# SUPPLEMENTARY EXERCISES

for Chapter 7 (symmetric functions) of

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1. [2] Find the number $f(n)$ of pairs $(\lambda,\mu)$ such that $\lambda \vdash n$ and $\mu$ covers $\lambda$ in Young's lattice $Y$. Express your answer in terms of $p(k)$, the number of partitions of $k$, for certain values of $k$. Try to give a direct bijection, avoiding generating functions, recurrence relations, induction, etc.

2. [2] Let $p_r(n)$ denote the number of partitions of $n$ of rank $r$. Find the generating function
$$F_r(t) = \sum_{n\geq 0} p_r(n)t^n.$$

3. [1] Express the symmetric function $p_1 e_\lambda$ in terms of elementary symmetric functions.

4. [2] Let
$$\begin{aligned} F_n(x) = (-x_1 + x_2 + x_3 + \cdots + x_n)(x_1 - x_2 + x_3 + \cdots + x_n) \\ \cdots (x_1 + x_2 + \cdots + x_{n-1} - x_n). \end{aligned}$$
Show that
$$F_n(x) = \sum_{k=2}^{n} (-1)^k 2^k e_k e_1^{n-k} - e_1^n,$$
in the ring $\Lambda_n$ of symmetric functions in $n$ variables.

5. [2] Show that
$$e_2^m = \frac{1}{2^m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} p_1^{2k} p_2^{m-k}.$$

6. [2+] Complete the "missing" expansion in Exercise 7.48(f) of EC2 by showing that
$$F_{\mathrm{NC}_{n+1}} = \sum_{\lambda\vdash n} \varepsilon_\lambda \frac{(n+2)(n+3)\cdots(n+\ell(\lambda))}{m_1(\lambda)!\cdots m_n(\lambda)!} h_\lambda.$$

7. [2+] For what real numbers $a$ is the symmetric formal power series $F(x) = \prod_i (1 + ax_i + x_i^2)$ $e$-positive, i.e., a nonnegative (infinite) linear combination of the $e_\lambda$'s?

8. [2+] Find all symmetric functions $f \in \Lambda^n$ that are both $e$-positive and $h$-positive.

9. (a) [5–] Let $f(n)$ (respectively, $g(n)$) denote the maximum (respectively, the absolute value of the minimum) of the numbers $\langle e_n, m_\lambda \rangle$, where $\lambda \vdash n$. For instance,

$$e_4 = h_1^4 - 3h_1^2 h_2 + 2h_1 h_3 + h_2^2 - h_4,$$

so $f(4) = 2$ and $g(4) = 3$. The values $f(3), f(4), \ldots, f(20)$ are 1, 2, 3, 6, 10, 15, 30, 60, 105, 168, 252, 420, 756, 1260, 2520, 5040, 9240, 15840. For $g(3), g(4), \ldots, g(20)$ we get 2, 3, 4, 6, 12, 20, 30, 42, 60, 140, 280, 504, 840, 1512, 2520, 4620, 7920, 13860. What can be said about these numbers? If an exact formula seems difficult, what about an asymptotic formula? Can one describe those $n$ for which $f(n) = g(n)$? Those $n$ satisfying $3 \le n \le 50$ with this property are 6, 9, 17, 21, 24, 48. It seems quite likely that $\lim_{n\to\infty} f(n)/g(n) = 1$.

   (b) [5–] Is the largest entry of the inverse Kostka matrix $(K^{-1}_{\lambda\mu})$ for $\lambda, \mu \vdash n$ equal to $f(n)$? Is the smallest entry equal to $-g(n)$?

10. [1+] Find all $f \in \Lambda^n$ for which $\omega f = 2f$.

11. (a) [2–] Let $\lambda, \mu \vdash n$. Show that $\langle e_\lambda, h_\mu \rangle \le \langle h_\lambda, h_\mu \rangle$.

    (b) [2+] When does equality occur?

12. [2] Let $P(x)$ be a polynomial satisfying $P(0) = 1$. Express $\omega \prod_i P(x_i)$ as an infinite product.

13. [2–] Let $j, k \ge 1$. Expand the monomial symmetric function $m_{\langle k^j \rangle}$ as a linear combination of power sums $p_\lambda$.

14. (a) [3–] Let $\Lambda^n_{\mathbb{Z}}$ denote the (additive) abelian group with basis $\{m_\lambda\}_{\lambda\vdash n}$. Let $\Pi^n_{\mathbb{Z}}$ denote the subgroup generated by $\{p_\lambda\}_{\lambda\vdash n}$. Thus by the Note after Corollary 7.7.2,

$$[\Lambda^n_{\mathbb{Z}} : \Pi^n_{\mathbb{Z}}] = \prod_{\mu\vdash n} d_\mu,$$

where $d_\mu = \prod_{i\geq 1} m_i(\mu)!$. Show that in fact

$$\Lambda^n_{\mathbb{Z}}/\Pi^n_{\mathbb{Z}} \cong \bigoplus_{\mu\vdash n} \mathbb{Z}/d_\mu\mathbb{Z}.$$

(b) [2] (for readers familiar with Smith normal form) Let $X_n$ denote the character table of $\mathfrak{S}_n$. Deduce from (a) that $X_n$ has the same Smith normal form as the diagonal matrix with diagonal entries $d_\mu$, $\mu \vdash n$.

15. [2–] Let $\alpha \in \mathbb{R}$ (or consider $\alpha$ to be an indeterminate). Expand the product $\prod_i (1+x_i)^\alpha$ as an (infinite) linear combination of the power sums $p_\lambda$.

16. [2] Let $f(x,y) \in \Lambda(x)\hat{\otimes}\Lambda(y)$, where the notation means that $f(x,y)$ is a formal power series that is symmetric with respect to $x_1, x_2, \dots$ and separately with respect to $y_1, y_2, \dots$. Let $\frac{\partial}{\partial p_k(x)} f(x,y)$ denote the partial derivative of $f(x,y)$ with respect to $p_k(x)$ when $f(x,y)$ is written as a polynomial in the $p_i(x)$'s (regard the $y_j$'s as constants). Find a simple formula for

$$\frac{\partial}{\partial p_k(x)} \prod_{i,j} (1-x_i y_j)^{-1}.$$

17. [2+] Fix $n \geq 1$. Find a simple formula for the number of pairs $(u,v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that $uv = vu$. Generalize to any finite group $G$ instead of $\mathfrak{S}_n$.

18. [2+] Let $p$ be prime, and let $f_p(n)$ denote the number of partitions $\lambda \vdash n$ for which $z_\lambda \not\equiv 0 \,(\mathrm{mod}\, p)$. Find a simple expression (expressed as an infinite product) for $F_p(x) = \sum_{n\geq 0} f_p(n) x^n$.

19. (a) [3] Fix $n \geq 1$, and let $S$ be an $n$-element subset of $\mathbb{P}$. Show that the *field* $\mathbb{Q}(p_1(x_1,\dots,x_n), p_2(x_1,\dots,x_n), \dots)$ of all rational symmetric functions over $\mathbb{Q}$ in the variables $x_1,\dots,x_n$ is generated by $\{p_i(x_1,\dots,x_n) : i \in S\}$ if $\mathbb{P} - S$ is closed under addition.

    (b) [5] Prove the converse.

20. (a) [2+] Show that the symmetric power series

$$T = \frac{\sum_{n\geq 0} h_{2n+1}}{\sum_{n\geq 0} h_{2n}}$$

is a power series in the *odd* power sums $p_1$, $p_3$, $p_5, \ldots$.

(b) [3–] Identify the coefficients when $T$ is written as a power series in the power sums.

21. [3] Let $f \in \Lambda_{\mathbb{Z}} \cap \mathbb{Q}[p_1, p_3, p_5, \ldots]$, and write $f = \sum_\lambda a_\lambda p_\lambda$. Show that when the rational number $a_\lambda$ is written in lowest terms, then the denominator is odd.

22. [2+] Let $k \geq 1$ and $\lambda \vdash n$ for some $n$. Find a simple formula for the scalar product

$$\langle (1 + h_1 + h_2 + \cdots)^k, h_\lambda \rangle.$$

23. (a) [2+] Let $p$ be a prime, and define the symmetric polynomial

$$F_p = F_p(x_1, \ldots, x_{2p-1}) = \sum_{\substack{S \subseteq [2p-1] \\ \#S = p}} \left( \sum_{i \in S} x_i \right)^{p-1},$$

where the first sum ranges over all $p$-element subsets of $1, 2, \ldots,$ $2p - 1$. Show that when $F_p$ is written as a linear combination of monomials, every coefficient is divisible by $p$.

(b) [2+] Deduce from (a) the Erdős-Ginzburg-Ziv theorem: given any $(2p-1)$-element subset $X$ of $\mathbb{Z}$, there is a $p$-element subset $Y$ of $X$ such that $\sum_{i \in Y} i \equiv 0 \pmod{p}$.

(c) [2+] Show that when $F_p$ is written as a linear combination of power sums $p_\lambda$, every coefficient is an integer divisible by $p$.

24. Given $f \in \Lambda^n_{\mathbb{Q}}$ and $k \in \mathbb{P}$, let $f(kx)$ denote the symmetric function $f$ in $k$ copies of each variable $x_1, x_2, \ldots$. Thus for instance $p_n(kx) = kp_n(x)$.

(a) [2–] Let $\{u_\lambda : \lambda \vdash n\}$ be a basis for $\Lambda^n_{\mathbb{Q}}$, and let

$$f(kx) = \sum_{\lambda \vdash n} c_\lambda(k) u_\lambda. \qquad (1)$$

Show that $c_\lambda(k)$ is a polynomial in $k$ (with rational coefficients). This allows us to use equation (1) to *define* $f(kx)$ for *any* $k$ (in some extension field $F$ of $\mathbb{Q}$, say).

(b) [2–] For any $j \in F$, let $g(x) = f(jx)$. For any $k \in F$ show that $g(kx) = f(jkx)$.

(c) [2] Express $f(-x)$ in terms of $f(x)$ and $\omega$.

25. [2+] Evaluate the scalar product $\langle h_{21^{n-2}}, h_{21^{n-2}} \rangle$.

26. [2+] Fix $n \geq 1$. Find the dimension of the subspace of $\Lambda_{\mathbb{Q}}^n$ spanned by $\{h_\lambda + e_\lambda : \lambda \vdash n\}$.

27. (a) [3–] Define a linear transformation $\varphi : \Lambda_{\mathbb{Q}}^n \to \Lambda_{\mathbb{Q}}^n$ by $\varphi(e_\lambda) = m_{\lambda'}$. Find the size of the largest block in the Jordan canonical form of $\varphi$.

(b) [5–] Find the entire Jordan canonical form of $\varphi$.

(c) [5–] Do the same for such linear transformations as $e_\lambda \mapsto m_\lambda$, $h_\lambda \mapsto m_\lambda$, $p_\lambda \mapsto m_\lambda$, $p_\lambda \mapsto h_\lambda$.

28. Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product on $\Lambda_{\mathbb{Q}}$. Two linear transformations $A, B : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ are *adjoint* if $\langle Af, g \rangle = \langle f, Bg \rangle$ for all $f, g \in \Lambda_{\mathbb{Q}}$.

(a) [2] Find the adjoint to $\omega + aI$, where $I$ denotes the identity transformation and $a$ is a constant.

(b) [2] Let $\frac{\partial}{\partial p_i} f$ denote the partial derivative of $f \in \Lambda_{\mathbb{Q}}$ with respect to $p_i$ when $f$ is written as a polynomial in $p_1, p_2, \ldots$. Define the linear transformation $M_j$ by $M_j(f) = p_j f$. Express the adjoint of $M_j$ in terms of the operators $\frac{\partial}{\partial p_i}$.

29. [2] Let $k \geq 2$. Compute the Kostka number $K_{(k,k,k),(k-1,k-1,1^{k+2})}$.

30. (a) [2+] Let $\lambda \vdash n$ and $\lambda \subseteq \langle k^n \rangle$. Give a bijective proof that

$$K_{\langle k^n \rangle / \lambda, \langle (k-1)^n \rangle} = f^\lambda.$$

(b) [2] Deduce from (a) that $K_{\langle k^n \rangle, \langle (k-1)^n, 1^n \rangle}$ is equal to the number of permutations in $\mathfrak{S}_n$ with no increasing subsequence of length $k+1$.

31. (a) [3–] Let $g(n)$ denote the number of odd Kostka numbers of the form $K_{\lambda,\langle 2^n\rangle}$, where $\lambda \vdash 2n$ and $\langle 2^n\rangle = (2,2,\ldots,2)$ ($n$ 2's). Show that

$$g(2^r) = \binom{2^r+1}{2}.$$

(b) [3] Let $n = 2^r + s$, where $1 \leq s \leq 2^{r-1} - 1$. Show that $g(n) = g(2^r)g(s)$.

(c) [5–] What about $g(2^r - 1)$ and $g(2^r + 2^{r-1})$?

(d) [5–] The values of $g(n)$ for $1 \leq n \leq 24$ are 1, 3, 5, 10, 10, 30, 50, 36, 36, 108, 180, 312, 312, 840, 1368, 136, 136, 408, 680, 1360, 1360, 6800, 4352. Can it be proved that $g(4n) = g(4n+1)$?

32. [2–] How many SYT of shape $(n^n)$ have main diagonal $(1, 4, 9, 16, \ldots, n^2)$?

33. [2] Let $\lambda \vdash n$. Define $g(\lambda) = \sum_{\mu \vdash n} K_{\lambda\mu}$, where $K_{\lambda\mu}$ denotes a Kostka number. Set $g(\emptyset) = 1$. Find a formula for the generating function $\sum_\lambda g(\lambda)s_\lambda$, where the sum ranges over all partitions $\lambda$ of all nonnegative integers. Your formula should be a simple infinite product.

34. (a) [2] Let $\lambda \vdash d \geq 1$. Find a simple expression for the average number of 1's in an SSYT of shape $\lambda$ and maximum part at most $n$.

(b) [3–] Let $m_1(T)$ be the number of 1's in the SSYT $T$, and let $B_{\lambda,n}(k)$ denote the $k$th binomial moment of $m_1(T)$ with respect to the uniform distribution on SSYT's of shape $\lambda \vdash d$ and maximum part at most $n$. That is, $B_{\lambda,n}(k)$ is the expected value of $\binom{m_1(T)}{k}$. Show that

$$B_{\lambda,n}(k) = \frac{(d)_k}{(n+k-1)_k}\frac{f^{\lambda/(k)}}{f^\lambda},$$

where $(m)_k$ is the falling factorial.

(c) [2] Find a simple expression for the average of the square of the number of 1's in an SSYT of shape $\lambda$ and maximum part at most $n$.

(d) [2] With $\lambda \vdash d$ as above, prove the identity

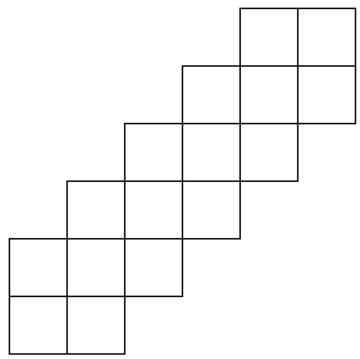
$$\frac{(d)_k}{(n+k-1)_k}\frac{f^{\lambda/(k)}}{f^\lambda} = \frac{\sum_m \binom{m}{k} s_{\lambda/m}(1^{n-1})}{s_\lambda(1^n)}.$$

NOTE. We always define $f^{\lambda/\mu} = s_{\lambda/\mu} = 0$ if $\mu \not\subseteq \lambda$.

35. [3] Let $\delta_m = (m-1, m-2, \ldots, 1)$. Define skew shapes

$$\begin{aligned}
\alpha_n &= (n, n, n-1, n-2, \ldots, 2)/\delta_{n-1} \\
\beta_n &= (n, n, n, n-1, n-2, \ldots, 2)/\delta_n \\
\gamma_n &= (n, n, n, n-1, n-2, \ldots, 1)/\delta_n.
\end{aligned}$$

For instance, the diagram below shows $\alpha_6$.



Show that

$$\begin{aligned}
f^{\alpha_n} &= \frac{(3n-2)!E_{2n-1}}{(2n-1)!2^{2n-2}} \\
f^{\beta_n} &= \frac{(3n-1)!E_{2n-1}}{(2n-1)!2^{2n-1}} \\
f^{\gamma_n} &= \frac{(3n)!(2^{2n-1}-1)E_{2n-1}}{(2n-1)!2^{2n-1}(2^{2n}-1)},
\end{aligned}$$

where $E_{2n-1}$ denotes an Euler number.

36. [3] Let $n \geq 1$. Show that

$$\begin{aligned}
\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq 3}} f^{2\lambda} &= \sum_{k=1}^{n} f^{(k,k,1^{2n-2k})} \\
\sum_{\substack{\lambda \vdash 2n+1 \\ \ell(\lambda) = 3 \\ \lambda_i > 0 \Rightarrow \lambda_i \text{ odd}}} f^{\lambda} &= \sum_{k=1}^{n} f^{(k,k,1^{2n-2k+1})}.
\end{aligned}$$

37. (a) [2] True or false? There exists a nonzero symmetric function $f$ for which $y := (2s_4 - s_{31} + s_{2,2} - s_{211} + 2s_{1111})f$ is Schur-positive, i.e., $\langle y, s_\lambda \rangle \geq 0$ for all $\lambda$.

(b) [5–] What can be said about the set of symmetric functions $g \in \Lambda^n_{\mathbb{R}}$ for which there exists $0 \neq f \in \Lambda$ such that that $fg$ is Schur-positive? Is there a finite algorithm for determining whether $f$ exists?

(c) [5–] What if we require $f$ to be Schur-positive in (b)?

38. [3+] Let $\lambda \vdash n$, $\mu \vdash k$, and $\ell = \ell(\lambda)$. Let $\mathrm{RT}(\mu, \ell)$ be the set of all *reverse* SSYT of shape $\mu$ and largest part at most $\ell$. For a square $u \in \mu$ let $c(u)$ denote its content. Write $(n)_k = n(n-1)\cdots(n-k+1)$. Show that

$$f^{\lambda/\mu} = \frac{f^\lambda}{(n)_k} \sum_{T \in \mathrm{RT}(\mu,\ell)} \prod_{u \in \mu} (\lambda_{T(u)} - c(u)),$$

where $T(u)$ denotes the entry in square $u$ of $T$.

*Example.* For $\mu = (2,1)$ we get

$$\begin{aligned} f^{\lambda/21} \quad = \quad & \frac{f^\lambda}{(n)_3} \left( \sum_{i<j\leq\ell} \lambda_j(\lambda_j - 1)(\lambda_i + 1) \right. \\ & + \sum_{i<j\leq\ell} \lambda_j(\lambda_i + 1)(\lambda_i - 1) + \sum_{i<j<k\leq\ell} \lambda_k(\lambda_j - 1)(\lambda_i + 1) \\ & \left. + \sum_{i<j<k\leq\ell} \lambda_k(\lambda_i - 1)(\lambda_j + 1) \right). \end{aligned}$$

39. (a) [2+] Let $\lambda \in \mathrm{Par}$ and $n \geq \lambda_1 + \lambda'_1$. Show that there are exactly $n$ ways to add a border strip of size $n$ to $\lambda$.

