_{\pi-\mathrm{pt},\mathcal{T}}(M \times \mathbb{S}; \mathbb{C}^N)$ with symbol $\tilde{a}$ and base family $\tilde{\beta}$. This is therefore a family of (fully) elliptic operators and hence has constant index

$$\operatorname{ind}(A_t) \in \mathbb{Z} \text{ is smooth, hence constant.} \tag{L23.35}$$

On the other hand we have also shown (L23.30), so it suffices to show that

$$\operatorname{ind}(\tilde{A}_1) = I(1), \tag{L23.36}$$

which reduces to proving the index formula (L23.10) under the additional assumption that

$$\sigma_0(A) \text{ is fibre constant on } S^*_+(M \times \mathbb{S}) \longrightarrow M \times \mathbb{S}. \tag{L23.37}$$

This just means that $\sigma_0(A) = \sigma_0(L)$ where $L \in \mathcal{C}^\infty(M \times \mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$ is a bundle isomorphism for our trivial bundle. As an isomorphism L has vanishing index, so we may compose A with the inverse and strengthen (L23.37) further and suppose that

$$\sigma_0(A) = \mathrm{Id}\,. \tag{L23.38}$$

Of course, we do have to check afterwards that the putative index formula behaves 'correctly' under multiplication by a bundle isomorphism.

Thus we are reduced to the case where the symbol is the identity. As we know, by a further (small) homotopy we may assume that the operator itself is of the form $\mathrm{Id} + B$ where $B \in \Psi^{-\infty,0}_{\pi-\mathrm{pt},\mathcal{T}}(M \times \mathbb{S}; \mathbb{C}^N)$, meaning in particular that its indicial family is

$$\beta : S^*M \longrightarrow G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N), \tag{L23.39}$$

so is of the form Id +smoothing and everywhere invertible. This group (classifying for K^{-1}) is contained in our contractible Toeplitz group, so one might think that the whole thing could be contracted away and the index would then be zero. However this cannot be done without deforming the symbol to be non-trivial again. In fact this approach works perfectly well and reduces the problem to the Atiyah-Singer theorem on $M \times \mathbb{S}$.

Rather than do this I will go in the 'opposite direction'. Namely we can further deform the base family, through invertibles of course, until it is a family of finite rank perturbations of the identity, just doing the deformation

$$(1-t)\beta + t\left(\mathrm{Id} - \pi_{(k)} + \pi_{(k)}\beta\pi_{(k)}\right),\ k \text{ large} \tag{L23.40}$$

where $\pi_{(k)}$ is projection onto the span of the first k terms in the Fourier expansion. It follows that the same deformation for the operator gives an elliptic family

$$(1-t)A + t\left((\mathrm{Id} - \pi_{(k)} + \pi_{(k)}A\pi_{(k)}\right) \tag{L23.41}$$

so we are reduced to the case that A acts as the identity on the span of all Fourier coefficients greater than k (of course with arbitrary coefficients in $\mathcal{C}^\infty(M)$).

This corresponds to the inclusion $\Psi^0(M, \mathbb{C}^M) \hookrightarrow \Psi^{-\infty,0}_{\pi-\mathrm{pt},\mathcal{T}}(M \times \mathbb{S}; \mathbb{C}^N)$ in which a pseudodifferential operator is lifted to a finite dimensional subbundle of $\mathcal{C}^\infty(M \times \mathbb{S}; \mathbb{C}^N)$ as a bundle over M. The symbol simply lifts to define the base symbol and the index to the index so we are finally reduced to the Atiyah-Singer theorem on M,

$$\operatorname{ind}(A) = \int_M \mathrm{Td}(M)\operatorname{Ch}_{\mathrm{odd}}(\sigma_0(A)),\ A \in \Psi^0(M; \mathbb{C}^M) \text{ elliptic.} \tag{L23.42}$$

By construction,[7] the eta 'character' lifts the odd Chern character, so indeed (L23.42) reduces to, and hence implies, (L23.10) in this case.

Finally then it remains to check what happens to the formula under composition with a bundle isomorphism; I certainly did not do this in the lecture!

23+. Addenda to Lecture 23

23+.1. Composition with bundle isomophisms.

23+.2. Non-Toeplitz extension. As a first simple generalization of the product-type index formula (L23.10) consider the case of a general elliptic element $A \in \Psi^{0,0}_{\pi-\mathrm{pt}}(M \times \mathbb{S}; \mathbb{C}^N)$ for the product fibration (L23.1). The obvious generalization of (L23.10) is to include 'both sides' of the symbol and to extend the eta invariant to pseudodifferential operators on $\mathbb{S}$.

