POSETS OF WIDTH TWO AND SKEW YOUNG DIAGRAMS

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Abstract. Let P be a finite poset of width two, i.e., with no three-element antichain. We associate with P a skew Young diagram $\Upsilon(P)$ and discuss some of the properties of the map Υ. In particular, if we regard $\Upsilon(P)$ as a poset in a standard way, then the linear extensions of P are in bijection with the order ideals of $\Upsilon(P)$.

1. Introduction

We follow [4][5] for terminology involving posets, Young diagrams, etc. Let P be a finite poset of width at most two, i.e., with no three-element antichain. We will associate with P a skew Young diagram (or skew shape) $\Upsilon(P) = \lambda/\mu$ with the property that the linear extensions w of P are in a natural bijection with the diagrams ν/μ contained in λ/μ, denoted $\nu/\mu = \Upsilon(w)$. Equivalently, regarding λ/μ as a poset in a standard way (defined in Section 3), the linear extensions of P correspond to the order ideals of λ/μ. The squares of λ/μ are in bijection with the incomparable pairs of elements of P. With this identification, the subdiagrams ν/μ are the inversion sets of the linear extensions w. Since there is a known determinantal formula for the generating function for subshapes of λ/μ according to size, the same is true for linear extensions of P according to number of inversions.

The map Υ is also well-behaved with respect to the descent sets of the linear extensions of P (with respect to a certain labeling of the elements of P). In particular, define a *corner square* of ν/μ to be a square $u \in \nu/\mu$ with no square $v \in \nu/\mu$ directly to the right or directly below u. Then the corner squares of $\Upsilon(w)$ correspond to the descents of w, and the diagonals on which these corner squares lie determine the descent set.

The proofs of our results are straightforward; the main point of the paper is just to point out the connection between width two posets and skew Young diagrams.

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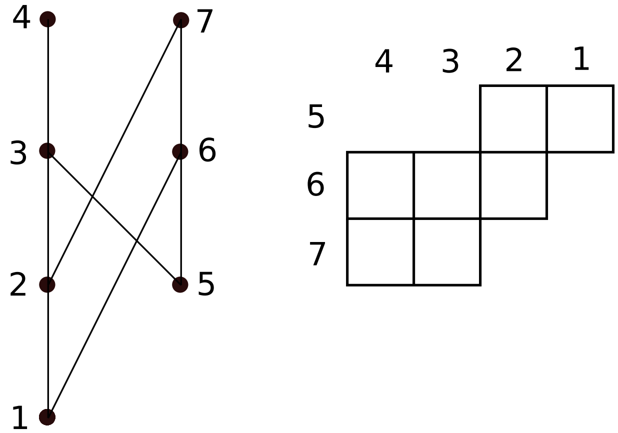

Figure 1. A poset P and its corresponding skew shape $\Upsilon(P)$

2. The correspondence Υ

A finite poset P of width at most two is obtained, up to isomorphism, by taking two disjoint chains $C_1 : 1 < 2 < \cdots < m$ and $C_2 : m+1 < m+2 < \cdots < m+n$ (one or both of which may be empty) and adjoining additional relations of the form $i < j$ or $i > j$ for $i \in C_1$ and $j \in C_2$. We call the triple (P, C_1, C_2) an *(m,n)-ladder*, It costs us nothing in our treatment to assume that every element of P is contained in a two-element antichain; such width two posets we call *full*.