(b) [2] Show that (a) is equivalent to the following statement. Let $f\colon \mathbb{Z} \to \{0,1\}$, such that $f(n) = 0$ for $n$ sufficiently small (that is, $-n$ is sufficiently large) and $f(n) = 1$ for $n$ sufficiently large. Let $a$ be the least integer for which $f(i) = 1$, and let $b$ be the greatest integer for which $f(j) = 0$. Then for $n > b - a$,

$$\#\{i \,:\, f(i) = 0,\ f(i+n) = 1\} = n.$$

40. [3–] Define an *outer square* of a skew shape to be a square $(i, j)$ of the shape for which $(i+1, j+1)$ is not in the shape. Define recursively a skew shape $\lambda/\mu$ to be *totally connected* as follows: (1) a (nonempty) border strip is totally connected, and (2) a skew shape $\lambda/\mu$ which is not a border strip is totally connected if its outer squares form a border strip whose removal results in a totally connected skew shape. The *depth* of any skew shape is the number of times the outer squares have to be removed until reaching the empty set. For instance, 7666653/43221 has depth three: remove the outer squares to get 555542/43221, then the outer squares to get (up to translation) 221/1, and finally the outer squares to get $\emptyset$.

    Let $b_k(n)$ be the number of totally connected skew shapes, up to translation, of size $n$ and depth $k$. Thus $b_1(n) = 2^{n-1}$, with generating function

$$B_1(x) := \sum_{n \geq 1} b_1(n) x^n = \frac{x}{1-2x}.$$

    Show that

$$\begin{aligned} B_k(x) &:= \sum_{n \geq 1} b_k(n) x^n \\ &= \frac{x^{k^2}}{(1-2x^k) \prod_{i=1}^{k-1} (1-2x^i)^2}. \end{aligned}$$

    *Example.* $b_2(4) = 1$ (corresponding to the skew shape 22, which happens to be an ordinary shape), while $b_2(5) = 4$ (corresponding to 222/1, 32, 33/1, and 221).

41. (a) [2–] For any partitions $\lambda$ and $\mu$, express $s_\lambda s_\mu$ as a skew Schur function.

    (b) [2] Show that every skew Kostka number $K_{\lambda/\mu,\nu}$ is equal to some Littlewood-Richardson coefficient $c^{\alpha}_{\beta\gamma}$.

42. (a) [3–] Let $H_\lambda$ denote the product of the hook lengths of $\lambda$. Show that $H_\lambda s_\lambda$ is $p$-integral. (The only known proof uses representation theory. It would be interesting to give a more elementary proof. The difficulty rating assumes a knowledge of the representation theory of finite groups.)

(b) [3] Let $\lambda \vdash n$ and $\mu \vdash k$. Show that $(n)_k f^{\lambda/\mu}/f^\lambda \in \mathbb{Z}$. You may assume (a).

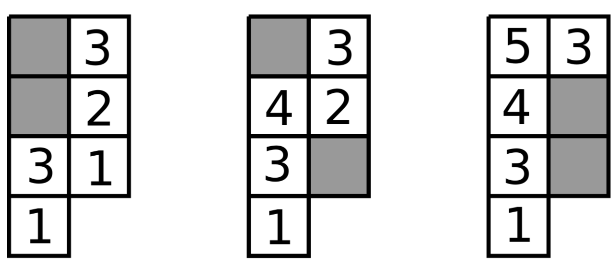
43. [3] Let $\lambda/\mu$ be a skew partition and $D$ a subset of $\lambda$ (identified with its Young diagram). A cell $(i, j) \in D$ is called *active* if $(i+1, j), (i, j+1)$, and $(i+1, j+1)$ are in $\lambda - D$. If $u$ is an active cell of $D$, then define $\alpha_u(D)$ to be the set obtained by replacing $(i, j)$ in $D$ by $(i+1, j+1)$. This replacement is called an *excited move*. An *excited diagram* $\lambda/\mu$ is a subdiagram of $\lambda$ obtained from $\mu$ by a sequence of excited moves on active cells. Let $\mathcal{E}(\lambda/\mu)$ be the set of excited diagrams of $\lambda/\mu$. (We allow the sequence of excited moves to be empty, so $\mu$ itself is always an excited diagram.)

Show that if $|\lambda/\mu| = n$ then

$$f^{\lambda/\mu} = n! \sum_{E \in \mathcal{E}(\lambda/\mu)} \frac{1}{\prod_{u \in \lambda - E} h(u)},$$

where $h(u)$ is the hook length of $u$ in the diagram $\lambda$.

*Example.* Let $\lambda/\mu = 2221/11$. There are three excited diagrams for this skew shape, shown below with the hook lengths of the complementary cells $\lambda - E$.




It follows that

$$f^{2221/11} = 5! \left( \frac{1}{3 \cdot 3 \cdot 2 \cdot 1 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 1} + \frac{1}{5 \cdot 4 \cdot 3 \cdot 3 \cdot 1} \right) = 9.$$

44. [3–] Let $\lambda/\mu$ be a skew shape and $k \geq 0$. Let $M_i$ be the set of all skew shapes obtained from $\lambda/\mu$ by removing a vertical strip of size $0 \leq i \leq k$ from $\mu$ (i.e., adding this strip to the inner boundary of $\lambda/\mu$) and adding

a horizontal strip of size $k-i$ to $\lambda$ (i.e., adding this strip to the outer boundary of $\lambda/\mu$). Show that

$$s_k s_{\lambda/\mu} = \sum_{i=0}^{k} (-1)^i \sum_{\rho \in M_i} s_\rho.$$

45. (a) [2+] Let $\lambda$ be a partition and $m \geq \lambda_1$. Let $\lambda \cup m$ denote the partition obtained by adding a part of length $m$ to $\lambda$. Let $e_i^\perp$ denote the linear operator on symmetric functions adjoint to multiplication by $e_i$. Show that

$$s_{\lambda \cup m} = \left( \sum_{i \geq 0} (-1)^i h_{m+i} e_i^\perp \right) s_\lambda.$$

(b) [2+] Let $1 \leq k \leq n/2$. Let $f_k(n)$ be the number of permutations $w = a_1 \cdots a_n \in \mathfrak{S}_n$ such that $a_1 < a_2 < \cdots < a_{n-k}$, and the longest increasing subsequence of $w$ has length exactly $n-k$. Show that

$$f_k(n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n)_{k-i}.$$

(c) [3–] Is there a "nice" proof of (b) based on the Principle of Inclusion-Exclusion?

46. [2–] Let $1 \leq k \leq n$ and $\lambda = (k, 1^{n-k})$ (called a *hook shape*). For any $\mu \vdash n$ find a simple formula for the Kostka number $K_{\lambda\mu}$.

47. [2+] Let $A$ be the $m \times n$ matrix of all 1's. If $A \xrightarrow{\text{rsk}} (P, Q)$, then describe the SSYT's $P$ and $Q$.

48. (a) [3] Let $\lambda$ be a partition with distinct parts. A *shifted standard Young tableau* (SHSYT) of shape $\lambda$ is defined just like an ordinary standard Young tableau of shape $\lambda$, except that each row is indented one space to the right from the row above. An example of an SHSYT of shape $(5, 4, 2)$ is given by

$$\begin{array}{|c|c|c|c|c|c} \hline 1 & 2 & 3 & 5 & 9 & \\ \hline \multicolumn{1}{c|}{} & 4 & 6 & 8 & 11 & \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & 10 & \multicolumn{1}{|c}{} & \\ \cline{3-4} \end{array}.$$

Call two permutations $u, v \in \mathfrak{S}_n$ *$W$-equivalent* if they belong to the same equivalence class of the transitive closure of the following relation: either (i) they have the same insertion tableau under the RSK-algorithm or (ii) $u(1) = v(2)$, $u(2) = v(1)$, and $u(i) = v(i)$ for $3 \le i \le n$. For instance, the $W$-equivalence classes for $n = 3$ are $\{123, 213, 231, 321\}$ and $\{312, 132\}$. Show that the number of $W$-equivalence classes in $\mathfrak{S}_n$ is equal to the number of SHSYT of size $n$.

(b) [5–] Can this be generalized in an interesting way?

49. Let $\lambda$ be a partition of $n$ with distinct parts, denoted $\lambda \models n$. Let $g^\lambda$ denote the number of shifted SYT of shape $\lambda$, as defined in Problem 48.

(a) [3–] Prove by a suitable modification of RSK that

$$\sum_{\lambda \models n} 2^{n-\ell(\lambda)} \left(g^\lambda\right)^2 = n!. \tag{2}$$

(b) [3] The "shifted analogue" of Corollary 7.13.9 is the following curious result. Let $\zeta = (1+i)/\sqrt{2} = e^{2\pi i/8}$. Let

$$u(n) = \sum_{\lambda \models n} \zeta^{\ell(\lambda)} 2^{(n-\ell(\lambda))/2} g^\lambda.$$

Show that

$$\sum_{n \ge 0} u(n) \frac{t^n}{n!} = e^{\zeta t + \frac{1}{2}t^2}. \tag{3}$$

50. (a) [2+] Let $\lambda$ be a partition with distinct parts. A *strict shifted SSYT* (or S4YT) of shape $\lambda$ is a way of filling the squares of the shifted diagram of $\lambda$ with positive integers such that each row and column is weakly increasing, and every diagonal from the upper left to lower right is strictly increasing. A *component* of an S4YT is a maximal connected set of equal entries. For instance, the diagram

$$\begin{array}{cccccc}
1 & 1 & 1 & 1 & 2 & 2 \\
 & 2 & 2 & 3 & 3 & 4 \\
 & & 3 & 3 & 4 & 4 \\
 & & & 4 & 7 &
\end{array}$$

is an S4YT with seven components, where we have outlined each component. Given an S4YT $T$, let $k(T)$ denote its number of components. Define

$$Q_\lambda(x) = \sum_T 2^{k(T)} x^T,$$

summed over all S4YT of shape $\lambda$. Show that $Q_\lambda(x)$ is a symmetric function. For instance

$$Q_{31}(x) = 4m_{31} + 8m_{22} + 16m_{211} + 32m_{1111}.$$

(b) [3] Show that

$$\sum_\lambda 2^{-\ell(\lambda)} Q_\lambda(x) Q_\lambda(y) = \prod_{i,j} \frac{1+x_i y_j}{1-x_i y_j},$$

where the sum is over all partitions of all $n \geq 0$.

(c) [3–] A *diagonal-strict plane partition* (DSPP) is a plane partition such that there are no $2 \times 2$ squares of equal *positive* entries. If $\pi$ is an DSPP, then define $k(\pi)$ as before (ignoring 0 entries). Let $|\pi|$ denote the sum of the parts of $\pi$. Use (b) to show that

$$\begin{aligned} \sum_\pi 2^{k(\pi)} q^{|\pi|} &= \prod_{j \geq 1} \left( \frac{1+q^j}{1-q^j} \right)^j \\ &= 1 + 2q + 6q^2 + 16q^3 + 38q^4 + 88q^5 + 196q^6 + \cdots, \end{aligned}$$

where $\pi$ ranges over all DSPP.

51. [3–] Let $f(n)$ be the number of plane partitions $(\pi_{ij})_{i,j \geq 1}$ of $n$ satisfying $\pi_{22} \leq 1$. Show that

$$\begin{aligned} \sum_{n \geq 1} f(n) x^n &= \frac{\sum_{k \geq 2} (-1)^k \binom{k}{2} x^{\binom{k}{2}}}{\prod_{j \geq 1} (1-x^j)^3} \\ &= x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + 86x^7 \\ &\quad + 159x^8 + 279x^9 + 488x^{10} + \cdots. \end{aligned}$$

52. [2+] Evaluate the sums

$$\sum_{\lambda\vdash n} f^{\lambda/2} f^{\lambda} \quad \text{and} \quad \sum_{\lambda\vdash n} \left(f^{\lambda/2}\right)^2.$$

Here $\lambda/2$ is short for the skew shape $\lambda/(2)$.

53. (a) [3–] Given an SYT $T$, define $\mathrm{sgn}(T)$ as in Problem 117(b). If $T$ has shape $\lambda$, then let $v(T) = v(\lambda) = \sum \lambda_{2i}$. Suppose that $w \in \mathfrak{S}_n$ and $w \stackrel{\mathrm{rsk}}{\rightarrow} (P, Q)$. Show that

$$\mathrm{sgn}(w) = (-1)^{v(P)} \mathrm{sgn}(P) \cdot \mathrm{sgn}(Q).$$

(b) [2] Let $w_T$ be the reading word of $T$ as in Problem 117(b), and let

$$I_\lambda(q) = \sum_T q^{\mathrm{inv}(w_T)},$$

summed over all SYT $T$ of shape $\lambda$, where $\mathrm{inv}(w_T)$ denotes the number of inversions of $w_T$. Show that

$$\sum_{\lambda\vdash n} (-1)^{v(\lambda)} I_\lambda(-1)^2 = 0.$$

54. Let $o(\lambda)$ denote the number of odd parts of the partition $\lambda$, and $d(\lambda)$ the number of distinct parts. Let $t_{n-1}$ denote the number of involutions in $\mathfrak{S}_{n-1}$.

(a) [2] Use the RSK algorithm to show that

$$\sum_{\lambda\vdash n} o(\lambda) f^\lambda = n t_{n-1}.$$

(b) [2+] Show that

$$\sum_{\lambda\vdash n} d(\lambda) f^\lambda = n t_{n-1}.$$

55. [3] With $o(\lambda)$ as in Problem 54(a), show that

$$\sum_{\lambda\vdash n} f^\lambda \left(\frac{1+q}{1-q}\right)^{o(\lambda)} = \sum_{\lambda\vdash n} f^\lambda \prod_{u\in\lambda} \frac{1+q^{h(u)}}{1-q^{h(u)}},$$

where $h(u)$ denotes the hook length of $u$. Equivalently,

$$\exp\left(\frac{1+q}{1-q}t+\frac{1}{2}t^2\right)=\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\lambda\vdash n}f^\lambda\prod_{u\in\lambda}\frac{1+q^{h(u)}}{1-q^{h(u)}}.$$

56. [2+] Given an SYT $T$, let $\sigma(T)$ be the largest integer $k$ such that $1,2,\dots,k$ appear in the first row of $T$. Let $E_n$ denote the expected value of $\sigma(\mathrm{ins}(w))$, where $w$ is a random (uniform) permutation in $\mathfrak{S}_n$ and $\mathrm{ins}(w)$ denotes the insertion tableau of $w$ under the RSK algorithm. Thus

$$E_n=\frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\sigma(\mathrm{ins}(w)).$$

Find $\lim_{n\to\infty}E_n$.

57. Let $p_{ij}(n)$ be the average value of the $(i,j)$-entry $P(i,j)$ of $P$ when $w\mapsto(P,Q)$ under RSK, for $w\in\mathfrak{S}_n$. (If $P$ has no $(i,j)$-entry, then set $P(i,j)=0$.) For instance, $p_{12}(3)=\frac{1}{6}(2+2+2+3+3+0)=2$. Set $v_{ij}=\lim_{n\to\infty}p_n(i,j)$.

   (a) [1] Find $p_{11}(n)$ for all $n\geq 1$.

   (b) [2+] Show that $v_{12}=e$.

   (c) [3–] Show that

$$v_{13}=e^2\sum_{n\geq 1}\frac{1}{(n-1)!(n+1)!}=5.090678\cdots.$$

   (d) [3–] Show that $v_{22}=1+v_{13}$. (The only known proof is computational. No simple reason is known.)

   (e) [5–] Show that

$$v_{ij}=e^{(i+j)^2+o((i+j)^2)}$$

   as $i,j\to\infty$. Can better information be found about $v_{ij}$?

58. (a) [3] Let $T$ be a random SYT of shape $(n,n)$ (uniform distribution on all $C_n$ such tableaux). Let $p_{ij}(n)$ be the expected value of the entry $T_{ij}$, where $1\leq i\leq 2$ and $1\leq j\leq n$. Let

$$\overline{p}_{ij}=\lim_{n\to\infty}p_{ij}(n).$$

Show that

$$
\begin{aligned}
p_{1,d-1} &= 2d - \frac{d\binom{2d}{d}}{4^{d-1}} \\
p_{2,d} &= 2d + \frac{d\binom{2d}{d}}{4^{d-1}}.
\end{aligned}
$$

(b) [5–] Are there analogous results for other shapes $n\lambda$ as $n \to \infty$?

59. [3] Show that as $n \to \infty$, for almost all (i.e., a $(1 - o(1))$-fraction) permutations $w \in \mathfrak{S}_n$ the number of bumping operations performed in applying RSK to $w$ is

$$(1 + o(1))\frac{128}{27\pi^2}n^{3/2}.$$

Moreover, the number of comparison operations performed is

$$(1 + o(1))\frac{64}{27\pi^2}n^{3/2}\log_2 n.$$

60. (a) [2+] Let $n = pq$, $w \in \mathfrak{S}_n$ and $w \xrightarrow{\text{rsk}} (P, Q)$. Suppose that the shape of $P$ and $Q$ is a $p \times q$ rectangle. Show that when the RSK algorithm is applied to $w$, every bumping path is vertical (never moves strictly to the left).

(b) [2] Let $P = (a_{ij})$ and $Q = (b_{ij})$ in (a). Deduce from (a) that $w(b_{ij}) = a_{p+1-i,j}$.

61. [2] Let $i, j, n \geq 1$. Evaluate the sum

$$f_n(i, j) = \sum_{\lambda \vdash n} s_\lambda(1^i)s_\lambda(1^j).$$

62. [3–] Let $y_n = \sum_{\lambda \vdash n} s_\lambda^2$. Find the generating function

$$F(x) = \sum_{n \geq 0}\langle y_n, y_n\rangle x^n.$$

Express your answer in terms of the generating function $P(x, t) = \prod_{i \geq 1}(1 - tx^i)^{-1}$ (for a suitable value of $t$).

63. [3] Let

$$f(n) = \left\langle \sum_{\mu \vdash n} s_\mu^2, \sum_{\lambda \vdash n} s_{2\lambda} \right\rangle,$$

where $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$. Thus

$$(f(0), f(1), \dots, f(10)) = (1, 1, 3, 5, 12, 20, 44, 76, 157, 281, 559).$$

Show that

$$\sum_{n \geq 0} f(n) q^n = \prod_{i \geq 1} \frac{1}{\sqrt{1 - 2q^i}} \cdot \prod_{j \geq 1} \frac{1}{(1 - q^{2j})^{2^{j-2}}}.$$

64. [2+] Find all symmetric functions $G \in \hat{\Lambda}_{\mathbb{R}}$ (the completion of $\Lambda_{\mathbb{R}}$ as defined on page 291 of EC2) such that $\langle G, fg \rangle = \langle G, f \rangle \cdot \langle G, g \rangle$ for all $f, g \in \Lambda_{\mathbb{R}}$.