Proposition 51. *For any fully elliptic element $A \in \Psi^{0,0}_{\pi-\mathrm{pt}}(M \times \mathbb{S}; \mathbb{C}^N)$*

(23+.43)
$$\operatorname{ind}(A) = -\frac{1}{2}\int_{S^*M} \operatorname{Td}(M)\wedge\eta(\beta)+\sum_{\Sigma=\pm}\int_{S^*_\Sigma(M\times\mathbb{S})} \operatorname{Td}(M)\wedge\operatorname{Ch}(a),\ \beta=\beta(A),\ a=\sigma_0(A).$$

Proof. For the moment I just assume that the right side of (23+.43) is homotopy invariant. This involves the extension of the analysis of the variation of the η forms to the full pseudodifferential calculus on the circle and is straightforward (there is an effective reversal of orientation between the two components of the cosphere bundle.

Granted this, it is enough to prove (23+.43) for some operator which is in the same path component of the full elliptic operators. Following the construction of homotopies above the operator may be deformed to one with symbol on in the positive half $S^*_+(M\times\mathbb{S})$ equal to that of a bundle isomorphism and then, composing with the inverse of this, to one with symbol equal to the identity on $S^*_+(M \times \mathbb{S})$. To proceed further we 'separate' the base family into the product of two Toeplitz families, one for the top and one for the bottom.

By a further small deformation we can assume that the full symbol of the base family is equal to the identity on the positive side of $S^*\mathbb{S}$, i.e. that the base family is of the form

(23+.44)
$$\beta(A) = \pi\operatorname{Id}\pi+(\operatorname{Id}-\pi)\alpha_- -(\operatorname{Id}-\pi)+\gamma,\ \gamma\in\mathcal{C}^\infty(B;\Psi^{-\infty}(\mathbb{S};\mathbb{C}^N)),\ \alpha_1\in\mathcal{C}^\infty(B;\Psi^0_{-\mathcal{T}}(\mathbb{S};\mathbb{C}^N))$$

and the whole operator is invertible. Since γ is a smoothing family

$$\pi_{\le k}\gamma\pi_{\le k}\longrightarrow\gamma\in\mathcal{C}^\infty(B;\Psi^{-\infty}(\mathbb{S};\mathbb{C}^N)) \tag{23+.45}$$

where $\pi_{\le k}$ is projection onto all modes (including the negative ones) less than k. Thus when the norm of the difference in (23+.45) is sufficiently small we can replace γ by $\pi_{\le k}\gamma\pi_{\le k}$, and in fact A by

$$\pi_{\le k}A\pi_{\le k} + \operatorname{Id}-\pi_{\le k}$$

and so arrange that A itself acts as the identity on all Fourier modes $\exp(il\theta)$, $l > k$. Now, conjugating the whole operator by the bundle isomophism $e^{i(k+1)\theta}$ gives a family

$$A' = e^{-i(k+1)\theta}Ae^{i(k+1)\theta}\in\mathcal{C}^\infty(B;\Psi^{0,0}_{\pi\text{-pt},-\mathcal{T}}(M\times\mathbb{S};\mathbb{C}^N)) \tag{23+.46}$$

[7] Modulo the constant chase

with values in the negative Toeplitz subalgebra, extended as the identity on the positive modes. Again from the invariance of the result under bundle isomorphisms it suffices to prove the formula for A'.

This reduces the problem to the previous case, with the orientation reversed. $\square$

CHAPTER 24

Index theorems and applications

Lecture 24: 13 December, 2005

L24.1. Inadequacies and extensions. Today I will sketch some conjectural extension of the theorem index theorem I talked about last time and some other applications of the things I have been talking about. In fact I can list a few extra lectures, or topics within lectures, that I would like to do or have done. Starting with the latter category are a couple of topics that I feel I have covered somewhat inadequately (but I will likely put something in the addenda to the notes).