Consider an $n \times m$ array R of squares (i,j), where the columns of R are indexed by $m, m-1, \ldots, 1$ from left-to-right and the rows by $m+1, m+2, \ldots, m+n$ from top-to-bottom. Define $\Upsilon(P, C_1, C_2)$ to be the subarray of R consisting of those pairs (squares) (i,j) with $i > j$, such that i and j are incomparable in P. Thus $i \in C_2$ and $j \in C_1$. When no confusion will result we write $\Upsilon(P)$ for $\Upsilon(P, C_1, C_2)$. Figure 1 shows a poset P and the corresponding set $\Upsilon(P)$ of squares. The squares in $\Upsilon(P)$ are $52, 51, 64, 63, 62, 74, 73$. Note that by definition, the number $\#\Upsilon(P)$ of squares in $\Upsilon(P)$ is the number of two-element antichains (or incomparable pairs of elements) of P. The assumption that P is full is equivalent to the statement that no row and no column of R is empty, i.e., every row and every column of R contains at least one square of $\Upsilon(P)$. We say that $\Upsilon(P)$ is a *full* subset of R.

Theorem 2.1. *$\Upsilon(P)$ is a skew Young diagram. Conversely, given a full skew Young diagram λ/μ contained in an $n \times m$ rectangle R, there is a unique full (m,n)-ladder (P, C_1, C_2) such that $\lambda/\mu = \Upsilon(P, C_1, C_2)$.*

Proof. To show that $\Upsilon(P)$ is a skew Young diagram, it suffices to prove the following three assertions.

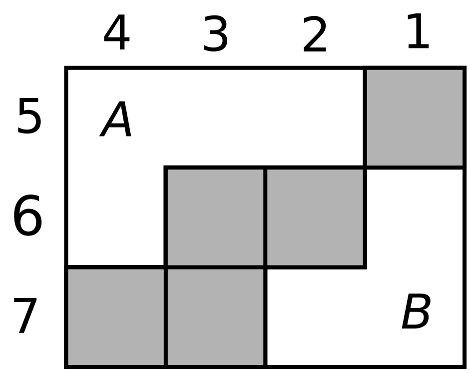

FIGURE 2. The regions defined by a full skew shape in a rectangle

(1) If $(i,a) \in \Upsilon(P)$, $(i,c) \in \Upsilon(P)$, and $a < b < c$, then $(i,b) \in \Upsilon(P)$.
(2) If $(a,j) \in \Upsilon(P)$, $(c,j) \in \Upsilon(P)$, and $a < b < c$, then $(b,j) \in \Upsilon(P)$.
(3) If $(i,j) \in \Upsilon(P)$ and $(k,h) \in \Upsilon(P)$ with $i < k$ and $j < h$, then $(i,h) \in \Upsilon(P)$ and $(k,j) \in \Upsilon(P)$.

Write $u \parallel v$ to denote that u and v are incomparable in a poset Q. For the first assertion, note that for any poset Q, with elements i and $a < b < c$, if $i \parallel a$ and $i \parallel c$ then $i \parallel b$. The second assertion is similar (or equivalent to the first by symmetry). In any poset P, if $i < k$, $h < j$, $i \parallel j$, and $h \parallel k$, then it is easy to check that $i \parallel h$ and $k \parallel j$. This proves that $\Upsilon(P$ is a skew Young diagram.

Conversely, let λ/μ be as in the statement of the theorem. Let A be the set of squares in R above λ/μ and B the set of squares below. See Figure 2 for an example. Define a poset P with elements $[m+n] = \{1, 2, \ldots, m_n\}$ as follows. If $(i,j) \in A$ then set $i < j$. If $(i,j) \in B$ then set $i > j$. It is straightforward to check that these relations define a poset P for which $\Upsilon(P) = \lambda/\mu$.

We claim that P is unique. Otherwise there is a different (m,n)-ladder (Q, C_1, C_2) with the same incomparable pairs, and hence the same comparable pairs $\{i,j\}$. Thus there exists $i \in C_1$ and $j \in C_2$ such that $i < j$ in P and $j < i$ in Q. If j is comparable to all $k > i$ in P then j is comparable to all elements of P; hence P is not full. Thus j is incomparable to some $k > i$ in P. But j cannot be incomparable to some $k > i$ in Q since $j < i$ in Q. This contradicts the assumption that P and Q have the same incomparable pairs.