65. [3+] Let $V_n = \prod_{1 \leq i < j \leq n} (x_i - x_j)$. Show that for $k \geq 0$,

$$\langle V_n^{2k}, V_n^{2k} \rangle_n = \frac{((2k+1)n)!}{(2k+1)!^n \, n!}.$$

The notation $\langle\, , \rangle_n$ indicates that the scalar product is taken in the ring $\Lambda_n$, i.e., the Schur functions $s_\lambda(x_1, \dots, x_n)$ with $\ell(\lambda) \leq n$ form an orthonormal basis.

66. [2+] Let

$$a_n = \left\langle h_2^n, \sum_{\lambda \vdash 2n} s_\lambda \right\rangle.$$

Find the generating function $F(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$. (A result in Chapter 5 may prove useful.)

67. (a) [3–] Find the number $f(n)$ of ways to move from the empty partition $\emptyset$ to $\emptyset$ in $n$ steps, where each step consists of either (i) adding a box, (ii) removing a box, or (iii) adding and then removing a box, always keeping the diagram of a partition (even in the middle

of a step of type (iii)). For instance, $f(3) = 5$, corresponding to the five sequences

$$\begin{array}{cccc} \emptyset & (1,\emptyset) & (1,\emptyset) & (1,\emptyset) \\ \emptyset & (1,\emptyset) & 1 & \emptyset \\ \emptyset & 1 & (2,1) & \emptyset \\ \emptyset & 1 & (11,1) & \emptyset \\ \emptyset & 1 & \emptyset & (1,\emptyset) \end{array}.$$

Express your answer as a familiar combinatorial number and not, for instance, as a sum.

(b) [3–] Given a partition $\lambda$, let $f_\lambda(n)$ be the same as in (a), except we move from $\emptyset$ to $\lambda$ in $n$ steps. Define

$$T_n = \sum_\lambda f_\lambda(n) s_\lambda.$$

For instance,

$$T_3 = 5 + 10s_1 + 6s_2 + 6s_{11} + s_3 + 2s_{21} + s_{111}.$$

Find $\langle T_m, T_n \rangle$. As in (a), express your answer as a familiar combinatorial number.

68. (a) [2+] Let $h(t) \in \mathbb{C}[[t]]$ with $h(0) \neq 0$, and let $g(t) \in \mathbb{C}[[t]]$. Write $p = p_1 = \sum x_i$. Let $\Omega$ be the operator on $\Lambda_{\mathbb{C}}$ defined by

$$\Omega = g(p) + h(p)\frac{\partial}{\partial p}.$$

Define

$$F(x,p) = \sum_{n \geq 0} \Omega^n(1)\frac{x^n}{n!}.$$

Show that

$$F(x,p) = \exp\left[-M(p) + M(L^{\langle -1 \rangle}(x + L(p)))\right],$$

where

$$\begin{aligned} L(t) &= \int_0^t \frac{ds}{h(s)} \\ M(t) &= \int_0^t \frac{g(s)ds}{h(s)}, \end{aligned}$$

and where $L^{\langle -1 \rangle}$ denotes the compositional inverse of $L$.

(b) [1+] Let $g(t) = t$ and $h(t) = 1$, so $\langle \Omega^n(1), s_\lambda \rangle$ is the number of oscillating tableaux of shape $\lambda$ and length $n$, as defined in Exercise 7.24(d). Show that

$$F(x,p) = \exp\left(px + \frac{1}{2}x^2\right).$$

(c) [2] Let $f_\lambda(n)$ be the number of ways to move from the empty partition $\emptyset$ to $\lambda$ in $n$ steps, where the steps are as in Problem 67(a). Use (a) to show that

$$\sum_{n\geq 0} \sum_{\lambda \in \mathrm{Par}} f_\lambda(n) s_\lambda \frac{t^n}{n!} = \exp\left(-1 - p + (1+p)e^x\right).$$

(d) [2–] Let $g_\lambda(n)$ be the number of ways to move from $\emptyset$ to $\lambda$ in $n$ steps, where each step consists of adding one square at a time any number $i$ of times (including $i = 0$) to the current shape and then either stopping or deleting one square (always maintaining the shape of a partition). Show that

$$\sum_{n\geq 0} \sum_{\lambda \in \mathrm{Par}} g_\lambda(n) s_\lambda \frac{x^n}{n!} = \exp\left(1 - p - \sqrt{(1-p)^2 - 2x}\right).$$

In particular,

$$\sum_{n\geq 0} g_\emptyset(n) \frac{x^n}{n!} = \exp\left(x + \sum_{k\geq 2} (2k-3)!! \frac{x^k}{k!}\right).$$

(e) [2–] Let $j_\lambda(n)$ be the number of ways to move from $\emptyset$ to $\lambda \vdash k$ in $n$ steps, where each step consists of adding one square at a time any number $i$ of times (including $i = 0$) to the current shape or else deleting one square (always maintaining the shape of a partition). Show that

$$j_\lambda(n) = n! \left(\!\!\binom{n}{k}\!\!\right) f^\lambda,$$

where as usual $f^\lambda$ denotes the number of SYT of shape $\lambda$.

69. [3] Let $w = a_1 a_2 \cdots a_{2n} \in \mathfrak{S}_{2n}$. Suppose that $a_i + a_{2n+1-i} = 2n+1$ for all $1 \leq i \leq n$. Show that the shape of the insertion tableau $\mathrm{ins}(w)$ can be covered with $n$ dominos.

70. [2–] Let $d, n \geq 1$ and $\zeta = e^{2\pi i/d}$, a primitive $d$th root of unity. Let $f \in \Lambda^n$. Show that $f(1, \zeta, \ldots, \zeta^{d-1}) = 0$ unless $d|n$.

71. [3–] Let $(n-3)/2 \leq m \leq n-1$. Show that

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} f^\lambda = t(n) - \sum_{\substack{i,j,l \geq 0 \\ 2i+j+2l=n-m-1}} \frac{(-1)^i (n)_{i+j}}{i!\, j!} t(j),$$

where $t(j)$ denotes the number of involutions in $\mathfrak{S}_j$.

72. [2] Let $u$ be a square of the skew shape $\lambda/\mu$. We can define the *hook* $H(u) = H_{\lambda/\mu}(u)$ just as for ordinary shapes, viz., the set of squares directly to the right of $u$ and directly below $u$, counting $u$ itself once. Similarly we can define the *hook length* $h(u) = h_{\lambda/\mu}(u) := \#H(u)$. Let $(\lambda/\mu)^r$ denote $\lambda/\mu$ rotated 180°, as in Exercise 7.56. Show that

$$\sum_{u \in \lambda/\mu} h_{\lambda/\mu}(u) = \sum_{u \in (\lambda/\mu)^r} h_{(\lambda/\mu)^r}(u).$$

73. (a) [2+] For any partition $\lambda \vdash n$, show that

$$\sum_{u \in \lambda} h(u)^2 = n^2 + \sum_{u \in \lambda} c(u)^2,$$

where $h(u)$ denotes the hook length and $c(u)$ the content of the square $u$.

(b) [5] Find a bijective proof.

74. (a) [3–] Let $\eta_k(\lambda)$ be the number of hooks of length $k$ of the partition $\lambda$. Show that

$$\sum_{\lambda \vdash n} \eta_k(\lambda) = k \sum_{\lambda \vdash n} m_k(\lambda).$$

As usual, $m_k(\lambda)$ denotes the number of parts of $\lambda$ equal to $k$. Note that Problem 1 is equivalent to the case $k = 1$. Is there a simple bijective proof similar to the solution to Problem 1?

(b) [3–] Part (a) can be rephrased as follows. For $u = (i, j) \in \lambda$, let $r(u) = \lambda_i$, the length of the row in which $u$ appears. Then the statistics $h(u)$ and $r(u)$ have the same distribution over all squares of all $\lambda \vdash n$, i.e.,

$$\sum_{\lambda\vdash n}\sum_{u\in\lambda} x^{h(u)} = \sum_{\lambda\vdash n}\sum_{u\in\lambda} x^{r(u)}.$$

Show in fact that $h(u)$ and $r(u)$ have a symmetric joint distribution, i.e., if

$$F(x,y) = \sum_{\lambda\vdash n}\sum_{u\in\lambda} x^{h(u)}y^{r(u)},$$

then $F(x,y)=F(y,x)$.

(c) [2] Let $n \geq 1$. For any real (or complex) number $z$, show that

$$\sum_{\lambda\vdash n}\sum_{u\in\lambda} h(u)^z = \sum_{\lambda\vdash n}\sum_{i} \lambda_i^{z+1}.$$

75. [3+] Given a partition $\lambda$ and $u \in \lambda$, let $a(u)$ and $\ell(u)$ denote the arm and leg lengths of $u$ as in Exercise 7.26. Define

$$\gamma(\lambda) = \#\{u \in \lambda \,:\, a(u) - l(u) = 0 \text{ or } 1\}.$$

Show that

$$\sum_{\lambda\vdash n} q^{\gamma(\lambda)} = \sum_{\lambda\vdash n} q^{\ell(\lambda)},$$

where $\ell(\lambda)$ denotes the length (number of parts) of $\lambda$.

76. [2–] Let $a(\lambda,n)$ be the degree of the polynomial $s_\lambda(1,q,\ldots,q^{n-1})$, and let $b(\lambda,n)$ be the exponent of the largest power of $q$ dividing this polynomial. Show that $a(\lambda,n)+b(\lambda,n)$ depends only on $|\lambda|$ and $n$.

77. [2] Let $\delta = (n-1,n-2,\ldots,0)$ as usual, and let $\lambda \in \mathrm{Par}$ with $n \geq \ell(\lambda)$. Find the Schur function expansion of the product

$$s_\delta(x_1,\ldots,x_n)s_\lambda(x_1^2,\ldots,x_n^2).$$

78. [2+] Let $t$ be an indeterminate. When $\left(\sum_\lambda s_\lambda\right)^t$ is expanded in terms of power sums, the coefficient of $p_\lambda$ will be a polynomial $P_\lambda(t)$. If $\lambda \vdash n$, then show that

$$n!P_\lambda(t) = \sum_{\substack{w\in\mathfrak{S}_n\\ \rho(w^2)=\lambda}} t^{\kappa(w)},$$

where $\kappa(w)$ denotes the number of cycles of $w$.

79. [2] Let $E_k$ denote an Euler number (the number of alternating permutations of $1, 2, \ldots, k$). Evaluate the determinants

$$A_n = \left| \frac{E_{2i+2j-1}}{(2i+2j-1)!} \right|_{i,j=1}^{n}$$

and

$$B_n = \left| \frac{E_{2i+2j-3}}{(2i+2j-3)!} \right|_{i,j=1}^{n}.$$

Hint. Use Exercise 7.40.

80. [2+] Let $f(n)$ be the number of permutations $w \in \mathfrak{S}_n$ such that both $w$ and $w^{-1}$ are alternating. Let

$$\begin{aligned} L(x) &= \frac{1}{2} \log \frac{1+x}{1-x} \\ &= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots . \end{aligned}$$

Use Corollary 7.23.8 and Exercise 7.64 to show that

$$\begin{aligned} \sum_{k \geq 0} f(2k+1) x^{2k+1} &= \sum_{k \geq 0} E_{2k+1}^2 \frac{L(x)^{2k+1}}{(2k+1)!} \\ \sum_{k \geq 0} f(2k) x^{2k} &= \frac{1}{\sqrt{1-x^2}} \sum_{k \geq 0} E_{2k}^2 \frac{L(x)^{2k}}{(2k)!}, \end{aligned}$$

where $E_n$ denotes an Euler number.

81. [3–] Let $a(n)$ denote the number of alternating involutions in $\mathfrak{S}_n$, i.e., the number of involutions in $\mathfrak{S}_n$ that are alternating permutations in the sense of the last two paragraphs of Section 3.16. Let $E_m$ denote an Euler number. Use Problem 126 below and Exercise 7.64 to show that

$$\begin{aligned} \sum_{k \geq 0} a(2k+1) x^{2k+1} &= \sum_{i,j \geq 0} \frac{E_{2i+2j+1}}{(2i+1)!\, j!\, 4^j} \left( \tan^{-1} x \right)^{2i+1} \left( \log \frac{1+x^2}{1-x^2} \right)^j \\ \sum_{k \geq 0} a(2k) x^{2k} &= \frac{1}{\sqrt[4]{1-x^4}} \sum_{i,j \geq 0} \frac{E_{2i+2j}}{(2i)!\, j!\, 4^j} \left( \tan^{-1} x \right)^{2i} \left( \log \frac{1+x^2}{1-x^2} \right)^j . \end{aligned}$$

82. [3+] Let $\lambda \vdash n$, and let $a, b, c, d$ be (commuting) indeterminates. Define

$$w(\lambda) = a^{\sum \lceil \lambda_{2i-1}/2 \rceil} b^{\sum \lfloor \lambda_{2i-1}/2 \rfloor} c^{\sum \lceil \lambda_{2i}/2 \rceil} d^{\sum \lfloor \lambda_{2i}/2 \rfloor}.$$

For instance, if $\lambda = (5, 4, 4, 3, 2)$ then $w(\lambda)$ is the product of the entries below in the diagram of $\lambda$:

$$\begin{array}{l} a\,b\,a\,b\,a \\ c\,d\,c\,d \\ a\,b\,a\,b \\ c\,d\,c \\ a\,b \end{array} \quad .$$

Let $y = \sum_\lambda w(\lambda) s_\lambda$, where $\lambda$ ranges over all partitions. Show that

$$\log(y) - \sum_{n \geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n \geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].$$

Note that if we set $a = qt$, $b = q^{-1}t$, $c = qt^{-1}$, $d = q^{-1}t^{-1}$ and then set $q = t = 0$, then $y$ becomes $\sum_\lambda s_\lambda$, where $\lambda$ ranges over all partitions such that each $\lambda_i$ and $\lambda'_i$ is even.

83. Let $\omega_y : \Lambda(x, y) \to \Lambda(x) \otimes \Lambda(y)$ be the algebra homomorphism defined by

$$\omega_y p_n(x, y) = p_n(x) + (-1)^{n-1} p_n(y). \tag{4}$$

Equivalently, $\omega_y$ is the automorphism $\omega$ acting on the $y$-variables only. Write $\omega_y f(x, y) = f(x/y)$. In particular, $s_\lambda(x/y)$ is called a *super Schur function*. Let

$$\Sigma = \mathrm{im}(\omega_y) = \{f(x/y) \,:\, f \in \Lambda\},$$

a subalgebra of $\Lambda(x) \otimes \Lambda(y)$.

(a) [2–] Show that

$$s_\lambda(x/y) = \sum_{\mu \subseteq \lambda} s_\mu(x) s_{\lambda'/\mu'}(y). \tag{5}$$

(b) [3–] Let $g(x, y) \in \Lambda(x) \otimes \Lambda(y)$, and let $t$ be an indeterminate. Show that $g \in \Sigma$ if and only if

$$g(x, y)\big|_{x_1 = t,\, y_1 = -t} = g(x, y)\big|_{x_1 = y_1 = 0}. \tag{6}$$

(c) [3] Prove the following "finite analogue" of (b). Let $g \in \Lambda(x_1, \ldots, x_m) \otimes \Lambda(y_1, \ldots, y_n)$. Then

$$g|_{x_1=t,\, y_1=-t} = g(x,y)|_{x_1=y_1=0}$$

if and only if $g$ is a polynomial in the "variables" $p_i(x_1, \ldots, x_m) + (-1)^{i-1} p_i(y_1, \ldots, y_n)$, $i \geq 1$.

(d) [2–] Show that for any $f \in \Lambda$, $f(x/x)$ is a polynomial in the odd power sums $p_1, p_3, p_5, \ldots$.

(e) [3–] Define a *supertableau* of shape $\lambda$ to be an array $T$ of positive integers of shape $\lambda$ such that (i) the rows and columns are weakly increasing, and (ii) the diagonals from the upper left to lower right are strictly increasing (equivalently, there is no $2 \times 2$ square of equal entries). A maximal rookwise-connected subset of equal entries is called a *component* of $T$. Let $c(T)$ denote the number of components. For instance, if $T$ is given by:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 2 | 3 |
| 1 | 2 | 2 | 2 | 3 | 4 | |
| 1 | 2 | 3 | 3 | 3 | 4 | |
| 2 | 2 | 4 | 4 | | | |
| 3 | 4 | 4 | | | | |

then $T$ has one component of 1's, two components of 2's, three components of 3's, and two components of 4's, so $c(T) = 8$. Show that

$$s_\lambda(x/x) = \sum_T 2^{c(T)} x^T,$$

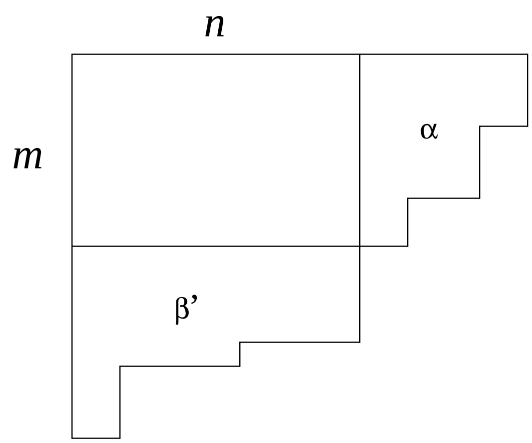
where $T$ ranges over all supertableaux of shape $\lambda$ and $x^T$ has its usual meaning.

(f) [2] Let $(n^m)$ denote an $m \times n$ rectangular shape. Show in two different ways that

$$s_{(n^m)}(x_1, \ldots, x_m / y_1, \ldots, y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j).$$

The first proof (easy) should use Exercises 7.41 and 7.42. The second proof should use (b) above (in the easy "only if" direction) but no RSK, Cauchy identity, etc.

(g) [3–] More generally, let $\alpha$, $\beta$ be partitions with $\ell(\alpha) \leq m$ and $\ell(\beta) \leq n$. Let $[m, n, \alpha, \beta]$ denote the partition obtained by adjoining $\alpha$ to the right of $(n^m)$ and $\beta'$ below $(n^m)$, as illustrated below.