- The discussion of stabilization.
- Isotropic calculus and proper coverage of Bott/Thom/Todd – respectively elements, isomorphism(s) and class. In particular at this stage I have not really described the Todd class at all. Give an oriented real vector bundle $V \longrightarrow M$, say over a compact manifold M, the Thom isomorphism in cohomology is the identification

(L24.1)
$$H^*_{\mathrm{c}}(V) \longrightarrow H^*(M)$$

given by fibre integration – it is always an isomorphism. On the other hand given a *complex* vector bundle, which we can also denote V, there is an extension of the Bott isomorphism (which is the case that V is trivial)

(L24.2)
$$K_{\mathrm{c}}(V) \longrightarrow K(M).$$

Both for the compactly supported K-theory and the K-theory of the base there are Chern character maps – as we have discussed. This gives a diagramme

(L24.3)
$$\begin{tikzcd} K_{\mathrm{c}}(V) \arrow[r, "\mathrm{Thom}"] \arrow[d, "\mathrm{Ch}"'] & K(M) \arrow[d, "\mathrm{Ch}"] \\ H^{\mathrm{evn}}_{\mathrm{c}}(V) \arrow[r, "\mathrm{Thom}", "\times\,\mathrm{Td}"'] & H^{\mathrm{evn}}(M). \end{tikzcd}$$

The problem here is that if we simply take the Thom isomorphisms top and bottom then the diagramme does not commute. This is not totally surprising, since the maps Thom isomorphism are defined under different conditions. To get an isomorphism we have to follow the Thom isomorphism on the bottom, in cohomology, by multiplication by a characteristic class. This is the Todd class, Td, of V. The class that appears in the

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Atiyah-Singer formula is the Todd class of M which is by definition the Todd class of the complexified Tangent bundle of M.
- The computation of $\mathrm{Ch}(P_A)$, the Chern class of the the 'extension' of the original elliptic family. This is very close to the relative Chern character that I have discussed and I will certainly try to put it in somewhere. The Todd class comes out quite naturally from this computation.
- General product-tye index theorem in K-theory.
- Product-type index theorem in cohomology.
- Odd families and determinants.
- Determinant line bundle.
- Gerbes.

I will briefly describe what I think is going on as regards the index theorem for product type operators and quickly indicate how to define the determinant line bundle and gerbes – I don't think I will have time to mention the odd index theorem and determinants, although that is very closely related to the eta invariant and so things I have been talking about. It is possible that I will be motivated enough to write out some more ficticious lectures.

L24.2. Product-type K-theory. So, to talk about the formulation of an index theorem, in K-theory, associated to product-type pseudodifferential operators, let me recall the Atiyah-Singer index theorem, again. This is really at least two theorems. In K-theory it states that for any fibration of compact manifolds

$$\begin{tikzcd} M \arrow[d, "\phi"] \\ B \end{tikzcd} \tag{L24.4}$$

there are two different maps in K-theory which are equal

$$K^0_{\mathrm{c}}(T^*(M/B)) \overset{\mathrm{ind_a}}{\underset{\mathrm{ind_t}}{\longrightarrow}} K^0(B). \tag{L24.5}$$

The top map, the analytic index, is defined by identifying elements of the K-group as triples $(\mathbb{E}, a)$ in which a may be identified as the symbol of a family of elliptic operators, $A \in \Psi^0(M/B; \mathbb{E})$ and then $\mathrm{ind_a}([(\mathbb{E}, a)])$ is the image of the (stabilized) index bundle in $K^0(B)$. The other map is defined via 'geometric trivialization', in which the fibration is embedded as a subfibration of a product fibration $\mathbb{S}^N \times B$.

To extend this to the product-type case we consider a compound fibration of compact manifolds

$$\begin{tikzcd} M \arrow[d, "\psi"'] \arrow[dd, bend left, "\phi=\Phi\psi"] \\ Y \arrow[d, "\Phi"'] \\ B. \end{tikzcd} \tag{L24.6}$$

Here the choice of notation indicates that it is the overall fibration which is analogous to (L24.4). Let me denote by $\Psi^{0,0}\psi - \mathrm{pt}(M/B; \mathbb{E})$ the space of product-type pseudodifferential operators for the compound fibration. For each point $b \in B$ this is an operaor on the fibre of ϕ which is of product type with respect to the fibration ψ (over $\Phi^{-1}(b)$. Given what we have done above, the definition should be fairly self-evident. Such an operator has two 'symbols' and ellipticity means invertibility of both

$$\begin{aligned}\sigma(A) &\in \mathcal{C}^\infty([S^*(M/B), S^*(X/B)]; \hom(\mathbb{E})) \\ \beta(A) &\in \Psi^0(\pi^* M/S^*(X/B); \mathbb{E})\end{aligned} \tag{L24.7}$$

where, with some abuse of notation, $\pi : S^*(X/B) \longrightarrow X$ and the fibration in the second case is the pull-back of $M \longrightarrow X$ to $S^*(X/B)$. In this case the analytic index is well defined by stabilization of any elliptic family, $\mathrm{ind_a}(A) \in K^0(B)$, and only depends on $\sigma(A)$ and $\beta(A)$.