□

Corollary 2.2. *Let $m, n \geq 1$. The following sets have equal cardinality.*

- *Full (m,n)-ladders.*
- *Full skew Young diagrams λ/μ inside an $n \times m$ rectangle R.*

Let us denote the cardinality in the previous corollary by $f(m,n)$. What can be said about this number? For instance,

$$\begin{aligned} f(1,n) &= 1 \\ f(2,n) &= \frac{1}{2}(n^2+3n-2) \\ f(3,n) &= \frac{1}{12}(n^4+8n^3+11n^2-20n+12) \\ f(4,n) &= \frac{1}{144}(n^6+15n^5+67n^4+45n^3-140n^2+300n-144). \end{aligned}$$

It's not hard to see that for fixed m, $f(m,n)$ is a polynomial in n of degree $2m-2$.

Given a full poset P of width two, there is another poset (in fact, a distributive lattice) that we can associate with P and whose elements are in bijection with the incomparable pairs of P. See [5, Exer. 3.72]. We don't know, however, of any connection with the present paper.

3. Inversions and order ideals

Given P as above, let $\mathcal{L}(P)$ denote the set of linear extensions of P, regarded as permutations $w = a_1a_2\cdots a_{m+n}$ in the symmetric group $\mathfrak{S}_{m+n}$. Thus if $a_i < a_j$ in P then $i < j$. An *inversion* of w is a pair (a_i, a_j) where $i < j$ and $a_i > a_j$. The *inversion set* $\mathcal{I}(w)$ is the set of all inversions of w. For instance, $w = 5126374$ is a linear extension of the poset P of Figure 1, with inversion set (abbreviating (a,b) as ab) $\mathcal{I}(w) = \{51, 52, 53, 54, 63, 64, 74\}$. Note that 53 and 54 will be inversions for any $w \in \mathcal{L}(P)$. If $\Upsilon(P) = \lambda/\mu$, then these pairs 53 and 54 index the squares of μ. The inversion set $\mathcal{I}(w)$ consists of the squares of the shape $\nu = (4,2,1)$ contained in λ and (necessarily) containing μ. In fact (as we will soon prove), this construction gives a bijection between linear extensions $w \in \mathcal{L}(P)$ and skew shapes ν/μ contained in λ/μ. The squares (i,j) of ν are just the inversions of w.

We can regard λ/μ as a poset in a standard way, namely, a square u is covered by a square v if v borders u on the right or on the bottom. The skew shapes ν/μ contained in λ/μ are then just the order ideals of λ/μ. Thus we have the curious fact that Υ converts linear extensions to order ideals.

Theorem 3.1. *Let $\Upsilon(P) = \lambda/\mu$. The map $\mathcal{I}$ is a bijection from $\mathcal{L}(P)$ to partitions ν satisfying $\mu \subseteq \nu \subseteq \lambda$, where we are identifying ν with the set of squares (i, j) of its diagram (using the indexing defined in Section 2).*

Proof. It follows directly from the definition of Υ that for all $w \in \mathcal{L}(P)$ and all $(i, j) \in \mu$, we have $(i, j) \in \mathcal{I}(w)$. Moreover, if $(i, j) \in \mathcal{I}(w)$ then $(i, j) \in \lambda$. To show that $\mathcal{I}(w)$ is an order ideal, it suffices to show that (1) if $(i, j) \in \mathcal{I}(w)$ and $i > m + 1$ then $(i - 1, j) \in \mathcal{I}(w)$, and (2) if $(i, j) \in \mathcal{I}(w)$ and $j < m$ then $(i, j + 1) \in \mathcal{I}(w)$. To show (1), since $(i, j) \in \mathcal{I}(w)$ we have that i precedes j in w and $i > j$. But $i - 1 < i$ in P since $m + 1 < m + 2 < \cdots < m + n$. Hence $i - 1$ precedes i and therefore also precedes j in w, so $(i - 1, j) \in \mathcal{I}(w)$. The proof of (2) is similar.

