Some Congruence Properties of Symmetric Group Character Values

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We follow symmetric function notation and terminology from [4, Ch. 7]. Let $\lambda \vdash n$, and let f^λ denote the number of standard Young tableaux of shape λ. Equivalently, f^λ is the dimension of the irreducible representation of the symmetric group $\mathfrak{S}_n$ indexed by λ. Let ℓ be a prime, and let

$$n = \alpha_0 + \alpha_1\ell + \alpha_2\ell^2 + \cdots,$$

with $0 \leq \alpha_i < \ell$, the base ℓ expansion of n. Let

$$P(x) = \prod_{n\geq 1}(1-x^n)^{-1}.$$

If $G(x)$ is a power series, then $[x^\alpha]G(x)$ denotes the coefficient of x^α in $G(x)$. Finally, write $m_\ell(n)$ for the number of partitions $\lambda \vdash n$ for which f^λ is relatively prime to ℓ. I. G. Macdonald [1] showed that

$$m_\ell(n) = \prod_{r\geq 0}[x^{\alpha_r}]P(x)^{\ell^r}. \qquad (1)$$

In particular, if each $\alpha_r = 0$ or 1, so $n = \ell^{k_1} + \ell^{k_2} + \cdots$ with $k_1 < k_2 < \cdots$, then

$$m_\ell(n) = \ell^{k_1+k_2+\cdots}. \qquad (2)$$

Equation (2) had earlier been conjectured by J. McKay for $\ell = 2$, inspiring Macdonald to write his paper.

In this note we give a simpler approach to equation (1) based on symmetric functions, allowing us to extend the result to some other irreducible character values of $\mathfrak{S}_n$.

Lemma 1. *Let $\lambda \vdash n$. The number of ways to add a border strip of size $m > n$ to λ is m.*

Proof. Straightforward. □

First we do the special case (2).

Proof of equation (2). If f, g are symmetric functions over $\mathbb{Z}$, then write $f \equiv g \,(\mathrm{mod}\, \ell)$ to mean that every coefficient of $f - g$ is divisible by ℓ. Thus $p_j^{\ell^r} \equiv p_{j\ell^r} \,(\mathrm{mod}\, \ell)$, so

$$\begin{aligned} p_1^n &= p_1^{\ell^{k_1}+\ell^{k_2}+\cdots} \\ &\equiv p_{\ell^{k_1}} p_{\ell^{k_2}} \cdots \,(\mathrm{mod}\, \ell) \end{aligned}$$

By the Murhanghan-Nakayama rule,

$$p_{\ell^{k_1}} p_{\ell^{k_2}} \cdots = \sum_B \mathrm{sgn}(B) s_{\mathrm{sh}(B)},$$

where B is obtained by beginning with a hook B_1 of size ℓ^{k_1}, then adjoining a border strip B_2 of size ℓ^{k_2}, etc. Here $\mathrm{sgn}(B) = \pm 1$ and $\mathrm{sh}(B)$ is the shape of B. By Lemma 1, there are ℓ^{k_1} choices for B_1, then ℓ^{k_2} choices for B_2, etc., so $N = \ell^{k_1+k_2+\cdots}$ choices in all. It is easy to see that all the shapes obtained in this way are distinct. Hence $p_{\ell^{k_1}} p_{\ell^{k_2}} \cdots$ is a linear combination of N Schur functions, each with sign ± 1. Now $p_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda$, so taking p_1^n modulo ℓ completes the proof. □

Proof of equation (1). Now we obtain

$$p_1^n \equiv p_1^{\alpha_0} p_\ell^{\alpha_1} p_{\ell^2}^{\alpha_2} \cdots \,(\mathrm{mod}\, \ell).$$