Show that

$$\begin{aligned} & s_{[m,n,\alpha,\beta]}(x_1, \ldots, x_m / y_1, \ldots, y_n) \\ &= s_\alpha(x_1, \ldots, x_m) s_\beta(y_1, \ldots, y_n) \cdot \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j). \end{aligned}$$

84. (a) [3] Define a grading on the ring $\Lambda$ of symmetric functions by setting $\deg(p_i) = 1$ for all $i \geq 1$. Thus $\deg(p_\lambda) = \ell(\lambda)$. Let $\hat{s}_\lambda$ denote the sum of the terms of least degree appearing in the expansion of $s_\lambda$ in terms of power sums, called a *bottom Schur function*. It is an easy consequence of Exercise 7.52 and the Murnaghan-Nakayama rule that this least degree is equal to $\operatorname{rank}(\lambda)$. For instance,

$$s_{221} = \frac{1}{24} p_1^5 - \frac{1}{12} p_2 p_1^3 - \frac{1}{6} p_3 p_1^2 + \frac{1}{8} p_2^2 p_1 + \frac{1}{4} p_4 p_1 - \frac{1}{6} p_3 p_2,$$

so

$$\hat{s}_{221} = \frac{1}{4} p_4 p_1 - \frac{1}{6} p_3 p_2.$$

Let $V_n$ denote the subspace of $\Lambda_{\mathbb{Q}}$ spanned by all $\hat{s}_\lambda$ such that $\lambda \vdash n$. Show that a basis for $V_n$ is given by $\{\hat{s}_\lambda : \operatorname{rank}(\lambda) = \ell(\lambda)\}$.

(b) [2] Deduce from (a) that $\dim V_n$ is the number of $\mu \vdash n$ whose parts differ by at least 2. (By the Rogers-Ramanujan identities, this is also the number of $\mu \vdash n$ whose parts are $\equiv \pm 1 \pmod{5}$.)

(c) [3] Define the *augmented monomial symmetric function* $\tilde{m}_\lambda = r_1!r_2!\cdots m_\lambda$, where $\lambda = \langle 1^{r_1}, 2^{r_2}, \dots\rangle$. Let $t_\lambda$ denote the result of substituting $ip_i$ for $p_i$ in the expansion of $\hat{s}_\lambda$ in terms of power sums. Suppose that $t_\lambda = \sum_\mu a_{\lambda\mu}p_\mu$. Show that

$$t_\lambda = \sum_\mu a_{\lambda\mu}\tilde{m}_\mu.$$

(d) [5–] Let $W_n$ denote the space of all $f \in \Lambda^n_{\mathbb{Q}}$ such that if $f = \sum_\mu a_{\lambda\mu}p_\mu$, then $f = \sum_\mu a_{\lambda\mu}\tilde{m}_\mu$. Find $\dim W_n$. Does $W_n$ have a nice basis?

(e) [5–] Let $\varphi_k(s_\lambda)$ denote the sum of the terms of the least $k$ degrees (that is, of degrees $\operatorname{rank}(\lambda)$, $\operatorname{rank}(\lambda)+1, \dots, \operatorname{rank}(\lambda)+k-1$) appearing in the expansion of $s_\lambda$ in terms of power sums, so in particular $\hat{s}_\lambda = \varphi_1(s_\lambda)$. Let $V_n^{(k)}$ denote the subspace of $\Lambda^n_{\mathbb{Q}}$ spanned by all $\varphi_k(s_\lambda)$. Show that a basis for $V_n^{(2)}$ is given by $\{\hat{s}_\lambda : \operatorname{rank}(\lambda) \geq \ell(\lambda)-1\}$

(f) [5–] Find a basis and/or the dimension of $V_n^{(k)}$ for $k \geq 3$. Note. It is *false* that a basis for $V_n^{(3)}$ is given by $\{\hat{s}_\lambda : \operatorname{rank}(\lambda) \geq \ell(\lambda)-2\}$.

85. (a) [2+] Show the $\mathbb{Z}$-linear span of all the augmented monomial symmetric functions $\tilde{m}_\lambda$ of Problem 84(c) is equal to the $\mathbb{Z}$-linear span of all power sum symmetric functions $p_\mu$.

(b) [5–] Let $\gamma$ be the linear transformation on $\Lambda_{\mathbb{Q}}$ defined by $\gamma(m_\lambda) = \tilde{m}_\lambda$. Does $\gamma$ have any interesting properties?

86. (a) [2] Let $s_k^\perp$ denote the adjoint to multiplication by $s_k$, so $s_k^\perp$ is the linear operator on $\Lambda$ defined by $s_k^\perp s_\lambda = s_{\lambda/k}$. Show that $s_k^\perp p_\lambda = \sum_\nu p_\nu$, where $\nu$ is obtained from $\lambda$ by removing a set of parts (regarding equal parts as distinguishable) summing to $k$. For instance, $s_3^\perp p_1^5 p_2 = 10p_1^2p_2 + 5p_1^4$.

(b) [2] Let $\psi$ be the linear operator on $\Lambda$ defined by $\psi f = \sum_{k\geq 0} s_k^\perp f$. Show that $\psi f = f|_{p_i \to p_i+1}$, i.e., expand $f$ as a polynomial in the $p_i$'s and substitute $p_i+1$ for each $p_i$. In particular, $\psi$ is an algebra automorphism.

87. [3] Let $t$ be an indeterminate. Let $\vartheta : \Lambda \to \Lambda[t]$ be the specialization (homomorphism) defined by

$$\vartheta(p_k) = t + \sum_{i=1}^{k} \binom{k}{i} p_i.$$

Show that

$$\vartheta(s_\lambda) = \sum_{\mu \subseteq \lambda} \frac{f^{\lambda/\mu}}{|\lambda/\mu|!} \left( \prod_{u \in \lambda/\mu} (t + c(u)) \right) s_\mu,$$

where $c(u)$ denotes the content of the square $u$.

88. [3–] Define a $\mathbb{Q}$-linear transformation $\varphi\colon \Lambda_{\mathbb{Q}} \to \mathbb{Q}[t]$ by

$$\varphi(s_\lambda) = \frac{\prod_{i=1}^{n} (t + \lambda_i + n - i)}{H_\lambda},$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \vdash n$ and $H_\lambda$ denotes the product of the hook lengths of $\lambda$. Show that for any $\mu \vdash n$ with $\ell(\mu) = \ell$ and $m_1(\mu) = m$ (the number of parts of $\mu$ equal to 1), we have

$$\varphi(p_\mu) = (-1)^{n-\ell} \sum_{i=0}^{m} \binom{m}{i} t(t+1) \cdots (t+i-1).$$

89. (a) [3] Define a $\mathbb{Q}$-linear transformation $\psi\colon \Lambda_{\mathbb{Q}} \to \mathbb{Q}[n]$ by

$$\psi(m_\lambda) = \left( \sum \lambda_i^2 \right) m_\lambda(1^n).$$

Let $\lambda \vdash d$. Show that

$$\begin{aligned} \psi(s_{1^d}) &= \frac{n(n-1)(n-2)\cdots(n-d+1)}{(d-1)!} \\ \psi(s_\lambda) &= \frac{(dn + a_\lambda) s_\lambda(1^n)}{(n+1)H_\lambda}, \ \ \lambda_1 \geq 2, \end{aligned}$$

where $H_\lambda$ denotes the product of the hook lengths of $\lambda$ and

$$a_\lambda = \sum_i \lambda_i^2 + 2 \sum_{i<j} \lambda_i' \lambda_j'.$$

(b) [3–] Show that

$$\psi(p_\lambda) = n^{\ell(\lambda)-1}\left(\left(\sum \lambda_i^2\right) n + d^2 - \sum \lambda_i^2\right).$$

(c) [3–] Show that

$$\psi(e_\lambda) = \frac{dn + 2\sum_{i<j}\lambda_i\lambda_j}{n\prod_i \lambda_i!}\prod_{i\geq 1}(n-i+1)^{\lambda'_i}.$$

(d) [3–] Show that

$$\psi(h_\lambda) = \frac{dn^2 + (2d^2 - d - 2\sum_{i<j}\lambda_i\lambda_j)n + 2\sum_{i<j}\lambda_i\lambda_j}{n(n+1)\prod_i \lambda_i!}\prod_{i\geq 1}(n+i-1)^{\lambda'_i}.$$

(e) [3–] Show that

$$\psi(\omega(m_\lambda)) = \frac{\varepsilon_\lambda\left(\left(\sum \lambda_i^2\right) n + 2d - \sum \lambda_i^2\right)}{\prod_i m_i(\lambda)!}\prod_{\substack{i=1\\ i\neq 2}}^{\ell(\lambda)}(n+i-1).$$

(f) [3–] Extend to

$$\psi_r(m_\lambda) = \left(\sum \lambda_i^r\right) m_\lambda(1^n).$$

where $r \geq 3$. In particular, if $\lambda_1 \geq r$ then show that $\psi_r(s_\lambda)$ is divisible by

$$\frac{s_\lambda(1^n)}{(n+1)(n+2)\cdots(n+r-1)}.$$

90. [5] Let $\mathcal{I}$ be a collection of subintervals $\{i, i+1, \dots, i+j\}$ of $[n]$. (Without loss of generality we may assume that $\mathcal{I}$ is an *antichain*, i.e., if $I, J \in \mathcal{I}$ and $I \subseteq J$, then $I = J$.) Define

$$f_{\mathcal{I}}(x) = \sum_{i_1 i_2 \cdots i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $i_1 i_2 \cdots i_n$ ranges over all $n$-tuples of positive integers such that if $j, k \in I \in \mathcal{I}$ and $j \neq k$, then $x_{i_j} \neq x_{i_k}$. Thus $f_{\mathcal{I}} \in \Lambda$. For instance, if $\mathcal{I} = \emptyset$ then $f_{\mathcal{I}} = e_1^n$, and if $\mathcal{I} = \{[n]\}$ then $f_{\mathcal{I}} = n!\, e_n$. Show that $f_{\mathcal{I}}$ is $e$-positive.

91. (a) [2+] Fix integers $1 \leq m \leq n$. Find simple formulas for the four sums

$$\begin{aligned}
a(m,n) &= \sum_{\mu\vdash m}\sum_{\nu\vdash n-m}\sum_{\lambda\vdash n} f^\mu f^\nu f^\lambda c^\lambda_{\mu\nu} \\
b(m,n) &= \sum_{\mu\vdash m}\sum_{\nu\vdash n-m}\sum_{\lambda\vdash n} f^\mu f^\nu c^\lambda_{\mu\nu} \\
c(m,n) &= \sum_{\mu\vdash m}\sum_{\nu\vdash n-m}\sum_{\lambda\vdash n} f^\nu f^\lambda c^\lambda_{\mu\nu} \\
d(m,n) &= \sum_{\mu\vdash m}\sum_{\nu\vdash n-m}\sum_{\lambda\vdash n} f^\lambda c^\lambda_{\mu\nu},
\end{aligned}$$

where $c^\lambda_{\mu\nu}$ denotes a Littlewood-Richardson coefficient. Some of the formulas may involve the number $t(k)$ of involutions in $\mathfrak{S}_k$ for certain $k$.

(b) [3] Suppose that $c^\lambda_{\mu\nu} = 2$. Show that $c^{n\lambda}_{n\mu,n\nu} = n+1$ for every positive integer $n$.

(c) [2+] Let

$$e(m,n) = \sum_{\mu\vdash m}\sum_{\nu\vdash n-m}\sum_{\lambda\vdash n} f^\nu c^\lambda_{\mu\nu}.$$

Show that

$$\sum_{m\geq 0}\sum_{k\geq 0} e(m,m+k)x^m\frac{y^k}{k!} = P(x)\exp\left(\frac{y}{1-x}+\frac{y^2}{2(1-x^2)}\right),$$

where $P(x) = \prod_{i\geq 1}(1-x^i)^{-1}$.

(d) [5–] Do something similar for

$$f(m,n) = \sum_{\mu\vdash m}\sum_{\nu\vdash n-m}\sum_{\lambda\vdash n} c^\lambda_{\mu\nu}.$$

92. (a) [3–] Show that

$$\sum_{\mu,\nu,\lambda}\left(c^\lambda_{\mu\nu}\right)^2 t^{|\mu|}q^{|\lambda|} = \frac{1}{\prod_{i\geq 1}(1-(1+t^i)q^i)}. \tag{7}$$

(b) [5–] Part (a) when $t = 1$ can be restated as follows: Let

$$f(n) = \sum_{\substack{\mu,\nu,\lambda \\ \lambda \vdash n}} \left(c^{\lambda}_{\mu\nu}\right)^2.$$

Then $f(n)$ is equal to the number of partitions $\lambda \vdash n$, with each part $\lambda_i > 0$ colored either red or blue. Find a bijective proof.

(c) [5–] Develop a theory of the largest or the typical Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$, where $\lambda \vdash n$, analogous to what was done for $f^{\lambda}$ (Exercise 7.109(e)). It follows from (a) that

$$\log_2 \max_{\substack{\lambda,\mu,\nu \\ \lambda \vdash n}} c^{\lambda}_{\mu,\nu} \sim \frac{n}{2},$$

but this gives no insight into what partitions $\lambda, \mu, \nu$ achieve the maximum value. It also follows from (a) that the maximum of $c^{\lambda}_{\mu,\nu}$ for $\lambda \vdash n$ occurs when $|\mu|$ and $|\nu|$ are both near $n/2$.

(d) [2] Generalize (a) as follows: let $k \geq 1$. Then

$$\sum_{\substack{\mu^1,\ldots,\mu^k,\lambda \\ \lambda \vdash n}} \langle s_{\mu^1} \cdots s_{\mu^k}, s_{\lambda} \rangle \, t_1^{|\mu^1|} \cdots t_k^{|\mu^k|} q^n = \frac{1}{\prod_{i \geq 1} \left(1 - (t_1^i + \cdots + t_k^i) q^i\right)}.$$

Here $(\mu_1, \ldots, \mu_k)$ runs over all $k$-tuples of partitions, which without loss of generality satisfy $\sum |\mu^i| = n$. (Difficulty rating of [2] assumes that (a) has been solved.)

(e) [2] Let $\psi_n$ denote the character of the action of $\mathfrak{S}_n$ on itself by conjugation. Show that for $\lambda \vdash n$,

$$\langle \mathrm{ch}\, \psi_n, h_{\lambda} \rangle = \sum_{\substack{\lambda \vdash n \\ \mu^1 \vdash \lambda_1, \mu^2 \vdash \lambda_2, \ldots}} \langle s_{\lambda}, s_{\mu^1} s_{\mu^2} \cdots \rangle^2.$$

93. [3–] Let $k \geq 1$ and

$$B_k(x) = \sum_{\ell(\lambda) \leq k} s_{\lambda}(x),$$

as in Exercise 7.16(a). Show that

$$B_k(x) = \frac{\sum_{\mu} (-1)^{c_{\mu}} s_{\mu}(x)}{\prod_i (1 - x_i) \cdot \prod_{i<j} (1 - x_i x_j)},$$

where $\mu$ ranges over all partitions whose Frobenius notation has the form

$$\mu = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1+k & a_2+k & \cdots & a_r+k \end{pmatrix},$$

and where $c_\mu = (|\mu| - rk + r)/2$.

94. [5–] Let $n$ be even. Show that the symmetric function $\sum_{i=0}^{n}(-1)^i e_2^i h_2^{n-i}$ is Schur positive.

95. (a) [3] For two skew shapes $\lambda/\mu$ and $\nu/\rho$ such that $\lambda+\nu$ and $\mu+\rho$ both have all even parts, show that

$$\left(s_{\frac{\lambda+\nu}{2}/\frac{\mu+\rho}{2}}\right)^2 \geq_s s_{\lambda/\mu}s_{\nu/\rho},$$

where $f \geq_s g$ means that $f-g$ is Schur-positive.

(b) [3] For two partitions $\lambda$ and $\mu$, let $\lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \dots)$ be the partition obtained by rearranging all parts of $\lambda$ and $\mu$ in weakly decreasing order. Let $\mathrm{sort}_1(\lambda,\mu) = (\nu_1, \nu_3, \nu_5, \dots)$ and $\mathrm{sort}_2(\lambda,\mu) = (\nu_2, \nu_4, \nu_6, \dots)$. Show that

$$s_{\mathrm{sort}_1(\lambda,\mu)}s_{\mathrm{sort}_2(\lambda,\mu)} \geq_s s_\lambda s_\mu.$$

96. [3+] Let $\lambda, \mu, \nu$ be partitions and $n \in \mathbb{P}$. Show that $c^\lambda_{\mu\nu} \neq 0$ if and only if $c^{n\lambda}_{n\mu,n\nu} \neq 0$.

97. [4–] Let $\lambda, \mu, \nu$ be partitions of length at most $n$. If $A$ is an $n \times n$ hermitian matrix with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, then write $\mathrm{spec}(A) = (\alpha_1, \dots, \alpha_n)$. Show that the following two conditions are equivalent:

- There exist $n \times n$ hermitian matrices $A, B, C$ such that $A = B + C$, $\mathrm{spec}(A) = \lambda$, $\mathrm{spec}(B) = \mu$, and $\mathrm{spec}(C) = \nu$.
- $c^\lambda_{\mu\nu} \neq 0$, where $c^\lambda_{\mu\nu}$ denotes a Littlewood-Richardson coefficient.

98. (a) [3+] If $\lambda \vdash n$ and $p$ is prime, then an *abelian $p$-group of type $\lambda$* is the direct sum $\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\lambda_2}\mathbb{Z} \oplus \cdots$. Show that the following two conditions are equivalent:

   - There exists an abelian $p$-group $G$ of type $\lambda$ and a subgroup $H$ of type $\mu$ such that $G/H$ has type $\nu$.

- $c^\lambda_{\mu\nu} \neq 0$, where $c^\lambda_{\mu\nu}$ denotes a Littlewood-Richardson coefficient.

(b) [5] Let $n \geq 2$. Suppose that $G$ has type $n\lambda$, $H$ has type $n\mu$, and $G/H$ has type $n\nu$. Is there a subgroup $K$ of $G$ of type $\lambda$ such that $K \cap H$ has type $\mu$ and $K/(K \cap H)$ has type $\nu$?

99. (a) [2–] Find all partitions $\lambda \vdash n$ such that $\chi^\lambda(\mu) \neq 0$ for all $\mu \vdash n$.

(b) [5–] Find all partitions $\mu \vdash n$ such that $\chi^\lambda(\mu) \neq 0$ for all $\lambda \vdash n$.

100. [2] Show that for every partition $\lambda$ there exists a partition $\mu$ for which $\chi^\lambda(\mu) = \pm 1$.

101. [2+] Given $\lambda \vdash n$, let $H_\lambda$ denote the product of the hook lengths of $\lambda$, so $H_\lambda = n!/f^\lambda$. Show that for $k \in \mathbb{N}$,

$$\sum_{\lambda \vdash n} H_\lambda^{k-2} = \frac{1}{n!} \#\{(w_1, w_2, \ldots, w_k) \in \mathfrak{S}_n^k \,:\, w_1^2 w_2^2 \cdots w_k^2 = 1\}.$$

Hint. Use Exercises 7.69(b) (or more precisely, its solution) and 7.70.