There is every reason (meaning I have not checked the details) to think that we can define an adequate replacement for $K^0_{\mathrm{c}}(T^*(M/B))$ in this setting. Namely, if we consider all invertible pairs (L24.7), subject to the compatibility condition that the symbol of β is the restriction of σ to the boundary, and then impose an equivalence condition corresponding to bundle isomorphism, stabilization and homotopy, then we arrive at an Abelian group which I will denote $K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)$. There will also be an odd version of this, $K^{-1}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)$. Of course the basic idea is that the analytic index descends to this space and defines

$$\mathrm{ind_a} : K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B) \longrightarrow K^0(B). \tag{L24.8}$$

Assuming this construction does work we will get a map generated by σ,

$$K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B) \longrightarrow K^0_{\mathrm{c}}(T^*(M/X). \tag{L24.9}$$

This comes from evaluating the symbol in the 'vertical' directions of the fibration. The manifold with boundary on which σ is defined fibres over $S^*(M/X)$ and this implies that the homotopy class of σ, ignoring β and the compatibility condition, is actually determined by the image in (L24.9). The families index theorem for the pull-back of fibration of M over X to $S^*(X/B)$ gives rise to a second map

$$\mathrm{ind} : K^0_{\mathrm{c}}(T^*(M/X) \longrightarrow K^0(S^*(X/B) \tag{L24.10}$$

which vanishes on the image of (L24.9), expressing the fact that the symbol *is* the symbol of a family of invertible operators there – namely the $\beta(A)$. The space $K^{-1}(S^*(X/B))$ can be identified with[1] homotopy classes of maps into a $G^{-\infty}$. We can take this to be smooth families β which therefore give a pair of symbols (L24.7) with true symbolic part the identity. Putting all this together we arrive at the *conjectural* 6-term exact sequence

$$\begin{tikzcd} K^{-1}(S^*(X/B)) \arrow[r, "\iota"] & K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[r, "\sigma"] & K^0_{\mathrm{c}}(T^*(M/X)) \arrow[d, "\mathrm{ind}"] \\ K^{-1}_{\mathrm{c}}(T^*(M/X)) \arrow[u, "\mathrm{ind}"] & K^{-1}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[l, "\sigma"] & K^0(S^*(X/B)) \arrow[l, "\iota"] \end{tikzcd} \tag{L24.11}$$

The analytic index should be compatible with this complex (it is also defined on $K^{-1}(S^*(X/B))$ and it should be possible to define a toplogical index by embedding and arrive at a (conjectural at this stage) extension of (L24.5):

$$K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \xrightarrow[\mathrm{ind_t}]{\mathrm{ind_a}} K^0(B). \tag{L24.12}$$

[1]In fact was defined here as

QUESTION 1. Is there some more traditional realization of the group $K^0_{c,\psi-\mathrm{pt}}(T^*(M/B))$ – the 6-term exact sequence (L24.11) should serve as a guide to what this might be.

L24.3. Product-type cohomology. In the standard case, the second form of the Atiyah-Singer theorem, or if you prefer 'the Atiyah-Singer formula', gives Chern character of the index bundle in terms of the Chern character of the symbol. So here we expect to get a Chern character

$$\mathrm{Ch}_{\psi-\mathrm{pt}} : K^0_{c,\psi-\mathrm{pt}}(T^*(M/B)) \longrightarrow H^{\mathrm{evn}}_{c,\psi-\mathrm{pt}}(T^*(M/B)) \tag{L24.13}$$

where both the map and the image space are yet to be determined. One thing we expect is that the 6-term exact sequence will be replicated at this level and in fact will correspond to a natural transformation (i.e. be functorial) from the K-theory complex to the 'cohomological complex':