By Lemma 1 it follows that

$$m_\ell(n) = \prod_{r \geq 0} m_\ell(\alpha_r \ell^r).$$

If we expand $p_{\ell^r}^{\alpha_r}$ in terms of Schur functions, the shapes λ that appear will be those partitions of $\alpha_r \ell^r$ with empty ℓ-core. Let $\mu_1, \ldots, \mu_{\alpha_r}$ be the ℓ^r-quotient of λ. Let $c_i = |\mu_i|$. Then by standard properties of cores and quotients [2, Exam. I.1.8, p. 12, and Exam. I.5.2(b), p. 75],

$$\langle p_{\ell^r}^{\alpha_r}, s_\lambda \rangle = \pm \binom{\sum \mu_i}{\mu_1, \mu_2, \ldots} f^{\mu_1} f^{\mu_2} \cdots .$$

Because $\alpha_r < p$, it follows easily that

$$\langle p_{\ell^r}^{\alpha_r}, s_\lambda \rangle \not\equiv 0 \, (\mathrm{mod}\, p).$$

Hence $m_\ell(\alpha_r \ell^r)$ is equal to the number of partitions of $\alpha_r \ell^r$ with empty ℓ^r-core. By [4, Exer. 7.59(e)] this number is $[x^{\alpha_r}]P(x)^{\ell^r}$, and the proof follows. □

Numerous generalizations suggest themselves.

- Can one determine for each $0 \le i < \ell$ the number of $\lambda \vdash n$ for which $f^\lambda \equiv i \, (\mathrm{mod}\, \ell)$?

- Rather than using $p_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda$, use

$$p_j^n = \sum_{\lambda \vdash jn} \chi^\lambda(\langle j^n \rangle) s_\lambda,$$

  where $\langle j^n \rangle$ denotes the partition with n parts equal to j. For instance, taking $j = \ell^k$ gives:

  Proposition 2. *Let $\lambda \vdash \ell^k n$. The number of character values $\chi^\lambda(\ell^k, \ell^k, \dots)$ (n terms equal to ℓ^k) that are not divisible by ℓ is equal to the number of $\mu \vdash \ell^k n$ for which f^μ is not divisible by ℓ (given by equation (1)).*

  What about other values of j, i.e., $j \ne \ell^k$?

- Use h_j instead of p_j. Use [4, Exer. 7.61] to expand $h_j^{\ell^r} \equiv h_j(x_1^{\ell^r}, x_2^{\ell^r}, \dots)$ in terms of Schur functions. This will give Kostka number congruences. For instance, let $g(n)$ denote the number of odd Kostka numbers $K_{\lambda, \langle 2^n \rangle}$, $\lambda \vdash 2n$. Since $h_2(x_1^n, x_2^n, \dots) = h_2[p_n]$ (plethysm) is a linear combination of $\binom{n+1}{2}$ Schur functions with coefficients ± 1, we get $g(2^r) = \binom{2^r+1}{2}$. We apparently have

$$\begin{aligned} g(2^r+1) &= \binom{2^r+1}{2} \\ g(2^r+2) &= 3\binom{2^r+1}{2} \\ g(2^r+3) &= 5\binom{2^r+1}{2}. \end{aligned}$$

  What about $g(2^r - 1)$? the values of $g(n)$ for $1 \le n \le 15$ are 1, 3, 5, 10, 10, 30, 50, 36, 36, 108, 180, 312, 312, 840, 1368 (I think).

- What about g^λ (shifted SYT) instead of f^λ? And projective characters of $\mathfrak{S}_n$ instead of ordinary ones? A relevant exercise might be [2, Exam. I.1.9, p. 14].

- What about differential posets? I.e., replace f^λ for $\lambda \vdash n$ with the number $e(x)$ of saturated chains from $\hat{0}$ to an element x of rank n. The Fibonacci differential poset in particular may be interesting. In this case $e(x)$ is the dimension of an irreducible representation of the Okada algebra $\mathcal{O}_n$ [3], so we can also ask about congruence properties of character values of $\mathcal{O}_n$.

References

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[4] R. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, New York/Cambridge, 1999.