102. [2+] For any partition $\lambda \neq \emptyset$, show that

$$\frac{s_{\lambda/1}(1, q, q^2, \ldots)}{(1-q)s_\lambda(1, q, q^2, \ldots)} = \sum_{u \in \lambda} q^{c_u},$$

where $c_u$ denotes the content of the square $u$ of $\lambda$.

103. (a) [2+] Show that

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} (f^\lambda)^2 \prod_{u \in \lambda} (t + c_u^2) \cdot \frac{x^n}{n!^2} = (1-x)^{-t},$$

where $c_u$ denotes the content of the square $u$ in the diagram of $\lambda$.

(b) [3+] Show that

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} (f^\lambda)^2 \prod_{u \in \lambda} (t + h_u^2) \cdot \frac{x^n}{n!^2} = \prod_{i \geq 1} (1 - x^i)^{-1-t},$$

where $h_u$ denotes the hook length of the square $u$ in the diagram of $\lambda$.

(c) [3] Show that for any $r \geq 0$ we have

$$\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2\sum_{u\in\lambda}\prod_{i=0}^{r-1}(c_u^2-i^2)=\frac{(2r)!}{(r+1)!^2}(n)_{r+1}. \tag{8}$$

(d) [2] Deduce from equation (8) that

$$\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2\sum_{u\in\lambda}c_u^{2k}=\sum_{j=1}^{k}T(k,j)\frac{(2j)!}{(j+1)!^2}(n)_{j+1},$$

where $T(k,j)$ is a *central factorial number* (Exercise 5.8).

(e) [3] Show that for any $r \geq 0$ we have

$$\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2\sum_{u\in\lambda}\prod_{i=1}^{r}(h_u^2-i^2)=\frac{1}{2(r+1)^2}\binom{2r}{r}\binom{2r+2}{r+1}(n)_{r+1}. \tag{9}$$

(f) [2] Deduce from (e) that

$$\frac{1}{n!}\sum_{\lambda\vdash n}f_\lambda^2\sum_{u\in\lambda}h_u^{2k}=\sum_{j=1}^{k+1}T(k+1,j)\frac{1}{2j^2}\binom{2(j-1)}{j-1}\binom{2j}{j}(n)_j,$$

where $T(k+1,j)$ is as in (d).

(g) [3] Let $F=F(x)\in\Lambda_{\mathbb{Q}}$ be a symmetric function. Define

$$\Phi_n(F)=\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2F(h_u^2\,:\,u\in\lambda).$$

Here $F(h_u^2\,:\,u\in\lambda)$ denotes substituting the quantities $h_u^2$, where $u$ is a square of the diagram of $\lambda$, for $n$ of the variables of $F$, and setting all other variables equal to 0. Show that $\Phi_n(F)$ is a polynomial function of $n$.

(h) [3] Let $G(x;y)$ be a power series of bounded degree (say over $\mathbb{Q}$) that is symmetric separately in the $x$ variables and $y$ variables. Let

$$\Psi_n(G)=\frac{1}{n!}\sum_{\lambda\vdash n}(f^\lambda)^2G(c_u\,:\,u\in\lambda;\ \lambda_i-i\,:\,1\leq i\leq n).$$

Show that $\Psi_n(G)$ is a polynomial function of $n$.

104. [2+] Let $k \geq 1$ and $p \geq 2$. Show that the number of $p$-cores (as defined in Exercise 7.59(d)) with largest part $k$ is $\binom{k+p-2}{p-2}$.

105. Let $p, q \geq 2$. A $(p,q)$-*core* is a partition that is both a $p$-core and a $q$-core. Assume now that $\gcd(p,q) = 1$.

   (a) [3–] Show that the number of $(p,q)$-cores is $\frac{1}{p+q}\binom{p+q}{p}$. For instance, there are seven $(5,3)$-cores, namely, $\emptyset, 1, 2, 11, 31, 211, 4211$.

   (b) [3] Let $c = \lfloor p/2 \rfloor$ and $d = \lfloor q/2 \rfloor$. Show that the number of self-conjugate $(p,q)$-cores is equal to $\binom{c+d}{c}$.

   (c) [3] Show that the largest $n$ for which some partition of $n$ is a $(p,q)$-core is equal to

$$\frac{(p^2-1)(q^2-1)}{24}.$$

   Moreover, this $(p,q)$-core is unique (and therefore self-conjugate).

   (d) [3+] Show that the average size of a $(p,q)$-core is equal to

$$\frac{(p+q+1)(p-1)(q-1)}{24}.$$

   (e) [3+] Show that the average size of a self-conjugate $(p,q)$-core is also equal to

$$\frac{(p+q+1)(p-1)(q-1)}{24}.$$

106. (a) [2+] For a positive integer $k$ and partitions $\lambda^1, \dots, \lambda^k \vdash n$, define

$$g_{\lambda^1,\lambda^2,\cdots,\lambda^k} = \langle \chi^{\lambda^1}\chi^{\lambda^2}\cdots\chi^{\lambda^k}, \chi^{(n)} \rangle.$$

   (Note that $\chi^{(n)}$ is the trivial character of $\mathfrak{S}_n$, with $\chi^{(n)}(w) = 1$ for all $w \in \mathfrak{S}_n$.) Show that

$$\begin{aligned} u_k(n) &:= \sum_{\lambda^1,\dots,\lambda^k \vdash n} (g_{\lambda^1,\lambda^2,\dots,\lambda^k})^2 \\ &= \sum_{\mu \vdash n} (z_\mu)^{k-2}. \end{aligned} \tag{10}$$

(b) [2+] Show that

$$\begin{aligned} v_k(n) &:= \sum_{\lambda^1,\dots,\lambda^k \vdash n} g_{\lambda^1,\lambda^2,\dots,\lambda^k} \\ &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{sq}(w)^k, \end{aligned}$$

where

$$\mathrm{sq}(w) = \#\{y \in \mathfrak{S}_n \,:\, y^2 = w\},$$

the number of square roots of $w$.

(c) [2+] For fixed $n$, let $M(n)$ be the maximum value of $g_{\lambda\mu\nu}$ as $\lambda$, $\mu$, $\nu$ range over partitions of $n$. Show that

$$\log M(n) = \frac{n}{2}\log n - \frac{n}{2} + \mathrm{O}(\sqrt{n}), \quad \text{as } n \to \infty.$$

107. Fix a partition $\mu \vdash k$, and define $N(n;\mu) = \sum_{\lambda \vdash n} f^{\lambda/\mu}$. Let $t(j)$ denote the number of involutions in $\mathfrak{S}_j$.

(a) [2+] Show that for all $n, k \geq 0$ we have

$$N(n+k;\mu) = \sum_{j=0}^{k} \binom{n}{j} \left( \sum_{\nu \vdash k-j} f^{\mu/\nu} \right) t(n-j).$$

(b) [3–] Let $\tilde{\nu}$ be the partition obtained from $\nu$ by replacing each even part $2i$ with $i, i$. Equivalently, if $w$ is a permutation of cycle type $\nu$, then $w^2$ has cycle type $\tilde{\nu}$. Show that for $n \geq k$,

$$N(n;\mu) = \sum_{j=0}^{k} \frac{t(n-j)}{(k-j)!} \sum_{\substack{\nu \vdash j \\ m_1(\nu)=m_2(\nu)=0}} z_\nu^{-1} \chi^\mu(\tilde{\nu}, 1^{k-j}).$$

For instance,

$$N(n;32) = N(n;221) = \frac{1}{24}(t(n) - 4t(n-3) + 6t(n-4)).$$

108. (a) [2] Let $\lambda \vdash pq$ and $\mu = \langle p^q \rangle$. Show that $\chi^\lambda(\mu) = 0$ unless $\lambda$ has an empty $p$-core.

(b) [3–] Let $\lambda \vdash pq$, and suppose that $\lambda$ has an empty $p$-core. Let $\mu = \langle p^q \rangle$. Show that when we use equation (7.75) to evaluate $\chi^\lambda(\mu)$, then every term $(-1)^{\mathrm{ht}(T)}$ has the same value.

(c) [2+] Let $Y$ denote Young's lattice and $Y_{p,\emptyset}$ the sublattice of $Y$ consisting of partitions with empty $p$-core. Let $\varphi\colon Y_{p,\emptyset} \to Y^p$ be the isomorphism of Exercise 7.59(e). Let $\lambda \in Y_{p,\emptyset}$ with $\lambda \vdash pq$. Suppose that $\varphi(\lambda) = (\lambda^1, \dots, \lambda^p)$, where $\lambda^i \vdash n_i$. With $\mu$ as above, show that

$$\chi^\lambda(\mu) = \pm \binom{q}{n_1, \dots, n_p} f^{\lambda^1} \cdots f^{\lambda^p}.$$

109. (a) [3–] Fix a partition $\mu \vdash k$. Given $\lambda \vdash n \geq k$, define

$$\widehat{\chi}^\lambda(\mu, 1^{n-k}) = \frac{(n)_k \chi^\lambda(\mu, 1^{n-k})}{\chi^\lambda(1^n)}.$$

Let $p \times q$ denote the partition with $p$ parts equal to $q$. Fix a partition $w_\mu \in \mathfrak{S}_k$ of cycle type $\mu$, and let $\kappa(w)$ denote the number of cycles of the permutation $w \in \mathfrak{S}_k$. Show that

$$\widehat{\chi}^{p\times q}(\mu, 1^{pq-k}) = (-1)^k \sum_{uv = w_\mu} p^{\kappa(u)} (-q)^{\kappa(v)},$$

where the sum ranges over all $k!$ pairs $(u, v) \in \mathfrak{S}_k \times \mathfrak{S}_k$ satisfying $uv = w_\mu$. HINT. Use the Murnaghan-Nakayama rule and Exercise 7.70.

(b) [3+] Suppose that (the diagram of) the partition $\lambda$ is a union of $m$ rectangles of sizes $p_i \times q_i$, where $q_1 \geq q_2 \geq \cdots \geq q_m$, as shown in Figure 1. Let $\mathfrak{S}_k^{(m)}$ denote the set of permutations $u \in \mathfrak{S}_k$ whose cycles are colored with $1, 2, \dots, m$. More formally, if $C(u)$ denotes the set of cycles of $u$, then an element of $\mathfrak{S}_k^{(m)}$ is a pair $(u, \varphi)$, where $u \in \mathfrak{S}_k$ and $\varphi : C(u) \to [m]$. If $\alpha = (u, \varphi) \in \mathfrak{S}_k^{(m)}$ and $v \in \mathfrak{S}_k$, then define a "product" $\alpha v = (w, \psi) \in \mathfrak{S}_k^{(m)}$ as follows. First let $w = uv$. Let $\tau = (a_1, a_2, \dots, a_j)$ be a cycle of $w$, and let $\rho_i$ be the cycle of $u$ containing $a_i$. Set

$$\psi(\tau) = \max\{\varphi(\rho_1), \dots, \varphi(\rho_j)\}.$$

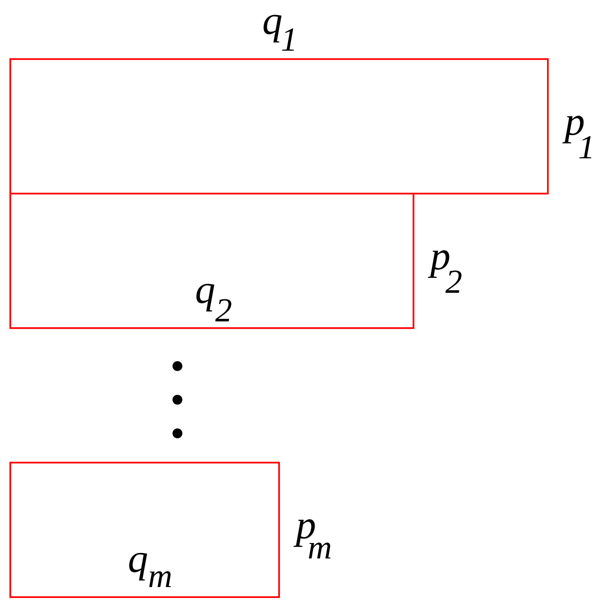



Figure 1: A union of $m$ rectangles

For instance (multiplying permutations from left to right),

$$(\overbrace{1,2,3}^{1})(\overbrace{4,5}^{2})(\overbrace{6,7}^{3})(\overbrace{\ 8\ }^{2})\cdot(1,7)(2,4,8,5)(3,6) = (\overbrace{1,4,2,6}^{3})(\overbrace{3,7}^{3})(\overbrace{5,8}^{2}).$$

Note that it is an immediate consequence of the well-known formula

$$\sum_{w\in\mathfrak{S}_k} x^{\kappa(w)} = x(x+1)\cdots(x+k-1)$$

that $\#\mathfrak{S}_k^{(m)} = (k+m-1)_k$.

Given $\alpha = (u,\varphi) \in \mathfrak{S}_k^{(m)}$, let $\boldsymbol{p}^{\kappa(\alpha)} = \prod_i p_i^{\kappa_i(\alpha)}$, where $\kappa_i(\alpha)$ denotes the number of cycles of $u$ colored $i$, and similarly $\boldsymbol{q}^{\kappa(\beta)}$, so $(-\boldsymbol{q})^{\kappa(\beta)} = \prod_i (-q_i)^{\kappa_i(\beta)}$.

Let $\lambda$ be the partition of $n$ given by Figure 1. Let $\mu \vdash k$ and fix a permutation $w_\mu \in \mathfrak{S}_k$ of cycle type $\mu$. Define

$$F_\mu(\boldsymbol{p};\boldsymbol{q}) = F_\mu(p_1,\ldots,p_m;q_1,\ldots,q_m) = \widehat{\chi}^\lambda(\mu, 1^{n-k}).$$

Show that

$$F_\mu(\boldsymbol{p};\boldsymbol{q}) = (-1)^k \sum_{\alpha w_\mu = \beta} \boldsymbol{p}^{\kappa(\alpha)}(-\boldsymbol{q})^{\kappa(\beta)},$$

where the sum ranges over all $(k+m-1)_k$ pairs $(\alpha,\beta) \in \mathfrak{S}_k^{(m)} \times \mathfrak{S}_k^{(m)}$ satisfying $\alpha w_\mu = \beta$. In particular, $F_\mu(\boldsymbol{p};\boldsymbol{q})$ is a polynomial function of the $p_i$'s and $q_i$'s with integer coefficients, satisfying

$$(-1)^k F_\mu(1,\dots,1;-1,\dots,-1) = (k+m-1)_k.$$

110. (a) [2+] Let $\kappa(w)$ denote the number of cycles of $w \in \mathfrak{S}_n$. Show that

$$P_n(q) := \sum_w q^{\kappa(w(1,2,\dots,n))} = \frac{1}{n(n+1)}\left((q+n)_{n+1} - (q)_{n+1}\right).$$

where $w$ ranges over all $(n-1)!$ $n$-cycles in $\mathfrak{S}_n$ and $w(1,2,\dots,n)$ denotes the product of $w$ with the $n$-cycle $(1,2,\dots,n)$. For instance,

$$\begin{aligned}\sum_{\rho(w)=(3)} q^{\kappa(w(1,2,3))} &= \frac{1}{12}((q+3)_4 - (q)_4)\\ &= q^3 + q.\end{aligned}$$

HINT. Use Exercise 7.70.

(b) [2+] Show that all the zeros of $P_n(q)$ have real part 0.

(c) [3–] It follows from (a) that

$$P_n(q) = \frac{1}{\binom{n+1}{2}} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} c(n+1,n-2i)q^{n-2i},$$

where $c(n+1,n-2i)$ denotes the number of permutations $w \in \mathfrak{S}_{n+1}$ with $n-2i$ cycles. Is there a bijective proof? (In fact, it isn't so obvious that $c(n+1,n-2i)$ is divisible by $\binom{n+1}{2}$. J. Burns has proved the stronger result that if $\lambda \vdash n+1$ and $\varepsilon_\lambda = -1$, then $(n+1)!/z_\lambda$ is divisible by $\binom{n+1}{2}$.)

(d) [3] Generalize (b) as follows. Fix $\lambda \vdash n$. Define

$$P_\lambda(q) = \sum_{\rho(w)=\lambda} q^{\kappa(w(1,2,\dots,n))},$$

where $w$ ranges over all permutations in $\mathfrak{S}_n$ of cycle type $\lambda$. Show that all the zeros of $P_\lambda(q)$ have real part 0.

111. (a) [3] Define two compositions $\alpha$ and $\beta$ of $n$ to be *equivalent* if $s_{B_\alpha} = s_{B_\beta}$ (as defined in §7.23). Describe the equivalence classes of this equivalence relation, showing in particular that the cardinality of each equivalence class is a power of two.

NOTE. A "trivial" equivalence is given by

$$(\alpha_1, \alpha_2, \dots, \alpha_k) \sim (\alpha_k, \dots, \alpha_2, \alpha_1).$$

It is surprising that an equivalence class can have more than two elements, e.g., $\{(1,2,1,3,2), (2,3,1,2,1), (2,1,2,3,1), (1,3,2,1,2)\}$.

(b) [3] Let $f(n)$ denote the number of different symmetric functions $s_{B_\alpha}$ for $\alpha \in \mathrm{Comp}(n)$. Thus $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 6$, $f(5) = 10$, $f(6) = 20$. Show that

$$f(n) = 2\left(2^{n-1} * 2^{\lfloor n/2 \rfloor} * \left(2^{n-1} + 2^{\lfloor n/2 \rfloor}\right)^{-1}\right),$$

where $*$ denotes Dirichlet convolution, defined by

$$(a * b)_n = \sum_{d \mid n} a_d b_{n/d},$$

and where $^{-1}$ denotes inverse with respect to Dirichlet convolution.

112. [3] Define the *rank* of a skew shape $\lambda/\mu$ to be the minimal number of border strips in a border strip tableau of shape $\lambda/\mu$. It it easy to see that when $\mu = \emptyset$ this definition agrees with that on page 289 of EC2. Let $|\lambda/\mu| = n$, and let $\nu$ be a partition of $n$ satisfying $\ell(\nu) = \mathrm{rank}(\lambda/\mu)$. Show that $\chi^{\lambda/\mu}(\nu)$ is divisible by $m_1(\nu)!\, m_2(\nu)! \cdots$. (Incidentally, note that by the definition (7.75) of $\chi^{\lambda/\mu}(\nu)$ we have $\chi^{\lambda/\mu}(\nu) = 0$ if $\ell(\nu) < \mathrm{rank}(\lambda/\mu)$.)