$$\begin{tikzcd}
H^{\mathrm{odd}}(S^*(X/B)) \arrow[r, "\iota"] & H^{\mathrm{evn}}_{c,\psi-\mathrm{pt}}(T^*(M/B)) \arrow[r, "\sigma"] & H^{\mathrm{evn}}_{c}(T^*(M/X)) \arrow[d, "\mathrm{ind}"] \\
H^{\mathrm{odd}}_{c}(T^*(M/X)) \arrow[u, "\mathrm{ind}"] & H^{\mathrm{odd}}_{c,\psi-\mathrm{pt}}(T^*(M/B)) \arrow[l, "\sigma"] & H^{\mathrm{evn}}(S^*(X/B)) \arrow[l, "\iota"]
\end{tikzcd} \tag{L24.14}$$

with the maps from (L24.11) all being the corresponding Chern characters
(L24.15)

$$\begin{tikzcd}
K^{-1}(S^*(X/B)) \arrow[rr, "\iota"] \arrow[dr, "\mathrm{Ch}_{\mathrm{odd}}"] & & K^0_{c,\psi-\mathrm{pt}}(T^*(M/B)) \arrow[rr, "\sigma"] \arrow[d, "\mathrm{Ch}_{\psi-\mathrm{pt}}"] & & K^0_c(T^*(M/X)) \arrow[dl, "\mathrm{Ch}_{\mathrm{evn}}"] \arrow[ddd, "\mathrm{ind}"] \\
& H^{\mathrm{odd}}(S^*(X/B)) \arrow[r, "\iota"] & H^{\mathrm{evn}}_{c,\psi-\mathrm{pt}}(T^*(M/B)) \arrow[r, "\sigma"] & H^{\mathrm{evn}}_{c}(T^*(M/X)) \arrow[d, "\mathrm{ind}"] & \\
& H^{\mathrm{odd}}_{c}(T^*(M/X)) \arrow[u, "\mathrm{ind}"] & H^{\mathrm{odd}}_{c,\psi-\mathrm{pt}}(T^*(M/B)) \arrow[l, "\sigma"] & H^{\mathrm{evn}}(S^*(X/B)) \arrow[l, "\iota"] & \\
K^{-1}_c(T^*(M/X)) \arrow[uuu, "\mathrm{ind}"] \arrow[ur, "\mathrm{Ch}_{\mathrm{odd}}"] & & K^{-1}_{c,\psi-\mathrm{pt}}(T^*(M/B)) \arrow[ll, "\sigma"] \arrow[u, "\mathrm{Ch}_{\psi-\mathrm{pt}}"] & & K^0(S^*(X/B)) \arrow[ll, "\iota"] \arrow[ul, "\mathrm{Ch}_{\mathrm{evn}}"]
\end{tikzcd}$$

So, what should $H^k_{c,\psi-\mathrm{pt}}(T^*(M/B))$ be? The anticipated form of the Chern character is the guide here. Let me try to be a little abstract here and consider a more general setting of a manifold with boundary Z with a fibration $\psi : \partial Z \longleftrightarrow Y$ of its boundary. In the present setting, $Z = [\overline{T^*(X/B)}, S^*X]$ with the 'old' boundary removed (because we really want to consider relative cohomology, i.e. with compact supports, as far as this part of the boundary is concerned.) Thus, we will not assume that Z is compact but we do assume that Y is compact. Then consider pairs

$$(u, \tau) \in \mathcal{C}^\infty_c(X; \Lambda^k) \times \mathcal{C}^\infty(Y; \lambda^j) \tag{L24.16}$$

where $k = j + d$, d being the fibre dimension of the fibration ψ. This pair is closed if $du = 0$ as a smooth form and

$$\iota^*_{\partial Z} u = \psi^* d\tau. \tag{L24.17}$$

The pair is exact if there exists $(u',\tau')\in\mathcal{C}^\infty_c(Z;\Lambda^{k-1})\times\mathcal{C}^\infty(Y;\Lambda^{j-1})$ such that $du'=u$ and $\iota^*_{\partial Z}u'=\psi^*\tau'$.

Note that this definition is modelled on relative cohomology.

Exercise 21. Check that in the case that $\psi:\partial Z\longrightarrow\partial Z$ on recovers the compactly supported cohomology of $\operatorname{int}Z=Z\backslash\partial Z$ which is the same as $H^k_{\mathrm{c}}(Z,\partial Z)$.

Problem 1. Show that this cohomology is well-defined and gives a 6-term exact sequence as in (L24.14). Try to identify the cohomology with intersection cohomology of the stratified space Z/ψ in which the boundary of Z is 'smashed' to Y.