It remains to show that if $\mu \subseteq \nu \subseteq \lambda$, then there is a $w \in \mathcal{L}(P)$ with $\mathcal{I}(w) = \nu$ (identifying ν with its set of squares (i, j)). Let $\nu' = (\nu'_1, \nu'_2, \ldots, \nu'_m)$ denote the conjugate partition to ν. Consider the sequence

(3.1)
$$v = (\lambda'_n + 1, \lambda'_{n-1} + 2, \ldots, \lambda'_1 + m, m - \lambda_1 + 1, m - \lambda_2 + 2, \ldots, m - \lambda_n + n).$$

For instance, if $m = 5$, $n = 4$, and $\nu = (4, 2, 2, 1)$, then $v = (1, 3, 4, 7, 9, 2, 5, 6, 8)$. It is straightforward to check that in general $v \in \mathfrak{S}_{m+n}$, and that $v^{-1} \in \mathcal{L}(P)$ with $\mathcal{I}(v^{-1}) = \nu$, completing the proof. □

Example 3.2. We illustrate the above proof by continuing the example $m = 5$, $n = 4$, $\nu = (4, 2, 2, 1)$, and $v = (1, 3, 4, 7, 9, 2, 5, 6, 8)$ (μ and λ are irrelevant). In Figure 3 the row or column indexed by k as described in Section 2 and illustrated in Figure 1 is now indexed by v_k as defined in equation (3.1), where $v = (v_1, \ldots, v_9)$. It's not hard to see why $v_1 < \cdots < v_5$ and $v_6 < \cdots < v_9$ and $\{v_1, \ldots, v_9\} = [9]$. Moreover, for a square (i, j) of the rectangle, we have $v_i < v_j$ if and only if $(i, j) \in \nu$. This implies that $v^{-1} \in \mathcal{L}(P)$ and $\mathcal{I}(v^{-1}) = \nu$ (regarded as a set of pairs (i, j)).

There is a determinantal formula due to Handa and Mohanty [3] (see also Gessel and Loehr [2]) for the sum

$$A_{\lambda/\mu} := \sum_{\mu \subseteq \nu \subseteq \lambda} q^{|\nu|}.$$

By Theorem 3.1 we can "transfer" this result to linear extensions of width two posets P, yielding the following corollary.

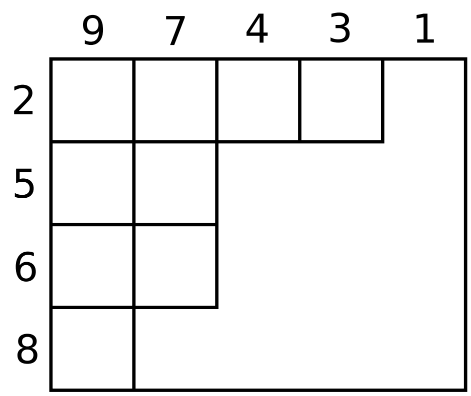

FIGURE 3. The shape $\nu = (4,2,2,1)$ from Example 3.2

Corollary 3.3. *Let (P, C_1, C_2) be an (m,n)-ladder with $\Upsilon(P) = \lambda/\mu$. Write* $\mathrm{inv}(w)$ *for the number of inversions of a permutation w. Then*

$$\sum_{w\in\mathcal{L}(P)} q^{\mathrm{inv}(w)} = \det\left[\binom{\boldsymbol{\lambda_i - \mu_j + 1}}{\boldsymbol{i-j+1}} q^{\binom{i-j+1}{2}+(i-j+1)\mu_j}\right]_{1\le i,j\le n}.$$

Here $\binom{\boldsymbol{\lambda_i-\mu_j+1}}{\boldsymbol{i-j+1}}$ *denotes a q-binomial coefficient, and we set* $\binom{\boldsymbol{a}}{\boldsymbol{b}} = 0$ *if* $b < 0$.