113. Let $\lambda/\mu$ be a skew shape, identified with its Young diagram $\{(i,j) : \mu_i < j \le \lambda_i\}$. We regard the points $(i,j)$ of the Young diagram as squares. An *outside top corner* of $\lambda/\mu$ is a square $(i,j) \in \lambda/\mu$ such that $(i-1,j), (i,j-1) \notin \lambda/\mu$. An *outside diagonal* of $\lambda/\mu$ consists of all squares $(i+p, j+p) \in \lambda/\mu$ for which $(i,j)$ is a fixed outside top corner. Similarly an *inside top corner*

of $\lambda/\mu$ is a square $(i,j) \in \lambda/\mu$ such that $(i-1,j), (i,j-1) \in \lambda/\mu$ but $(i-1,j-1) \notin \lambda/\mu$. An *inside diagonal* of $\lambda/\mu$ consists of all

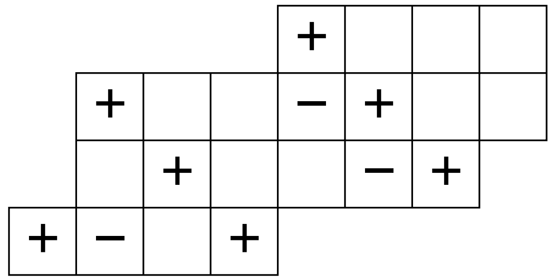


Figure 2: Outside and inside diagonals of the skew shape 8874/411

squares $(i+p, j+p) \in \lambda/\mu$ for which $(i,j)$ is a fixed inside top corner. If $\mu = \emptyset$, then $\lambda/\mu$ has one outside diagonal (the main diagonal) and no inside diagonals. Figure 2 shows the skew shape 8874/411, with outside diagonal squares marked by $+$ and inside diagonal squares by $-$. Let $d^+(\lambda/\mu)$ (respectively, $d^-(\lambda/\mu)$) denote the total number of outside diagonal squares (respectively, inside diagonal squares) of $\lambda/\mu$.

Generalizing the code $C_\lambda$ of Exercise 7.59, define the *code* $C_{\lambda/\mu}$ of $\lambda/\mu$ to be the two-line array whose top line is $C_\lambda$ and whose bottom line is $C_\mu$, where the indexing is "in phase." For instance,

$$C_{8874/411} = \begin{array}{cccccccccccccccc} \cdots & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots \end{array}.$$

[3–, for the first four] Show that the following numbers are equal:

- The rank of $\lambda/\mu$, as defined in Exercise 112 above.
- $d^+(\lambda/\mu) - d^-(\lambda/\mu)$
- The number of rows in the Jacobi-Trudi matrix for $\lambda/\mu$ (i.e., the matrix of Theorem 7.16.1) which don't contain a 1.
- The number of columns of $C_{\lambda/\mu}$ equal to $\begin{smallmatrix}1\\0\end{smallmatrix}$ (or to $\begin{smallmatrix}0\\1\end{smallmatrix}$).
- [3+] The largest power of $t$ dividing the polynomial $s_{\lambda/\mu}(1^t)$.

114. [2+] Let $\kappa(w)$ denote the number of cycles of $w \in \mathfrak{S}_n$. Regard $\kappa$ as a class function on $\mathfrak{S}_n$. Let $\lambda \vdash n$. Show that

$$\langle \kappa, \chi^\lambda \rangle = \begin{cases} \sum_{i=1}^n \frac{1}{i}, & \text{if } \lambda = (n) \\ (-1)^{n-p-q} \frac{p-q+1}{(n-q+1)(n-p)}, & \text{if } \lambda = (p, q, 1^{n-p-q}),\ q > 0 \\ 0, & \text{otherwise.} \end{cases}$$

115. (a) [3] Define a class function $f_n : \mathfrak{S}_n \to \mathbb{Z}$ by

$$f_n(w) = n!(\kappa(w)+1)^{\kappa(w)-1},$$

where $\kappa(w)$ denotes the number of cycles of $w$. Show that $f_n$ is a character of $\mathfrak{S}_n$.

(b) [3–] Let $F(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, so $F(x)^{\langle -1\rangle} = x^{1/x}$. Let the Taylor series expansion of $F(x)$ about $x=1$ be given by

$$\begin{aligned} F(x) &= \sum_{n\geq 0} a_n \frac{(x-1)^n}{n!} \\ &= 1 + u + 2\frac{u^2}{2!} + 9\frac{u^3}{3!} + 56\frac{u^4}{4!} + 480\frac{u^5}{5!} + 5094\frac{u^6}{6!} + \cdots, \end{aligned}$$

where $u = x-1$. Show that $\langle f_n, \mathrm{sgn}\rangle = a_n$, where sgn denotes the sign character of $\mathfrak{S}_n$.

(c) [2] Show that for $n \geq 1$,

$$a_n = \sum_{k=1}^{n} s(n,k)(k+1)^{k-1},$$

where $s(n,k)$ is a Stirling number of the first kind.

116. Let $E(\lambda)$ (respectively, $O(\lambda)$) be the number of SYT of shape $\lambda$ whose major index is even (respectively, odd).

(a) [2+] Express the symmetric function

$$R_n = \sum_{\lambda\vdash n} (E(\lambda) - O(\lambda)) s_\lambda$$

in terms of the power sum symmetric functions.

(b) [2+] Deduce from (a) that if $\lambda \vdash n$, then $E(\lambda) = O(\lambda)$ if and only if one cannot place $\lfloor n/2 \rfloor$ disjoint dominos (i.e., two squares with an edge in common) on the diagram of $\lambda$.

(c) [2+] Show that (b) continues to hold for skew shapes $\lambda/\mu$ when $|\lambda/\mu|$ is even, but that the "only if" part can fail when $|\lambda/\mu|$ is odd.

(d) [2+] Let $p$ be prime. Generalize (a)–(c) to the case $A_0(\lambda) = A_1(\lambda) = \cdots = A_{p-1}(\lambda)$, where $A_i(\lambda)$ denotes the number of SYT $T$ of shape $\lambda$ satisfying $\mathrm{maj}(T) \equiv i \,(\mathrm{mod}\, p)$.

117. (a) [5] A problem superficially similar to 116(b) is the following. We can regard an SYT of shape $\lambda$ (or more generally, a linear extension of a finite poset $P$) as a permutation of the squares of $\lambda$ (or the elements of $P$), where we fix some particular SYT $T$ to correspond to the identity permutation. Define an *even* SYT to be one which, regarded as a permutation, is an even permutation, and similarly *odd* SYT. For which $\lambda$ is the number of even SYT the same as the number of odd SYT? (It's easy to see that the answer does not depend on the choice of the "identity SYT" $T$.) This problem has been solved for rectangular shapes by a difficult argument (rating [3] or even [3+]).

(b) [3] Given an SYT $T$ with $n$ squares, let $w_T$ be the permutation of $[n]$ obtained by reading the elements of $T$ in the usual reading order (left-to-right, top-to-bottom). Write $\mathrm{sgn}(T) = \mathrm{sgn}(w_T)$, i.e., $\mathrm{sgn}(T) = 1$ if $w_T$ is an even permutation, and $\mathrm{sgn}(T) = -1$ if $w_T$ is an odd permutation. Show that

$$\sum_T \mathrm{sgn}(T) = 2^{\lfloor n/2 \rfloor},$$

where $T$ ranges over all SYT with $n$ squares.

118. Let $X_G$ denote the chromatic symmetric function of the graph $G$, as defined in Exercise 7.47. Define

$$F_n = \sum_G X_G,$$

where $G$ ranges over the incomparablity graphs of all $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free $n$-element posets, up to isomorphism. (The number of such posets is the Catalan number $C_n$; see Exercise 6.19(ddd).) Write $F_n = \sum_{\lambda \vdash n} c_\lambda e_\lambda$.

(a) [2] Show that $\langle F_n, p_1^n \rangle = n! C_n$. Equivalently, if we regard $F_n$ as the Frobenius characteristic of a character $\psi_n$, then $\dim \psi_n = n! C_n$.

NOTE. A priori $\psi_n$ is the Frobenius characteristic of a virtual character. However, it follows from Exercise 7.47(h) that $\psi_n$ is in fact the Frobenius characteristic of an actual character of $\mathfrak{S}_n$.

(b) [2] Show that $c_{\langle 1^n \rangle} = 1$.

(c) [2+] Show that $c_{\langle 2,1^{n-2} \rangle} = 3n - 4$.

(d) [3–] Show that $\sum_{\lambda \vdash n} c_\lambda = (2n-1)!!$.

(e) [3–] Show that $c_{(n)} = n(2n-3)!!$.

(f) [3–] Note that $(2n-1)!!$ is the number of (complete) matchings on the vertex set $[2n]$. Find a combinatorial interpretation of $c_\lambda$ as the number of such matchings with a suitable property indexed by partitions of $n$.

(g) [3–] Let $\omega F_n = \sum_{\lambda \vdash n} d_\lambda p_\lambda$. It follows from Exercise 7.47(d,e) that $d_\lambda \in \mathbb{N}$, and it is easy to see from (e) above that $\sum d_\lambda = (2n - 1)!!$. Find a combinatorial interpretation of $d_\lambda$ as the number of matchings on the vertex set $[2n]$ with a suitable property indexed by partitions of $n$.

(h) [5–] Is there a "natural" action of $\mathfrak{S}_n$ on a space of dimension $n!C_n$ with Frobenius characteristic $F_n$?

The symmetric functions $F_n$ for $1 \leq n \leq 5$ are given by

$$F_1 = e_1 = p_1$$

$$F_2 = e_1^2 + 2e_2 = 2p_1^2 - p_2$$

$$F_3 = e_1^3 + 5e_2e_1 + 9e_3 = 5p_1^3 - 7p_2p_1 + 3p_3$$

$$\begin{aligned} F_4 &= e_1^4 + 8e_2e_1^2 + 6e_2^2 + 30e_3e_1 + 60e_4 \\ &= 14p_1^4 - 37p_2p_1^2 + 30p_3p_1 + 9p_2^2 - 15p_4 \end{aligned}$$

$$\begin{aligned} F_5 &= e_1^5 + 11e_2e_1^3 + 53e_3e_1^2 + 21e_2^2e_1 + 259e_4e_1 + 75e_3e_2 + 525e_5 \\ &= 42p_1^5 - 176p_2p_1^3 + 204p_3p_1^2 + 122p_2^2p_1 - 196p_4p_1 - 100p_3p_2 + 105p_5. \end{aligned}$$

119. [3–] Let $P$ be a finite poset with $n$ elements, and let $\mathfrak{S}_P$ denote the set of all permutations of elements of $P$. Given $w = w_1w_2 \cdots w_n \in \mathfrak{S}_P$, define the *$P$-descent set* of $w$ by

$$\mathrm{PDes}(w) = \{1 \leq i \leq n-1 \,:\, w_i >_P w_{i+1}\}.$$

Set

$$Y_P = \sum_{w \in \mathfrak{S}_P} L_{\alpha_{\mathrm{PDes}(w)}},$$

where $L$ denotes the fundamental quasisymmetric function of equation (7.89), and $\alpha_S$ denotes the composition of $n$ corresponding to the set $S \subseteq [n-1]$ (page 356 of EC2). Show that

$$Y_P = \omega X_{\mathrm{inc(P)}},$$

where $X_G$ is the chromatic symmetric function of the graph $G$ (Exercise 7.47), and $\mathrm{inc}(P)$ is the incomparability graph of $G$.

120. Given $n \in \mathbb{P}$, let $X$ be any subset of $\{(i,j) : 1 \leq i \leq n,\ 1 \leq j \leq n,\ i \neq j\}$. An $X$-*descent* of a permutation $w = a_1 \cdots a_n \in \mathfrak{S}_n$ is an index $1 \leq i \leq n-1$ for which $(a_i, a_{i+1}) \in X$. The $X$-*descent set* $\mathrm{Xdes}(w)$ of $w$ is the set of all $X$-descents of $w$. If $\alpha$ is a composition of $n$, then we write $L_S$ for the fundamental quasisymmetric function $L_\alpha$, where if $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathrm{Comp}(n)$, then $S = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\} \subseteq [n-1]$. Define a quasisymmetric function $U_X$ by

$$U_X = \sum_{w \in \mathfrak{S}_n} L_{\mathrm{XDes}(w)}.$$

(a) [2+] Show that $U_X$ is a $p$-integral symmetric function, i.e., a symmetric function whose power sum expansion has integer coefficients.

(b) [2+] Let

$$\overline{X} = \{(i,j) : 1 \leq i \leq n,\ 1 \leq j \leq n,\ i \neq j\} - X.$$

What is the relationship between $U_X$ and $U_{\overline{X}}$?

(c) [3] The *record set* $\mathrm{rec}(w)$ consists of all indices $0 \leq i \leq n-1$ for which $a_{i+1}$ is a left-to-right maximum. Thus always $0 \in \mathrm{rec}(w)$. If $\mathrm{rec}(w) = \{r_0, \dots, r_j\}_<$, then define the *record partition* $\mathrm{rp}(w)$ to be the numbers $r_1 - r_0, r_2 - r_1, \dots, n - r_j$ arranged in decreasing order. Let $X$ have the property that if $(i,j) \in X$ then $i > j$. Show that

$$U_X = \sum_w p_{\mathrm{rp}(w)},$$

where $w$ ranges over all permutations in $\mathfrak{S}_n$ with no $X$-descents.

(d) [2+] Is it always true that $U_X$ is $p$-positive, as is the case in (c)?

(e) [2] Let $P$ be a partial ordering of $[n]$. Let $Y = \{(i,j) : i >_P j\}$. Show that $U_Y = \omega X_{\mathrm{inc}(P)}$, where $X_{\mathrm{inc}(P)}$ is the chromatic symmetric function of the incomparability graph $\mathrm{inc}(P)$ of $P$.

(f) [2+] Let $X = \{(2,1),(3,2),\ldots,(n,n-1)\}$. Set $f_n = \#\{w \in \mathfrak{S}_n : \mathrm{XDes}(w) = \emptyset\}$. Show that

$$U_X = \sum_{i=1}^{n} f_i s_{i,1^{n-i}}. \tag{11}$$

For instance, when $n=4$ we have $U_X = 11s_4 + 3s_{31} + s_{211} + s_{1111}$.
Note. It is known that

$$\sum_{n\geq 0} f_n \frac{x^n}{n!} = \frac{e^{-x}}{(1-x)^2}.$$

(g) [2+] (a $q$-analogue of (f)) Let $X$ be as in (f). Define

$$U_X(q) = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{des}(w^{-1})} L_{\mathrm{XDes}(w)},$$

where des denotes the number of (ordinary) descents. Set

$$f_n(q) = \sum_{w} q^{\mathrm{des}(w^{-1})},$$

where $w$ ranges over all permutations in $\mathfrak{S}_n$ with $\mathrm{XDes}(w) = \emptyset$. For instance,

$$\begin{aligned} f_1(q) = f_2(q) &= 1 \\ f_3(q) &= 1+2q \\ f_4(q) &= 1+8q+2q^2 \\ f_5(q) &= 1+22q+28q^2+2q^3. \end{aligned}$$

Show that

$$U_X(q) = \sum_{i=1}^{n} q^{n-i} f_i(q) s_{i,1^{n-i}}.$$

(h) [2+] Find analogues of (f) and (g) for $X = \{(2,1),(3,2),\dots,(n,n-1),(1,n)\}$.

121. (a) [2–] In the previous problem, we can consider $X$ to be a directed graph on the vertex set $[n]$, with an edge from $i$ to $j$ if $(i,j) \in X$. A *Hamiltonian path* of $X$ is a permutation $i_1 i_2 \cdots i_n \in \mathfrak{S}_n$ such that $(i_k, i_{k+1}) \in X$ for $1 \leq k \leq n-1$. Let $\mathrm{ham}(X)$ denote the number of Hamiltonian paths in $X$. For a symmetric function $f$ write $f(1)$ for $f(1,0,0,\dots)$. Show that $\mathrm{ham}(X) = U_{\overline{X}}(1)$ and $\mathrm{ham}(\overline{X}) = U_X(1)$.

(b) [2] Show that for any $p$-integral symmetric function $f$ we have $f(1) \equiv \omega f(1) \pmod 2$.

(c) [2] Deduce that

$$\mathrm{ham}(X) \equiv \mathrm{ham}(\overline{X}) \pmod 2.$$

122. (a) [2+] A *tournament* is a digraph $X$ (as in the previous problem) such that for all $1 \leq i < j \leq n$, either $(i,j) \in X$ or $(j,i) \in X$, but not both. Assume for the rest of this problem that $X$ is a tournament. Show that $\omega U_X = U_X$.

(b) [3] Extending (a), show that

$$U_X = \sum_w 2^{\mathcal{O}(w)} p_{\rho(w)}, \tag{12}$$

where $w$ ranges over all permutations in $\mathfrak{S}_n$ for which every cycle length is odd and every nonsingleton cycle of $w$ is a cycle of $X$, and where $\mathcal{O}(w)$ is the number of nonsingleton cycles of $w$.

(c) [2+] Deduce from (b) that every tournament has an odd number of Hamiltonian paths.

123. Let $A = (a_{ij})$ be an $n \times n$ matrix, say over $\mathbb{R}$. Define the "symmetric function determinant" $\mathrm{sfdet}(A)$ by

$$\mathrm{sfdet}(A) = \sum_{w \in \mathfrak{S}_n} a_{1,w(1)} \cdots a_{n,w(n)} p_{\rho(w)}.$$

(a) [2–] Show that $\mathrm{sfdet}(A)$ specializes to the usual determinant by applying the homomorphism $\varphi\colon \Lambda_{\mathbb{R}} \to \mathbb{R}$ satisfying $\varphi(p_i) = (-1)^{i-1}$.

(b) [3] Show that $\mathrm{sfdet}(A)$ is Schur positive if $A$ is positive definite.

(c) [5–] Show that $\mathrm{sfdet}(A)$ is $h$-positive if $A$ is positive definite.

(d) [3+] A real matrix is *totally nonnegative* if every minor (determinant of a square submatrix) is nonnegative. Show that if $A$ is totally nonnegative, then $\mathrm{sfdet}(A)$ is Schur positive.

(e) [5] Show that if $A$ is totally nonnegative, then $\mathrm{sfdet}(A)$ is $h$-positive.

(f) [3] Show that (d) implies that the answer to Exercise 7.47(j) is affirmative.

124. (a) [3–] Let $g_\lambda = \sum_\pi x_1^{c_1(\pi)} x_2^{c_2(\pi)} \cdots$, where $\pi$ ranges over all reverse plane partitions of shape $\lambda$, and $c_i(\pi)$ is the number of columns of $\pi$ that contain the part $i$. Show that $g_\lambda$ is an (inhomogeneous) symmetric function whose highest degree part is $s_\lambda$.

(b) [3] Define an *elegant* SSYT of skew shape $\lambda/\mu$ to be an SSYT of shape $\lambda/\mu$ for which the numbers in row $i$ lie in the interval $[1, i-1]$. In particular, there are no elegant SSYT of shape $\lambda/\mu$ if the first row of $\lambda/\mu$ is nonempty. Let $f_\lambda^\mu$ be the number of elegant SSYT of shape $\lambda/\mu$. Show that

$$g_\lambda = \sum_{\mu \subseteq \lambda} f_\lambda^\mu s_\mu.$$

In particular, $g_\lambda$ is Schur positive.