The idea of this definition is that the Chern character (L24.13) is supposed to be given by that pair $(u,\tau)=(\mathrm{Ch}(\sigma),\eta(\beta)$ where $\mathrm{Ch}(\sigma)$ is the relative Chern charcter *form* (i.e. the same formula as before but smoothly up to the boundary and η [2] is the eta form as discussed above. The relationship $\iota^*\partial u=\psi^*\tau$ is just what is I showed last in the special case of a product with a circle, but the argument should go over in general when connections are put in.

It is also natural to expect that in the case of a fibration[3], as for $T^*(M/B)$ where the structure is fibrewise, there should be a pushforward map

$$H^{\mathrm{evn}}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B))\longrightarrow H^{\mathrm{evn}}(B) \tag{L24.18}$$

L24.4. Determinant bundle. Next, a few words about the determinant bundle.

The numerical index is the 0-dimensional part of the Chern character of the index in the standard families case. In the odd case (which I have not discussed) the Chern character maps to odd-degree cohomology and the 1-dimensional part can be thought of as a 'spectral flow' of the phase of a determinant. Back in the usual even setting, the 2-dimensional part of the Chern character corresponds to the determinant bundle, as I will discuss. In the odd case the 3-dimensional part corresponds to the curvature of a gerbe, which I had hoped to get to. The 4-dimensional part of the even Chern character should correspond to a 2-gerbe, but this is not very well understood geometrically.

Back in the usual families case we consider and elliptic family $P\in\Psi^0(M/B;\mathbb{E})$ and for simplicity we assume that the numerical index

$$\#-\operatorname{ind}(P)=\dim\operatorname{null}(P)-\dim\operatorname{null}(P^*)=0. \tag{L24.19}$$

It is of course constant. This is not strictly necessary but definitely simplifies the construction.

The vanishing of the numerical index means that for each point $b\in B$ the operator $P_b\in\Psi^0(Z_b;\mathbb{E}_b)$ can be perturbed by a smoothing operator to be invertible. This allows us to define a big bundle over B where the fibre at b is

$$\mathcal{P}_b=\{P_b+Q_b;Q_b\in\Psi^{-\infty}(Z_b;\mathbb{E}_b);(P_b+Q_b)^{-1}\in\Psi^0(Z_b;\mathbb{E}^-_b). \tag{L24.20}$$

Not only is each fibre non-empty, but it is a principal space for the action of the group $G^{-\infty}(Z_b;E_{b,-})$. Namely, two elements P_b+Q_b, $P_b+Q'_b\in\mathcal{P}_b$ must be related by

$$P_b+Q_b=(\operatorname{Id}+R_b)(P_b+Q'_b),\ R_b\in G^{-\infty}(Z_b;E_{b,-})\subset\Psi^{-\infty}(Z_b;E_{b,-}). \tag{L24.21}$$

[2] Remember I am not claiming that the normalization is correct (yet).

[3] With oriented fibres

Thus we can think of

$$\begin{array}{c}\mathcal{P}\\ \Big\downarrow\\ B\end{array} \tag{L24.22}$$

as a 'principal bundle' although the groups acting on the fibres are actually varying and form a bundle of groups, $G^{-\infty}(M/B;E_-)$. In the discussion earlier on the stabilization of the index bundle I faced[4] the issue of showing that there are 'exhausting' smooth families of projections in $\Psi^{-\infty}(M/B;E_-)$. Using these one can see that the space of components of the sections, $g\in\mathcal{C}^\infty(B;G^{-\infty}(M/B;E_-)$ is actually canonicaly equal to $K^{-1}(B)$. That is, the bundle of groups is at least 'weakly trivial'. Thus (L24.22) does behave very much as a principal bundle.

Recall that the Fredholm determinant is a multiplicative function

$$G^{-\infty}(M/B);E_-)\longrightarrow\mathbb{C}^* \tag{L24.23}$$

defined globallly and invariantly. In this sense it is a 1-dimensional representation of our bundle of groups, or a character if you prefer. As such the principal bundle (L24.22) induces a line bundle (a 1-dimensional vector bundle) over B. The fibre at $b\in B$ is

(L24.24)
$B_b=(\mathcal{P}_b{\times}\mathbb{C})/\sim,\ (P_b{+}Q_b,z)\sim(P_b{+}Q'_b,z')\Longleftrightarrow$ (L24.21) holds and $z=\det(\mathrm{Id}+R_b)z'$.

24+. Addenda to Lecture 24

[4] And shelved for some time.

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