For instance, if P is given by Figure 1, then

$$\begin{aligned}\sum_{w\in\mathcal{L}(P)} q^{\mathrm{inv}(w)} &= \det\begin{bmatrix} \binom{\mathbf{3}}{\mathbf{1}}q^2 & 1 & 0 \\ q^5 & \binom{\mathbf{4}}{\mathbf{1}} & 1 \\ 0 & \binom{\mathbf{3}}{\mathbf{2}}q & \binom{\mathbf{3}}{\mathbf{1}} \end{bmatrix} \\ &= q^9 + 3q^8 + 4q^7 + 5q^6 + 4q^5 + 4q^4 + 2q^3 + q^2.\end{aligned}$$

The set $J(Q)$ of order ideals of any finite poset Q, ordered by inclusion, forms a finite distributive lattice [5, §3.4]. There is a nice way to see from the skew shape $\lambda/\mu = \Upsilon(P)$ what is the Hasse diagram of $J(P)$ for an (m,n)-ladder P. Adjoin to the $n \times m$ rectangle R containing $\Upsilon(P)$ another row at the top and column at the right to form an $(n+1)\times(m+1)$ rectangle R'. Given a square u of R', let K_u be the largest subrectangle of R' for which u is the lower left-hand corner. Call a square u of R' *sticky* if $\lambda/\mu \cup K_u$ is a skew diagram (contained in R'). Partially order the set $\mathcal{S}$ of sticky squares by defining t to cover s if t borders s on the left or on the bottom. Thus the upper-right corner square of R', which is always sticky, is the minimal element $\hat{0}$ of this partial ordering. We omit the straightforward proof of the following result.

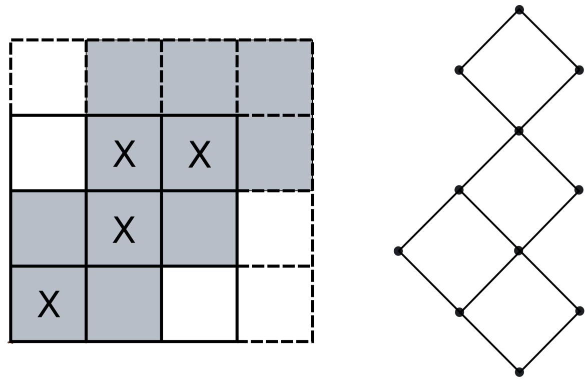

FIGURE 4. The distributive lattice corresponding to the skew shape 321/11

Theorem 3.4. *The poset $\mathcal{S}$ is isomorphic to $J(P)$.*

An example of Theorem 3.4 is given in Figure 4. We take $m = n = 3$ and $\lambda/\mu = (3, 2, 1)/(1, 1)$. The squares of λ/μ are marked with an X. The sticky squares are shaded. The Hasse diagram of the distributive lattice $J(P)$ is shown on the right.

There is a further corollary to Theorem 3.1. The *weak (Bruhat) order* $W(\mathfrak{S}_N)$ of the symmetric group $\mathfrak{S}_N$ may be defined by $v \leq w$ if $\mathcal{I}(v) \subseteq \mathcal{I}(w)$. Thus from Theorem 3.4 we obtain the following result. (We give $\Upsilon(P)$ the poset structure defined preceding Theorem 3.1.) We omit the details of the proof.

Corollary 3.5. *Let P be a poset of width two (not necessarily full) on $[m + n]$ containing the two chains $1 < 2 < \cdots < m$ and $m + 1 < m + 2 < \cdots < m + n$. Then the set $\mathcal{L}(P)$ is an interval in the weak order isomorphic to the distributive lattice $J(\Upsilon(P))$.*

The fact that $\mathcal{L}(P)$ is a distributive lattice is also a consequence of the characterization by Stembridge [6, Thm. 3.2] of intervals in $W(\mathfrak{S}_n)$ (or more generally in the weak order of any Coxeter group) that are distributive lattices, together with the characterization by Billey-Jockusch-Stanley [1, Thm. 2.1] of fully commutative elements of $\mathfrak{S}_n$ as the 321-avoiding permutations.