*Example.* Let $\lambda = (2,1)$. Then there is one elegant SSYT of the empty shape $(2,1)/(2,1)$ and one elegant SSYT of shape $(2,1)/(2)$. Hence $g_{2,1} = s_{2,1} + s_2$.

(c) [3] For $k \geq 0$ and $n \geq 1$, let $g_n^{(k)} = \sum_{j=0}^n \binom{k-1+j}{j} h_{n-j}$. For instance, $g_n^{(0)} = h_n$ and $g_n^{(1)} = h_n + h_{n-1} + \cdots + h_1 + 1$. Set $g_0^{(k)} = 1$ and $g_{-n}^{(k)} = 0$ for $n > 0$. Show that if $\lambda = (\lambda_1, \ldots, \lambda_m)$, then

$$g_\lambda = \det\left( g_{\lambda_i - i + j}^{(i-1)} \right)_{i,j=1}^m.$$

125. (a) [2] A *set-valued tableau* of shape $\lambda/\mu$ is a filling of the diagram of $\lambda/\mu$ with nonempty finite subsets of $\mathbb{P}$ such that if each subset is

replaced by one of its elements, then an SSYT always results. If $T$ is a set-valued tableau, then let $x^T = x_1^{c_1(T)} x_2^{c_2(T)} \cdots$, where $c_i(T)$ is the number of boxes of $T$ containing $i$. Set $|T| = \sum c_i(T)$, the total number of elements appearing in all the boxes. Define

$$G_{\lambda/\mu}(x) = \sum_T (-1)^{|T|-|\lambda/\mu|} x^T,$$

where $T$ ranges over all set-valued tableaux of shape $\lambda/\mu$. For instance,

$$G_{1^n} = e_n - ne_{n+1} + \binom{n+1}{2} e_{n+2} - \binom{n+2}{3} e_{n+3} + \cdots .$$

Show that $G_{\lambda/\mu}$ is a symmetric formal power series (i.e., an element of the completion $\hat{\Lambda}$ of the ring $\Lambda$ of symmetric functions) whose least degree part is $s_{\lambda/\mu}$.

(b) [3] Let $f_\lambda^\mu$ have the meaning of the previous problem. Show that

$$s_\mu = \sum_{\lambda \supseteq \mu} f_\lambda^\mu G_\lambda.$$

For instance,

$$s_{1^n} = e_n = G_n + nG_{n+1} + \binom{n+1}{2} G_{n+2} + \binom{n+2}{3} G_{n+3} + \cdots .$$

(c) [2] Deduce from (a) and (b) that $\langle g_\lambda, G_\mu \rangle = \delta_{\lambda\mu}$, where $g_\lambda$ has the meaning of the previous problem.

(d) [3] For $k \geq 0$ and $n \geq 1$, let $G_n^{(k)} = \sum_{i,j\geq 0} (-1)^j \binom{k+i-2}{i} s_{(n+i,1^j)}$. For instance, $G_n^{(1)} = s_n - s_{(n,1)} + s_{(n,1,1)} - \cdots$. Set $G_0^{(k)} = 1$ and $G_{-n}^{(k)} = 0$ for $n > 0$. Show that if $\lambda = (\lambda_1, \ldots, \lambda_m)$, then

$$G_\lambda = \det \left( G_{\lambda_i - i + j}^{(m-i+1)} \right)_{i,j=1}^m .$$

126. (a) [3] Let $L_\lambda$ be the symmetric function of Exercise 7.89(f), where $\lambda \vdash n$. Let $\alpha \in \mathrm{Comp}(n)$, and let $B_\alpha$ be the corresponding border strip (as defined on page 383). Show that

$$\langle L_\lambda, s_{B_\alpha} \rangle = \#\{w \in \mathfrak{S}_n \,:\, \rho(w) = \lambda,\ D(w) = S_\alpha\},$$

where $D(w)$ denotes the descent set of $w$. Equivalently, writing $F_\alpha$ for the fundamental quasisymmetric function which we denoted by $L_\alpha$ in Corollary 7.23.6 and elsewhere in EC2, it follows from this corollary that

$$L_\lambda = \sum_{\substack{w\in\mathfrak{S}_n\\ \rho(w)=\lambda}} F_{\mathrm{co}(w)}. \tag{13}$$

(b) [2] Let $\lambda = \langle 1^{m_1}2^{m_2}\cdots\rangle \vdash n$ with $m_1 = m_2 = 0$ and $m_4 = 1$. Let $\mu$ be obtained from $\lambda$ by changing the part equal to 4 into $2, 1, 1$. Show that for any $S \subseteq [n-1]$,

$$\begin{aligned}&\#\{w \in \mathfrak{S}_n \,:\, \rho(w) = \lambda,\ D(w) = S\}\\ &\quad = \#\{w \in \mathfrak{S}_n \,:\, \rho(w) = \mu,\ D(w) = S\}.\end{aligned}$$

Is there a combinatorial proof?

(c) [5–] Find the dimension $f(n)$ of the $\mathbb{Q}$-span of all $L_\lambda \in \Lambda^n_{\mathbb{Q}}$.

127. (a) [2+] Let $L_\lambda$ be as in the previous exercise. Show that

$$\sum_{\lambda\vdash n} L_\lambda = p_1^n.$$

(b) [3–] Show that

$$\sum_{\lambda\in\mathrm{Par}} L_\lambda(x)p_\lambda(y) = \exp \sum_{m,n,d\geq 1} \frac{1}{mnd}\mu(d)p_{nd}(x)^m p_{md}(y)^n,$$

where $\lambda$ ranges over all partitions of all nonnegative integers. Note that this formula makes Exercise 7.89(g) obvious.

(c) [3–] Show that

$$\sum_{\substack{\lambda\vdash n\\ \varepsilon_\lambda=1}} L_\lambda = \begin{cases} \frac{1}{2}\left(p_1^n + p_2^{n/2}\right), & \text{if } n \text{ is even}\\ \frac{1}{2}\left(p_1^n + p_1 p_2^{(n-1)/2}\right), & \text{if } n \text{ is odd,}\end{cases}$$

where $\lambda$ ranges over all partitions of $n$ satisfying $\varepsilon_\lambda = 1$.

128. (a) [2+] Let $\lambda \vdash k$. Show that

$$\sum_{n\geq 0} L_\lambda(1^n)t^n = \frac{f_\lambda(t)}{(1-t)^{k+1}},$$

where

$$f_\lambda(t) = \sum_{\substack{w\in\mathfrak{S}_k\\ \rho(w)=\lambda}} t^{\mathrm{des}(w)+1}.$$

Here $\rho(w)$ is the cycle type of $w$, and $\mathrm{des}(w)$ is the number of descents of $w$.

(b) [2+] Let $\lambda = \langle 1^{m_1}, \dots, k^{m_k}\rangle \vdash k$. Define

$$f_j(n) = \frac{1}{j}\sum_{d|j} \mu(d) n^{j/d}.$$

Show that

$$L_\lambda(1^n) = \prod_{j=1}^{k} \left(\!\!\binom{f_j(n)}{m_j}\!\!\right).$$

(c) [2+] Let $k \geq 1$. Define the *even Eulerian polynomial* $A_k^*(t)$ by

$$A_k^*(t) = \sum_{w\in\mathfrak{A}_k} t^{\mathrm{des}(w)+1},$$

where $\mathfrak{A}_k$ denotes the alternating group of degree $k$. Show that

$$\frac{1}{2}\sum_{n\geq 1}\left(n^k + n^{\lceil k/2\rceil}\right)t^n = \frac{A_k^*(t)}{(1-t)^{k+1}}.$$

129. (a) [3–] Let $S = \{i_1, \dots, i_k\}_< \subseteq [n-1]$. Write $B_S$ for the border strip $B_\alpha$ (as defined on page 383 of EC2), where $\alpha = (i_1, i_2 - i_1, i_3 - i_2, \dots, n - i_k) \in \mathrm{Comp}(n)$. Now suppose that $n$ is even, and let $B_S$ be a border strip of size $n$. There is a unique way to tile $B_S$ with $n/2$ dominos. If we shrink each of these dominos to a single square (a *monimo*), we obtain a border strip of size $n/2$ which we denote by $B_{S/2}$. For instance, if $S = \{3, 5, 6, 7\}$ then $S/2 = \{3\}$ (for any even $n \geq 8$). Let $v(B_S)$ be the number of vertical dominos in the

domino tiling of $B_S$. For $S \subseteq [n-1]$, let $\beta_n(S)$ be the number of permutations in $\mathfrak{S}_n$ with descent set $S$, and let $\gamma_n(S)$ be the number of such permutations that are even (i.e., belong to the alternating group $\mathfrak{A}_n$). Show that

$$\gamma_n(S) = \frac{1}{2}\left(\beta_n(S) + (-1)^{v(B_S)}\beta_{n/2}(S/2)\right).$$

(b) [3–] What is the corresponding result when $n$ is odd?

(c) [2] Let $E_n^*$ be the number of even alternating permutations $w = a_1a_2\cdots a_n$ in $\mathfrak{S}_n$, i.e., $w \in \mathfrak{A}_n$ and $a_1 > a_2 < a_3 > a_4 < \cdots$. Show that

$$E_n^* = \begin{cases} \frac{1}{2}\left(E_n + (-1)^{n/2}\right), & \text{if } n \text{ is even} \\ \frac{1}{2}E_n, & \text{if } n \text{ is odd } (n>1), \end{cases}$$

where $E_n$ denotes an Euler number.

(d) [2+] Let $E_n'$ be the number of even reverse alternating permutations $a_1a_2\cdots a_n$ in $\mathfrak{S}_n$, i.e., $w \in \mathfrak{A}_n$ and $a_1 < a_2 > a_3 < a_4 > \cdots$. Show that

$$E_n' = \begin{cases} \frac{1}{2}\left(E_n + 1\right), & \text{if } n \text{ is even} \\ \frac{1}{2}E_n, & \text{if } n \text{ is odd } (n>1). \end{cases}$$

(e) [2] Show that if $n$ is even and $S \subseteq \{1,3,5,\dots,n-1\}$, then

$$\gamma_n(S) = \frac{1}{2}\left(\beta_n(S) + (-1)^{\#S}\right).$$

130. [3–] (the Equivariant Exponential Formula) In Corollary 5.1.6 (the Exponential Formula) think of $f(n)$ as the number of structures of a certain type that can be put on an $n$-element set $S$. The allowable structures depend only on $n$, not on the elements of $S$. Assume that every structure is a unique disjoint union of connected structures. The symmetric group $\mathfrak{S}_n$ acts on the $n$-element structures by permuting the elements of $S$. Let $F_n$ denote the Frobenius characteristic symmetric function of this action. The group $\mathfrak{S}_n$ also acts on the set $\mathcal{C}_n$ of all connected $n$-element structures. Let $G_n(p_1, p_2, \dots)$ be the Frobenius

characteristic symmetric function, regarded as a polynomial in the $p_i$'s, of the action of $\mathfrak{S}_n$ on $\mathcal{C}_n$. Show that

$$\sum_{n\geq 0} F_n t^n = \exp \sum_{k,n\geq 1} \frac{1}{n} G_k(p_n, p_{2n}, p_{3n}, \dots) t^{kn}.$$

*Example.* Suppose that the only connected structure is a single edge between two vertices. Thus an arbitrary structure is a complete matching on $2m$ vertices. Example A2.9 on pages 449–450 becomes a special case of the present problem.

131. (a) [2+] Fix $n \geq 1$. Given $S, T \subseteq [n-1]$, let

$$\beta(S,T) = \#\{w \in \mathfrak{S}_n \,:\, D(w) = S,\ D(w^{-1}) = T\}.$$

Let $f(n) = \max_{S,T\subseteq[n-1]} \beta(S,T)$. Show that there is some $S \subseteq [n-1]$ for which $f(n) = \beta(S,S)$.

(b) [5–] Show that $f(n) = \beta(S,S)$, where $S = \{1,3,5,\dots\} \cap [n-1]$.

132. (a) [2+] Let $L_\lambda$ be as in #127 above, and let $\mu \vdash n$. Show that $\langle L_\lambda, h_\mu \rangle$ is equal to the number of permutations of the multiset $\{1^{\mu_1}, 2^{\mu_2}, \dots\}$ that standardize (in the sense of §1.7) to a permutation of cycle type $\lambda$.

(b) [3] The symmetric group $\mathfrak{S}_n$ acts on the set of all parking functions of length $n$ (defined in Exercise 5.49) by permuting coordinates. Define the *parking function symmetric function* $\mathrm{PF}_n$ to be the Frobenius characteristic of this action (or of its character). Show that $\mathrm{PF}_n = \omega F_{\mathrm{NC}_{n+1}}$, using the notation of Exercise 7.48(f).

(c) [2] Show that $\langle L_\lambda, \mathrm{PF}_n \rangle$ is the number of parking functions of length $n$ that standardize to a permutation of cycle type $\lambda$.

(d) [2] Show that the number $f(n)$ of parking functions of length $n$ that standardize to an $n$-cycle is given by

$$f(n) = \frac{1}{n} \sum_{d|n} \mu(d)(n+1)^{\frac{n}{d}-1}.$$

133. (a) [2] With $\mathrm{PF}_n$ as in the previous exercise (with $\mathrm{PF}_0 = 1$), define $\mathrm{PF}_\lambda = \mathrm{PF}_{\lambda_1}\mathrm{PF}_{\lambda_2}\cdots$. Show that the set $\{\mathrm{PF}_\lambda \,:\, \lambda \vdash n\}$ is a $\mathbb{Z}$-basis for $\Lambda^n$.

(b) [3–] Show that if $\lambda \vdash n$, then

$$\langle \mathrm{PF}_n, \mathrm{PF}_\lambda \rangle = \frac{1}{n+1} \prod_{i \geq 1} \frac{1}{\lambda_i + 1} \binom{(n+1)(\lambda_i+1)+\lambda_i-1}{\lambda_i}.$$

In particular,

$$\langle \mathrm{PF}_n, \mathrm{PF}_n \rangle = \frac{1}{(n+1)^2} \binom{n(n+3)}{n}.$$

(c) [5–] Is there a nice formula for, or combinatorial interpretation of, $\langle \mathrm{PF}_\lambda, \mathrm{PF}_\mu \rangle$? In general, it has large prime factors.

(d) [3–] Write $d_i$ for the number of parts of $\mu$ equal to $i$. Show that

$$\begin{aligned} e_n &= \frac{1}{n+1} \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \binom{n+\ell(\mu)}{d_1, d_2, \ldots, n} \mathrm{PF}_\mu \\ p_n &= \sum_{\mu \vdash n} (-1)^{\ell(\mu)+1} \binom{n+\ell(\mu)-1}{d_1, d_2, \ldots, n-1} \mathrm{PF}_\mu \\ h_n &= \frac{1}{n-1} \sum_{\mu \vdash n} (-1)^{\ell(\mu)+1} \binom{n+\ell(\mu)-2}{d_1, d_2, \ldots, n-2} \mathrm{PF}_\mu, \ n \geq 2. \end{aligned}$$

(e) [2+] Write $\mathrm{PF}_\lambda = \sum_\mu r_{\lambda\mu} e_\mu$. Let $R_n = [r_{\lambda\mu}]_{\lambda,\mu \vdash n}$, the transition matrix from the $e_\mu$ basis to the $\mathrm{PF}_\lambda$ basis in degree $n$. Show that $R_n^2 = I$, the identity matrix.

(f) [2] Let $\mathrm{PF}^*_\mu$ denote the dual basis to $\mathrm{PF}_\lambda$. Show that $\mathrm{PF}^*_\mu = \sum_\lambda r_{\lambda\mu} f_\lambda$, where $f_\lambda$ is a forgotten symmetric function.

(g) [2+] Show that $\mathrm{PF}^*_n = p_n$ and $\mathrm{PF}^*_{a,b} = p_a p_b - (a+b+1) p_{a+b}$ if $a > b$, while $\mathrm{PF}^*_{a,a} = \frac{1}{2}(p_a^2 - (2a+1)p_{2a})$.

(h) [2] Show that

$$r_{n,\mu} = \frac{\varepsilon_\mu}{n+1} \binom{n+\ell(\mu)}{d_1, d_2, \ldots, n}.$$

(i) [2+] Show that $r_{\lambda,1^n} = \prod_{i \geq 1} C_{\lambda_i}$, where $C_m$ denotes a Catalan number.

(j) [2+] Show that $r_{\lambda,21^{n-2}} = -\frac{1}{2}(n - \ell(\lambda)) \prod_{i \geq 1} C_{\lambda_i}$.

(k) [5–] What else can be said about the numbers $r_{\lambda\mu}$? It is easy to see that $\varepsilon_\mu r_{\lambda\mu} \geq 0$. Does $\varepsilon_\mu r_{\lambda\mu}$ have a nice combinatorial interpretation? In general, it has large prime factors.

(l) [5–] Since $\{e_\lambda\}$ and $\{\mathrm{PF}_\lambda\}$ are multiplicative bases, the linear map $\varphi\colon \Lambda \to \Lambda$ defined by $\varphi(e_\lambda) = \mathrm{PF}_\lambda$ is an algebra automorphism (as well as an involution). What can be said about $\varphi$? For instance, is there a "nice" $\mathbb{Z}$-basis for $\Lambda$ that is permuted by $\varphi$? Is there a "nice" $\mathbb{Q}$-basis for $\Lambda_{\mathbb{Q}}$ consisting of eigenvectors for $\varphi$?

(m) [3–] Show that

$$\sum_{n\geq 0} \langle \mathrm{PF}^*_{1^n}, \mathrm{PF}^*_{1^n}\rangle t^n = \exp \sum_{m\geq 1} \binom{2m-1}{m}^2 \frac{t^m}{m}.$$

134. [3–] Let $y := \sum_\lambda s_\lambda$. Show that

$$y * y = \exp\left(\sum_{n\geq 1} \frac{p_{2n-1}}{(2n-1)(1-p_{2n-1})}\right) \cdot \left(\prod_{n\geq 1}(1-p_n^2)\right)^{-1/2},$$

where $*$ denotes internal product.

135. [2+] Let $n \geq 1$, and let $M_n$ be the matrix $M_n = [s_\lambda * s_\mu]_{\lambda,\mu\vdash n}$. Show that the eigenvalues of $M_n$ are the power sums $p_\nu$, $\nu \vdash n$. What is the eigenvector corresponding to $p_\nu$?

136. (a) [3–] The "defining representation" of $\mathfrak{S}_n$ is the usual definition of $\mathfrak{S}_n$ as the set of all permutations of $[n]$. The Frobenius characteristic of the character of this representation is $s_n + s_{n-1,1}$. Show that for any $k \geq 1$, we have

$$(s_n + s_{n-1,1})^{*k} = \sum_{i=1}^{k} S(k,i) s_1^i s_{n-i}, \tag{14}$$

where $^{*k}$ denotes the $k$-fold internal product and $S(k,i)$ is a Stirling number of the second kind.

(b) [5–] The action of $\mathfrak{S}_n$ on two-element subsets of $[n]$ has Frobenius characteristic $s_n + s_{n-1,1} + s_{n-2,2}$. Find a formula analogous to equation (14) for $(s_n + s_{n-1,1} + s_{n-2,2})^{*k}$.

(c) [5–] Generalize.

137. [5–] Let $|\lambda/\mu| = n$ and

$$\begin{aligned} f^{\lambda/\mu}(q) &= (1-q)(1-q^2)\cdots(1-q^n)s_{\lambda/\mu}(1,q,q^2,\dots) \\ &= \sum_T q^{\mathrm{maj}(T)}, \end{aligned}$$

where $T$ ranges over all skew SYT of shape $\lambda/\mu$. (See Proposition 7.19.11.) We can regard $f^{\lambda/\mu}(q)$ as the "natural" $q$-analogue of $f^{\lambda/\mu}$. Investigate when $f^{\lambda/\mu}(q)$ has unimodal coefficients. This isn't always the case (e.g., $\lambda = (2,2), \mu = \emptyset$) but it does seem to be unimodal in certain cases, such as when $\mu = \emptyset$ and $\lambda$ is an arithmetic progression ending with 1.

138. [3–] For any sequence $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^k$, we can define the Schur function $s_\gamma(x_1, \dots, x_n)$ by the bialternant formula of Theorem 7.15.1. It is clear by permuting the rows of $a_{\gamma+\delta}$ that $s_\gamma(x_1, \dots, x_n)$ is either equal to 0 or to $\pm s_\lambda(x_1, \dots, x_n)$ for some partition $\lambda$. Given a composition $\alpha = (\alpha_1, \dots, \alpha_k)$ with $k \leq n$, let $\overline{\alpha}$ denote $\alpha$ with $n-k$ 0's appended at the end, so $\overline{\alpha}$ has length $n$. Let $L_\alpha$ denote the fundamental quasisymmetric function of Section 7.19. Let $f(x_1, \dots, x_n)$ be a homogeneous quasisymmetric function of degree $n$, with $L$-expansion

$$f(x_1, \dots, x_n) = \sum_{\alpha \in \mathrm{Comp}(n)} c_\alpha L_\alpha(x_1, \dots, x_n),$$

where $\mathrm{Comp}(n)$ is the set of all compositions of $n$ and $c_\alpha \in \mathbb{C}$ (or any commutative ring in place of $\mathbb{C}$). Suppose that $f(x_1, \dots, x_n)$ is a symmetric function. Show that

$$f(x_1, \dots, x_n) = \sum_{\alpha \in \mathrm{Comp}(n)} c_\alpha s_{\overline{\alpha}}(x_1, \dots, x_n).$$

139. [2–] We follow the notation of Sections 7.19 and 7.23. Let $\alpha \in \mathrm{Comp}(n)$ and $\lambda \vdash n$. Show that $\langle s_{B_\alpha}, s_\lambda \rangle$ is equal to the number of SYT of shape $\lambda$ and descent set $S_\alpha$.

140. (a) [2–] For a sequence $u = u_1 \cdots u_n$ of positive integers, define the *descent set* $D(u)$ in analogy to permutations, i.e.,

$$D(u) = \{i \,:\, u_i > u_{i+1}\} \subseteq [n-1].$$

Given $S \subseteq [n-1]$, define

$$f_S = \sum x_{u_1} \cdots x_{u_n},$$

where $u_1 \cdots u_n$ ranges over all sequences $u$ of positive integers of length $n$ satisfying $D(u) = S$. Show that $f_S = s_{B_{\mathrm{co}(S)}}$, using the notation of Sections 7.19 and 7.23.

(b) [2+] Let $\mathcal{S}_k$ denote the set of all finite sequences $u_1 u_2 \cdots u_n$ of positive integers containing no strictly decreasing factor of length $k$, i.e., we never have $u_i > u_{i+1} > \cdots > u_{i+k-1}$. Show that

$$\sum_{u_1 \cdots u_n \in \mathcal{S}_k} x_{u_1} \cdots x_{u_n}$$

$$= \frac{1}{1 - e_1 + e_k - e_{k+1} + e_{2k} - e_{2k+1} + e_{3k} - e_{3k+1} + \cdots}.$$

141. (a) [3–] Let $L_\alpha$ be as in equation (7.89). (Don't confuse with the $L_\lambda$ of Exercise 7.89.) Suppose that $s_\lambda = f + g$, where $f, g \in \Lambda$ and $f, g$ are $L$-positive. Show that $f = 0$ or $g = 0$.

(b) [2+] Give an example of an $L$-positive symmetric function that isn't $s$-positive.

142. [2] Let $\mathfrak{A}_n$ denote the alternating group of degree $n$ (regarded as a subgroup of $\mathfrak{S}_n$). Express the cycle index polynomial $Z_{\mathfrak{A}_n}$ as a linear combination of Schur functions.

143. [2+] Let $\chi$ be a character of $\mathfrak{S}_n$. Let $\mathrm{ch}(\chi) = \sum_{\mu \vdash n} c_\mu m_\mu$. Show that

$$c_\mu = \langle \chi|_\mu, 1_{\mathfrak{S}_\mu} \rangle,$$

the multiplicity of the trivial character $1_{\mathfrak{S}_\mu}$ of the Young subgroup $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots$ in the restriction $\chi|_\mu$ of $\chi$ to $\mathfrak{S}_\mu$. In particular, if $\chi$ is a permutation representation then $c_\mu$ is the number of orbits of $\mathfrak{S}_\mu$.

144. (a) [1+] Let $X$ be a nonempty subset of $\mathfrak{S}_n$. Suppose that the cycle indicator $Z_X$ is $s$-positive. Show that $X$ contains the identity element of $\mathfrak{S}_n$.

(b) [5] What can be said about subsets $X$ of $\mathfrak{S}_n$ for which $Z_X$ is $s$-positive or $h$-positive? (See equation (7.120), Exercise 7.111(c,d), and Problem 145 below for some information.)

145. Let $G$ be a subgroup of $\mathfrak{S}_n$ for which the cycle indicator $Z_G$ is $h$-positive.

(a) [2+] Show that $Z_G = h_\lambda$ for some $\lambda \vdash n$.

(b) [3–] Show in fact that $G$ is conjugate to the Young subgroup $\mathfrak{S}_\lambda$.

146. [2] Let $\mathfrak{I}_n$ denote the set of all indecomposable permutations in $\mathfrak{S}_n$, as defined in EC1, second ed., Exercise 1.128(a). Let $\tilde{Z}_{\mathfrak{I}_n}$ denote the augmented cycle indicator of $\mathfrak{I}_n$, as defined in Definition 7.24.1. Show that

$$\sum_{n\geq 1} \tilde{Z}_{\mathfrak{I}_n} x^n = 1 - \frac{1}{\sum_{n\geq 0} n! h_n x^n},$$

a direct generalization of Exercise 1.128(a) (second ed.).

147. (a) [3] let $y = \sum_{w\in\mathfrak{S}_n} a_w w \in \mathbb{R}\mathfrak{S}_n$ (the real group algebra of $\mathfrak{S}_n$). Suppose that the action of $y$ on $\mathbb{R}\mathfrak{S}_n$ by right multiplication has only nonnegative (real) eigenvalues. Show that the symmetric function $\sum_{w\in\mathfrak{S}_n} a_w p_{\rho(w)}$ is Schur-positive.

(b) [3] Let $\pi = \{A_1, \ldots, A_j\}$ and $\sigma = \{B_1, \ldots, B_k\}$ be two partitions of the set $[n]$. Let $\chi$ and $\psi$ be any characters of $\mathfrak{S}_n$. Define

$$f := \sum_{u\in\mathfrak{S}_{A_1}\times\cdots\times\mathfrak{S}_{A_j}} \ \sum_{v\in\mathfrak{S}_{B_1}\times\cdots\times\mathfrak{S}_{B_k}} \chi(u)\psi(v)p_{\rho(uv)}.$$

Show that $f$ is Schur positive.

(c) [2] Show that the analogue of (b) for *three* partitions of $[n]$ is false, even when the three characters are the trivial characters.

148. [3+] For any $1 \leq k \leq n$, show that the symmetric function

$$C_k(x_1, \ldots, x_n) = \prod_{1\leq i_1<i_2<\cdots<i_k\leq n} (x_{i_1} + x_{i_2} + \cdots + x_{i_k})$$

is Schur-positive.

149. (a) [3–] Let $A_1,\dots,A_k$ be subsets of $[n]$ satisfying $\bigcup A_i = [n]$ and $A_i \cap A_j = \{1\}$ for all $i < j$. Set $a_i = \#A_i$. Show that the symmetric function

$$G_{a_1,\dots,a_k} := \sum_{w_1\in\mathfrak{S}_{A_1}} \cdots \sum_{w_k\in\mathfrak{S}_{A_k}} p_{\rho(w_1\cdots w_k)} \tag{15}$$

is equal to $\prod(a_i-1)!$ times the coefficient of $x_1^{a_1}\cdots x_k^{a_k}$ in

$$\left(\sum_{i_1,\dots,i_k\geq 1} p_{i_1+\cdots+i_k-k+1} x_1^{i_1}\cdots x_k^{i_k}\right) H(x_1)\cdots H(x_k),$$

where $H(t)=\sum_{n\geq 0} h_n t^n$.

(b) [3–] Show that

$$\begin{aligned} G_{a,b} &= (a-1)!\,(b-1)! \sum_{j=0}^{\min(a-1,b-1)} (a-j)(b-j)s_{a+b-1-j,j} \\ &= (a-1)!\,(b-1)! \sum_{j=0}^{\min(a-1,b-1)} (a+b-1-2j)h_{a+b-1-j}h_j. \end{aligned}$$

(c) [2] Show that $G_{a_1,a_2,a_3}$ need not be $s$-positive.

(d) [5–] Let $A,B\subseteq[n]$ such that $\#A=a$, $\#B=b$, and $\#(A\cap B)=m$. Show that

$$\begin{aligned} &\frac{1}{(a-m)!\,(b-m)!}\sum_{u\in\mathfrak{S}_A}\sum_{v\in\mathfrak{S}_B} p_{\rho(uv)} \\ &= \sum_{j=0}^{\min(a-m,b-m)} \left(\prod_{i=1}^{m}(a-m+i-j)(b-m+i-j)\right) s_{a+b-m-j,j} \\ &= m!\,(m-1)! \\ &\cdot \sum_{j=0}^{\min(a-m,b-m)} \binom{a-1-j}{a-m-j}\binom{b-1-j}{b-m-j}(a+b-m-2j)h_{a+b-m-j}h_j. \end{aligned}$$

(e) [5–] Extend to other sets $A_1,\dots,A_k$.

150. [3–] Give a super-analogue of Theorem 7.24.4 (Pólya's theorem). More precisely, when $Z_G(x/y)$ is expanded as a linear combination of the $m_\lambda(x)m_\mu(y)$'s, give a combinatorial interpretation of the coefficients.

151. (a) [2+] Let $T$ be an SYT of shape $\lambda \vdash n$. We can regard the tableau $\mathrm{evac}(T)$ (as defined in Appendix 1) as a permutation of the entries $1, 2, \ldots, n$ of $T$. Show that this permutation is even if and only if the integer $\binom{n}{2} + (\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))/2$ is even, where $\mathcal{O}(\mu)$ denotes the number of odd parts of the partition $\mu$. (Note that this condition depends only on the shape $\lambda$ of $T$.)

(b) [3] Let $e(n)$ denote the number of partitions $\lambda \vdash n$ for which $\mathrm{evac}(T)$ is an even permutation of $T$, for some (or every) SYT $T$ of shape $\lambda$. Let $p(n)$ denote the total number of partitions of $n$. Show that $e(n) = (p(n) + (-1)^{\binom{n}{2}} f(n))/2$, where

$$\sum_{n\geq 0} f(n)x^n = \prod_{i\geq 1} \frac{1 + x^{2i-1}}{(1 - x^{4i})(1 + x^{4i-2})^2}.$$

152. [2] Express $\mathrm{ex}\, f[g]$ in terms of $\mathrm{ex}\, f$ and $\mathrm{ex}\, g$, where $\mathrm{ex}$ denotes the exponential specialization and $f[g]$ denotes plethysm.

153. [2+] Expand the plethysm $h_2[h_n]$ in terms of Schur functions.

154. [2+] Express the plethysm $e_n[e_1^2]$ in terms of sums and products of Schur functions. For instance, when $n = 1$ either $s_1^2$ or $s_2 + s_{11}$ are acceptable answers.

155. [2] Define $\Phi\colon \hat{\Lambda} \to \hat{\Lambda}$ by

$$\Phi(f) = (1 + h_1 + h_2 + h_3 + \cdots)[f],$$

where brackets denote plethysm. Show that

$$\Phi(f + g) = \Phi(f)\Phi(g). \tag{16}$$

Equivalently,

$$h_n[f + g] = \sum_{k=0}^{n} h_k[f]h_{n-k}[g].$$

156. The *plethystic inverse* of $f \in \Lambda$ is a symmetric function $g \in \Lambda$ satisfying $f[g] = g[f] = p_1$ (the identity element of the operation of plethysm). (See Exercise 7.88(d).) It is easy to see that if $g$ exists, then it is unique. Moreover, $g$ exists if and only if $f$ has constant term 0 and $[p_1]f \neq 0$.

(a) [2] Describe the plethystic inverse of $f = \sum_{n\geq 1} a_n p_1^n$, where $a_1 \neq 0$, in terms of "familiar" objects.

(b) [2] Let $f = \sum_{n\geq 1} a_n p_n$, where $a_1 \neq 0$. Describe the plethystic inverse of $f$ in terms of Dirichlet convolution. The *Dirichlet convolution* $f * g$ of two functions $f, g : \mathbb{P} \to \mathbb{C}$ is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

157. Let $\mathrm{Lie}_n$ denote the symmetric function $\mathrm{ch}\,\psi_1$ of Exercise 7.88, so

$$\mathrm{Lie}_n = \frac{1}{n}\sum_{d|n} \mu(d) p_d^{n/d}.$$

(a) [3–] Show that the symmetric functions $\frac{\sum_{n\geq 0} e_{2n+1}}{\sum_{n\geq 0} e_{2n}}$ and $\sum_{n\geq 0} \mathrm{Lie}_{2n+1}$ are plethystic inverses.

(b) [3–] Show that the symmetric functions $\frac{\sum_{n\geq 0}(-1)^n e_{2n+1}}{\sum_{n\geq 0}(-1)^n e_{2n}}$ and $\sum_{n\geq 0}(-1)^n \mathrm{Lie}_{2n+1}$ are plethystic inverses.

158. [2–] The group $\mathrm{GL}(n, \mathbb{C})$ acts on the space $\mathrm{Mat}(n, \mathbb{C})$ of $n \times n$ complex matrices by left multiplication. Express the character of this action as a linear combination of irreducible characters.

## CHRONOLOGY OF NEW PROBLEMS (beginning 4/13/02)

96. April 13, 2002

97. April 13, 2002

151. May 5, 2002

92. June 8, 2003

14. June 10, 2003

114. October 6, 2003

115. October 6, 2003

82. October 10, 2003

84. October 10, 2003

134. October 10, 2003

111. October 13, 2003

4. July 3, 2004

113. August 17, 2004

91. (b) January 1, 2005

47. February 13, 2005

60. April 16, 2005

59. April 17, 2005

68. December 13, 2005

80. December 13, 2005

146. December 31, 2005

81. January 3, 2006

140. August 2, 2006

23. October 22, 2006

93. August 7, 2007

35. September 4, 2007

124. September 29, 2007

125. September 29, 2007

30. November 22, 2007

143. February 13, 2008

124(c). March 14, 2008

125(d). March 14, 2008

104. March 25, 2008

131. April 26, 2008

103. June 29, 2008

3. July 11, 2008

103. (expanded) July 15, 2008

88. July 15, 2008

54. October 9, 2008

55. February 20, 2009

50. February 20, 2009

44. March 22, 2009

45. March 22, 2009

74(b). April 4, 2009

19. November 17, 2009

111(b). August 17, 2010

102. August 14, 2013
11. August 30, 2013
18. August 30, 2013
22. August 30, 2013
31. August 30, 2013
57. August 31, 2013
105. August 31, 2013
42. December 11, 2013
9. December 18, 2013
37. December 18, 2013
5. December 21, 2013
95. December 21, 2013
38. December 23, 2013
106. December 23, 2013
76. December 23, 2013
78. December 23, 2013
40. December 23, 2013
106. December 3, 2014 (corrected)
135. December 3, 2014
92. December 3, 2014 (part (a) refined)
105. August 6, 2015 (difficulty rating of (d) and (e))
36. September 21, 2015
31. September 21, 2015 (part (b) modified)

6. October 23, 2015

147. March 2, 2017

149. March 3, 2017

108. December 25, 2017

64. February 14, 2018

89. July 29, 2018

64. August 4. 2018 (corrected)

36. August 31, 2018 (updated and corrected)

73. October 14, 2018

37(b). November 6, 2018 (typo corrected)

57. May 11, 2019 (slightly modified)

58. May 18, 2019

148. January 10, 2020

132. January 15, 2020

118. May 6, 2020

157. May 6, 2020

118. (expanded) May 18, 2020

133. May 19, 2020

132. (expanded) May 22, 2020

149. (expanded) May 24, 2020

106. (expanded) May 26, 2020

33. May 26, 2020 (from a quiz for the course 18.315, fall 2013)

100. May 26, 2020 (from a quiz for the course 18.315, fall 2013)

154. May 26, 2020 (from a quiz for the course 18.315, fall 2013)

34. May 26, 2020 ((a) and (c) from a quiz for the course 18.315, fall 2013)

34. (expanded) May 28, 2020

133. (expanded) June 2, 2020

118. (expanded) June 2, 2020

136. June 14, 2020

138. June 18, 2020

136. (expanded) June 19, 2020

133. (expanded) July 5, 2020

39. July 26, 2020

94. July 26, 2020

41. (expanded) July 31, 2020

92. (expanded) August 29, 2020

92. (expanded) September 3, 2020

51. September 15, 2020

98. October 3, 2020

98. (expanded) October 5, 2020

115. (expanded) October 13, 2020

86. November 15, 2020

123. February 24, 2021

123. February 28, 2021 (expanded)

43. March 14, 2021

155. August 17, 2021

130. August 17, 2021
21. September 15, 2021
127. September 23, 2021 (expanded)
128. September 23, 2021
129. September 23, 2021
119. October 17, 2021
120. November 10, 2021
85. November 10, 2021
126. November 23, 2021 (expanded)
133. November 27, 2021 (expanded)
53. December 20, 2021
74. December 23, 2021 (expanded)
53. December 26, 2021
129. January 3, 2022 (expanded)
120. January 7, 2022 (expanded)
121. April 18, 2022
122. April 19, 2022