4. DESCENT SETS AND CORNER SQUARES

In addition to the inversion set $\mathcal{I}(w)$ for $w \in \mathcal{L}(P)$, it is also easy to determine from $\Upsilon(P)$ the descent set of w. Recall that if $w = a_1 \cdots a_{m+n} \in \mathfrak{S}_{m+n}$, then the *descent set* $\mathrm{Des}(w)$ is defined as

$$\mathrm{Des}(w) = \{1 \leq k \leq m + n - 1 \colon a_k > a_{k+1}\}.$$

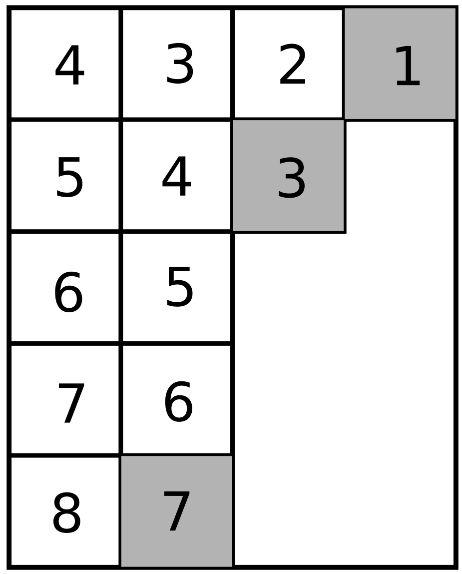

FIGURE 5. The corner squares of $\nu = (4, 3, 2, 2, 2)$

Let $D = \lambda/\mu$ be a skew diagram in an $n \times m$ rectangle R, with rows and columns indexed as before. We may assume that D is full, i.e., R has no empty row or column. For a square $u = (i, j) \in D$, define $e(u) = i + j - m - 1$. Thus the the top right corner square u has $e(u) = 1$.

Theorem 4.1. *Let $\mathcal{I}(w)$ occupy the squares of the partition ν. Let $C(\nu)$ be the set of corner squares of ν. Then*

$$\mathrm{Des}(w) = \{e(u) \colon u \in C(\nu)\}.$$

In particular, the number $\mathrm{des}(w)$ *of descents of w is equal to the number $\#C(\nu)$ of corner squares of ν.*

Example 4.2. Let $m = 4$, $n = 5$, and $\nu = (4, 3, 2, 2, 2)$. Then Figure 5 shows that $\mathrm{Des}(w) = \{1, 3, 7\}$. The three corner squares are shaded.

Proof of Theorem 4.1. Suppose that $w = w_1 \cdots w_N$ is any sequence of integers, and there are numbers $a \leq b < c \leq d$ such that $w_a > w_d$ and $w_b > w_c$. We then say that the inversion (w_b, w_c) is *inside* (w_a, w_d). Note that if (w_a, w_d) is an inversion, then there is a *descent* (w_c, w_{c+1}) inside (w_a, w_d). If $(i, j), (k, h) \in \Upsilon(P)$, then (k, h) is inside (i, j) if and only if $k \geq i$ and $j \leq h$ (using $1 < 2 < \cdots < m$ and $m+1 < \cdots < m+n$ in P). It follows that the inversion $(i, j) \in \Upsilon(P)$ corresponds to a descent in w if and only if (i, j) is a corner square.

Now let (i, j) be a corner square corresponding to the descent r of w, i.e., $i = w_r > w_{r+1} = j$. The elements of P less than j and preceding j in w are $1, 2, \dots, j-1$. The elements of P greater than j and preceding j in w are $m+1, m+2, \dots, i$. Thus the total number of elements preceding j is $i+j-m-1 = e(i,j)$, completing the proof. □

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